ROUTING GAMES

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How to split the demand of i=1,…,I classes into K parallel links

- x_k^i flow of class i on link k
- Demand constraint: $\sum_{k=1}^{K} x_k^i = \varphi_i$, $x_k^i \geq 0$
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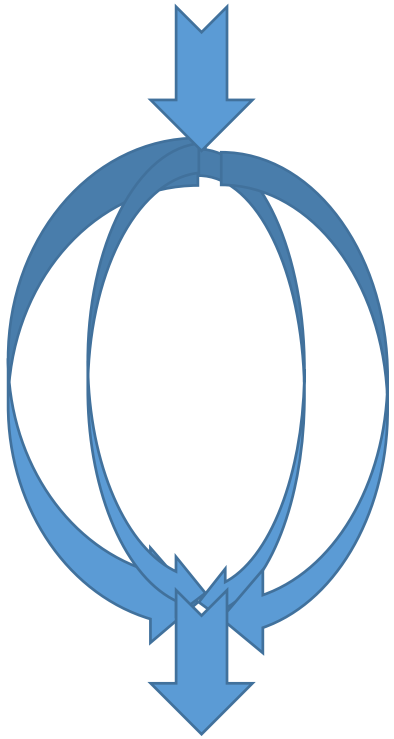
- Total flow $x_k = \sum_{i=1}^{N} x_k^i$ over link k
- Cost $f_k(x_k)$ of link k



Class optimisation and global optimisation

- Cost of class i is the sum of link costs averaged over the demand of that class J(i,x)=$\sum_{k=1}^{K} x_k^i\, f_k(x_k)$

- Minimizing average delay: J(x)=$\sum_{k=1}^{K} x_k\, f_k(x_k)$=:$\sum_{k=1}^{K} J_k(x_k)$

Demand constraints KKT Theorem:

- Single class, J convex.
- Define the Lagrangian L(x, Υ) = J(x) + $\Upsilon(\sum_{k=1}^{K} x_k - \varphi)$
- Then

x* minimizes J(x) s.t. $\sum_{k=1}^{K} x_k = \varphi, \quad x_k \geq 0$ If and only if

there exists some Υ s.t. x* minimizes L(x, Υ) and

$\Upsilon(\sum_{k=1}^{K} x_k - \varphi)$=0

- Proof: $\min_{x\geq 0} \max_{\Upsilon}$ L(x, Υ) = $\min_{x\geq 0}$ J(x) s.t. $\sum_{k=1}^{K} x_k = \varphi$

Now use the maxmin theorem to change order of min and max

KKT conditions on the derivative

- Assume J differentiable. Then

$$\text{x* minimizes} \quad J(x) \quad \text{s.t.} \quad \sum_{k=1}^{K} x_k = \varphi, \qquad x_k \geq 0$$

If and only if

there exists some ϓ s.t. at x=x*,

$$\frac{\partial L(x, \text{ϓ})}{\partial x_k} = \frac{\partial J(x)}{\partial x_k} + \text{ϓ} = 0, \qquad k=1,\ldots,K$$

$$\text{ϓ}\left(\sum_{k=1}^{K} x_k - \varphi\right) = 0$$

Non-atomic game: WARDROP equilibrium

WARDROP EQ. is defined as a class configuration x* satisfying the demand (and conservation) constraints as well as the following conditions:

- All Routes that carry strictly positive flow have the same cost ϓ
- The cost of a route that carries zero flow is at least ϓ

A user here is infinitesimally small. A user cannot gain by unilateral deviation. WE is a NE for a game with a continuum set of players.

Beckmann potential

- By the definition, X* is a Wardrop Eq if

  there exists ϓ s.t. $J_k(x_k) + ϓ \geq 0$, $x_k(J_k(x_k) + ϓ)$=0

If $J_k(x_k)$ were the derivative of some cost then we recognize here the computation of the coresponding global optimal average cost.

Framework

Various frameworks

- **F1** Infinite populations of players, each of which has a negligible impact on the utility of other players [Wardrop 52] (Non-atomic game)
- **F2** Each of finitely many players controls the fraction of its flow over each path. Framework of [ORS 93] (Splittable atomic game)
- **F3** Finitely many players each sending one unit of flow. [Rosenthal 1972] (Atomic, non-splittable)
- **F4** Finitely many players each sending one or more units of flow

- Cost of a path is the sum of costs of links upon the path
- Related frameworks: mixed equilibria, crowding games, delegation, aggregative games…

Wardrop Equilibrium general topology

- Consider a directed graph (V,L) where V are nodes and L are directed links.
- There are K classes of flows. A class is characterized by a source, destination, and demand
- Link costs $f_l\ (x_l)$ depend on the sum of the link flows $x_l = \sum_i \mathrm{x}_l^i$
- For each node v and each player, there are flow conservation constraints involving the inflow and outflow to the node. Thus there is a Lagrange multiplier per node and player
- Each node can be on an input link to a node v and on an output link of a node u. Thus each link (u,v) appears in two node constraints per player

Conservation constraints: Link Routing

Link routing: the incoming traffic at each node can be split among the outgoing links. The flow conservation equations become:

$$r_v^i + \sum_{j \in In(v)} x_j^i = \sum_{j \in Out(v)} x_j^i \tag{4}$$

with:

$$r_v^i = \begin{cases} d_i & , \text{ if } v = s(i) \\ -d_i & , \text{ if } v = d(i) \\ 0 & , \text{ otherwise} \end{cases} \tag{5}$$

Paths and Links

- Path cost: sum of link costs along the path

- Thus for each node u and player i there is a constant λ_u^i $s.t.$ for any link (uv),

  $f_{uv}(x_{uv}) \geq \lambda_u^i - \lambda_v^i$ with equality if $x_{uv}^i > 0$. (*)

  Then x is called an equilibrium

- Choose i and let $p = (u_1, \ldots, u_n)$ be any path between s(i) and d(i). There is a_i such that the path cost is at least a_i with strict equality if all links in p carry positive flows.
- $a_i = \lambda_u - \lambda_v$ where u=d(i) and v=s(i)

Beckmann's potential

- The characterization of the equilibrium

$f_{uv}(x_{uv}) \geq \lambda_u^i - \lambda_v^i$ with equality if $x_{uv}^i > 0$

can be interpreted as the derivative form of KKT conditions related to the global optimization of

$$J(x) = \sum_l \int_0^{x_l} f_l(s)ds$$

Subject to conservation constraints + nonnegativity of flows.

Ex: minimize average link cost: $J(x) = \sum_l x_l f_l(x_l)$

Equivalent to Wardrop with link costs of

$$x_l f_l'(x_l) + f_l(x_l)$$

Cournot Nash equilibrium

Class = player

We assume the following cost structure:

- $J_l^i(\mathbf{x})$ cost of player i on link l
- The cost is additive over links: $J^i(\mathbf{x}) = \sum_l J_l^i(\mathbf{x})$
- There exists a positive, strictly increasing, convex and continously differentiable cost density $t_l(x_l) \geq 0$ such that $J_l^i(x_l^i, x_l) = x_l^i t_l(x_l)$.

Theorem

The Nash equilibrium converges to the Wardrop equilibrium, in the following senses:

- *Let* $\mathbf{x}^m$ *be an equilibrium that corresponds to the replacement of each player* i *by* m *symmetrical copies. Then any limit of a converging subsequence is a Wardrop equilibrium*
- *The Wardrop equilibrium is an* ϵ*-equilibrium for the* m*-th game for all* m *large enough (i.e. no player can gain more than* ϵ *by deviating)*
- *For all* m *large enough, an equilibrium in the* m*-th game is an* ϵ*-Wardrop equilibrium*

Potential in Cournot Nash equilibrium

- Player i minimizes $\sum_l [\, x_l^i \, t_l \, (\, x_l)]$

- The B.R. faced by player i is equivalent to the existence of Lagrange multipliers such that for every (uv),

$t_{uv} \, (\, x_{uv}) + [\, x_{uv}^i \, t_{uv}{}' \, (\, x_{uv})] \geq$ λ(i,u) - λ(i,v)

with strict equality if $x_{uv}^i > 0$

Summing over I gives

$I \; t_{uv} \, (\, x_{uv}) + x_{uv} t_{uv}{}' \, (\, x_{uv}) \geq$ λ(u) - λ(v)

Equivalently

$(I-1)\, t_{uv}\,(\,x_{uv})$ + $t_{uv}\,(\,x_{uv})$ + $x_{uv}\, t_{uv}{}'\,(\,x_{uv})$ ≥ λ(u) - λ(v)

Or

$t_{uv}\,(\,x_{uv})$ + $(I-1)^{-1}[x_{uv} + t_{uv}\,(\,x_{uv})]$' ≥ λ(u) - λ(v)

This looks like the KKT conditions associated with the global minimization of

$\sum_l \int_0^{x_l} t_l(s)ds$ + $(I-1)^{-1}[x_{uv} + t_{uv}\,(\,x_{uv})]$'

- Problem: this was obtained by summing

- $t_{uv}\ (x_{uv})$ +$x^{i}_{uv}\ \ t_{uv}{}'\ (x_{uv})$ ≥ λ(i,u) - λ(i,v)

- But sum of complementarity conditions does not give the complementarity for the sum

- For example, if player 1 sends flow to a link and player 2 does not, the inequality can be strict (due to player 2) and yet the sum of flows is positive.

All positive flow [ORS]: if one player sends flow on a link k then all players send flow on k. Under APF we have a potential (in the sense of Beckman).

Convergence to Wardrop

1. Strict Diagonal Concavity (SDC) framework

- **Rosen (1965):** Consider an n-player game.

  Let G_{ij} be the 2nd order derivative of the utility of player i with respect to the actions of players I and j. If G+G^T is strictly positive definite then the equilibrium is unique

Haurie, Marcott [1986] applied to the case of finitely many players to show convergence of Cournot Nash to Wardrop

For what type of network topology do we have SDC ?

- **Orda Rom and Shimkin** (1993) write a pioneering paper on routing games. They apply Rosen to a network of 2 players and 2 parallel links with convex costs. It satisfies SDC only when demand is very light demand. Yet equilibrium is always unique.
- Conclusion: SDC is restrictive.

The Idea of alternative proof of uniqueness

- 1. Replace each player i by m identical subplayers i(1) …. i(m) with total demand fix
- There exists a NE where i(k) have same policy.
- 2. Write the KKT condition for each subplayer i(k)
- Take the the sum of conditions for subplayers of i
- Thus the complementarity condition holds and we can replace i(k)’s by a single player with a “partial” or “local” potential.
- The equilibrium of the game with mI players equals to that of the game of I players where each player maximizes the local potential
- The local potential converges to Beckman’s potential uniformly over the policies of all other players

Framework F3: Congestion games

- A directed graph (G,L,V) with a set of links L and verteces V.
- Each of a finite set of players has to ship a single packets from its source to the destination.
- The cost of a path is the sum of the costs of the links over the path.
- A link cost is an incrneasing function of the number of users that use the link.
- Rosenthal showed that this is a potential game

References

- *E.Altman, R. Combes, Z. Altman and S. Sorin,* Routing Games in the many players regime GAMECOMM , May 16, Paris, France, 2011

- *E. Altman, O. Pourtallier , A. Haurie, F. Moresino, ,* Approximating Nash equilibria in nonzero-sum games , *International Game Theory Review,* Vol 2, Nos. 2-3, pp. 155-172, 2000.

Non-monotonicity in networks

Adding a link, or adding capacity to a link, increases delay for all users: **Braess Paradox**.

The paradox occurs in competitive routing (several users optimise). But also in the case of network optimisation using Bellman-Ford type algorithms (RIP).

Implication: Upgrade by adding capacity to a bottleneck may deteriorate performance

References:

•D. Braess, "Uber ein paradoxen der werkehrsplannung", *Unternehmenforschung*, **12**, 256-268, 1968.

•Y. A. Korilis, A. A. Lazar and A. Orda, "Avoiding the Braess Paradox in Noncooperative Networks", *Proc. IEEE Conference on Decision & Control (San Diego)*, 864-878, 1997.

1 Original Braess paradox

Adding a link, or adding capacity to a link, increases delay for all users

Assume: route 1-3, as well as 2-4 are used.

$$g(x_1) + f(x_3) = f(x_2) + g(x_4). \qquad (1)$$

We now express x_5:

$$x_1 - x_3 = -x_2 + x_4.$$

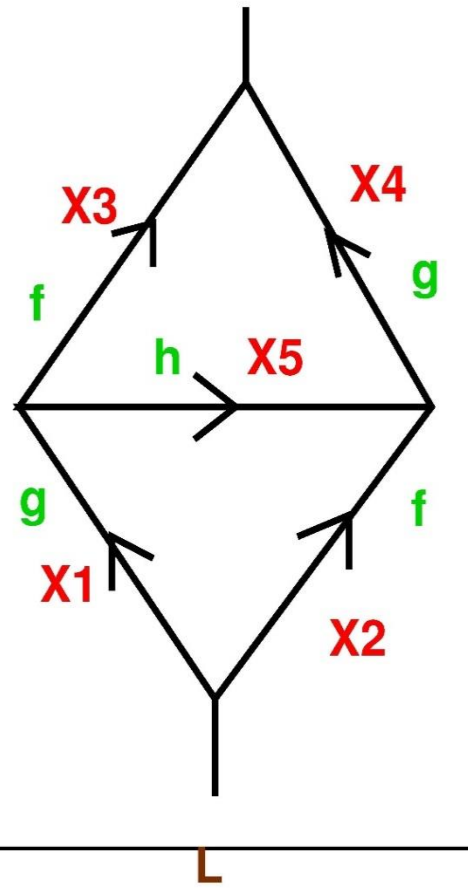
If f is linear then this implies

$$f(x_1) - f(x_3) = -f(x_2) + f(x_4).$$

Summing with (1), we get

$$f(x_1) + g(x_1) = f(x_4) + g(x_4).$$

If $f+g$ is strictly increasing then $x_1 = x_4$. Hence also $x_2 = x_3$.



Now, $x_2 = L - x_1$. Hence

$$x_5 = x_1 - x_3 = x_1 - x_2 = 2x_1 - L.$$

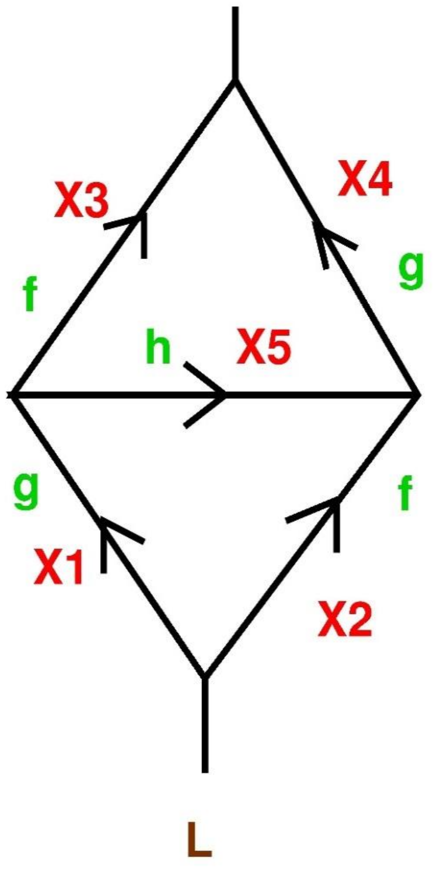
If route 1-5-4 is also used then

$$g(x_1) + h(x_5) = f(x_2).$$

We conclude that

$$g(x_1) + h(2x_1 - L) = f(L - x_1).$$

This gives x_1.

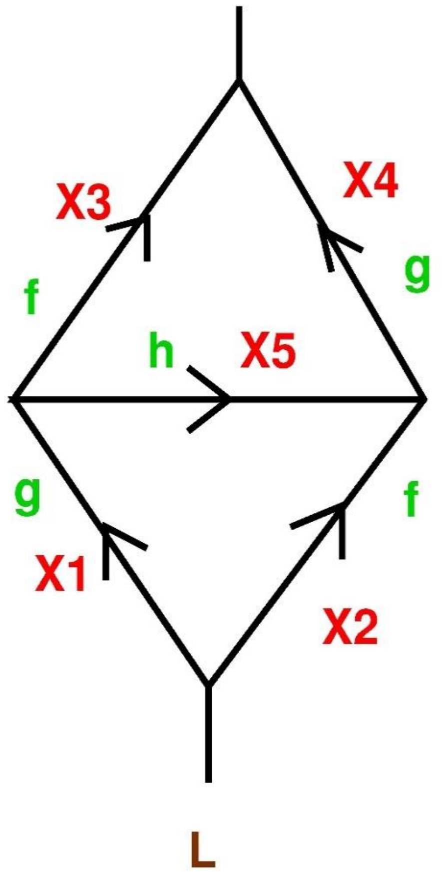


Original Example of Braess

•Choose $L = 6$, $f(x) = 50 + x$, $g(x) = 10x$, $h(x) = \infty$.
Then $x_1 = x_2 = 3$,

$$D_{13} = D_{24} = 83.$$

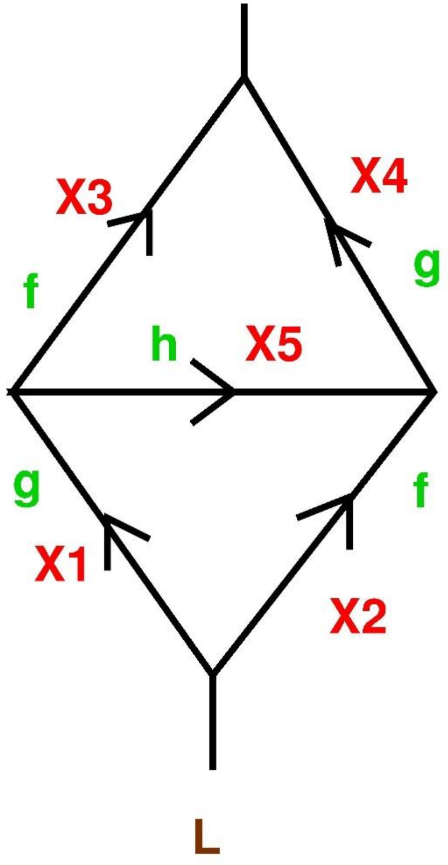
•Take $h = 10 + x$.
With $x_1 = x_3 = 3$, $D_{154} = 70 < D_{13}$. Not Wardrop Equilibrium!



•Suppose 1 unit moves from 2-4 to 1-5-4.
Then

$$D_{1-3} = 40{+}53 = 93, \quad D_{2-4} = 52{+}30 = 82,$$

$$D_{1-5-4} = 40 + 11 + 40 = 91.$$

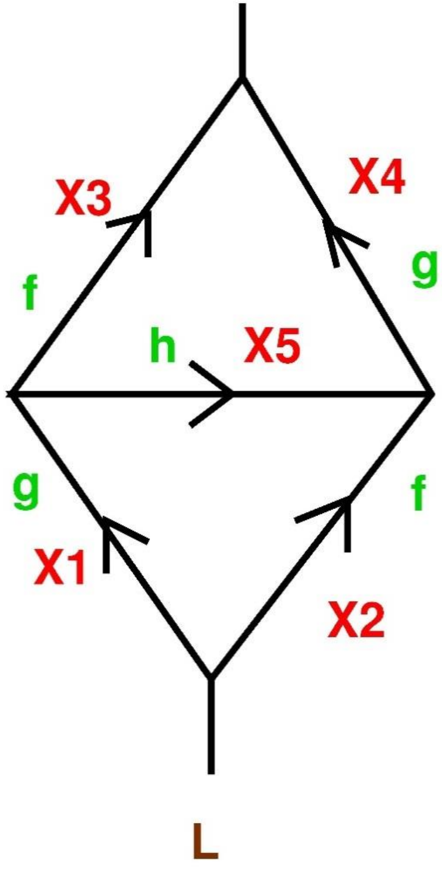


•Suppose 1 unit moves from 1-3 to 1-5-4.
We have $x_1 = 4, x_2 = 2, x_5 = 2.$
Then

$$D_{1-3} = 40{+}52 = 92, \quad D_{2-4} = 52{+}40 = 92,$$

$$D_{1-5-4} = 40 + 12 + 40 = 92.$$

This satisfies Wardrop conditions!



We now check the equation:

$$g(x_1) + h(2x_1 - L) = f(L - x_1).$$

We get:

$$10x_1 + (10 + 2x_1 - 6) = (50 + 6 - x_1)$$

Hence $x_1 = 4$, $x_2 = L - x_1 = 2$, $x_5 = x_1 - x_2 = 2$.

PROJECTS in Routing Games
Game Theory for Loss Networks
Course 2020

Proposed by Eitan Altman

DESCRIPTION OF TASKS

Each group of students will be asked to study networks with one of the topologies that I propose

In each project there is a specific optimisation criterion related to the loss probabilities of packets. The students are asked to

1 Model the problem as a game

2 Compute the Nash and Wardrop equilibrium as well as global optimum

3 Compute bounds on the price of anarchy and price of stability

4 To provide numerical study of convergence of best response algorithm

5 Establish existence of a unique equilibrium or present counter example

6 Is the policy where only direct path are used, an equilibrium?

How to derive the costs

Consider packets that arrive at a source according to a Poisson process with rate ϕ. Each packet requires an exponentially distributed service time with parameter μ. Service times are i.i.d. Two type of losses:

- **Independent losses**: I.I.D. Random losses with probability q
- A Poisson process with rate ϕ subject to iid losses with probability q results in a Poisson process with rate q ϕ

- **Collisions**: A loss occurs if there is an arrival during the service of another packet. In that case a packet is lost. The probability of this event is $\phi /(\phi +\mu)$

The superposition of K Poisson processes with rates r(1),…,r(K) is a Poisson process with rate r=r(1)+…+r(K)

Routing on parallel links over collision channels

- N players. Player i splits its demand φ_i to K links.
- Service time at link k is exponentially distributed with rate μ_k.
- Arrival to link k from player i is Poisson with rate λ_k^i.
- Total arrival process to link k is Poisson with rate

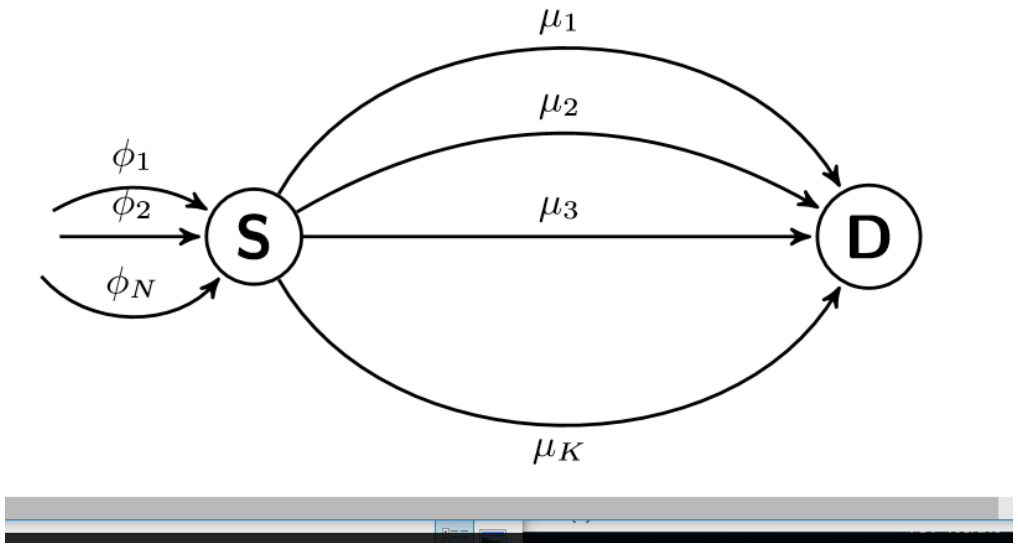
  $\lambda_k = \sum_{i=1}^{K} \lambda_k^i$

- Loss occurs if a packet arrives during the service time of another one, and then the arrival is lost.
- Loss probability of packets at link k is

  $P(k)=\lambda_k/(\lambda_k+ \mu_k)$

- Loss probability of packets at link k corresponding to player i is

Loss rate for player i is R(i)= $\sum_{i=1}^{K} \lambda_k^i \, P(i,k)$



Routing on a triangle with relay and overflow cost

- 2N players: N ship their demand from node 1 to 3 and N ship from 2 to 3. A player can split its traffic between a direct path 13 and 23 or an indirect one 123 and 213.
- There is a fixed cost of using link d.
- The other two links m=1,2 are modeled as MM1K queus
- The probability that an arrival is lost at link m is

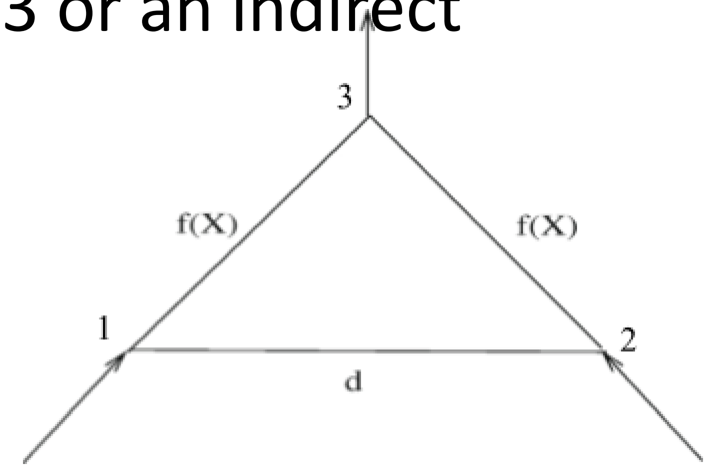


Fig. 1. Load balancing topology

$P_m = \frac{(\rho_m)^K}{G(m,K)}$ *where* $G(m,K) = \sum_{k=0}^{K} (\rho_m)^k$ and

$\rho_m = \frac{\lambda_m}{\mu_m}$. Loss rate of player i on link m is $\lambda_m^i P_m$

Cost for player i is $\sum_{m=1}^{2} \lambda_m^i \, (P_m + d1\,(i,m))$ where 1(i,m) is the indicator that equals 1 if m is on the indirect path for player i

Triangular topology - losses at the relay

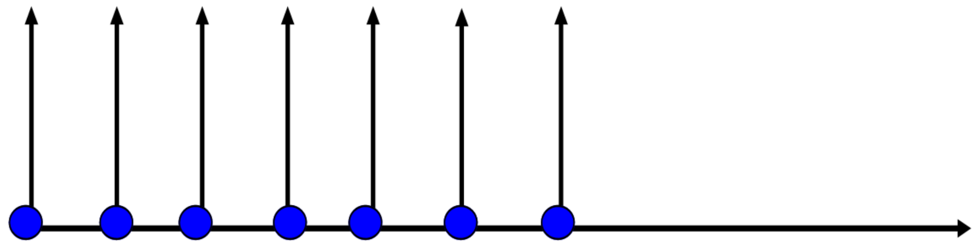
- Ref1: Eitan Altman, Joy Kuri, Rachid El-Azouzi. A routing game in networks with lossy links. 7th International Conference on NETwork Games COntrol and OPtimization (NETGCOOP 2014), Oct 2014, Trento, Italy. hal-01066453
- Ref2: Eitan Altman, Corinne Touati. Load Balancing Congestion Games and their Asymptotic Behavior. [Research Report] Inria. 2015. hal-01249199

- **PROJ 1:**

Study the game with triangle topology where the cost of a player is a weighted sum of her loss probability and relay cost,

Then study the problem with the relay cost d replaced by a collision channel where losses of packets are iid with probability q. This models a noisy wireless channel.

Routing on the line



PROJ 2. Traffic on a line goes from left to right. At each node i there are n players connected. Each has the choice of splitting its demand between

(1) going directly to the destination (vertical arrow)

(2) relaying the traffic to node i+1 at some cost d and then go to the destination.

Each of the vertical links is an MMKK queue and has buffer overflow according to the MMKK loss formula. Player i minimises the loss probability plus relay cost averaged over the amount it sends to each path. Objective: obtain a difference equation for the node i as a function of decisions of nodes i-1 and i+1. Check if not using the relay is an equilibrium; Find other equilibria. Study splitable routing game on the line topology with loss probability criteria and where relay costs is replaced by iid losses

References on Line topologies

M. Haddad, E. Altman and J. Gaillard, "Sequential routing game on the line: Transmit or relay?," 2012 International Conference on Communications and Information Technology (ICCIT), Hammamet, 2012, pp. 297-301.

Manjesh Kumar Hanawal, Eitan Altman, Rachid El-Azouzi and Balakrishna Prabu, " Spatio-temporal control for Dynamic Routing Games, GameNets 2011 (Sanghai, China, April 2011)

Abdelillah Karouit, Majed Haddad, Eitan Altman, Abdellatif Matar. Routing game on the line: The case of multi-players. UNet’2017 - Third International Symposium on Ubiquitous Networking, May 2017, Casablanca, Morocco. hal-01536349

Routing on the circle

PROJ 3. I groups of n players each decide what fraction of their flow to send clockwize and what shoul go anti-clockwize. Each node i is a source for the traffic originating from n players. The destination is the node i+1 (modulo the number of nodes I). Objective: obtain a difference equation for the node i as a function of decisions of nodes i-1 and i+1. Is not using the relay is an equilibrium? Are there other equilibria?

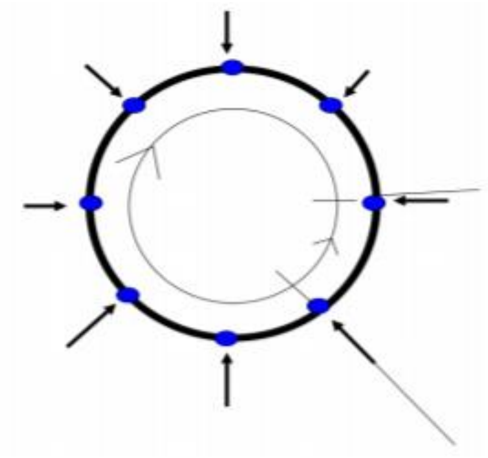


Fig. 1.   Competitive routing on a bidirectional circular network.

PROJ 4. The same description of players and nodes also holds in the second model, but now the destination is a common central node c. A player i has to decide what fraction of its traffic it sends directly to C and what fraction uses an indirect path: first it relays the traffic to node i+1 and then it it goes to C;

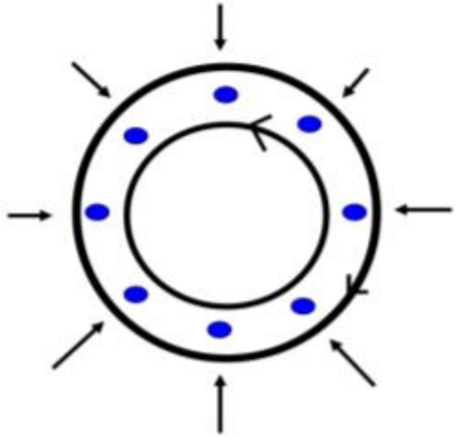


Fig. 2.   Competitive routing between two unidirectional circular paths.

In boh projects, study splitable routing on the ring topologies with loss probability criteria. Relays will induce fixed cost d, and the direct links will create collision losses. Then study instead the overflow losses of MM1K queue.

Routing on a ring/circle

Ref1: Ramya Burra, Chandramani Singh, Joy Kuri, Eitan Altman. Routing on a Ring Network. Song, Ju Bin; Li, Husheng; Coupechoux, Marceau. Game Theory for Networking Applications, Springer International Publishing, pp.25-36, 2019, 978-3-319-93057-2. 10.1007/978-3-319-93058-9_3. hal02417278

Ref2: Eitan Altman, Alejadra Estanislao, Manoj Panda. Routing Games on a Circle. NetGCOOP 2011 : International conference on NETwork Games, COntrol and OPtimization, Telecom SudParis et Université Paris Descartes, Oct 2011, Paris, France. hal-00644364

PROJ 3 corresponds to the topology of Ref1,

PROJ 4 choose one of the topologies ifrom Ref2

Experimental project: Path Recommendation

- Platforms such as googlemap or waze recommend routes from source to destination.
- Plan and perform experiments to check coherence between
- Time prediction to the destination
- The proposed path
- Identify on a map where network paradoxes are likely to occur
- Compute mixed equilibrium with some atomic splitable sources that model the platforms, and some classzs tht are atomless

4 Other models of losses

We saw a loss related to access and to buffer overflow. Here are other loss models

- 1 Service has been assumed to continue even if the served packet was corrupted. Assume next that it is aborted if there is an arrival
- 2 Upon collision , both arrival and served packekts are lost. But the service stops

Computing the loss rate due to multiple access at a link k: In both models above, we define renewal cycle C = I+B where I is the idle period and B is the busy period. The idle period is exponentially dist with parameter λ_k for both models.

In model 1, the busy period is exponentially distributed with parameter μ_k. Indeed, B is unchanged if we replace the loss of served packet by a loss of an arrival packet.

Thus the loss probability is B/(I+B) by Pasta. where E[I]= $1/\lambda_k$, $\mathrm{E}[\mathrm{B}] = 1/\ \mu_k$. Thus by PASTA, the loss probability equals

$$\frac{1/\mu_k}{1/\lambda_k + 1/\mu_k} = \frac{\rho_k}{1+\rho_k} \quad (*)$$

More generally, the loss probabiity of a packet in an MM1K queue is

$$\frac{(\rho_k)^K (1-\rho_k)}{1-(\rho_k)^{K+1}}$$

which reduces to (*) for K=1

All packets lost

Next assume that upon collision , both arrival and served packets are lost and service is aborted. Then

$E[I]=\frac{1}{\lambda_k}$, $E[B]=\frac{1}{\lambda_k+\mu_k}$ and the probability of finding a packet in service is

$$p= E[B]/(E[B]+E[I])=\frac{1/(\lambda_k+\mu_k)}{1/\lambda_k+1/(\lambda_k+\mu_k)}=\frac{\lambda_k}{2\,\lambda_k+\mu_k}$$

The probability of a loss is

$$p+(1-p)\,\frac{\lambda_k}{\lambda_k+\mu_k}$$

Other loss models

- Redundancy : Send packets at higher rate. The additional packets are for coding. Assume that any N well received packets out of N+k packets allow one to retrieve all N+k packets. For example use XoR for k=1. Then we can still retrieve a packet even if it is lost provided that all other are well received.

- Use outage probability formulae to determine loss probabilities.

In practice when there is a collision then there is some probability of losing both packets involve, some probability of losing just the one or the other, and there is some small probability that both packets will be transfered successfully. There are many formulae for these outage probabilities since they strongly depend on the type of modulation used for transmitting the packets

Network Engineering Games

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Framework F1 (W.E.)

- Only direct paths are used in the symmetric case.
- In the asymmetric case, if an indirect path is used in one direction then it is not used in the other directions.
- This follows the LOOP FREE PROPERTY
- Unique equilibrium

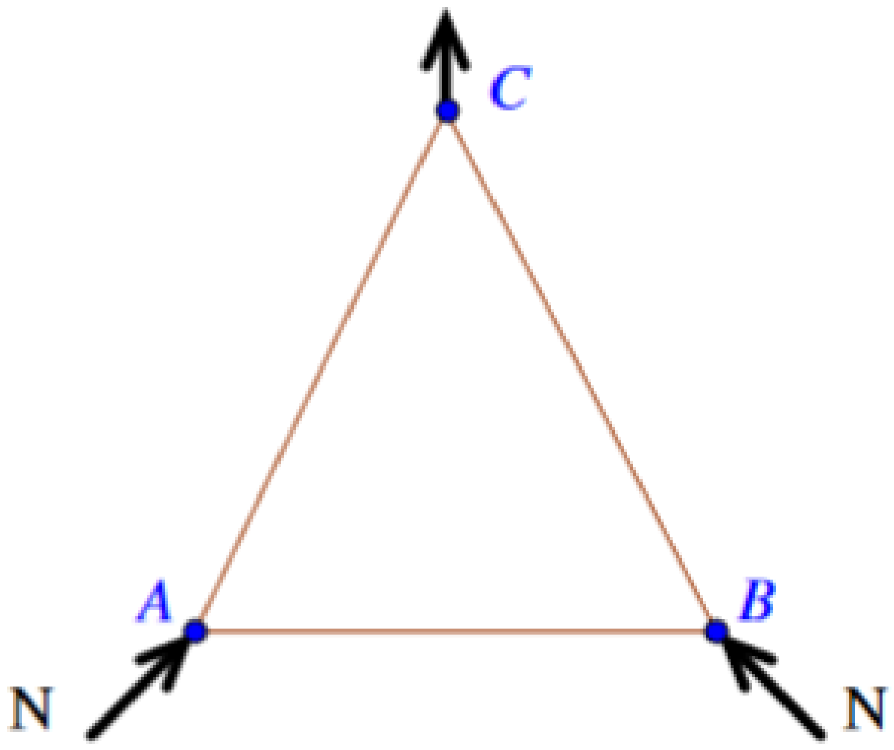

Fig. 1 Physical System

Example 1: Symmetric Load balancing game F2

- N players send a flow of tasks to A and N send to B.
- Each player has a fixed demand
- The decision of a player is what fraction of its flow to process locally and what fraction to send over to be processed at the other access point.
- After receiving service, the tasks leave the system (node C)

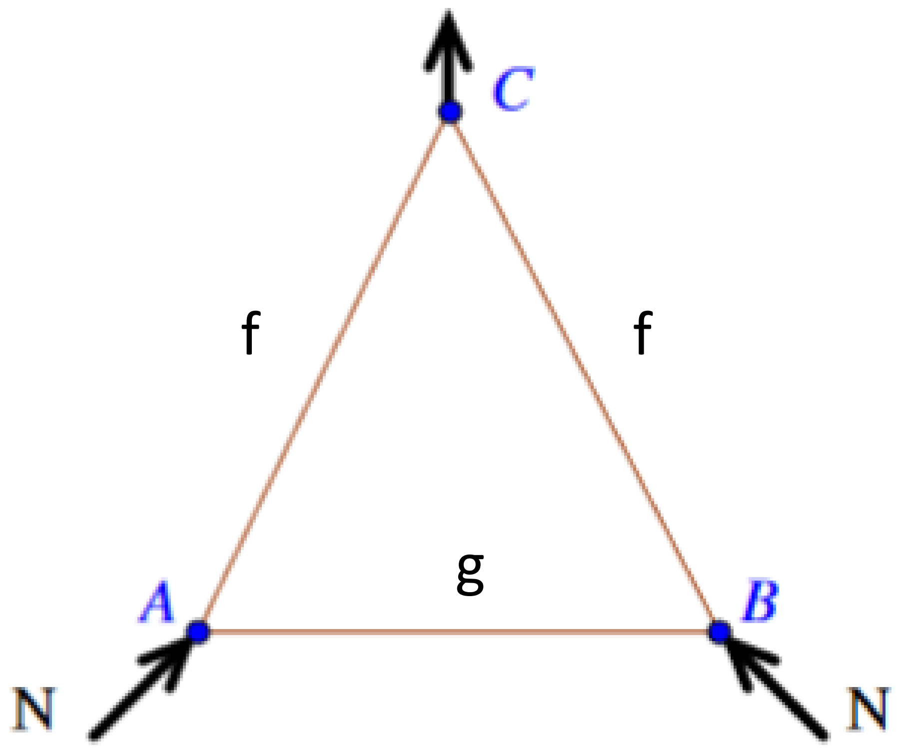

Fig. 1 Physical System

Load balancing routing game: Delay cost F2

- Processing incurs a load dependent delay: in each computer, the delay experienced by a task is a function f of the total amount of tasks processed in that computer.
- f is increasing convex (not strictly)
- Similarly with the forward delay g() on AB
- Each player minimizes the average delay of its packets.

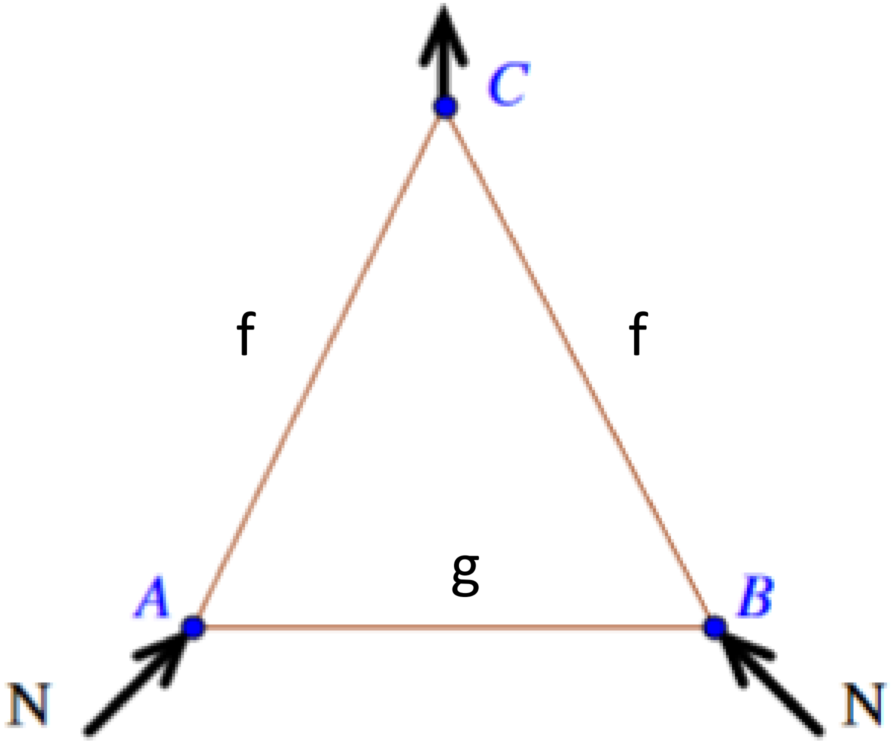

Fig. 1 Physical System

Price of collusion

- Defined [E Tardosh] as the ratio between the cost for noncolliding players before and after collusion
- Kameda’s paradox shows that collosion can be harmful for the colliding players as well
- Need another definition

Framework F2

- If AB incurs no delay then we have a routing game to parallel links.
- The equilibrium is unique. Price of anarchy is bounded.
- No Braess paradox

- With nonzero delay on AB, Prof Kameda and coauthors showed
  - Uniqueness of equilibrium
  - Unbounded price of anarchy
  - There is a surprising Braess type paradox
  - For given total demand, as N increases the paradox disappears
  - This contrasts the standard Braess paradox that holds in that regime

Potential game

- A game has a potential P if for every player i, and any action vectors a and b that differ only in the action of player i,

$$P(a)-P(b)=U(i,a)-U(i,b)$$

A game with an ordinal potential

the equality is replaced by an inequality

$$P(a)-P(b) > 0 \quad \text{iff} \quad U(i,a)-U(i,b) > 0$$

If the potential has a local maximum then it is an equilibrium.

Limits of best (or of better) responses are equilibria of the game [Shapely and Monderer]

Potential

Theorem 4. *For any finite number of players, the game is a potential game [8] with the potential function:*

$$
\begin{aligned}
F(f_{ABC}, f_{BAC}) = \\
&bN(f_{ABC} - f_{BAC})^2 \\
&+\frac{aN}{2}\left(f_{ABC} + f_{BAC}\right)\left(f_{ABC} + f_{BAC} + 1/N\right).
\end{aligned} \tag{2}
$$

Theorem 6. *The price of stability of the game is* 1 *and the price of anarchy is* $1 + \frac{b}{2aN^2}$.

of pure equil F3 Load Balancing, linear costs

- These are necessary and sufficient conditions.
- Some Algebra shows that symmetric equilibria exist.
- For any symmetric policy and in particular at equilibrium, **the direct paths carry N packets each.**
- Thus at equilibrium a packet taking direct path cannot benefit from deviation.
- The EQ condition – no profit by deviating **from an indirect path**
- Let Y(N) be a symm. policy that sends 2N packets over indirect paths
- The cost for any of these packet is a(2N) + b(N)
- This is an EQ if this cost does not exceed the one for a deviation to a direct path given by b(N+1)   THUS THE EQ CONDITION IS 1 <= b/(2a) .

Corollary

The number of pure equilibria equals the to integer part of b/(2a)

Prf: If the EQ condition holds for some N0 then it also hold for all integer in [1,N0)

Discrete setting: why many equilibria?

- In F1 and F2 no such problems.
- To understand why several equilibria appear in F3 recall that F3 is equivalent to a congestion game and therefore equilibria are local minima of a potential.
- Local minima are avoided if we minimize a convex function f on a convex compact set S. But what is a convex function on a discrete set? What is a convex subset of a discrete set?
- Suggestion: find a convex function F on an Eucledean set, whose projection on S is f. Does there exist such a function?
- Research in this direction could bring together the community of discrete optimization and that of convex optimization.

F3 with Linear COSTs

$$\begin{cases} C_{[AB]} = a(f_{BAC} + f_{ABC}), \\ C_{[AC]} = b(f_{BAC} + f_{AC}), \\ C_{[BC]} = b(f_{BC} + f_{ABC}). \end{cases}$$

and then:

$$C_{AB} = C_{[AB]}, \quad C_{ABC} = C_{[AB]} + C_{[BC]},$$
$$C_{BC} = C_{[BC]}, \quad C_{BAC} = C_{[AB]} + C_{[AC]}.$$

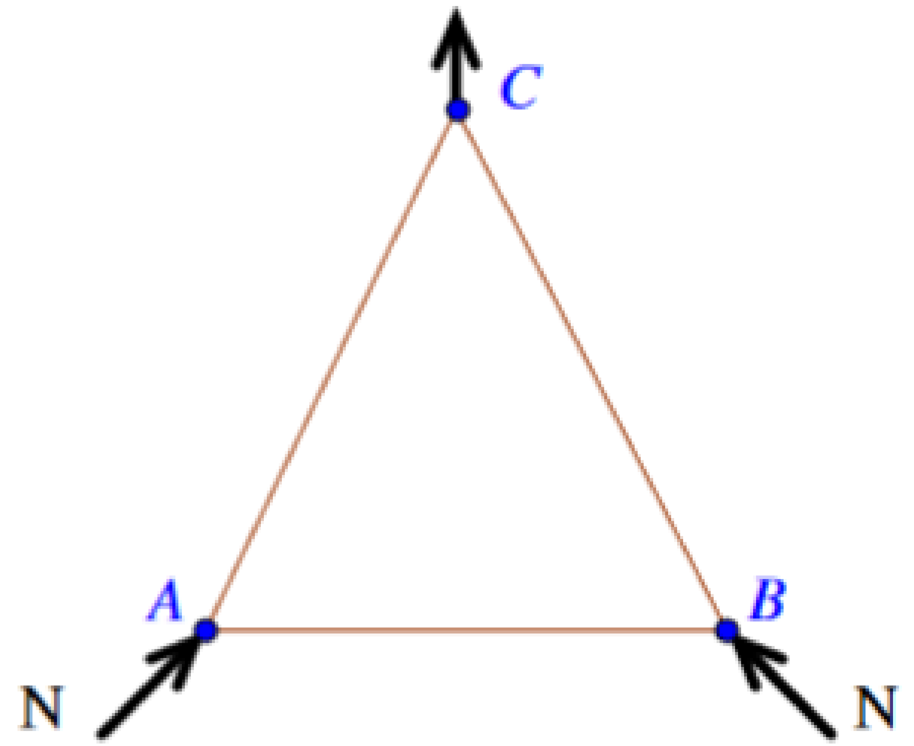

Fig. 1 Physical System

- The cost for a user in **F2-F4** is the average of path costs weighted by the fraction that the player sends over each of the paths. For framework **F3** a single packet is sent by each player so the cost for the player is the cost for the path that it takes.

Computing Wireless losses

- Assume all players send a fraction x of packets to be processed locally and 1-x to the indirect path
- **Losses due to noisy link:** We assume that packets are lost at the noisy link independently with probability 1-v
- Then at each access point there is a rate of arrival of NL(x+v(1-x))
- **Losses due to collisions** several possibilities of losses when arrival occurs during service
  - Packets in service are lost
  - New arrivals are lost
  - Both are lost
  - Outage probability according to formulae for modulation dependent losses

Loss probabilities

- Assume losses if there are arrivals during service time.
- Assume one player deviates from x to y
- Then the rate of arrivals to his local AP is

$$R1(x,y)=L(Nv(1-x) + (N-1)x + y)$$

- The rate of arrival to the other AP is

$$R2(x,y)=L(Nv + v(N-1)(1-x) + v(1-y))$$

- The loss probability is

$$P(x,y) = y\, s^*(R1) + (1-y)(v\, s^*(R2) + 1-v)$$

Paradoxes in a Multi-criteria Routing Game

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Majed Haddad and Nadia Oukid

GameNets 2017

System Model

- Three nodes: two source nodes S_r and S_l and one common destination node D.
- There are 2N sources of flows. Each flow consists of an independent Poisson distributed point process with a rate ϕ.
- Packets from source i = 1, . . . , N arrive at node S_l (left), whereas packets from source i = N + 1, . . . , 2N arrive at node S_r (right).

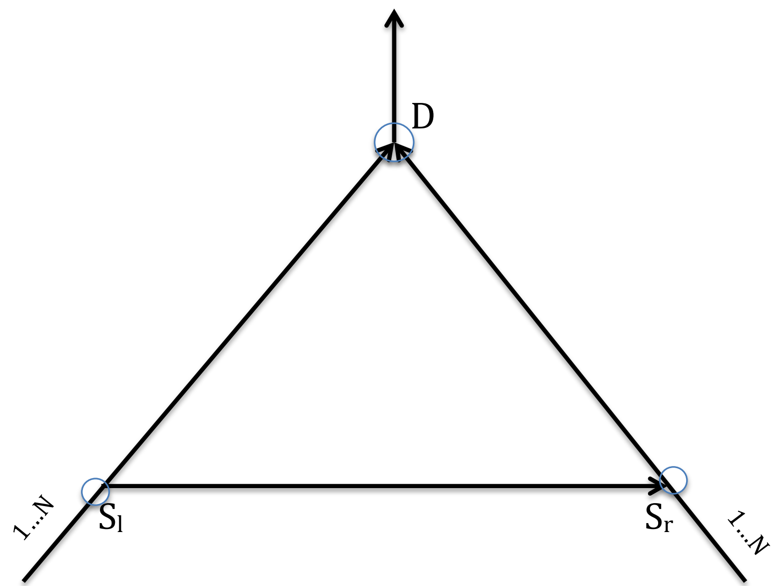


Fig. 1. The competitive routing model.

The load balancing topology

- Source i = 1, . . . , N can split its flow between its direct path $S_l D$ with probability p_i and the indirect one $S_l S_r D$.
- Source i = N +1,...,2N can split its flow between its direct path $S_r D$ with probability p_i and the indirect one $S_r S_l D$.
- Links $S_r S_l$ and $S_l S_r$ are assumed to be wireless so that packets sent over $S_r S_l$ and $S_l S_r$ suffer independent losses with some fix probability q.
- The delay over these links is assumed to be a constant denoted by δ.

The cost function

- The cost for source *i* is a weighted sum of the average delay of its flow and its loss rate:

$$J_i(p) = \frac{\phi p_i}{C - \phi \sum_{j=1}^{N} p_j - \phi q \sum_{i=N+1}^{2N} (1 - p_j)} +$$

$$\frac{\phi(1-q)(1-p_i)}{C - \phi \sum_{j=N+1}^{2N} p_j - \phi q \sum_{i=1}^{N} (1 - p_j)} +$$

$$\phi\delta(1-q)(1-p_i) + \gamma\phi q(1-p_i).$$

Global optimum calculation

The global optimal solution is obtained by solving:

$$\frac{\partial}{\partial p}\sum_{i=1}^{2N} J_i(p) = 0$$

We obtain the unique solution:

$$p = \frac{1}{2}\frac{a\phi N(2q^2 - 2q + 1) + bq - \gamma q - \delta}{\phi a q N(q-1)}.$$

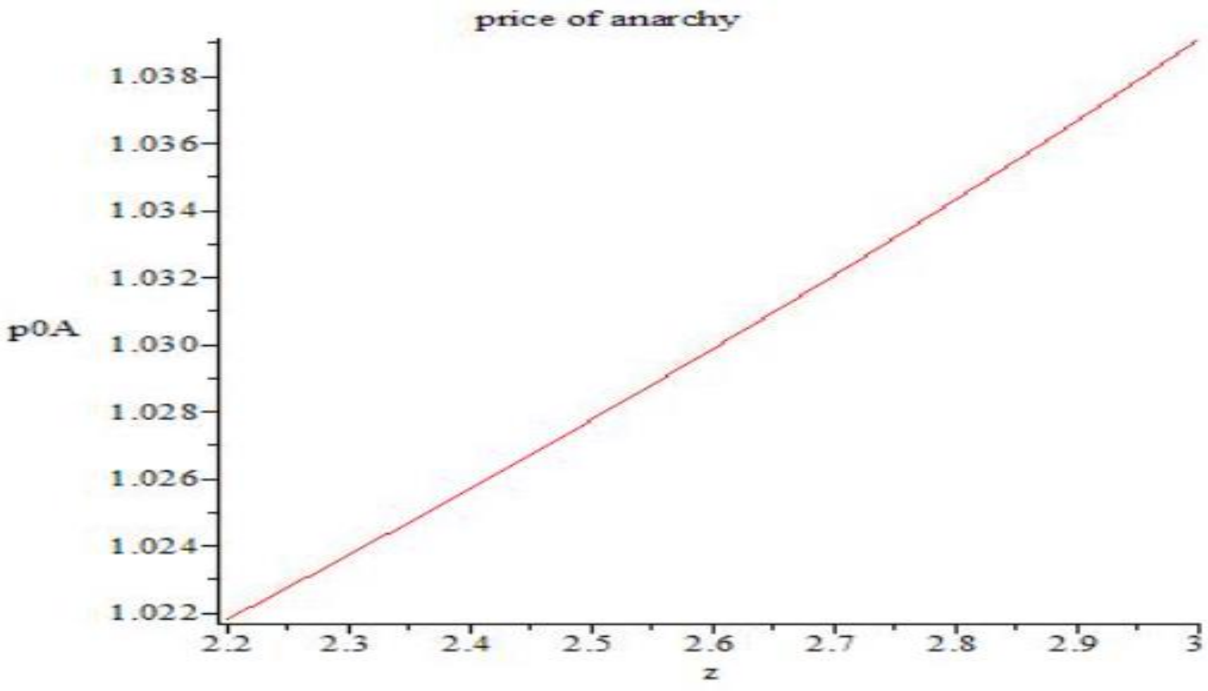
Nash equilibrium calculation

$$\hat{p} = \frac{a\phi N q^2 + a\phi(1-q) + qb - z}{a\phi q(qN - N - 1)}.$$

where

$$z = \gamma q + \delta$$

Price of anarchy



$$PoA = \frac{2NJ_i(\hat{p})}{2NJ_i(p)}$$

Numerical results (1)

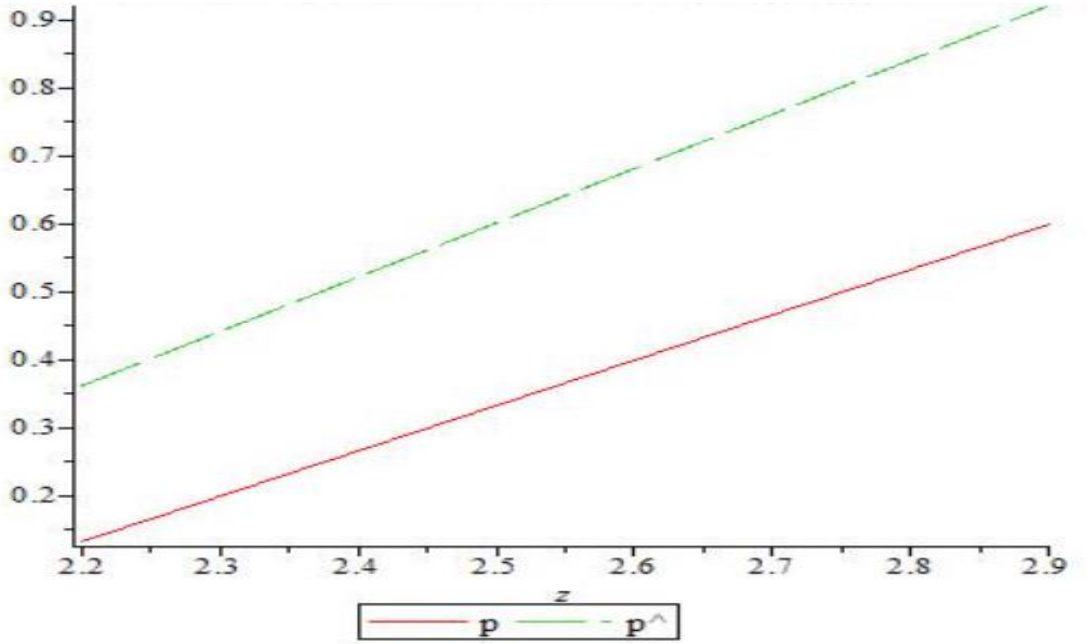


Fig. 2. The optimal solution and the equilibrium as a function of z.

Numerical results (2)

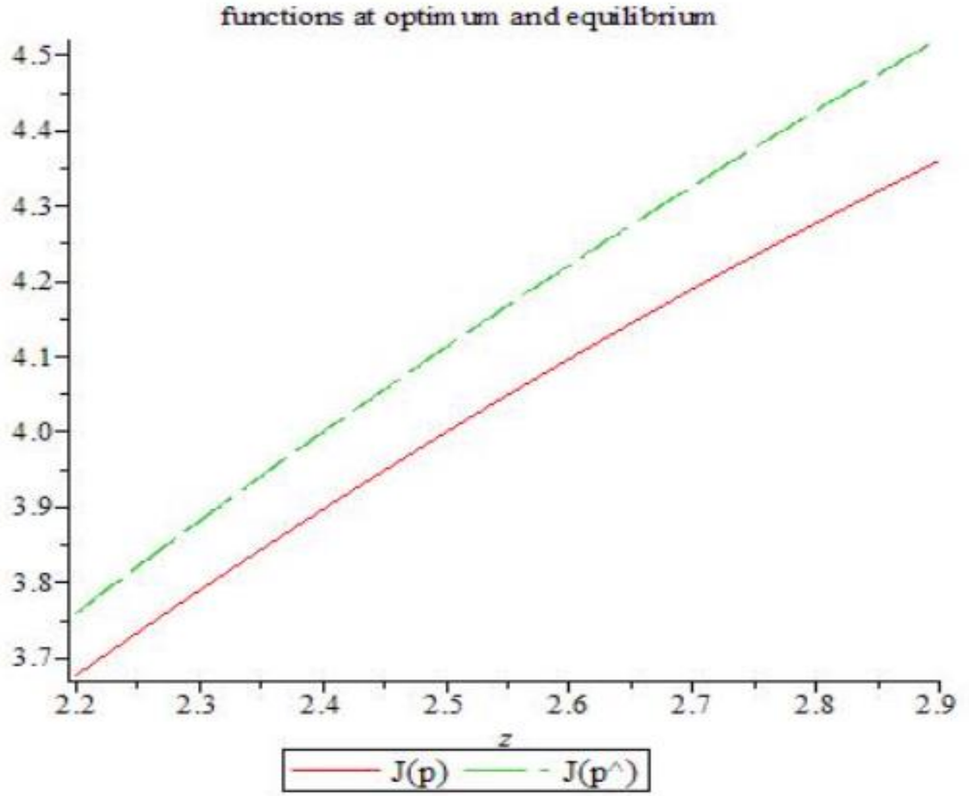


Fig. 3. The cost function at the optimal solution and the equilibrium as a function of z.

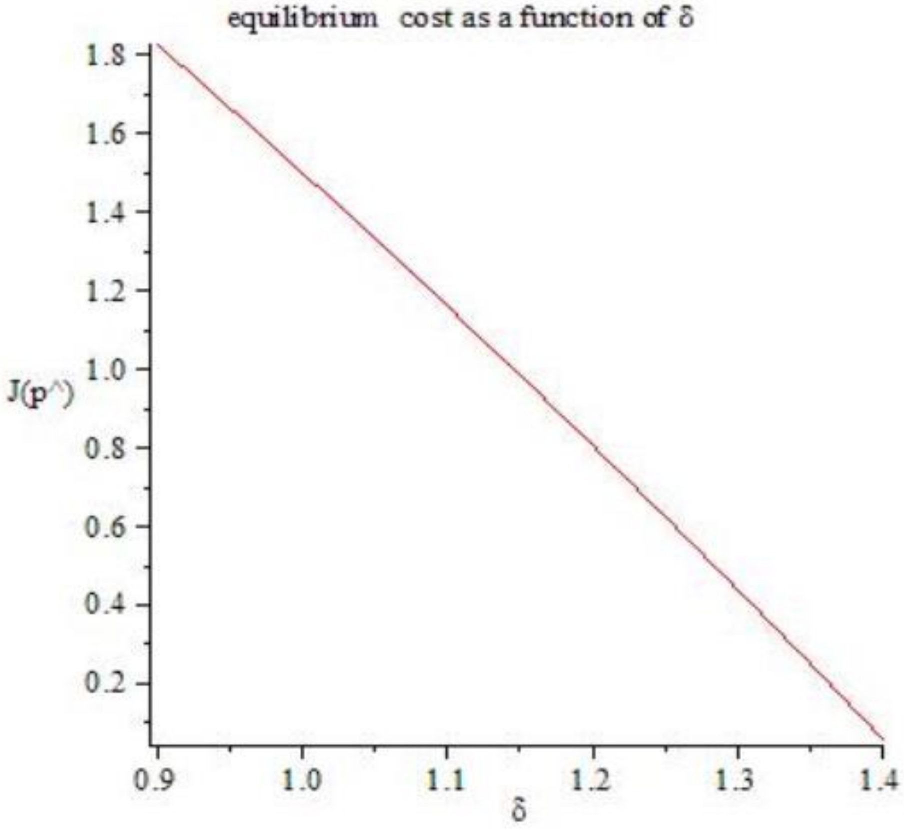
Paradoxes

- The condition for this type of paradox is then that the cost function J at equilibrium is decreasing in the network parameter (e.g., in the delay δ).
- The derivative of J at equilibrium should be decreasing where the latter is given by

$$J'(\delta) = \frac{a\phi + a\phi N^2(1+q^2) + 2a\phi Nq(1-N) + 2bq - 2z}{aq(qN - N - 1)^2}$$

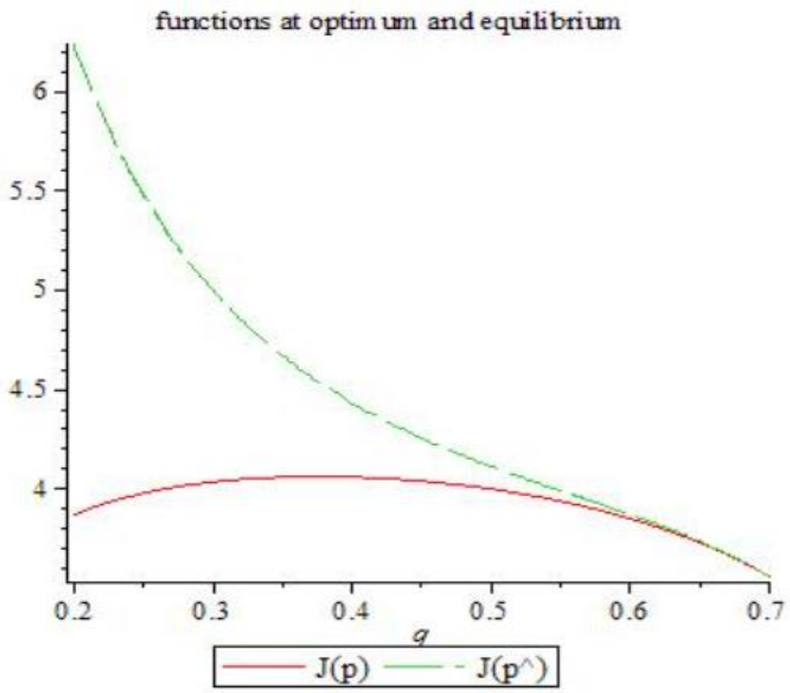
Kameda-paradox

We obtain the paradox in which larger link delay are beneficial for all users. Investing in faster links increases the delay and deteriorates the performance for all players.



New paradox multiobjective optimisation

- We identify a new type of paradox: the cost is seen not to be monotone in the quality of the link (the loss probability q).
- This phenomenon is due to the particular multi-objective structure of our problem.
- higher q increases the cost related to losses, but contributes to decreasing the global cost as more losses results in lower congestion and thus in lower delays.



Example 2 - Tullock rent-seeking game

- Player m makes a bid x(m) for purchassing some good.
- $$U(m) = \frac{x(m)}{\sum_{n=1}^{M} x(n)} K - x(m)$$
- Note that U depends on decisions of other players only through the sum of their bids. Agggregative game.
- X(m) controlled by player m. Continuity problems at x=0
- Monderer and Shapely: there exist an ordinal potential

Resource allocation game

- I resources.
- Player m with budget constraint B(m) splits its budget and bids x(m,i) on resource i.
- $$U(m) = \sum_{i=1}^{I} V_i \left(\frac{x(m,i)}{\sum_{n=1}^{M} x(n,i)}\right) - x(m,i)$$
- Aggregative game. If resource is time then we get a repeated game
- Can be viewed as a routing game with a parallel link topology

Discrete versions of Tullock game and resource allocation games

- Unique equilibrium
- Inherits the ordinal potential property
- If a player cannot split the traffic then there is a potential

Is the equilibrium unique?

- A multi strategy is called local optimum if it cannot be improved by a single player
- A strategy is a local optimum of the potential if and only if it is a Nash equilibrium of the original game
- Even if a potential is concave in each direction, the potential may have local optima.
- Need a method to establish uniqueness

Application: Blockchain game between miners

The blockchain is a distributed secure database containing validated blocks of transactions.

- A block is validated by special nodes called miners and the validation of each new block is done via the solution of a computationally difficult problem, which is called the proof-of-work puzzle.

- The miners compete against each other and the first to solve the problem announces it, the block is then verified by the majority of miners in this network, trying to reach consensus.

Blockchain game between miners

- After the propagated block reaches consensus, it is added to the distributed database.

- The miner who found the solution receives a reward either in the form of cryptocurrencies or in the form of a transaction reward.

- Because of the huge energy requirement necessary to be the first to solve the puzzle, blockchain mining is typically executed in specialized hardware.

- An Edge computing Service Provider (ESP) is introduced to support proof-of-work puzzle offloading by using its edge computing nodes

Example of advertisement for mining in cloud

Our work addresses the following two questions:

- given a single blockchain, how should rational users contribute to the mining process, possibly counting on third-party ESPs or mining pools to offload infrastructure costs?
- given multiple blockchains, e.g., in a multi-cryptocurrency ecosystem, how should rational miners distribute their monetary and/or computational budget towards mining?

Splittable approach for resource allocation game

- We assume next competition over splittable resources at a single ESP and single currency
- miner i decides how much to invest
- Its utility from investing x(i) is the payoff minus cost:

$$U(i)=x(i)/x - gx(i)$$

Where x is the total investment. This is the Tullock rent seeking game. It has a unique Nash equilibrium

Related to KELLEY MECHANISM in networking in which the goal of the network is to find a pricing g that will guarntee that the equilibrium will be globally optimal w.r.t. the payoff, and will meet some capacity constraint on the sum of x(i). The g is interpreted as lagrange multipliers. But do not depend on i

Splittable approach for Crypto Currency game

- We next assume that a miner i has a fixed budget B(i) that it can split between various crypto currencies. Its utility from investing x(i,k) in ccurrency k is

$$U(i,k)=x(i,k)/x(k) \; - \; g(k)x(i,k) \qquad \text{with} \qquad U(i)=\sum_{k=1}^{K} U(i,k)$$

Where x(k) is the total investment in currency k

This is a variation of Tullock rent seeking game. It has a unique Nash equilibrium.

Related to Kelly mechanism where there are K resources to split

Constraints

- We ask similar question as Kelley but our goal is to find prices which induce an equilibrium

- Each player may have own budget constraints.
- These are orthogonal constraints
- There may be further non orthogonal constraints on each ccurrency k of the form
- $\sum_{i=1}^{N} x(i,k) \leq V(k)$
- V may represent energy constraint on a currency
- Infinite number of equilibria
- How to select one

Constraints and normalized equilibrium

- $\sum_{i=1}^{N} x(i,k) \leq$ V(k)
- By KKT for each player i and policies x(-i) there exists r(i,k,x(-i)) such that the best response for player i is the solution of

$$\max U(i) + \sum_k r(i,k) \left(V(k) - \sum_{i=1}^{N} x(i,k) \right)$$

With complementarity constraints. If we set r as pricing then it guarantees that the argmax is feasible. The fix point is an equilibrium.

But this pricing is not scalable.

Does there exist an equilibrium x for which the vector r that DOES NOT DEPEND on i nor on x? If yes this is called normalized equilibrium

Main results

- THM1. The blockchain game has a unique normalized equilibrium
- Prf: Strict Diagonal Concavity holds

THM2. There is a primal dual learning scheme that guarantees that the constraints are met during the whole learning process

Stochastic ESP Association Game

- We now investigate a situation in which the number of miners varies in time.
- Consider a Poisson disributed arrival process of miners.
- Upon arrival, say at time t, a miner observes the number $N(t)$ of competing miners present.
- The time to compute a puzzle by a miner is exponentially distributed with mean $1/\mu$ if it is the only one attached to the ESP. When there are n miners attached then the service rate is n times slower. We model the service rate of a given miner at time t as a processor sharing with rate $\mu/N(t)$.
- Should the miner participate or not in the puzzle
- The utility for participaing in the mining depends on futur arrivals and their decision

Equilibrium structure [Kushner]

- We model this as a game.
- For each r there is anoher threshold (l,q)_r
- A type r arrival at time t joins the mining if N(t)>l(r). It does not join if N(t)<l(r) and it joins with probabiity q(r) id N(t)=l
- (l,q) is an equilibrium if it is the optimal threshold given that every one else uses that same threshold.

- Learning based on stochastic approximation

Learning

- How can a player learn if only makes one decision?
- There is a statitics list shared among the players. Each time a player leaves an entry in the table is updated
- The updated entry corresponds to the estimated reward of players who 1. found n miners when they joined and 2. have already left
- Upon arrival a player decides to join at state n if the estimated reward is positive
- We assume 1. Call admission: bounded number of players and 2. a small rate of uncontrolled arrivals

Reference

- *E. Altman and N. Shimkin* , Individually Optimal Dynamic Routing in a Processor Sharing System: Stochastic Game Analysis , *EE Pub No. 849* , August 1992. A later version can be found in *Operations Research* , pp. 776--784, 1998.

Non splittable Model

- There are K crypto-currencies and a single ESP
- There are N miners
- A puzzle related to crypto currency k requires from a miner an exponentially distributed time with expectation $1/\mu(k)$; we assume that a new puzzle is available at crypto currency k immediately after the previous puzzle at this crypto-currency was was solved
- Denote by $L(k)$ the number of miners that compete over the k-th currency. The fastest solves the puzzle after an exponentially distributed time with expectation $L(k)/\mu(k)$

Constant number of miners

- L is vued as a strategy.
- The utility for a miner to solve puzle k is

$$U(\, k, L(k)\,) \;=\; \mu(k)/L(k) - g(k)$$

- where g(k) is the cost for using the ESP for solving a puzzle related to the k-th crypto currency
- L is an eqilibrium if for all k for which L(k)>0 and all k'

$$U(\, k, L(k)\,) \geq U(\, k', L(k')+1)$$

- In other words no player can gain by deviating from k to k'

Non-splittable elastic ESP association game

- We assume a single currrency and R user classes
- A miner of class r pays g(r) for using the ESP per attempted puzzle
- We assume that a miner participates only if the utility is non-negative
- Let L be an R dimensional vector of loads.
- The utility for a class r user to join the miners is

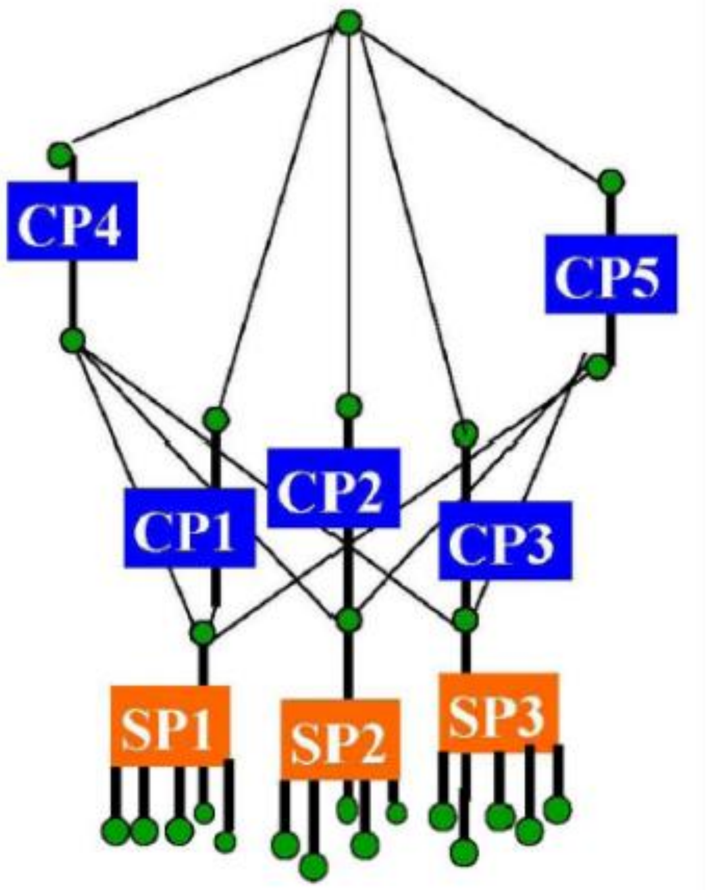
$$U(r,L)=1/|L| - g(r) \quad \text{If } L(r)>g(r) \text{ else } 0$$

- This is a crowding game and has pure equilibria

Application of routing games to Net Neutrality

BLUE – CPs
ORANGE – GLOBAL ISPs
LEAVES – Local ISPs (players)

A local ISP i has a demand, has
to decide how much to fetch from
Each CP
Link cost: delays at CPs, fixed link
cost of Dij on traffic between SPi
and CPj.
Objective: study collusions Dii=0
These are vertical Cartels

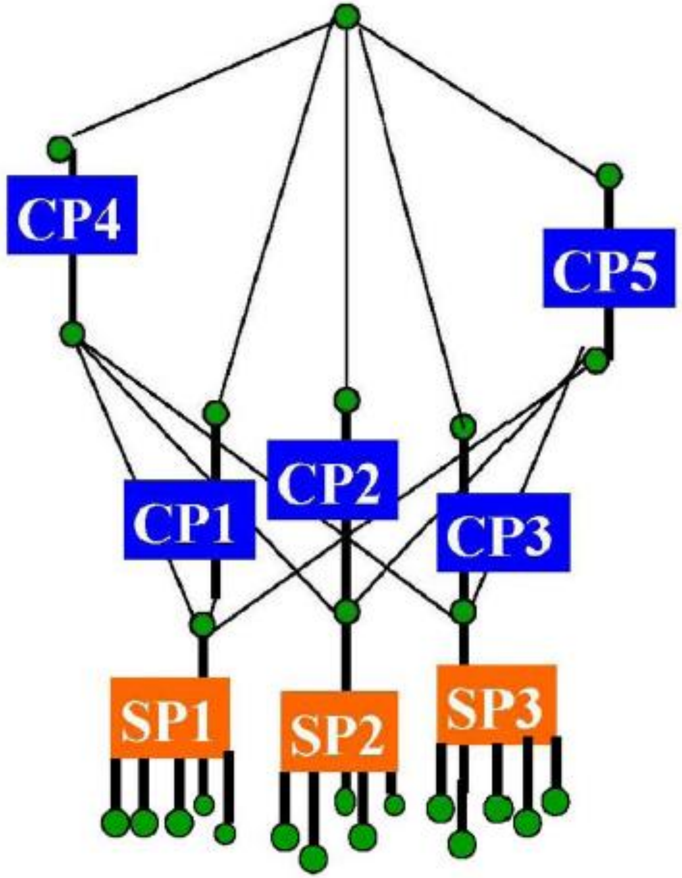


- We consider Users, ISPs and CPs.
- m ISPs and k are independent, n are combined

  ISP+CP.
- Subscribers of any ISP i can download content from any CP j at a cost of d[ij]
- Cost function: Delay + monetary cost

$$C^i(\underline{\mathbf{x}}) = \sum_{j=1}^{n+m} x_j^i (D_{cp}^j(x_j) + d_{ij})$$

- X[ij] is the amount of demand that user i requests from ISP j.



- We are interested in symmetric equilibria

- X = equilibrium rate x[ii] if the ISP of user i is in collusion with the CPi
- Y = the amount it fetches from any other non-independent CP
- Z = the amount it requests from independent CP
- The amount requested from subscribers of independent ISPs from independent CPs is v
- The amount requested from indep ISPs is

KKT conditions

- I users per ISP
- d[ii]=0
- Lagrangian:

$$L^i(\underline{\mathbf{x}}) = C^i(\underline{\mathbf{x}}) - \lambda^i(\sum_j x_j^i - \phi^i)$$

- KKT:

$$0 \leq \frac{\partial L^i(\mathbf{x})}{\partial x_j^i} = D_{cp}(x_i) + d_{ij} + x_j^i D'_{cp}(x_j) - \lambda^i$$

We have $x + (n-1)y + mz = n\xi + m\zeta = \phi$.

Let $\rho = Ix + I(n-1)y + nI\zeta$ be the amount of traffic at a super CP and let $\eta = nIz + kI\zeta$ be the amount of traffic from the independent CPs.

Assume first that x, y, z, ζ, η is an interior

$$0 = D_{cp}(\rho) + xD'_{cp}(\rho) - \lambda^i$$

$$0 = D_{cp}(\rho) + d + yD'_{cp}(\rho) - \lambda^i$$

$$0 = D_{cp}(\nu) + \delta + zD'_{cp}(\nu) - \lambda^i$$

For a subscriber of an independent ISP we have

$$0 = D_{cp}(\rho) + d + \zeta D'_{cp}(\rho) - \lambda^i$$

$$0 = D_{cp}(\nu) + \delta + \eta D'_{cp}(\nu) - \lambda^i$$

We conclude that

$$x - y = \frac{d}{D'_{cp}(\rho)}$$

We thus get

$$y = \frac{1}{n}\left(\phi - \frac{d}{D'_{cp}(I\phi)}\right)$$

$$x = \frac{1}{n}\left(\phi + (n-1)\frac{d}{D'_{cp}(I\phi)}\right)$$

This is compatible with the assumption of the theorem if

$$d \leq \phi D'_{cp}(I\phi)$$

If this is not sataisfied then at equilibrium, $y = 0$.

The cost at equilibrium is

$$C^i(x) = xD(\rho)+(n-1)y(d+D(\rho)) = \phi D(\rho)+(n-1)yd$$

$$= \phi D(\rho) + d\frac{n-1}{n}\left(\phi - \frac{d}{D'_{cp}(I\phi)}\right)$$

The value of d for which it is the largest is

$$d = \frac{\phi D'_{cp}(I\phi)}{2}$$

for which the equilibrium cost is

$$\phi D(\rho) + \frac{n-1}{n}\frac{1}{4}\phi^2 D'_{cp}(\rho)$$

The globally optimal solution is obtained at $y = 0$ for which the value is $\phi D(\rho)$.

The price of anarchy is given by

$$PoA = 1 + \frac{(n-1)\phi D'_{cp}(I\phi)}{4nD_{cp}(I\phi)}$$

In particular, let $D_{cp}(\rho) = \exp(4nsF(\rho)/(n-1))$ for some F. Then

$$D'_{cp}(\rho) = 4snD_{cp}(\rho)F'(\rho)/(n-1)$$

so that

$$PoA = 1 + \phi sF'(\rho)$$

Thus the PoA is unbounded.