Quantitative
Modeling
In
Parallel
Systems

G-Networks with multiple class of negative and positive customers *

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Abstract Recently [5] a new class of queueing networks with "negative and positive" customers was introduced and shown to have a non-standard product form. Positive customers are identical to the usual customers of a queueing network, while a negative customer which arrives to a queue simply destroys one positive customer. We call these generalized queueing networks G-Networks. In this paper we extend the model of [5] to the case of multiple class of positive customers, and multiple class of negative customers.

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As in known multiple class queueing networks a positive customer class is characterized by the routing probabilities and the service rate parameter at each service center while negative customers of different class may have diffecent deletion abilities. In the present paper all service time distributions are exponential. The service centers can be of the following types: FIFO (first-in-first-out), LIFO/PR (last-in-first-out with preemption), PS (processor sharing), with class dependent service rates.

1 Introduction

In a recent paper [5], a new class of queueing networks in which customers are either"negative" or "positive" was introduced. Positive customers enter a queue and receive service as ordinary queueing network customers, while a negative customer will vanish if it arrives to an empty queue and will reduce by one the number of positive customers in queue otherwise. Negative customers do not receive service. Positive customers which leave a queue to enter another one can become negative or remain positive. It has been shown [5] that networks of queues with a single class of positive and negative customers have a product form solution if the external positive or negative customer arrivals are Poisson, the service times of positive customers are exponential and independent, and if the movement of customers between queues is Markovian.

We shall call these generalized queueing networks "G-Networks" in order to distinguish them from the usual queueing network models. The single server queue with negative and positive customers has been examined in [6].

G-Networks can be used to represent different systems. The initial model in [5] was motivated by the analogy with neural networks [4]; each queue represents a neuron, and customers represent excitation (positive) or inhibition (negative) signals. Another possible application is to multiple resource systems: positive customers can be considered to be resource requests, while negative customers can correspond to decisions to cancel such requests.

Here we investigate networks with **multiple class of positive customers and one or more class of negative customers**. The number of negative customer classes allowed will depend on the type of service center considered. We consider three types of service centers which correspond to the following service disciplines:

- Type 1 : first-in-first-out (FIFO),
- Type 2 : processor sharing (PS),
- Type 4 : last-in-first-out with preemptive resume priority (LIFO/PR).

Here we are using the standard nomenclature of the BCMP theorem [1] in which Type 3 service centers, **which are not covered by our results**, have an infinite number of servers.

In Section 2 we prove that these multiple class G-Networks have product form. The stability conditions (i.e. necessary and sufficient conditions for the existence of a solution) for such networks have been solved recently in [7], and the result is discussed in Section 3. Indeed, it was shown in [5] that the customer flow equations of this type of network are non-linear, so that the existence of their solutions is not an obvious matter except for feed-forward networks. Thus the network stability conditions and the existence of the solutions to customer flow equations cannot be handled in a routine manner as with BCMP networks [1].

2 The model

As in [5] we are only interested in open G-Networks. Indeed, if the system is closed, then the total number of customers will decrease as long as there are negative customers in the network. We consider networks with an arbitrary number N of queues, an arbitrary number of positive customer classes K, and an arbitrary number of negative customer classes S. External arrival streams to the network are independent Poisson processes concerning positive customers of some class k or negative customers of some class c. We denote by $\Lambda_{i,k}$ the external arrival rate of positive customers of class k to queue i and by $\lambda_{i,m}$ be the external arrival rate of negative customers of class m to queue i.

Only positive customers are served, and after service they may change class, service center and nature (positive to negative), or depart from the system. The movement of customers between queues is represented by a Markov chain.

At its arrival in a non empty queue, a negative customer selects a positive customer in the queue in accordance with the service discipline at this station. If the queue is empty , then the negative customer simply disappears. Once the target is selected, the negative customer tries to destroy the selected customer. A negative customer of some class m succeed to destroy the selected positive customer of some class k at service center i with probability $K_{i,m,k}$. And with probability $(1 - K_{i,m,k})$ it does not succeed. A negative customer disappears as soon as it tries to destroy its targeted customer. Recall that a negative customer is either exogenous, or is obtained by the transformation of a positive customer as it leaves a queue.

A positive customer of class k which leaves queue i (after finishing service) goes to queue j as a positive customer of class l with probability $P^+[i,j]\ [k,l]$, or as a negative customer of class m with probability $P^-[i,j]\ [k,m]$. It may also depart from the network with probability $d[i,k]$.

We shall assume that $P^+[i,i][k,l] = 0$ and $P^-[i,i][k,m] = 0$ for all i,k,l,m so that customers are not allowed to return to the queue they have just left.Though this asssumption is without loss of generality for positive customers, it is indeed a restriction of the model for negative customers since it avoids double simultaneous customer "departures" at a queue.

Obviously we have for all i,k

$$\sum_{j=1}^{N}\sum_{l=1}^{R} P^+[i,j][k,l] \ + \ \sum_{j=1}^{N}\sum_{m=1}^{S} P^-[i,j][k,m] \ + \ d[i,k] \ = \ 1 \tag{1}$$

We assume that all service centers have exponential service time distributions. In the three types of service centers, each class of positive customers may have a distinct service rate $\mu_{i,k}$. However, when the service center is of Type 1 (FIFO) the following quantity, which takes into account the service rate, must be a constant for all class k of positive customer.

$$\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} = c_i \tag{2}$$

The following constraints on the deletion discipline and deletion probability are assumed to exist.

- In a Type 4 server, the only restriction is that the positive customer selected by a negative customer is the one in service.

- For a Type 2 server, the probability that any one positive customer of the queue is selected by the arriving negative customer is $1/c$ if c is the total number of customers in queue.
- For a Type 1 server, an arriving negative customer selects the one being served; and the following constraint must hold for all stations i of type 1 and classes of negative customers m such that $\sum_{j=1}^{N}\sum_{l=1}^{R} P^{-}[j,i][l,m] > 0$

$$\text{for all classes of positive customers} \quad \alpha, \beta, K_{i,m,\alpha} = K_{i,m,\beta} \tag{3}$$

This constraint implies that a negative customer of some class m arriving from the network does not "distinguish" between the positive customer classes it will try to delete, and that it will treat them all in the same manner

For Type 1 service centers, one may consider the following sufficient conditions which are simpler than equations 2 and 3. Note however that these conditions are more restrictive.

$$\begin{aligned} \forall \;\; k,p \text{ classes of positive customers,} \qquad & \mu_{i,k} = \mu_i \\ \forall \;\; m \text{ class of negative customers,} \qquad & K_{i,m,k} = K_{i,m,p} \end{aligned} \tag{4}$$

2.1 State representation

We shall denote the state at time t of the queueing network by the vector $x(t) = (x_1(t), ..., x_N(t))$. Here $x_i(t)$ represents the state of service center i. The vector $x = (x_1, ..., x_N)$ will denote a particular value of the state. $|x_i|$ will be the total number of customers in queue i.

For Type 1 and Type 4 servers, the instantaneous value of the state x_i of server i is represented by the vector $(r_{i,j})$ whose length is the number of customers in the queue and whose jth element is the class index of the jth customer in the queue. Furthermore, the customers are ordered according to the service order (FIFO or LIFO); it is always the customer at the head of the list which is in service. We denote by $r_{i,1}$ the class number of the customer in service and by $r_{i,\infty}$ the class number of the last customer in the queue.

For a PS (Type 2) service station, the instantaneous value of the state x_i is represented by the vector $(x_{i,k})$ which is the number of customers of class k in queue i.

2.2 Theorems

Let $\Pi(x)$ denote the stationary probability distribution of the state of the network, if it exists. The following result establishes the product form solution of the network being considered.

Theorem 1 *Consider a G-Network with the restrictions indicated above. If the system of non-linear equations*

$$q_{i,k} = \frac{\Lambda_{i,k} + \Lambda^{+}_{i,k}}{\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}[\lambda_{i,m} + \lambda^{-}_{i,m}]} \tag{5}$$

$$\Lambda^+_{i,k} = \sum_{j=1}^{N}\sum_{l=1}^{R} P^+[j,i][l,k]\mu_{j,l}q_{j,l} \quad (6)$$

$$\lambda^-_{i,m} = \sum_{j=1}^{N}\sum_{l=1}^{R} P^-[j,i][l,m]\mu_{j,l}q_{j,l} \quad (7)$$

has a solution such that

$$\textit{for each pair } i,k: \quad 0 < q_{i,k} \quad \textit{and} \quad \textit{for each station } i: \quad \sum_{k=1}^{R} q_{i,k} < 1$$

then the stationary distribution of the network state is

$$\Pi(x) = G \prod_{i=1}^{N} g_i(x_i) \quad (8)$$

where each $g_i(x_i)$ depends on the Type of service center i. The $g_i(x_i)$ in (5) have the following form :

FIFO *If the service center is of Type 1, then*

$$g_i(x_i) = \prod_{n=1}^{|x_i|} [q_{i,r_{i,n}}] \quad (9)$$

PS *If the service center is of Type 2, then*

$$g_i(x_i) = |x_i|! \prod_{k=1}^{R} \frac{(q_{i,k})^{x_{i,k}}}{x_{i,k}!} \quad (10)$$

LIFO/PR *If the service center is of Type 4, then*

$$g_i(x_i) = \prod_{n=1}^{|x_i|} [q_{i,r_{i,n}}] \quad (11)$$

and G is the normalization constant.

Sketch of the proof :

The proof of the theorem is based on algebraic manipulations of global balance equations. Note that it is not possible to use the "local balance" equations for customer classes at stations because of the effect of negative customer arrivals. We begin the proof by some technical lemmas.

Lemma 1 *The following flow equation is satisfied.*

$$\sum_{i=1}^{N}\sum_{k=1}^{R} q_{i,k}\mu_i(1-d[i,k]) = \sum_{i=1}^{N}\sum_{k=1}^{R} \Lambda^+_{i,k} + \sum_{i=1}^{N}\sum_{m=1}^{S} \lambda^-_{i,m} \quad (12)$$

Proof : by algebraic manipulations

Consider equation (6) and do the summation for all the stations and all the classes and exchange the order of summations in the right part of the equation :

$$\sum_{i=1}^{N}\sum_{k=1}^{R}\Lambda^{+}_{i,k} = \sum_{j=1}^{N}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}(\sum_{i=1}^{N}\sum_{k=1}^{R}P^{+}[j,i][l,k])$$

Similarly, using equation (7)

$$\sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m} = \sum_{j=1}^{N}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}(\sum_{i=1}^{N}\sum_{m=1}^{S}P^{-}[j,i][l,m])$$

And,

$$\sum_{i=1}^{N}\sum_{k=1}^{R}\Lambda^{+}_{i,k} + \sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m} = \sum_{j=1}^{N}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}(\sum_{i=1}^{N}\sum_{k=1}^{R}P^{+}[j,i][l,k] + \sum_{i=1}^{N}\sum_{m=1}^{S}P^{-}[j,i][l,m])$$

According to the definition of the routing matrix P (equation 1), we have

$$\sum_{i=1}^{N}\sum_{k=1}^{R}\Lambda^{+}_{i,k} + \sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m} = \sum_{j=1}^{N}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}(1-d[j,l])$$

Thus the proof of the lemma is complete.

□

To allow the algebraic manipulations of the Chapman Kolmogorov equations, we introduce some notations and first develop some intermediate results :

- The state dependent service rates for customers at service center j will be denoted by $M_{j,l}(x_j)$ where x_j refers to the state of the service center and l is the class of the customer concerned. From the definition of the service rate $\mu_{j,l}$, we obtain for the three types of stations :

  FIFO and LIFO/PR $M_{j,l}(x_j) = \mu_{j,l}1_{\{r_{j,1}=l\}}$,

  PS $M_{j,l}(x_j) = \mu_{j,l}\frac{x_{j,l}}{|x_j|}$.

- $N_{j,l}(x_j)$ is the deletion rate of class l positive customers due to external arrivals of all the classes of negative customers

  FIFO and LIFO/PR $N_{j,l}(x_j) = 1_{\{r_{j,1}=l\}}\sum_{m=1}^{S}K_{j,m,l}\lambda_{j,m}$

  PS $N_{j,l}(x_j) = \frac{x_{j,l}}{|x_j|}\sum_{m=1}^{S}K_{j,m,l}\lambda_{j,m}$.

- $A_{j,l}(x_j)$ is the condition which establishes that it is possible to reach state x_j by an arrival of a positive customer of class l

FIFO $A_{j,l}(x_j) = 1_{\{r_{j,\infty}=l\}}$,

LIFO/PR $A_{j,l}(x_j) = 1_{\{r_{j,1}=l\}}$,

PS $A_{j,l}(x_j) = 1_{\{|x_{j,l}|>0\}}$.

- $Z_{j,l,m}(x_j)$ is the probability that a negative customer of class m, arriving from the network, will delete a positive customer of class l.

  FIFO and LIFO/PR $Z_{j,l,m}(x_j) = 1_{\{r_{j,1}=l\}} K_{j,m,l}$

  PS $Z_{j,l,m}(x_j) = \frac{x_{j,l}}{|x_j|} K_{j,m,l}$.

- $Y_{j,m}(x_j)$ is the probability that a negative customer of class m which enters a non empty queue, will not delete a positive customer.

  FIFO and LIFO/PR $Y_{j,m}(x_j) = \sum_{l=1}^{R} 1_{\{r_{j,1}=l\}}(1 - K_{j,m,l})$

  PS $Y_{j,m}(x_j) = \sum_{l=1}^{R}(1 - K_{j,m,l})\frac{x_{j,l}}{|x_j|}$.

Denote by $(x_j + e_{j,l})$ the state of station j obtained by **adding into the server a positive customer of class** l. Denote by by $(x_i - e_{i,k})$ the state obtained by removing from the end of the list a class k customer (if it exists, since otherwise $(x_i - e_{i,k})$ will not be defined).

Lemma 2 *For any Type (1,2,4) of service center, the following relations hold*

$$M_{j,l}(x_j + e_{j,l})\frac{g_j(x_j + e_{j,l})}{g_j(x_j)} = \mu_{j,l} q_{j,l} \tag{13}$$

$$N_{j,l}(x_j + e_{j,l})\frac{g_j(x_j + e_{j,l})}{g_j(x_j)} = \sum_{m=1}^{S} K_{j,m,l} q_{j,l} \tag{14}$$

$$Z_{j,l,m}(x_j + e_{j,l})\frac{g_j(x_j + e_{j,l})}{g_j(x_j)} = K_{j,m,l} q_{j,l} \tag{15}$$

The proof is purely algebraic.

□

Remark : As a consequence, we have from equations (6), (7) and (13):

$$\Lambda^{+}_{i,k} = \sum_{j=1}^{N}\sum_{l=1}^{R} M_{j,l}(x_j + e_{j,l})\frac{g_j(x_j + e_{j,l})}{g_j(x_j)} P^{+}[j,i][l,k] \tag{16}$$

and

$$\lambda^{-}_{i,m} = \sum_{j=1}^{N}\sum_{l=1}^{R} M_{j,l}(x_j + e_{j,l})\frac{g_j(x_j + e_{j,l})}{g_j(x_j)} P^{-}[j,i][l,m] \tag{17}$$

Lemma 3 *Consider an arbitrary station which satisfies the constraints, and let us denote* $\Delta_i(x_i)$ *as the following :*

$$\begin{aligned}\Delta_i(x_i) = & \sum_{m=1}^{S} \lambda_{i,m}^{-} Y_{i,m}(x_i) \\ - & \sum_{k=1}^{R} (M_{i,k}(x_i) + N_{i,k}(x_i)) \\ + & \sum_{k=1}^{R} A_{i,k}(x_i)(\Lambda_{i,k} + \Lambda_{i,k}^{+}) \frac{g_i(x_i - e_{i,k})}{g_i(x_i)}\end{aligned}$$

Then for the three types of service centers, $1_{\{|x_i|>0\}}\Delta_i(x_i)$ *is equal to* $\sum_{m=1}^{S} \lambda_{i,m}^{-} 1_{\{|x_i|>0\}}$ *.*

The proof of the lemma is in the appendix.

□

Let us now proceed with the proof of the theorem. Consider the global balance equation for an arbitrary G-Network.

$$
\begin{aligned}
&\Pi(x)[\sum_{j=1}^{N}\sum_{l=1}^{R}(\Lambda_{j,l} + M_{j,l}(x_j)1_{\{|x_j|>0\}} + N_{j,l}(x_j)1_{\{|x_j|>0\}})] \\
&= \sum_{j=1}^{N}\sum_{l=1}^{R} \Pi(x - e_{j,l})\Lambda_{j,l}A_{j,l}(x_j)1_{\{|x_j|>0\}} \\
&+ \sum_{j=1}^{N}\sum_{l=1}^{R} \Pi(x + e_{j,l})N_{j,l}(x_j + e_{j,l}) \\
&+ \sum_{j=1}^{N}\sum_{l=1}^{R} \Pi(x + e_{j,l})M_{j,l}(x_j + e_{j,l})d[j,l] \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{R}\sum_{l=1}^{R} M_{j,l}(x_j + e_{j,l})\Pi(x - e_{i,k} + e_{j,l})P^{+}[j,i][l,k]A_{i,k}(x_i)1_{\{|x_i|>0\}} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{R}\sum_{l=1}^{R}\sum_{m=1}^{S} M_{j,l}(x_j + e_{j,l})\Pi(x + e_{i,k} + e_{j,l})P^{-}[j,i][l,m]Z_{i,k,m}(x_i + e_{i,k}) \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} M_{j,l}(x_j + e_{j,l})\Pi(x + e_{j,l})P^{-}[j,i][l,m]Y_{i,m}(x_i)1_{\{|x_i|>0\}} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} M_{j,l}(x_j + e_{j,l})\Pi(x + e_{j,l})P^{-}[j,i][l,m]1_{\{|x_i|=0\}}
\end{aligned}
$$

We divide both sides by $\Pi(x)$ and we assume that there is a product form solution. Then, we apply lemma 2.

$$
\begin{aligned}
&\sum_{j=1}^{N}\sum_{l=1}^{R}(\Lambda_{j,l} + M_{j,l}(x_j)1_{\{|x_j|>0\}} + N_{j,l}(x_j)1_{\{|x_j|>0\}}) \\
&= \sum_{j=1}^{N}\sum_{l=1}^{R} \frac{g_j(x_j - e_{j,l})}{g_j(x_j)}\Lambda_{j,l}A_{j,l}(x_j)1_{\{|x_j|>0\}} \\
&+ \sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} \lambda_{j,m}K_{j,m,l}q_{j,l} \quad + \quad \sum_{j=1}^{N}\sum_{l=1}^{R} \mu_{j,l}q_{j,l}d[j,l] \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{R}\sum_{l=1}^{R} \mu_{j,l}q_{j,l}P^{+}[j,i][l,k]A_{i,k}(x_i)\frac{g_i(x_i - e_{i,k})}{g_i(x_i)}1_{\{|x_i|>0\}} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{R}\sum_{l=1}^{R}\sum_{m=1}^{S} \mu_{j,l}q_{j,l}P^{-}[j,i][l,m]K_{i,m,k}q_{i,k} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} \mu_{j,l}q_{j,l}P^{-}[j,i][l,m]Y_{i,m}(x_i)1_{\{|x_i|>0\}} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} \mu_{j,l}q_{j,l}P^{-}[j,i][l,m]1_{\{|x_i|=0\}}
\end{aligned}
$$

After some subsitution, we group the first and the fourth terms of the right side of the equation.

$$
\begin{aligned}
&\sum_{j=1}^{N}\sum_{l=1}^{R}(\Lambda_{j,l}+M_{j,l}(x_j)1_{\{|x_j|>0\}}+N_{j,l}(x_j)1_{\{|x_j|>0\}})\\
&= \sum_{j=1}^{N}\sum_{l=1}^{R}1_{\{|x_j|>0\}}\frac{g_j(x_j-e_{j,l})}{g_j(x_j)}A_{j,l}(x_j)(\Lambda_{j,l}+\Lambda^{+}_{j,l})\\
&+ \sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S}\lambda_{j,m}K_{j,m,l}q_{j,l}\\
&+ \sum_{j=1}^{N}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}d[j,l]\\
&+ \sum_{i=1}^{N}\sum_{k=1}^{R}\sum_{m=1}^{S}\lambda^{-}_{i,m}K_{i,m,k}q_{i,k}\\
&+ \sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m}Y_{i,m}(x_i)1_{\{|x_i|>0\}}\\
&+ \sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m}1_{\{|x_i|=0\}}
\end{aligned}
$$

We add to both sides the quantity $\sum_{j=1}^{N}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}(1-d[j,l])$ and factorize three terms in the rigth side

$$
\begin{aligned}
&\sum_{j=1}^{N}\sum_{l=1}^{R}(\Lambda_{j,l}+M_{j,l}(x_j)1_{\{|x_j|>0\}}+N_{j,l}(x_j)1_{\{|x_j|>0\}})+\mu_{j,l}q_{j,l}(1-d[j,l])\\
&= \sum_{j=1}^{N}\sum_{l=1}^{R}1_{\{|x_j|>0\}}\frac{g_j(x_j-e_{j,l})}{g_j(x_j)}A_{j,l}(x_j)(\Lambda_{j,l}+\Lambda^{+}_{j,l})\\
&+ \sum_{j=1}^{N}\sum_{l=1}^{R}q_{j,l}(\mu_{j,l}+\sum_{m=1}^{S}\lambda_{j,m}K_{j,m,l}+\sum_{m=1}^{S}\lambda^{-}_{j,m}K_{j,m,l})\\
&+ \sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m}Y_{i,m}(x_i)1_{\{|x_i|>0\}}\\
&+ \sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m}1_{\{|x_i|=0\}}
\end{aligned}
$$

We substitute the value of $q_{i,k}$ in the second term. Then, we cancel the term $\Lambda_{j,l}$ which appears on both sides and we group terms to obtain :

$$
\sum_{j=1}^{N}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}(1-d[j,l])=\sum_{j=1}^{N}\sum_{l=1}^{R}\Lambda^{+}_{j,l}+\sum_{i=1}^{N}1_{\{|x_i|>0\}}\Delta_i(x_i)+\sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m}1_{\{|x_i|=0\}} \quad (18)
$$

where

$$
\begin{aligned}
\Delta_i(x_i) &= \sum_{m=1}^{S} \lambda_{i,m}^{-} Y_{i,m}(x_i) \\
&- \sum_{k=1}^{R} M_{i,k}(x_i) - \sum_{k=1}^{R} N_{i,k}(x_i) \\
&+ \sum_{k=1}^{R} A_{i,k}(x_i)(\Lambda_{i,k} + \Lambda_{i,k}^{+}) \frac{g_i(x_i - e_{i,k})}{g_i(x_i)}
\end{aligned}
$$

In Lemma 3, we have shown that $1_{\{|x_i|>0\}}\Delta_i(x_i)$ is equal to $\sum_{m=1}^{S} \lambda_{i,m}^{-} 1_{\{|x_i|>0\}}$ for the three types of service centers. Thus,

$$
\begin{aligned}
&\sum_{j=1}^{N}\sum_{l=1}^{R} \mu_{j,l} q_{j,l}(1 - d[j,l]) = \\
&\quad \sum_{j=1}^{N}\sum_{l=1}^{R} \Lambda_{j,l}^{+} \; + \; \sum_{i=1}^{N}\sum_{m=1}^{S} \lambda_{i,m}^{-}(1_{\{|x_i|=0\}} + 1_{\{|x_i|>0\}})
\end{aligned}
$$

Finaly, Lemma 1 shows that this flow equation is satisfied. This concludes the proof.

□

As in the BCMP [1] theorem, we can also establish the steady state distribution of the number of customers of each class in each queue. Define y_i as the vector $(y_{i,k})$ of the number of customers of class k in station i. And let us denote by y the vector of the state (y_i).

Theorem 2 *If the system of equations (5), (6) and (7) has a solution then, the steady state distribution $\pi(y)$ is given by*

$$\pi(y) = \prod_{i=1}^{N} h_i(y_i) \tag{19}$$

where the marginal probabilities $h_i(y_i)$ have the following form :

$$h_i(y_i) = (1 - \sum_{k=1}^{R} q_{i,k})|y_i|! \prod_{k=1}^{R} [(q_{i,k})^{y_{i,k}} / y_{i,k}!] \tag{20}$$

3 Existence of the Solution

Unlike BCMP or Jackson networks [1], the customer flow equations (5), (6) and (7) of the model we consider are non-linear. Therefore in general it is not easy to determine the necessary and sufficient conditions for their existence. The stability conditions of the network, i.e. the conditions under which the stationary solution exists, are closely related to the existence conditions for the flow equations.

Finaly, one may compute the values of $q_{i,k}$ using the following iterative scheme. This algorithm is based on the iterative computation of upper and lower bounds, respectively denoted as $\overline{[q_{i,k}]}_n$ and $\underline{[q_{i,k}]}_n$.

$$\begin{aligned}
\overline{[q_{i,k}]}_{n+1} &= min(1, \frac{\Lambda_{i,k} + \sum_{j=1}^{N} \sum_{l=1}^{R} P^{+}[j,i][l,k]\overline{[q_{j,l}]}_n}{\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}[\lambda_{i,m} + \sum_{j=1}^{N} \sum_{l=1}^{R} P^{-}[j,i][l,m]\underline{[q_{j,l}]}_n]}) \\
\underline{[q_{i,k}]}_{n+1} &= min(1, \frac{\Lambda_{i,k} + \sum_{j=1}^{N} \sum_{l=1}^{R} P^{+}[j,i][l,k]\underline{[q_{j,l}]}_n}{\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}[\lambda_{i,m} + \sum_{j=1}^{N} \sum_{l=1}^{R} P^{-}[j,i][l,m]\overline{[q_{j,l}]}_n]})
\end{aligned} \quad (21)$$

And we initialize the sequences of bounds with the following trivial bounds ($\overline{[q_{i,k}]}_0 = 1$ and $\underline{[q_{i,k}]}_0 = 0$). In [2], the author has proved that both bounds tend to the same limit. Furthermore, if the solution of equation 5 satisfies the stationnarity constraint $\sum_{k=1}^{R} q_{i,k} < 1$, then it is proved to be equal to the limit of the iterative scheme.

4 Conclusions

In this paper we have considered networks of queues with multiple class of positive customers and multiple class of negative customers. We have shown that this new type of networks has product form when service centers are of BCMP type, except for the infinite server case which we do not cover, with class dependent exponential service time distributions (except at FIFO service centers).

Further extensions of these results to more complex service distributions or more complex interactions between positive and negative customers remain to be done.

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5 Appendix

Proof of Lemma 3 : The proof consists in algebraic manipulations for the three types of stations.

LIFO/PR First consider an arbitrary LIFO station et remember the definition of Δ_i :

$$\begin{aligned}
1_{\{|x_i|>0\}}\Delta_i(x_i) &= 1_{\{|x_i|>0\}} \sum_{k=1}^{R} A_{i,k}(x_i)(\Lambda_{i,k}+\Lambda_{i,k}^{+})\frac{g_i(x_i-e_{i,k})}{g_i(x_i)} \\
&- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} M_{i,k}(x_i) \;-\; 1_{\{|x_i|>0\}} \sum_{k=1}^{R} N_{i,k}(x_i) \\
&+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda_{i,m}^{-} Y_{i,m}(x_i)
\end{aligned}$$

Then, we substitute the values of $Y_{i,m}$, $M_{i,k}$, $N_{i,k}$ and $A_{i,k}$ for a LIFO station :

$$\begin{aligned}
1_{\{|x_i|>0\}}\Delta_i(x_i) &= 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}\ (\Lambda_{i,k}+\Lambda_{i,k}^{+})/q_{i,k} \\
&- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}\ \mu_{i,k} \\
&- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} \\
&+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda_{i,m}^{-} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}(1-K_{i,m,k})
\end{aligned}$$

We use the value of $q_{i,k}$ from equation 5 to obtain after some cancellations of terms :

$$\begin{aligned}
1_{\{|x_i|>0\}}\Delta_i(x_i) &= 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}\left(\sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m}^{-} + \sum_{m=1}^{S} \lambda_{i,m}^{-}(1-K_{i,m,k})\right) \\
&= 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda_{i,m}^{-} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}
\end{aligned}$$

and as $1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} = 1_{\{|x_i|>0\}}$, we finaly get the result :

$$1_{\{|x_i|>0\}}\Delta_i(x_i) = 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda_{i,m}^{-} \tag{22}$$

FIFO Consider now an arbitrary FIFO station :

$$\begin{aligned}1_{\{|x_i|>0\}}\Delta_i(x_i) &= 1_{\{|x_i|>0\}} \sum_{k=1}^{R} A_{i,k}(x_i)(\Lambda_{i,k}+\Lambda^{+}_{i,k})\frac{g_i(x_i-e_{i,k})}{g_i(x_i)} \\ &- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} M_{i,k}(x_i) \quad - \quad \sum_{k=1}^{R} 1_{\{|x_i|>0\}} N_{i,k}(x_i) \\ &+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda^{-}_{i,m} Y_{i,m}(x_i)\end{aligned}$$

Similarly, we substitute the values of $Y_{i,m}$, $M_{i,k}$, $N_{i,k}$, $A_{i,k}$ and $q_{i,k}$:

$$\begin{aligned}1_{\{|x_i|>0\}}\Delta_i(x_i) &= 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,\infty}=k\}} \ (\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} + \sum_{m=1}^{S} K_{i,m,k}\lambda^{-}_{i,m}) \\ &- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} \ \mu_{i,k} \quad - \quad 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} \\ &+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda^{-}_{i,m} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}(1-K_{i,m,k})\end{aligned}$$

We divide in two parts the last term and regroup terms

$$\begin{aligned}1_{\{|x_i|>0\}}\Delta_i(x_i) &= 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,\infty}=k\}} \ (\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} + \sum_{m=1}^{S} K_{i,m,k}\lambda^{-}_{i,m}) \\ &- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} \ (\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} + \sum_{m=1}^{S} K_{i,m,k}\lambda^{-}_{i,m}) \\ &+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda^{-}_{i,m} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}\end{aligned}$$

But the assumptions of theorem 1 implies that we have the following relation :

$$\begin{aligned}&\sum_{k=1}^{R} 1_{\{r_{i,\infty}=k\}} \ (\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} + \sum_{m=1}^{S} K_{i,m,k}\lambda^{-}_{i,m}) = \\ &\quad \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} \ (\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} + \sum_{m=1}^{S} K_{i,m,k}\lambda^{-}_{i,m})\end{aligned}$$

Thus, as $1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} = 1_{\{|x_i|>0\}}$, we finaly get the expected result :

$$1_{\{|x_i|>0\}}\Delta_i(x_i) = 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda^{-}_{i,m} \tag{23}$$

PS Consider now an arbitrary PS station :

$$
\begin{aligned}
1_{\{|x_i|>0\}}\Delta_i(x_i) &= 1_{\{|x_i|>0\}} \sum_{k=1}^{R} A_{i,k}(x_i)(\Lambda_{i,k}+\Lambda^+_{i,k})\frac{g_i(x_i-e_{i,k})}{g_i(x_i)} \\
&- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} M_{i,k}(x_i) \quad - \quad \sum_{k=1}^{R} 1_{\{|x_i|>0\}} N_{i,k}(x_i) \\
&+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda^-_{i,m} Y_{i,m}(x_i)
\end{aligned}
$$

As usual, we subtitute the values of $Y_{i,m}$, $M_{i,k}$, $N_{i,k}$, $A_{i,k}$:

$$
\begin{aligned}
1_{\{|x_i|>0\}}\Delta_i(x_i) &= 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{|x_{i,k}|>0\}} \ \frac{(\Lambda_{i,k}+\Lambda^+_{i,k})}{q_{i,k}} \frac{x_{i,k}}{|x_i|} \\
&- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} \ \mu_{i,k} \ \frac{x_{i,k}}{|x_i|} \\
&- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} \frac{x_{i,k}}{|x_i|} \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} \\
&+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \sum_{k=1}^{R} \lambda^-_{i,m} \frac{x_{i,k}}{|x_i|}(1-K_{i,m,k})
\end{aligned}
$$

Then, we apply equation 5 to substitute $q_{i,k}$. After some cancelations of terms we obtain :

$$
\begin{aligned}
1_{\{|x_i|>0\}}\Delta_i(x_i) &= 1_{\{|x_i|>0\}} \sum_{k=1}^{R} \frac{x_{i,k}}{|x_i|} \sum_{m=1}^{S} K_{i,m,k}\lambda^-_{i,m} \\
&+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \sum_{k=1}^{R} \lambda^-_{i,m} \frac{x_{i,k}}{|x_i|}(1-K_{i,m,k})
\end{aligned}
$$

And finaly,

$$
1_{\{|x_i|>0\}}\Delta_i(x_i) = 1_{\{|x_i|>0\}} \sum_{k=1}^{R} \frac{x_{i,k}}{|x_i|} \sum_{m=1}^{S} \lambda^-_{i,m} \tag{24}
$$

As $1_{\{|x_i|>0\}} \sum_{k=1}^{R} \frac{x_{i,k}}{|x_i|} = 1_{\{|x_i|>0\}}$, once again, we establish the relation we need. This concludes the proof of lemma 3.

□