Use q_{ij} to denote the (i,j)-entry of the matrix Q $(i,j=1,2)$. Then $\Omega(Q)$ reads as

$$\Omega(Q)=\begin{bmatrix} -(1+\sqrt{-5})r-2 & -(2-\sqrt{-5})r+(1+\sqrt{-5}) \\ 2r+(1-\sqrt{-5}) & -(1+\sqrt{-5})r-2 \end{bmatrix} \quad (2)$$

where

$$r=(-1+\sqrt{-5})q_{11}-(2+\sqrt{-5})q_{12}+2q_{21}+(-1+\sqrt{-5})q_{22}. \quad (3)$$

Because any stabilizing controller can be obtained from the (2, 1)- and (2, 2)-entries of (2), we are interested in the set of r's of (3), that is, the ideal generated by $(-1+\sqrt{-5}), -(2+\sqrt{-5})$, and 2. By letting $q_{11}=q_{22}=0, q_{12}=\sqrt{-5}, q_{21}=-2+\sqrt{-5}$, we see that r is equal to 1. This implies that the set of r's of (3) is identical to $\mathcal{A}$. Hence, the parameterization of stabilizing controllers of p is given as

$$\frac{2r+(1-\sqrt{-5})}{-(1+\sqrt{-5})r-2} \quad (4)$$

where r is a parameter of $\mathcal{A}$. The denominator of (4) cannot be zero because in this case, $r=-(1/3)(1-\sqrt{-5})$, which is not in $\mathcal{A}$.

The form of (4) is similar to the Youla–Kučera parameterization. However, we note that as polynomials of r, the numerator and the denominator of the fraction have constant terms that are not coprime, and the same is the case for the coefficients of r.

IV. Conclusion

All stabilizing controllers of the example given in [1] can be parameterized by only one parameter even though it does not have a coprime factorization. From this result, even in the case where there is no doubly coprime factorization, we observe that the controller parameterization may be in the form of the Youla–Kučera parameterization and that the number of parameters may be smaller than $(m+n)^2$.

Based on the result of this note, we need to further investigate under what condition the parameterization adopts the form of the Youla-Kučera parameterization as in (4). Moreover, the minimum number of parameters should be explained. So far we know that the number of parameters is less than or equal to $(m+n)^2$ [4], but we do not yet have a method of determining the minimum number.

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Constrained Traffic Equilibrium in Routing

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***Abstract*—We study noncooperative routing in which each user is faced with a multicriterion optimization problem, formulated as the minimization of one criterion subject to constraints on others. We address the questions of existence and uniqueness of equilibrium. We show that equilibria indeed exist but uniqueness may be destroyed due to the multicriteria nature of the problem. We obtain uniqueness in some weaker sense under appropriate conditions: we show that the link utilizations are uniquely determined at equilibrium. We further study the normalized constrained equilibrium and apply it to pricing.**

***Index Terms*—Nash equilibria, networking games, pricing, quality of service (QoS), routing.**

I. Introduction

The current Internet routing is based on a single metric, related to the delay or distance between source and destination. Consequently, routing algorithms are used to choose routes for packets so as to minimize the number of hops. For real-time traffic, however, an application may have several criteria for quality of service. It might be sensitive to delays, to losses, or it might seek to minimize some cost imposed on the use of network resources. In the presence of several users that determine individually the routes for flows they control, each with several objectives, this gives rise to a noncooperative multicriteria game. Quality of service is often given through a bound on some performance measure (delay, loss rate, or jitter; see, e.g., [3]). In this note, we consider such cases which can be expressed as constraints on the load at each link. In many cases, performance measures are monotone in the link load, which then implies that bounds on these measures are obtained by bounding the link load. Such problems are known as games where the users strategy sets are not independent but coupled. Games of this kind are called coupled constraint games [9] and the constrained (or the coupled) Nash equilibrium is the corresponding solution concept.

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One could try to get rid of the constraints by incorporating them into the main objective cost function: that function would become infinite when the constraints are violated. Yet except for very special cases this will give rise to noncontinuous costs; moreover, the left limit of the cost as we approach the boundary might be finite whereas the value at the boundary is infinite. This discontinuity makes the uniqueness results for routing into parallel links from [8] inapplicable here, and we present an example that shows that there may be several equilibria.

Our first objective is to study the existence and uniqueness of coupled Nash equilibria. We show in a simple example of parallel links that there may be several such equilibria. In the same example, in absence of side constraints there would be a single equilibrium [8]. We establish existence results for a general topology, obtain a weak uniqueness result for the parallel link topology, and show that although the equilibrium may not be unique, the links' utilization at equilibrium is unique under some conditions. Our second objective is to study the normalized Nash equilibrium, a subclass of all equilibria which has attractive properties for pricing. The uniqueness of this equilibrium notion has been established in [9] for cases in which constrained Nash equilibria were not unique, under conditions that turn out not to hold in general in our setting (see [8]). In spite of that, we establish the uniqueness of the normalized Nash equilibrium for the case of parallel links, and we study its properties. We use some properties of this equilibrium to design an appealing pricing mechanism that would enforce a unique Nash equilibrium.

The structure of the note is as follows. In the next section we introduce the model and assumptions. In Section III, we establish existence of coupled Nash equilibrium and normalized equilibrium for general topology and motivate its use for decentralized pricing. In Section IV, we study the uniqueness of equilibria in the parallel link topology. In Section V, we study the uniqueness of normalized Nash equilibrium, and the last section concludes this note.

II. Model

Consider a network $\mathcal{G} = (\mathcal{N}, \mathcal{L})$ where $\mathcal{N}$ is a finite set of nodes and $\mathcal{L}$ is a set of links. We consider an extension of directional links (see [2]) where a link may carry traffic in both directions, but the direction for each user is fixed. We are given a set $\mathcal{I} = \{1, 2, \ldots, I\}$ of users sharing the network $\mathcal{G}$. With each user i, we associate a unique source $s(i)$ and destination $d(i)$, and a throughput demand r^i. Let f_l^i denote the amount of flow that user i sends over link l, which is constrained to be nonnegative, satisfy the flow conservation law, i.e., for each node v

$$f_l^i \geq 0, \qquad \sum_{l \in \mathrm{Out}(v,i)} f_l^i = \sum_{l \in \mathrm{In}(v,i)} f_l^i + r_v^i \tag{1}$$

where $\mathrm{Out}(v,i)$ is the set of outgoing links from node v available to user i, and $\mathrm{In}(v,i)$ is the corresponding set of links in-going to node v, $r_{s(i)}^i = r^i$, $r_{d(i)}^i = -r^i$, and $r_v^i = 0$ for $v \notin \{s(i), d(i)\}$. Further define $\mathbf{f_l} := \{f_l^1, \ldots, f_l^I\}$, $f_l := \sum_{i=1}^I f_l^i$, $\mathbf{f^i} := \{f_l^i\}_{l \in \mathcal{L}}$, $\mathbf{f}^{-i} = \{\mathbf{f}^1, \ldots, \mathbf{f}^{i-1}, \mathbf{f}^{i+1}, \ldots, \mathbf{f}^I\}$, $\mathbf{f} := \{\mathbf{f_l}\}_{l \in \mathcal{L}}$. We consider a situation where extra side constraints are imposed. These may represent constraints on quality of service which may be user dependent, and are given by

$$g_k(\mathbf{f}) \leq 0, \qquad k \in \mathcal{K} \tag{2}$$

where $\mathcal{K}$ is a finite index (e.g., formed by subsets of $\mathcal{I}, \mathcal{N}, \mathcal{L}$), and g_k: $\mathbb{R}_+^{|\mathcal{L}| \times I} \rightarrow \mathbb{R}$, $k \in \mathcal{K}$. Introduce the function h: $\mathbb{R}_+^{|\mathcal{L}| \times I} \rightarrow \mathbb{R}^m$ to describe the constraints (1)–(2), where m is the number of constraints. Hence, the allowed strategy will be limited by the requirement that $\mathbf{f}$ be selected from a set $\mathcal{R}$, where $\mathcal{R} = \{\mathbf{f}, h(\mathbf{f}) \leq 0\}$. We will say that $\mathcal{R}$ is a coupled constraint set.

The performance objective of user i is quantified by means of a cost function $J^i(\mathbf{f})$. User i aims to find a strategy $\mathbf{f}^i$ that minimizes her/his cost. This optimization depends on the routing decisions of the other users, described by the strategy profile $\mathbf{f}^{-i}$, since J^i is a function of the system flow configuration $\mathbf{f}$, and the constraints (2) are coupled.

Definition II.1: (Cost Functions and Nash Equilibrium): Let $J^i(\mathbf{f})$ be the cost for user i when the flows of all users are given by $\mathbf{f} \in \mathcal{R}$. A coupled Nash equilibrium of the routing game is a strategy profile from which no user finds it beneficial to unilaterally deviate. Thus, we seek for a coupled Nash equilibrium (CNE) $\tilde{\mathbf{f}}$, that is an $\tilde{\mathbf{f}} \in \mathcal{R}$ satisfying

$$\begin{aligned} J^i(\tilde{\mathbf{f}}) &= \min_{(\mathbf{f^i}, \tilde{\mathbf{f}}^{-\mathbf{i}}) \in \mathcal{R}} J^i(\tilde{\mathbf{f}}^{-i}, \mathbf{f}^i) \quad \text{where;} \\ J^i(\tilde{\mathbf{f}}^{-i}, \mathbf{f}^i) &:= J^i(\tilde{\mathbf{f}}^{\mathbf{1}}, \ldots, \tilde{\mathbf{f}}^{\mathbf{i-1}}, \mathbf{f}^{\mathbf{i}}, \tilde{\mathbf{f}}^{\mathbf{i+1}}, \ldots \mathbf{f}^{\mathbf{I}}). \end{aligned} \tag{3}$$

We make the following assumptions on the cost function J^i for all users $i \in \mathcal{I}$.

- **G1** J^i is given as the sum of link costs $J_l^i(\mathbf{f}_l)$: $J^i(\mathbf{f}) = \sum_{l \in \mathcal{L}} J_l^i(\mathbf{f}_l)$.
- **G2** J_l^i: $[0, \infty)^I \rightarrow [0, \infty)$ is continuous.
- **G3** J_l^i is convex in f_l^i and g_k is convex in f_l^i.
- **G4** J_l^i are continuously differentiable in f_l^i and g_k are continuously differentiable in f_l^i. We set $K_l^i := \partial J_l^i(\mathbf{f}_l)/\partial f_l^i, l \in \mathcal{L}$.
- **G5** The feasible set of (1) and (2) is nonempty. Moreover, for any user i and any strategy of the other users, the set of feasible strategies for player i contains a point that is strictly interior to every nonlinear constraint.

Functions that comply with the aforementioned assumptions shall be referred to as *type*-$\mathbf{G}$ functions.

We will use the following set of assumptions (only slightly different from those in [8, p. 512] for all users $i \in \mathcal{I}$).

- **A1** Assumptions $\mathbf{G}$ are satisfied, and J_l^i depends on the vector $\mathbf{f}_l$ only through user i's flow on link l and the total flow on that link. In other words, it can be written (with some abuse of notation) as $J_l^i(\mathbf{f_l}) = J_l^i(f_l^i, f_l)$.
- **A2** J_l^i is increasing in each of its arguments.
- **A3** Viewing $K_\ell^i = K_\ell^i(f_\ell^i, f_\ell)$ now as a function of two arguments, whenever J_l^i is finite, $K_l^i(f_l^i, f_l), l \in \mathcal{L}$, is increasing in each of its two arguments, and strictly increasing in the first one.

As in [8], we refer to functions that comply with these three assumptions as *type*-$\mathbf{A}$ functions.

Remark II.1: Cost functions used in real networks are either related to actual pricing, or they are related to some performance measure such as expected delay. In the first case, a frequently used cost is that of linear link costs, i.e., for each user i, $J^i(\mathbf{f}) = \sum_l f_l^i T_l(f_l)$ where $T_l(f_l) = a_l f_l + b_l$ [7]. When the costs represent delays they typically have the same form but with $T_l(f_l) = (c_l - f_l)^{-1} + d_l$. d_l represents the propagation delay related to link l, where as the first term represents queueing delay. This is the delay of an M/M/1 queue operating under the first-input–first-output regime (packets are served at arrival order; see [8]) or of an M/G/1 queue operating under the processor sharing regime. c_l has the interpretation of the queueing capacity. Other more complicated costs can be found in [1].

III. Existence of Equilibria and Pricing

A. Characterization of Equilibria and Normalized Nash Equilibrium

If assumptions $\mathbf{G}$ hold, it follows that the minimization in (3) is equivalent to the following Kuhn–Tucker conditions: for every $i \in \mathcal{I}$, there exists a set of (Lagrange multipliers) $\{\lambda_u^i\}_{u\in\mathcal{N}}$ and $\{\beta_k^i\}_{k\in\mathcal{K}}$ such that, for every link $(u,v)\in\mathcal{L}$

$$K_{uv}^i(\mathbf{f}_{uv})+\lambda_v^i-\lambda_u^i+\sum_{k\in\mathcal{K}}\beta_k^i\frac{\partial g_k(\mathbf{f})}{\partial f_{uv}^i}=0 \quad \text{if } f_{uv}^i>0 \tag{4}$$

$$K_{uv}^i(\mathbf{f}_{uv})+\lambda_v^i-\lambda_u^i+\sum_{k\in\mathcal{K}}\beta_k^i\frac{\partial g_k(\mathbf{f})}{\partial f_{uv}^i}\geq 0 \tag{5}$$

$$\beta_k^i g_k(\mathbf{f})=0, \qquad \beta_k^i\geq 0, \qquad k\in\mathcal{K}. \tag{6}$$

B. Normalized Nash Equilibrium and Pricing

Now, we define a subclass of CNE whose corresponding Lagrange multipliers have some special properties.

Definition III.1: The coupled Nash equilibrium $\mathbf{f}$ is a normalized Nash equilibrium [9] associated with some vector $\vec{\boldsymbol{\alpha}}>\mathbf{0}$ where $\vec{\boldsymbol{\alpha}}=(\alpha^1,\ldots,\alpha^I)$ and where $\mathbf{0}$ is a vector of zeros, if there exist some constants $\beta_k\geq 0$ $k\in\mathcal{K}$ such that (4)–(6) are satisfied where

$$\beta_k^i=\beta_k/\alpha^i, \qquad k\in\mathcal{K}, \qquad i\in\mathcal{I}. \tag{7}$$

Notice that if a user's weight α^i is greater than those of his competitors, then his corresponding Lagrange multipliers are smaller.

The normalized Nash equilibrium can be used in relation to an appealing pricing scheme in which additional congestion costs are imposed by the network. Congestion pricing will allow us to relax the original constraints $g_k(\mathbf{f})\leq 0$; yet the resulting equilibrium will have the following three appealing properties.

1) It will be a CNE for the original problem.
2) Nonzero congestion prices will only be imposed for saturated constraints: such constraints represent congestion, and in absence of congestion, no congestion cost is imposed.
3) The most interesting feature of this pricing is that congestion costs may be chosen to be user independent. This allows us to implement them in a decentralized way without requesting a per-flow information.

More precisely, assume that the utility of user i can be written as $-J^i(\mathbf{f})-(1/a^i)\sum_{k\in\mathcal{K}}C_k(\mathbf{f})$. $C_k(\mathbf{f})$ is a cost function that user i is charged due to congestion related to the kth constraint. Let $(\beta_l^i)^*$ be Lagrange multipliers that correspond to a CNE induced by taking in (7) $\vec{\boldsymbol{\alpha}}=(a^1,\ldots,a^I)$. Let β_l^* be defined as in (7). We set $C_k(\mathbf{f})=\beta_k^*\cdot g(\mathbf{f}_l)$. With this cost function we may now consider a competitive routing problem in which we ignore constraints (2). The obtained equilibrium is a CNE for the original constrained model, and the complementary slackness conditions imply that at the normalized equilibrium, no user actually pays any congestion cost. Under various conditions, there is a unique Nash Equilibrium [8] to the pricing game (where constraints (2) are removed) and the corresponding Kuhn–Tucker conditions obviously coincide with our original ones. We conclude that a simple pricing can replace the quality of service (QoS) constraints and yet force users to choose a CNE (so the constraints still hold). Since the pricing does not depend on the user (except for a multiplicative constant α_i which can be chosen to be the same for all users), the charging can be performed in a distributed way without need for per flow information. The existence of an equilibrium induced by such a pricing is, thus, equivalent to the existence of a normalized Nash equilibrium.

C. Existence of Equilibria

Assumption **G5** is a sufficient condition for the Kuhn–Tucker constraint qualification [4]. Hence, the routing game (3) using the cost functions of *type*-$\mathbf{G}$ is equivalent to a convex game in the sense of [9] and, thus the existence of a CNE as well as a normalized equilibrium is guaranteed [9, Th. 1].

Theorem III.1: Consider the cost function of *type*-$\mathbf{G}$. There exists a normalized Nash equilibrium point for every specified vector $\vec{\boldsymbol{\alpha}}>\mathbf{0}$ (componentwise) where $\vec{\alpha}=(\alpha^1,\alpha^2,\ldots,\alpha^I)$.

IV. CNE in a Parallel Links Topology

In this section, we present an example which shows that uniqueness of the CNE fails despite of condition $\mathbf{A}$. We then establish the uniqueness of the links' utilization in a parallel link topology. For the parallel links, we impose a quality of service constraint on each link of the form (2), where $\mathcal{K}=\mathcal{L}$ and g_l depends on the flows only through total flow on link l. Hence, the constraints can be written as $g_l(f_l)\leq 0$. g_l is assumed to be strictly increasing in f_l. Hence, g_l^{-1} exists and the above constraints becomes $f_l\leq d_l$, where $d_l=g^{-1}(0)$ (positive real). Note that now all constraints in (1) and (2) are linear so that condition **G5** becomes trivial.

Example of Nonuniqueness of Nash Equilibrium: Consider a network of two parallel links connecting a common source node to a common destination node. Link 1 has a capacity constraint of 1 and link 2 has capacity constraint of 10. There are two users, each with a throughput demand $r^1=r^2=1$ between source node and destination. Let $J^i, i=1,2$ be the cost function of user i such that $J^i(\mathbf{f})=\sum_{l=1}^2 f_l^i T_l(f_l)$ where $T_1(f_1)=f_1$ and $T_2(f_2)=f_2+10$ (T_l is the "link cost").

One first Nash equilibrium is: For the first user: $f_1^1=1, f_2^1=0$, for the second user: $f_1^2=0, f_2^2=1$. Another one is $f_1^1=0, f_2^1=1$ for the first user, and for the second user: $f_1^2=1, f_2^2=0$. In fact, any convex combination of the equilibria is also an equilibrium. We note that the costs we chose are of *type*-$\mathbf{A}$ and thus in absence of the side constraint there would be a single Nash equilibrium [8].

The same result is obtained if the link costs are replaced with the M/M/1 type costs plus some constants. For example, $T_1(f_1)=(2-f_1)^{-1}$ and $T_2(f_2)=(2-f_1)^{-1}+10$ give the same multiple equilibria. ■

Uniqueness of the Utilization at Nash Equilibrium: The following result establishes under some conditions the uniqueness of the total link flows for a parallel link topology: several parallel links connect two nodes: 1 and 2.

Theorem IV.1: Consider the cost functions of *type*-$\mathbf{A}$. Let $\mathbf{f}$ and $\hat{\mathbf{f}}$ be two coupled Nash equilibria. Let $\{\beta_l^i\}$ and $\{\hat{\beta}_l^i\}$ be corresponding Lagrange multipliers. Assume that for each link $l\in\mathcal{L}, \hat{\beta}_l^i\leq\beta_l^i, \forall i\in\mathcal{I}$ or $\hat{\beta}_l^i\geq\beta_l^i, \forall i\in\mathcal{I}$. Then, the link utilizations are the same under $\mathbf{f}$ and $\hat{\mathbf{f}}$.

Proof: Let $\mathbf{f}$ and $\hat{\mathbf{f}}$ be two CNEs. Then, we have from (4)

$$K_l^i\left(f_l^i,f_l\right)+\beta_l^i\geq\lambda^i; \quad K_l^i\left(f_l^i,f_l\right)+\beta_l^i=\lambda^i \quad \text{if } f_l^i>0 \tag{8}$$

$$K_l^i(\hat{f}_l^i,\hat{f}_l)+\hat{\beta}_l^i\geq\hat{\lambda}^i; \quad K_l^i(\hat{f}_l^i,\hat{f}_l)+\hat{\beta}_l^i=\hat{\lambda}^i \quad \text{if } \hat{f}_l^i>0 \tag{9}$$

where λ^i represents $\lambda_{s(i)}^i-\lambda_{d(i)}^i$, and $s(i),d(i)\in\{1,2\}$. We begin to show the following relations:

i) $\{\hat{\beta}_l^i<\beta_l^i, \hat{f}_l\geq f_l\}\Rightarrow\hat{f}_l=f_l$ moreover if $\hat{\lambda}^i\geq\lambda^i$ then $\hat{f}_l^i\geq f_l^i$ and the last inequality is strict if $\hat{f}_l^i>0$;
ii) $\{\hat{\beta}_l^i>\beta_l^i, \hat{f}_l\leq f_l\}\Rightarrow\hat{f}_l=f_l$ moreover if $\lambda^i\geq\hat{\lambda}^i$ then $\hat{f}_l^i\leq f_l^i$ and the last inequality is strict if $f_l^i>0$;
iii) $\{\hat{\lambda}^i\leq\lambda^i, \hat{\beta}_l^i\geq\beta_l^i, \hat{f}_l\geq f_l\}\Rightarrow\hat{f}_l^i\leq f_l^i$;
iv) $\{\hat{\lambda}^i\geq\lambda^i, \hat{\beta}_l^i\leq\beta_l^i, \hat{f}_l\leq f_l\}\Rightarrow\hat{f}_l^i\geq f_l^i$.

We will show only i) and iii), since ii) and iv) are symmetric. Assume that $\hat{\beta}_l^i<\beta_l^i$ and $\hat{f}_l\geq f_l$. Note that the last inequality implies that

$$g_l(\hat{f}_l)\geq g_l(f_l). \tag{10}$$

In other words, since $\hat{\beta}_l^i < \beta_l^i$ then from (6), $g_l(f_l) = 0$ and from (2) $g_l(\hat{f}_l) \leq 0$, thus $g_l(f_l) \geq g_l(\hat{f}_l)$, it follows by (10) that $g_l(f_l) = g_l(\hat{f}_l)$ and $f_l = \hat{f}_l$. Moreover if $\hat{\lambda}^i \geq \lambda^i$ then (i) holds trivially if $f_l^i = 0$. Otherwise, if $f_l^i > 0$, then (8) and (9) together with our assumptions imply that

$$K_l^i\left(f_l^i, f_l\right) + \beta_l^i = \lambda^i \leq \hat{\lambda}^i \leq K_l^i\left(\hat{f}_l^i, \hat{f}_l\right) + \hat{\beta}_l^i < K_l^i\left(\hat{f}_l^i, f_l\right) + \beta_l^i$$

where the last inequality follows from the monotonicity of K_l^i in its second argument and $\hat{\beta}_l^i < \beta_l^i$. Thus $K_l^i(f_l^i, f_l) \leq K_l^i(\hat{f}_l^i, f_l)$. Now, since K_l^i is nondecreasing in its first argument, this implies that $\hat{f}_l^i > f_l^i$. This establishes i).

Now, we assume that $\hat{\lambda}^i \leq \lambda^i, \hat{\beta}_l^i \geq \beta_l^i$ and $\hat{f}_l \geq f_l$. Note that iii) holds trivially if $\hat{f}_l^i = 0$. Otherwise, if $\hat{f}_l^i > 0$, then (8) and (9) together with our assumptions imply that

$$K_l^i\left(\hat{f}_l^i, \hat{f}_l\right) + \hat{\beta}_l^i = \hat{\lambda}^i \leq \lambda^i \leq K_l^i\left(f_l^i, f_l\right) + \beta_l^i \leq K_l^i\left(f_l^i, \hat{f}_l\right) + \hat{\beta}_l^i.$$

where the last inequality follows from the monotonicity of K_l^i in its second argument and $\beta_l^i \leq \hat{\beta}_l^i$. Thus $K_l^i(f_l^i, f_l) \leq K_l^i(\hat{f}_l^i, \hat{f}_l)$. Now, since K_l^i is nondecreasing in its first argument, this implies that $\hat{f}_l^i \leq f_l^i$, and iii) is established.

Let $\mathcal{L}_1 = \{l\colon \hat{f}_l > f_l\}$. Also, denote $\mathcal{I}_1 = \{i\colon \hat{\lambda}^i > \lambda^i\}, \mathcal{L}_2 = \{l\colon \hat{f}_l \leq f_l; \hat{\beta}_l^i \leq \beta_l^i\}$ and $\mathcal{L}_3 = \{l\colon \hat{f}_l \leq f_l; \hat{\beta}_l^i > \beta_l^i\}$. We observe that $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$. Assume that $\mathcal{L}_1 \neq \emptyset$, it follows by iv) that for $i \in \mathcal{I}_1$

$$\begin{aligned}\sum_{l\in\mathcal{L}_1} \hat{f}_l^i &= r^i - \sum_{l\in\mathcal{L}_2} \hat{f}_l^i - \sum_{l\in\mathcal{L}_3} \hat{f}_l^i \\ &\leq r^i - \sum_{l\in\mathcal{L}_2} f_l^i - \sum_{l\in\mathcal{L}_3} \hat{f}_l^i \leq \sum_{l\in\mathcal{L}_1} f_l^i + \sum_{l\in\mathcal{L}_3}\left(f_l^i - \hat{f}_l^i\right).\end{aligned}$$

Noting that i) implies that $\{l \in \mathcal{L}_1/\hat{\beta}_l^i < \beta_l^i\} = \emptyset$, hence, iii) implies that $\hat{f}_l^i \leq f_l^i$ for $l \in \mathcal{L}_1$ and $i \notin \mathcal{I}$, it follows that

$$\begin{aligned}\sum_{l\in\mathcal{L}_1} \hat{f}_l &= \sum_{l\in\mathcal{L}_1}\sum_{i\in\mathcal{I}_1} \hat{f}_l^i + \sum_{l\in\mathcal{L}_1}\sum_{i\notin\mathcal{I}_1} \hat{f}_l^i \\ &\leq \sum_{l\in\mathcal{L}_1}\sum_{i\in\mathcal{I}_1} f_l^i + \sum_{l\in\mathcal{L}_3}\sum_{i\in\mathcal{I}_1}\left(f_l^i - \hat{f}_l^i\right) + \sum_{l\in\mathcal{L}_1}\sum_{i\notin\mathcal{I}_1} f_l^i \\ &= \sum_{l\in\mathcal{L}_1} f_l + \sum_{l\in\mathcal{L}_3}\sum_{i\in\mathcal{I}_1}\left(f_l^i - \hat{f}_l^i\right) \\ &= \sum_{l\in\mathcal{L}_1} f_l + \sum_{l\in\mathcal{L}_3}(f_l - \hat{f}_l) - \sum_{l\in\mathcal{L}_3}\sum_{i\notin\mathcal{I}_1}\left(f_l^i - \hat{f}_l^i\right) \\ &< \sum_{l\in\mathcal{L}_1} f_l.\end{aligned}$$

The last inequality follows from ii), since for $l \in \mathcal{L}_3, \hat{f}_l = f_l$ and for $l \in \mathcal{L}_3$ and $i \notin \mathcal{I}_1 f_l^i > \hat{f}_l^i$.

The inequality (11) obviously our definition of $\mathcal{L}_1$, which implies that $\mathcal{L}_1$ is an empty set. By symmetry, it may also be concluded that the set $\{l\colon \hat{f}_l < f_l\}$ is empty. Thus, has been established that: $\hat{f}_l = f_l, \forall l \in \mathcal{L}$. ■

V. NORMALIZED NASH EQUILIBRIUM

A. Uniqueness of the Normalized Nash Equilibrium

The following result establishes the uniqueness of normalized Nash equilibrium for every given $\vec{\boldsymbol{\alpha}} > \mathbf{0}$. We note that the normalized Nash equilibria for a specified $\vec{\boldsymbol{\alpha}} > \mathbf{0}$ complies with the conditions of the last theorem. Thus, we have the following.

Theorem V.1: In a network of parallel links where the cost function of each user is of *type*-$\mathbf{A}$, the normalized Nash equilibrium for every specified $\vec{\alpha} > 0$ is unique.

Proof: Assume that for some $\vec{\boldsymbol{\alpha}} > \mathbf{0}$ we have two normalized equilibrium points $\hat{\mathbf{f}}$ and $\mathbf{f}$. Then, we have from (4), for all $(i, l) \in \mathcal{I} \times \mathcal{L}$

$$\begin{aligned}&\alpha^i K_l^i\left(f_l^i, f_l\right) + \beta_l \geq \alpha^i \lambda^i; \\ &\alpha^i K_l^i\left(f_l^i, f_l\right) + \beta_l^i = \alpha^i \lambda^i, \qquad \text{if } f_l^i > 0 \\ &\alpha^i K_l^i\left(\hat{f}_l^i, \hat{f}_l\right) + \hat{\beta}_l \geq \alpha^i \hat{\lambda}^i; \\ &\alpha^i K_l^i\left(\hat{f}_l^i, \hat{f}_l\right) + \hat{\beta}_l = \alpha^i \hat{\lambda}^i, \qquad \text{if } \hat{f}_l^i > 0.\end{aligned}$$

By contradiction, assume that there exists $(l_0, i) \in \mathcal{L} \times \mathcal{I}$ such that $\hat{f}_{l_0}^i \neq f_{l_0}^i$ and without loss of generality assume that $\hat{f}_{l_0}^i < f_{l_0}^i$. Hence, the last equations with our assumptions imply that

$$\begin{aligned}\alpha^i \lambda^i &= \alpha^i K_{l_0}^i\left(f_{l_0}^i, f_{l_0}\right) + \beta_{l_0} > \alpha^i K_{l_0}^i\left(\hat{f}_{l_0}^i, \hat{f}_{l_0}\right) + \beta_{l_0} \\ &\geq \alpha^i \hat{\lambda}^i + \beta_{l_0} - \hat{\beta}_{l_0}.\end{aligned} \tag{11}$$

Since $\sum_{l\in\mathcal{L}} \hat{f}_l^i = \sum_{l\in\mathcal{L}} f_l^i = r^i$, then there exist a link $l_1 \in \mathcal{L}$ such that $\hat{f}_{l_1}^i > f_{l_1}^i$. Similarly, we have

$$\begin{aligned}\alpha^i \hat{\lambda}^i &= \alpha^i K_{l_1}^i\left(\hat{f}_{l_1}^i, \hat{f}_{l_1}\right) + \hat{\beta}_{l_1} > \alpha^i K_{l_1}^i\left(f_{l_1}^i, f_{l_1}\right) + \hat{\beta}_{l_1} \\ &\geq \alpha^i \lambda^i + \hat{\beta}_{l_1} - \beta_{l_1}.\end{aligned} \tag{12}$$

Summing (11) and (12), we get $\beta_{l_1} - \hat{\beta}_{l_1} > \beta_{l_0} - \hat{\beta}_{l_0}$. Since $\hat{f}_{l_1} = f_{l_1}$ and $\hat{f}_{l_1}^i > f_{l_1}^i$, then there exists a user $j \in \mathcal{I}$ such that $\hat{f}_{l_1}^j < f_{l_1}^j$. By the same procedure we show that there exists $l_2 \in \mathcal{L}$ such that $\beta_{l_2} - \hat{\beta}_{l_2} > \beta_{l_1} - \hat{\beta}_{l_1}$. Proceeding inductively, we construct a monotonically increasing sequence $(\gamma_n)_{n\in\mathbb{N}}$ where $\gamma_n = \beta_{l_n} - \hat{\beta}_{l_n}$, we have a contradiction since the set of links is finite. We therefore conclude that $\hat{\mathbf{f}} = \mathbf{f}$. ■

B. Properties of the Normalized Nash Equilibrium

We will now investigate the dependence of the normalized equilibrium point on the value of vector $\vec{\alpha}$. We will show that in a certain sense the equilibrium value of J^i is a monotone decreasing function of α^i.

Theorem V.2: Consider the cost function of type-$\mathbf{A}$ and two weights $\vec{\alpha}$ and $\vec{\gamma}$ such that $\alpha^j = \gamma^j, j \neq i$, and $\tilde{\alpha}^i > \alpha^i$ for some i. Let $\mathbf{f}$ and $\tilde{\mathbf{f}}$, with $\mathbf{f} \neq \tilde{\mathbf{f}}$ be the corresponding unique normalized Nash equilibria. Then, the directional derivative of J^i along the ray $(\tilde{\mathbf{f}}^{\mathbf{i}} - \mathbf{f}^{\mathbf{i}})$ is negative.

Proof: By using the Kuhn–Tucker conditions corresponding to the normalized equilibrium $\mathbf{f}$, we have

$$(\alpha^j - \gamma^j)K_l^j\left(f_l^j, f_l\right) + \gamma^j K_l^j\left(f_l^j, f_l\right) + \beta_l \geq \alpha^j \lambda^j \tag{13}$$

$$(\alpha^j - \gamma^j)K_l^j\left(f_l^j, f_l\right) + \gamma^j K_l^j\left(f_l^j, f_l\right) + \beta_l = \alpha^j \lambda^j \quad \text{if } f_l^j > 0. \tag{14}$$

Multiplying by $(\tilde{f}_l^j - f_l^j)$, we obtain

$$(\alpha^j - \gamma^j)\left(\tilde{f}_l^j - f_l^j\right)K_l^j\left(f_l^j, f_l\right) + \gamma^j\left(\tilde{f}_l^j - f_l^j\right)K_l^j \times \left(f_l^j, f_l\right) + \left(\tilde{f}_l^j - f_l^j\right)\beta_l \geq \alpha^j\left(\tilde{f}_l^j - f_l^j\right)\lambda^j. \tag{15}$$

Indeed, (15) holds trivially if $f_l^j > 0$. Otherwise, if $f_l^j = 0$, then $(\tilde{f}_l^i - f_l^j)$ is nonnegative and by multiplying (14) by $\tilde{f}_l^i - f_l^j$ we obtain (15). Now, multiplying the Kuhn–Tucker conditions corresponding to the normalized equilibrium $\tilde{\mathbf{f}}$ by $(f_l^j - \tilde{f}_l^j)$, we obtain similarly

$$\gamma^j\left(f_l^j - \tilde{f}_l^j\right)K_l^j\left(\tilde{f}_l^j, f_l\right) + \left(f_l^j - f_l^j\right)\beta_l \geq \gamma^j\left(f_l^j - \tilde{f}_l^j\right)\tilde{\lambda}^j. \tag{16}$$

Summing (15) and (16), we now get

$$
\begin{aligned}
&(\alpha^j - \gamma^j)\left(\tilde{f}_l^j - f_l^j\right) K_l^j\left(f_l^j, f_l\right) \\
&\quad + \gamma^j \left(\tilde{f}_l^j - f_l^j\right)\left(K_l^j\left(f_l^j, f_l\right) - K_l^j\left(\tilde{f}_l^j, f_l\right)\right) \\
&\quad + \left(\tilde{f}_l^j - f_l^j\right)(\beta_l - \tilde{\beta}_l) \geq \left(\tilde{f}_l^j - f_l^j\right)(\alpha^j \lambda^j - \gamma^j \tilde{\lambda}^j).
\end{aligned}
$$

Since K_l^j is strictly increasing in the first argument, then (17) yields

$$
\begin{aligned}
(\alpha^j - \gamma^j)\left(\tilde{f}_l^j - f_l^j\right) K_l^j\left(f_l^j, f_l\right) + \left(\tilde{f}_l^j - f_l^j\right)(\beta_l - \tilde{\beta}_l) \\
> \left(\tilde{f}_l^j - f_l^j\right)(\alpha^j \lambda^j - \gamma^j \tilde{\lambda}^j). \quad (17)
\end{aligned}
$$

Recall that $\sum_{l \in \mathcal{L}}(\tilde{f}_l^j - f_l^j) = r^j - r^j = 0$. Thus, by summing up over $j \in \mathcal{I}$ and over $l \in \mathcal{L}$, we obtain

$$
\begin{aligned}
\sum_{j \in \mathcal{I}} \sum_{l \in \mathcal{L}} (\alpha^j - \gamma^j)\left(\tilde{f}_l^j - f_l^j\right) K_l^j\left(f_l^j, f_l\right) \\
+ \sum_{l \in L} (\tilde{f}_l - f_l)(\beta_l - \tilde{\beta}_l) > 0. \quad (18)
\end{aligned}
$$

Moreover, if $\tilde{f}_l > f_l (\mathrm{resp}\, \tilde{f}_l < f_l)$ then $\tilde{\beta}_l \geq \beta_l (\mathrm{resp}\, \tilde{\beta}_l \leq \beta_l)$, it follows that the second term of (18) is nonpositive, hence, we have $\sum_{j \in \mathcal{I}} \sum_{l \in \mathcal{L}} (\alpha^j - \gamma^j)(\tilde{f}_l^j - f_l^j) K_l^j(f_l^j, f_l) > 0$. Since $\alpha^j = \gamma^j$ for $j \neq i$ and $\alpha^i < \gamma^i$, the last inequality yields

$$
\sum_{l \in \mathcal{L}} \left(\tilde{f}_l^i - f_l^i\right) K_l^i\left(f_l^i, f_l\right) < 0. \quad (19)
$$

This is exactly the directional derivative of J^i along the ray $(\tilde{\mathbf{f}}^{\mathbf{i}} - \mathbf{f}^{\mathbf{i}})$. ■

An interpretation of Theorem V.2 is obtained if $\|\tilde{\mathbf{f}}^i - \mathbf{f}^i\|$ is sufficiently small, then it follows from (19) that $\sum_{l \in \mathcal{L}} J_l^i(\tilde{f}_l^i, \tilde{f}_l + \tilde{f}_l^i - f_l^i) < J^i(\mathbf{f})$ and since J^i is continuous then for $\|\tilde{\mathbf{f}}^{\mathbf{i}} - \mathbf{f}^{\mathbf{i}}\|$ sufficiently small we have $J^i(\tilde{\mathbf{f}}) < J^i(\mathbf{f})$.

In the sequel, we assume throughout that the costs are of *type*-$\mathbf{A}$, and that furthermore, *the cost functions and weights of all users are the same*, i.e., $K_l^i = K_l$ and $\alpha^i = \alpha, \forall i \in \mathcal{I}$ where α is positive real. For simplicity of notation and without loss of generality, we assume that $\alpha = 1$.

Lemma V.1: Assume that $f_{\hat{l}}^i > f_{\hat{l}}^j$ holds for some link $\hat{l}$ and users i and j. Then $f_l^i \geq f_l^j$ for all $l \in \mathcal{L}$; moreover, the last inequality is strict if $f_l^j > 0$.

Proof: Choose an arbitrary link l. If $f_l^j = 0$ then the implication is trivial. Otherwise, if $f_l^j > 0$. From the Kuhn–Tucker conditions, we have that $\lambda^j = \beta_l + K_l(f_l^j, f_l) \leq \beta_{\hat{l}} + K_{\hat{l}}(f_{\hat{l}}^j, f_{\hat{l}})$ and, since $f_{\hat{l}}^i > f_{\hat{l}}^j$ implies $f_{\hat{l}}^i > 0$, then we have $\lambda^i = \beta_{\hat{l}} + K_{\hat{l}}(f_{\hat{l}}^i, f_{\hat{l}}) \leq \beta_l + K_l(f_l^i, f_l)$. Thus, we have $\beta_l + K_l(f_l^j, f_l) \leq \beta_{\hat{l}} + K_{\hat{l}}(f_{\hat{l}}^j, f_{\hat{l}}) < \beta_{\hat{l}} + K_{\hat{l}}(f_{\hat{l}}^i, f_{\hat{l}}) \leq \beta_l + K_l(f_l^i, f_l)$, i.e., $K_l(f_l^j, f_l) < K_l(f_l^i, f_l)$ which implies $f_l^j < f_l^i$. ■

Theorem V.3: Consider the identical *type*-$\mathbf{A}$ cost functions. Assume that $r^i > r^j$. Then $f_l^i \geq f_l^j$ for all link $l \in \mathcal{L}$ and we have strict inequality of all links used by user i. Moreover, if $r^i = r^j$ then $f_l^i = f_l^j$ for all $l \in \mathcal{L}$.

Proof: Follows directly from Lemma V.1. ■

VI. Concluding Comments

We have considered in this note Nash equilibria arising in networks with additional side constraints (CNEs). We have first shown that the extra constraints may cause multiple equilibria in scenarios in which a single equilibrium would exist in their absence. We then advocated the use of the more refined equilibrium concept of normalized Nash equilibrium and showed its usefulness for simple pricing mechanisms. We further showed that it is unique in the parallel link topology.

The Normalized equilibrium is related to some parameters α^i of user i. Suppose that the J^i represent some performance measure, say the delay. If we go back to the pricing interpretation of the normalized Nash equilibrium, we conclude that we can differentiate users by incorporating different α^i's in the congestion pricing scheme; by changing α^i, user i can receive smaller delays at equilibrium. Some insight on how the change of α^i influences the performance follows from Theorem V.2 (for the case of parallel links). The α^i's are chosen by the network provider and could be related to different grades of service offered to users.

In the future, we plan to extend our results to more complex topologies and other forms of constraints.

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