1 Introduction

Timed event graphs can be seen as linear dynamic systems in the so-called $(\max,+)$-algebra [2]. The algebraic formalism used in the above approach provides a rich variety of results on stability, control and computational problems arising in both deterministic and stochastic event graphs. The aim of the present paper is to show that a similar algebraic approach can partially be generalized to free choice nets. The whole analysis is based on a decomposition of the net into its maximal $(\max,+)$-linear (i.e. event graph) subnetworks and a non-linear "routing" subnet; this decomposition is introduced together with basic definitions on Petri nets in Section 2. Several results obtained in the paper bear on general free choice nets, including the decomposition and evolution equations for the timed case. However, the main focus of the paper is the class of single-input free choice nets (SI-FCN), where all event-graph subnets arising in the decomposition are single-input $(\max,+)$-linear systems.

This decomposition is first used in Section 3, in order to analyze structural properties of *unbounded* free choice nets. In the case of SI-FCNets, we show that Commoner's characterization of liveness in terms of siphons and traps [6] (i.e. the network is live iff each siphon contains a trap with a non-empty marking), when added to the above decomposition, leads to polynomial time algorithms (in the size of the net) for checking liveness. The analysis is mainly based on graph theory.

The decomposition is then used in Section 4 to derive the evolution equations for the timed case. Up to a minor transformation, each free choice net admits a canonical representation in terms of a coupling of two linear systems. The first one pertains to the event graph components, and satisfies a non-autonomous $(\min,+)$-linear evolution equation with an input from the second system, whereas the second one is "essentially" a nonautonomous $(+,\times)$-linear system with an input from the first system. A few basic properties of this class of evolution equations are established, and in particular monotonicity and conservation properties which play a key role in the analysis of stability.

The last problem which is analyzed in the paper is stability (Sections 5 and 6). Stability is the property that a timed (resp. stochastic) unbounded Petri net may have, and which implies that its marking remains bounded (resp. bounded in probability). This is not a structural property in the sense that it strongly depends both on the initial marking and on the timing variables used in the net. The main result is a set of quasi necessary and sufficient conditions for stability for non-autonomous SI-FCNets. The conditions are expressed in terms of inequalities between two types of linear characteristics of the net: the $(\max,+)$-Perron Frobenius maximal eigenvalues (resp. the Lyapunov exponents) of the $(\max,+)$-linear subsystems on one side, and certain mean values pertaining to the $(+,\times)$ subsystem on the other side. Whenever this condition is fulfilled, we also show that the marking process converges to a periodic (resp. stationary) integer-valued process (resp. stochastic process). The main tools used in this part are the results already known for $(\max, +)$-linear systems and an ergodic theory construction which relies on the monotonicity properties of the evolution equations. A few special cases of this stability result are already known in the literature: for Jackson queueing networks, which happen to be SISO FCNets (in fact, all linear subsystems are here single input, single output, one dimensional systems), the stability condition boils down to comparing the mean service rate of each queue with the solution of the so called traffic equation (see [3]). So, the set of inequalities which characterize stability in the general SI-FCNet case can also be seen as a generalization of the well known conditions for stability in the Jackson network case. First order ergodic theorems (showing the existence of rates under general conditions) were also obtained in the SISO case in [7].

2 Timed Petri net

2.1 Dynamics of Petri Nets

A Petri net is a t-uple $(\mathcal{P}, \mathcal{T}, \mathcal{C}, \mathcal{M}_0)$ where $\mathcal{P}$ is the set of places, $\mathcal{T}$ is the set of transitions, $\mathcal{C}$ the set of arcs between places and transitions or between transitions and places ($\mathcal{C}$ is a subset of $\mathcal{P} \times \mathcal{T} \cup \mathcal{T} \times \mathcal{P}$). $\mathcal{M}_0$ is the initial marking in the places. We denote by ${}^\bullet t$ the set $\{p \in \mathcal{P} : (p, t) \in \mathcal{C}\}$ (i.e. the set of all input places of t). We define similarly the sets $t^\bullet$, ${}^\bullet p$, $p^\bullet$ as the set of output places of t, the set of input transitions of p and the set of output transitions of p, respectively.

A timed Petri net is a Petri net with temporal data attached to transition: $\sigma^t(n)$ is a data which gives the duration of the n-th firing of transition t. This means that if transition t begins to fire for the n-th time at epoch e, this firing will end at epoch $e + \sigma^t(n)$; tokens are then taken out of input places and put in output places of t according to the firing rule of the untimed Petri net.

For more on the matter, and in particular for the definitions of *deadlocks, liveness, structural liveness* etc. which are used in the paper, the reader is advised to consult the survey paper by Murata [11].

2.2 Free Choice Nets

Free choice nets (FCN) are Petri nets verifying the following conditions: $\forall p \in \mathcal{P}, t_1, t_2 \in p^\bullet, t_1 \neq t_2, \ {}^\bullet t_1 = {}^\bullet t_2 = \{p\}$. In other words, whenever two transitions share an input place, they have no other input place.

Free choice nets have been extensively studied in the 70's [6] and have regained interest recently [8], [12] because they constitute a nice compromise between power of description and tractability of problems.

Several 'semantics' can be used for the resolution of conflicts. The most common is called the *race policy.* Another one called the *routing policy* was introduced in [1].

Let p be a place with several output transitions.

In the case of the race policy, the resolution of conflictsis purely based on the temporal data. For free choice nets, this policy boils down to the following: as soon as the n-th token enters place p, the transitions of $p^\bullet$ immediately start their n-th firing (this is possible since they have no other pre-conditions by hypothesis), which takes $\sigma^t(n)$ for transition t. The transition which completes first wins the race for the n-th token, and consumes this token.

In the case of the routing policy, *routing data* are attached with each place with several output transitions. Place p has a routing sequence $\nu^p : \mathbb{N} \rightarrow p^\bullet$, where $\nu^p(n)$ gives the transition $t \in p^\bullet$ to which the n-th token to enter place p is routed. The routing sequences can be periodic or random. If this token is the k-th to be routed to transition $t \in p^\bullet$, then this token is immediately consumed by the transition (due to the FC property) where it experiences a firing time of $\sigma^t(k)$.

In fact, *in the free choice case*, both policies are equivalent. Assume that the net evolves according to the race policy. then the n-th token to enter place p is routed to the transition

$$\nu^p(n) = \mathrm{argmin}_{t \in p^\bullet} \sigma^t(n),$$

(at least whenever this argument is unique). This function can also be seen as predefined routing data (since the σ^t variables are part of the network data), and so an adequate renumbering of the firing times allows to view race as a special case of routing. The converse construction is also easy to do.

In what follows, we shall adopt the routing semantics.

2.3 Decomposition into Marked Graph Components

A place p in a FCN F is *serial* if $|^\bullet p| = |p^\bullet| = 1$. Let $\mathcal{A}$ be the set of transitions such that all their upstream places are serial, and let $\mathcal{B}$ be the set of transitions which do not belong to $\mathcal{A}$.

First we define a relation $\mathcal{L}$ by: $t, t' \in T$, $t\mathcal{L}t'$ if there is a serial place p verifying $\{^\bullet p, p^\bullet\} = \{t, t'\}$. Let $\mathcal{K}$ be the transitive closure of $\mathcal{L}$. $\mathcal{K}$ is a parallelism relation. We partition the set of transitions $\mathcal{T}$ into its maximal $\mathcal{K}$-classes, $\mathcal{T}_1, \cdots, \mathcal{T}_n$. We construct a decomposition of F in the following way: $\mathcal{P}_i = \{p \in \mathcal{P} | p \text{ serial and } ^\bullet p, p^\bullet \in \mathcal{T}_i\}$, for all i.

The marked graph component (MGC) G_i of F is the sub-Petri net of F $(\mathcal{P}_i, \mathcal{T}_i, \mathcal{C} \cap (\mathcal{P}_i \times \mathcal{T}_i \cup \mathcal{T}_i \times \mathcal{P}_i))$. One can easily check that G_i is a marked graph and is maximal in the sense that no marked graph included in F contains G_i, except G_i itself. A marked graph component G_i is degenerated if G_i is reduced to a single transition (with no places).

The places which do not belong to any component G_i are the places with several input transitions and/or several output transitions. These places will be called *routing* places in the following. The set of routing places is denoted $\mathcal{R}$.

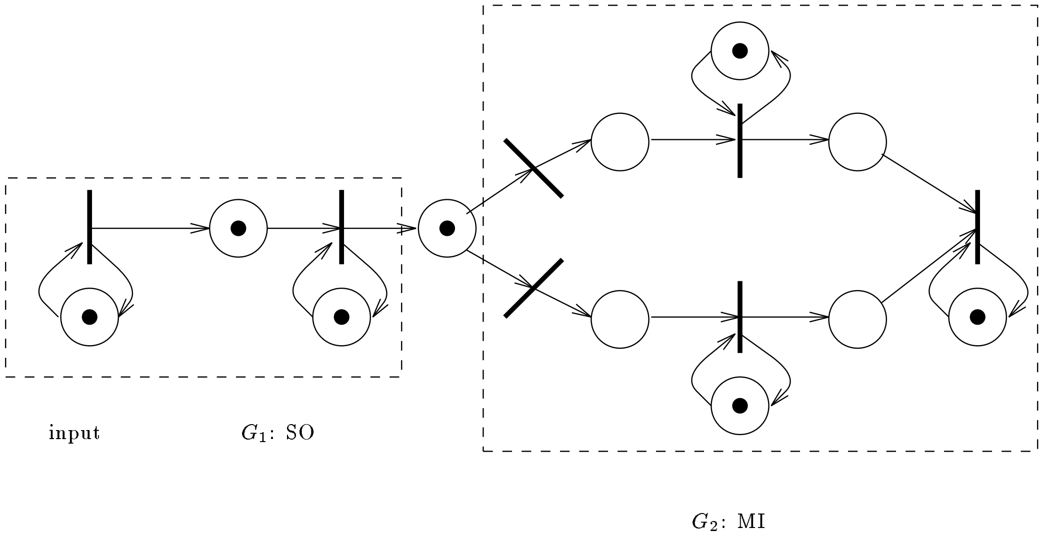


Figure 1: This FCN is decomposed into two MGC's.

In a first step, we will only consider networks with an exogeneous input into some places (although the case when input can also take place in transitions, will also be used and analyzed in the paper). Any place with such an input will be put into the set of routing places, by definition.

Some examples of decompositions are given in figures 1 and 2.

2.4 The Routing Net

If $F = (\mathcal{P}, \mathcal{T}, \mathcal{C})$ be a free choice net, we define its *routing graph* $H = (\mathcal{R}, \mathcal{G}, \mathcal{C}', w)$ as follows: $\mathcal{R}$ is the set of routing places, $\mathcal{G}$ is the set of MGC's and $\mathcal{C}'$ is the set of arcs in $\mathcal{C}$ between routing places and MGC's. The routing net is a weighted Petri net which is obtained by replacing each MGC by a transition and keeping only the routing places. If there is an arc from the routing place p to MGC G (resp. from G to p) in the original net, we put an arc from p to transition G in the routing net (resp. from G to p). The weight $w_{p,G}$ (resp. $w_{G,p}$) of this arc equals the number of arcs from p to G, i.e. $\#(p^\bullet \cap G)$ (resp. from G to p) in the original FCN.

Note that G denotes a transition in the routing net as well as a MGC in the original net. Similarly, p denotes a place in the routing net as well as in the original one. We also use the notation $G \in p^\star$ (resp. $G \in^\star p$), if there is an arc from p to G (resp. from G to p) in the routing net.

Finally, If E is a set of places in the original FCN, E_r denotes its projection on the routing graph, i.e. $E_r = E \cap R$.

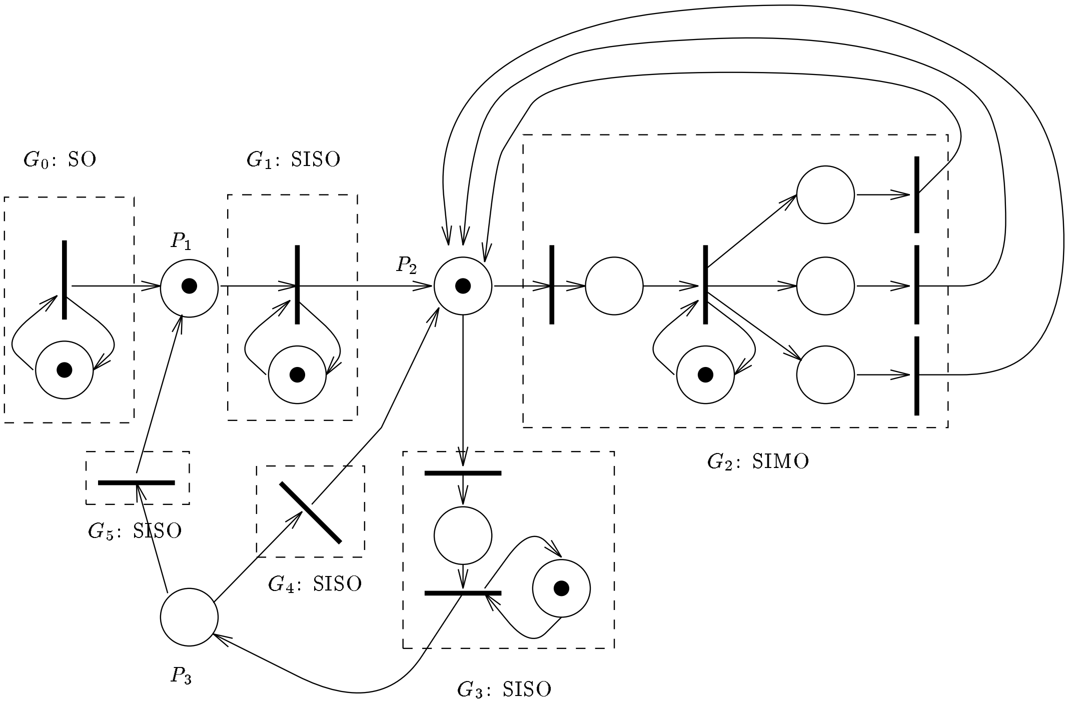


Figure 2: This is a SI-FCN. Indeed all its MGC's are SI.

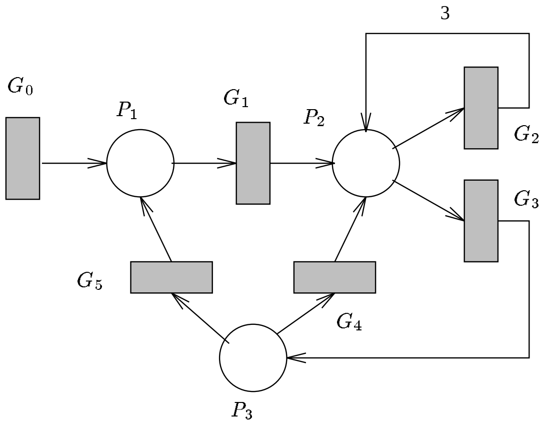


Figure 3: Since the graph F is a SI-FCN, its routing graph can be interpreted as a weighted free choice net. The default weight on an arc is one and only the weights different than 1 are explicitly written.

The routing net of the FCN depicted in Figure 2 is shown in Figure 3. Note that the routing net of a FCN is a FCN.

2.5 Classification of Free Choice Nets

We propose a classification of the marked graph components of a FCN based upon its links with the routing places.

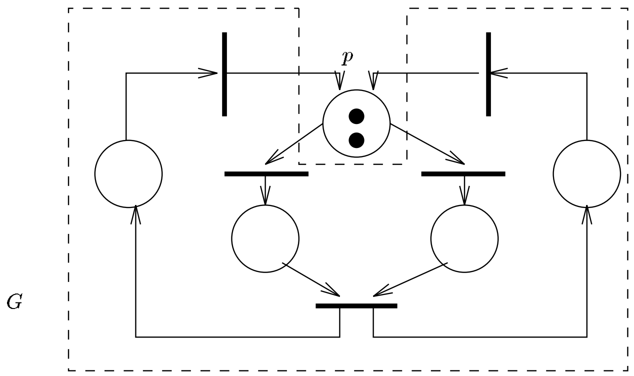
A MGC G_i is said *Single Input* (SI) if $\#\{t \in \mathcal{T}_i, {}^\bullet t \not\subset \mathcal{P}_i\} = 1$. In this case, we will denote $t^{(i)}$ and call *input transition* this unique transition of G_i. G_i is said *Multiple Input* (MI) if $\#\{t \in \mathcal{T}_i, {}^\bullet t \not\subset \mathcal{P}_i\} > 1$. A MGC G_i is said *Single Output* (SO) if $\#\{t \in \mathcal{T}_i, t^\bullet \not\subset \mathcal{P}_i\} = 1$. A MGC G_i is said *Multiple Output* (MO) if $\#\{t \in \mathcal{T}_i, t^\bullet \not\subset \mathcal{P}_i\} > 1$.

Thus all MGC's of a FCN can be put in one of the four classes, SISO, SIMO, MISO, MIMO. A FCN is said SI (resp. SO) if all its MGC's are SI (resp. SO) and MI (resp. MO) otherwise.

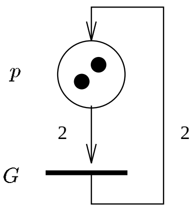
Note that if F is a SI-FCN, then its routing net H is a SI FCN.

3 Structural Properties of SI-FCN

In this section we show that certain structural properties of a FCN can be checked on its routing net. We will mainly focus on structural liveness (see [11]), but other properties like boundedness can also be tested at the routing net level only.



A non live MIMO-FCN



Its routing graph is live

Figure 4: This FCN is composed by only one routing place p and one MGC G. The routing net is live and the original net is not.

Note that if the FCN is not in the SI class, the routing net does not help to decide on the liveness of the original FCN. We illustrate this property the example in Figure 4, which depicts a FCN and its routing net. The FCN is not live whereas the routing net is live.

3.1 Traps, Siphons and Commoner's Liveness Condition

In [6], a necessary and sufficient condition of liveness for simple FCN is given. This condition uses the notions of siphons and traps.

Definition 1 *A siphon is a set of places S verifying ${}^\bullet S \subset S^\bullet$. A trap is a set of places T verifying $T^\bullet \subset {}^\bullet T$.*

Commoner established the following theorem:

Theorem 1 *A (non-weighed) FCN is live if and only if every siphon contains a marked trap.*

The structural liveness of a FCN can be characterized by checking whether *every siphon contains a trap.*[1]

A siphon is minimal if it does not contain a smaller siphon. Since the union of two traps is a trap, we can talk about the maximal trap included in a set of places. The structural version of Theorem 1 can easily be reduced to the following equivalent version:

A simple FCN is structurally live if and only if the maximal trap included in every minimal siphon is not the empty set.

Once a siphon S is given, one can test in *quadratic* time (in the number of places of the siphon) whether this siphon is structurally live (i.e. whether the maximal trap included in S is the empty set or not), by running the following (at most quadratic) procedure on siphon S:

1 Tag all the places which are not in S.

2 Tag all the places of S which have one downstream transition leading only to already tagged places. (i.e. all the places $p \in S$ s.t. there exists one transition t in $p^\bullet$ with $t^\bullet$ made of tagged places only.

3 Repeat step 2 until no new place is tagged.

Lemma 1 *The places which remain untagged when the run is complete form the maximal trap U included in siphon S.*

Proof: First we prove that U is a trap. If U is empty, then U is a trap. If U is not empty, let $p \in U$. By construction of U, for each $t \in p^\bullet$, $t^\bullet \cap U \neq \emptyset$. This is the definition of a trap.

Then, we prove that this trap is maximal. We prove that if p is tagged then no trap containing p is included in S. The proof is by induction on i, the step at which p has been tagged.

If p was tagged at step 1, p does not belong to S, therefore, a trap containing p cannot be included in S.

Suppose that a place that has been tagged at step j, $j \leq i$ does not belong to a trap included in S. Let p be a place tagged at step $i+1$. Then by construction, there exists a transition in $p^\bullet$ such that the whole set $t^\bullet$ has been tagged at previous steps. But all traps containing p must contain at least one place in

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[1] Unfortunately, checking this condition can be exponential in the size of the net. In [8], polynomial conditions of liveness and boundedness of FCN are derived under the form of an integer linear program. However, to the best of our knowledge, nothing is known in the case of unbounded FCN's.

$t^\bullet$ and from the induction assumption, this trap is not included in S. ∎

This lemma also allows one to test the liveness of a minimal siphon in quadratic time (in the number of places included in the siphon). Indeed, for this, it is necessary and sufficient to check whether the maximal trap included in the siphon is marked.

In the following section, we show that in the class SI-FCN, we can check (structural) liveness by looking at the (structural) liveness of the routing net.

3.2 Liveness of SI-FCNets

In the following, we will focus on SI-FCN. Let $F = (\mathcal{P}, \mathcal{T}, \mathcal{C})$ be a SI-FCN and $H = (\mathcal{R}, \mathcal{G}, \mathcal{C}')$ its routing net. We distinguish two classes of minimal siphons in F.

(K_1) A minimal siphons S belongs to class K_1 if it contains a cycle included in a MGC. In this case, the siphon is reduced to this cycle since a cycle in a marked graph is a siphon.

(K_2) A minimal siphon S belongs to K_2 if it contains routing places. Note that a K_2-siphon cannot contain a cycle included in a MGC, because it would not be minimal. So, the set of places of such a siphon which belong to MGC G_i can only consists of places in acyclic paths of G_i leading from the input-transition of G_i to the output-transitions in ${}^*S$.

Lemma 2 *If S is a minimal siphon in K_2, then S_r, its projection on the routing net ($S_r = S \cap \mathcal{R}$) is a siphon.*

Proof: Let p be a place in $\mathcal{R} \cap S$ and G a transition in ${}^*p$. We have to prove that there is a place q such that

$$q \text{ belongs to } \mathcal{R} \cap S \text{ and } G \in q^*. \tag{1}$$

Since G belongs to ${}^*p$, there is a transition t which belongs to $G \cap {}^\bullet p$. Since S is a siphon, there is a place q_0 in S such that $q_0^\bullet$ contains t. If q_0 is not in G, then G belongs to q_0^* and condition (1) is satisfied.

If q_0 is in G then it is serial, and there exists a place q_1 in S such that $q_0 \in q_1^{\bullet\bullet}$. More generaly, consider the sequence $q_2, \cdots, q_n, \cdots$, constructed in the same way (namely q_k is a place of S, and $q_{k-1} \in q_k^{\bullet\bullet}$), and stopped at the first place which does not belong to G. Since S cannot contain any cycle in G (S is a K_2-siphon), the last sequence stops after a finite number of steps, and so it ends with a place q_m in S, which does not belong to G. So G belongs to q_m^*, and place q_m satisfies condition (1).

This is true for all MCG in ${}^\bullet p$ and for all places p in S_r. So S_r is a siphon. ∎

Lemma 3 *Let S be a minimal siphon in the SI-FCN F of type K_2 and S_r its restriction to the routing net H. If U is a trap included in S for F, then $U \cap \mathcal{R}$ is a trap included in S_r for H. If V is trap included in S_r for H, then $U = V \cup \{p \in V^\star : p$ belongs to a path in S from a place in V to a place in $V\}$*[2] *is a trap included in S for F.*

Proof: First we prove that if U is a trap included in S, then $U_r = U \cap \mathcal{R}$ is a trap included in S_r. The proof is similar to that of the previous lemma.

We have to prove that if p is a place in U_r, then for all transitions G in H such that $G \in p^*$, there is a place q in U_r s.t. G^* contains q.

Let G be such that $G \in p^*$. Then there exists $t \in G$ s.t. t belongs to $p^\bullet$. Since U is a trap, there is a place in ${}^\bullet t \cap U$, say q_0. If q_0 is not in G, then $G \in {}^*q_0$ and so we can take $q = q_0$. If q_0 is in G, then it is serial, and there exists a place q_1 in U, such that $q_0 \in {}^{\bullet\bullet}q_1$. We construct similarly a finite sequence $q_1, q_2, \cdots, q_n$, of places belonging to U, with $q_{k-1} \in {}^{\bullet\bullet}q_k$, and such that all places of the sequence but the last, q_n belong to G. (since S contains no cycle in G, neither does U). We then take $q = q_n$.

We now prove the second part of the lemma. Let p be a place in U and t be any transition in $p^\bullet$. In order to prove that $U = V \cup \{p \in S : p \in V^*\}$ is a trap, we have to show that there is a place p' in U s.t. t belongs to ${}^\bullet p'$.

First consider the case when p belongs to $\mathcal{R}$. In this case, there is a G in p^* with $t \in G$. Now, since V is a trap, there is a place q in V s.t. $G \in {}^*q$. From the definition of U, all the places in $G \cap S$ are in U. Since S is a siphon, for all transitions in $G \cap {}^\bullet q$, there is a place q_0 in S with $q \in q_0^{\bullet\bullet}$. If q_0 is not in G, then by the single input property of G, $q_0 = p$ and $p' = q$ answers the question. If q_0 is in G, construct the sequence $q_1, \cdots, q_{n-1}$ of places in $G \cap S$ with $q_{k-1} \in q_k^{\bullet\bullet}$. As noted previously, all these places are distinct because S does not contain any cycle in G. So if n is such that q_{n-1} is the last place to be in G, then ${}^{\bullet\bullet}q_{n-1}$ is reduced to a single place: p and we can choose $p' = q_{n-1}$.

If p does not belong to $\mathcal{R}$, then p belongs to a MGC (say G) and to S. Note that since V is a trap, ${}^\star G \in V$ and $G^\star \cap V \neq \emptyset$. The places in $G \cap S$ form paths of G, each path leading from the input transition of G to a transition in $G \cap {}^\star S$. Indeed let $q \in G \cap S$. If $q^{\bullet\bullet} \notin S$, then $S - \{q\}$ is a siphon and S would not be minimal.

Now, p belongs to one of those paths, leading to a place in V. Let p' be the successor of p on this path. If $p' \in G$, then it verifies all the conditions of places of U. If $p' \notin G$, then p' is a routing place and is in $V \subset U$. Therefore, p has a successor in U, so U is a trap. ■

Note from the definition that $V = U_r$.

The previous lemmas also hold for general FCN if we replace each MGC by a complete bipartite graph between its entry-transitions and its exit-transitions

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[2] By $p \in V^\star$ we mean that p belongs to a MGC G such that $G \in V^\star$.

instead of replacing it by a single transition. Unfortunately, in this case, the following lemma does not hold which makes this new kind of reduction quite useless to test liveness.

Lemma 4 *The number of minimal siphons in H is polynomial in $|\mathcal{R}|$.*

Proof: Let S be a minimal siphon containing place p. Then for all input transitions of p, S must contain one of their input places. Since H is SI, there is only one input place per transition. Thus one has no choice and there is at most one siphon containing place p. Therefore the number of siphons in the routing net is linear in the number of places and we can find all the minimal siphons in $O(|\mathcal{R}|^2)$. ■

Theorem 2 *Checking (structural) liveness of a SI-FCN is polynomial.*

Proof: For any minimal siphon S in F, S_r is a siphon in H (lemma 2) and S contain a trap U is equivalent to S_r contains a trap U_r (lemma 3). Therefore, the structural liveness of F and H are equivalent, since all the MGC which are marked graphs are always structurally live.

The following algorithm checks the structural liveness of a SI-FCN:

1 Construct the routing net. $[O(|\mathcal{T}|^2)]$

2 For each siphon in the routing net check whether the maximal included trap is not the empty set. $[O(|\mathcal{R}|^3)$ using lemmas 1 and 4$]$

To check liveness, one has to go one more step:

3 Make sure that for each siphon its maximal trap is marked. If the trap in H is empty, we go back to the original graph, delete all places with tokens in each MGC and check whether there is still a path from the entry-transition to all exit-transitions with at least one place in the trap as output. $[\ O(|\mathcal{T}|^3)]$

4 Evolution Equations for Timed Free Choice Nets

In the following, we will focus on timed FCNets. A transition is in *starvation* if during the evolution of the net, it fires only a finite number of times. A *deadlock* is a situation where all the transitions are starving. In particular, a deadlock is a state of the system under which no firing will ever take place. For a FCNet, a deadlock is a marking D under which no transition is enabled (i.e $\forall t \in \mathrm{T} \exists p \in {}^\bullet t, Marking(p) = 0$).

4.1 Transformation of a FCNet

The FC nets we will consider will be assumed to satisfy the following assumption:

A transition $t \in \mathcal{B}$ with more than one incoming arc (i.e. with an and-convergence) is never preceded by a place p with more than one incoming arc (i.e. with an or-convergence)

This restriction introduces no loss of generality as any FC net can be transformed into an *equivalent* net which satisfies this assumption: because of the FC constraint, a transition t as above cannot be preceded by a place with multiple outcoming arcs; so, each place p as above can be replaced by a triple p', t', p'', where ${}^\bullet p' = {}^\bullet p$, $p''^\bullet = p^\bullet = t$, and where $p'^\bullet = t'$, ${}^\bullet t' = p'$, $t'^\bullet = p''$, ${}^\bullet p'' = t'$, without altering the time evolution of the net (provided t' is given a firing time equal to zero).

Note that under the above assumption,each transition of $\mathcal{B}$ has exactly one upstream place, and that this place is necessarily non-serial. This place may either have an or-divergence (in which case it precedes several transitions of $\mathcal{B}$) or an or-convergence, or both.

4.2 State Vectors

Let $X^t(u)$ denote the counter associated with t, namely, the number of firings initiated by transition t by time u. We will consider the version of this process which is continuous to the right. Let $Y(u)$ be the vector $\{X^t(u),\ t \in \mathcal{A}\}$, where the transitions are arranged in some order, and let $Z(u)$ be the vector $\{X^t(u),\ t \in \mathcal{B}\}$.

We shall first consider the case when firing times are constant, positive, and all multiple of a common number, which will be taken equal to 1 without loss of generality. We will show in Section 7 how to address the case with varying (and in particular stochastic) firing times, which can be treated with a similar method. We will denote M the (integer-valued) upper-bound on the firing times. We will denote $\nu^p(m)$ the m-th routing decision from place p ($\nu^p(m) \in p^\bullet$) and $\Pi^t(m)$ the sum

$$\Pi^t(m) = \sum_{l=1}^{m} 1_{\nu^{\bullet t}(l)=t}, \quad t \in \mathcal{B}, \tag{2}$$

where $\nu^p(.)$ is the routing function associated with place p.

As mentioned in the introduction, our main focus concerning stability will be the non-autonomous case. We will denote $\{T_n\}$ the input process. We will consider the case when the arrival times T_n (or equivalently the jump times of the associated counter, which we denote $R(t)$) are integer-valued.

Lemma 5 *Under the above assumptions, for all integers k, the counting vectors*

$\{Y(k), Z(k)\}$ *satisfy the following evolution equation*[3]*, which is valid for all* $k > M$*:*

$$Y(k) = \bigoplus_{l=1}^{M} (A_l \otimes Y(k-l) \oplus B_l \otimes Z(k-l)) \tag{3}$$

$$Z(k) = \Pi\left(\sum_{l=1}^{M} (P_l \times Z(k-l) + Q_l \times Y(k-l)) + R(k)\right), \tag{4}$$

where $(\oplus, \otimes)$ *is* $(\min, +)$ *(see [2]),* $(+, \times)$ *is the conventional algebra, and*

- *The matrix* A_l *on* $\mathcal{A} \times \mathcal{A}$ *is defined by* $A_l(t, t') = c$ *if the firing time of* $t \in \mathcal{A}$ *is* l *and there is a serial place with* c *initial tokens between* $t' \in \mathcal{A}$ *and* t*;* ∞ *otherwise. If there are more than one serial place between* t' *and* t*, we take* c *equal to the minimum of their initial markings.*
- *The matrix* B_l *on* $\mathcal{A} \times \mathcal{B}$ *is defined by* $B_l(t, t') = c$ *if the firing time of* t *is* l *and there is a serial place with* c *initial tokens between* $t' \in \mathcal{B}$ *and* $t \in \mathcal{A}$*;* ∞ *otherwise.*
- *The matrix* P_l *on* $\mathcal{B} \times \mathcal{B}$ *is defined by* $P_l(t, t') = 1$ *if the firing time of* $t \in \mathcal{B}$ *is* l *and there is a place connecting* t' *to* t*;* 0 *otherwise.*
- *The matrix* Q_l *on* $\mathcal{B} \times \mathcal{A}$ *is defined by* $Q_l(t, t') = 1$ *if the firing time of* $t \in \mathcal{B}$ *is* l *and there is a place connecting* t' *to* t*;* 0 *otherwise.*
- $R(0)$ *is the* $\mathbb{N}^{|\mathcal{B}|}$*-vectorof initial markings in the places of* ${}^\bullet\mathcal{B}$*:* $R(0)(t) = c$ *if* $t \in \mathcal{B}$ *is such that* ${}^\bullet t$ *has an initial marking of* c*. More generally,* $R(k)(t)$ *is the cumulated external input in that place up to time* k*.* $R(k)(t) = R(k-1)(t) + I(k)(t)$*, where* $I(k)(t) \neq 0$ *iff* $k \in \{T_n\}_n$ *and the exogeneous token is sent to place* ${}^\bullet t$*.*
- *For all vectors of integers* $Z = (Z_1, \ldots, Z_q)$*, where* $q = |\mathcal{B}|$*,* $\Pi(Z)$ *is the vector of integers* $(\Pi^1(Z_1), \ldots, \Pi^q(Z_q))$*.*

Proof Equation (3) is obtained as in [2]. For instance, the number of firings initiated by transition $t \in \mathcal{A}$ at time k cannot exceed the minimum of the number of tokens arrived in the places of ${}^\bullet t$ by time k, which is exactly

$$\bigoplus_{l=1}^{M} (A_l \otimes Y(k-l) \oplus B_l \otimes Z(k-l))_t .$$

---

[3] Even if the initial net is such that all its firing times are positive, the transformation of FC nets which was made at the beginning of § 4 may make it necessary to consider the case with some zero firing time transitions; this may translate into an extra term of the form $B_0 \otimes Z(k)$ in (3).

Furthermore, $Y^t(k)$ is equal to this quantity because transitions are assumed to fire as soon as they are enabled. For obtaining Equation (3), the key observation is that, due to our preliminary assumption, a transition t which belongs to $\mathcal{B}$ has at most one input arc, which allows us to write (4), and so the number of firings it initiates by time k is simply the 'Π^t-filtering' of the total number of arrivals into place ${}^\bullet t$ up to time k, that is

$$\left(\sum_{l=1}^{M} (P_l \times Z(k-l) + Q_l \times Y(k-l)) + R(k) \right) (t).$$

■

Remark The Π function satisfies the relation

$$\Pi^t(m+n) = \Pi^t(m) + \Pi^t(n) \circ \tau_m, \tag{5}$$

where $\Pi^t(.) \circ \tau_m$ is the Π^t function associated with the shifted sequence $\tau_m \nu^p(.)$, namely the sequence

$$\tau_m \nu^p(n) = \nu^p(n+m). \tag{6}$$

For instance, whenever the sequence $\nu^p(.)$ is random and i.i.d. (i.e. made of random variables which are independent and identically distributed), then $E[\Pi^t(n) \circ \tau_m] = E[\Pi^t(n)]$ so that

$$E[\Pi^t(m+n)] = E[\Pi^t(m)] + E[\Pi^t(n)]. \tag{7}$$

So, Π is 'linear in expectation'.

4.3 Total Number of Firings

Let $Z = Z(\infty)$ and $Y = Y(\infty)$ denote the vectors counting the total number of firings of the transitions. One can easily check the absence of deadlocks (a deadlock is a marking where no transition can fire) and related properties, from Y and Z: for instance, the system is deadlocked for the initial marking if and only if Z and Y are finite whenever the input process is such that $\forall k,\ R(k) = R(0)$.

Lemma 6 *The integer-valued vectors Z and Y satisfy the system of equations*

$$\begin{aligned} Y &= A \otimes Y \oplus B \otimes Z & (8) \\ Z &= \Pi \left(P \times Z + Q \times Y + R \right), & (9) \end{aligned}$$

where $A = \bigoplus_{l=1}^{M} A_l$, $B = \bigoplus_{l=1}^{M} B_l$, $P = \sum_{l=1}^{M} P_l$, $Q = \sum_{l=1}^{M} Q_l$ and $R = \lim_{k\to\infty} R(k)$.

A striking property is that this system does not depend on the variables σ^t anymore: in other words, all properties like starvation, liveness, deadlock and intermediates are associated with the switching functions, the topology, the initial marking and the R vector only, and *not* with timing variables (neither firing times nor arrival epochs of exogeneous tokens).

This allows us to establish relations between the liveness of the underlying untimed Petri net and the absence of deadlock or starvation in the timed version of the net.

In particular, if the untimed FCN is live, then the timed version never reaches a deadlock (this is not true for general Petri nets). Futhermore, if the untimed net is live and if all the functions Π^t are equitable[4] then, there will be no starvation in the timed version, that is, all the transitions will fire infinitely often. For more on this see [9].

5 Stability Analysis of FCNets

This section gives results on general FC nets. Some of the general results obtained here translate into more practical results in the SI case, studied in the next section.

5.1 Reference Deadlock

In this section, we will consider the class of networks which admit a *reference deadlock* (a deadlock is a marking where no transition can fire). Roughly speaking, the initial marking of the network should be this reference deadlock, and for all finite input processes, the networks should eventually enter the reference deadlock.

More precisely, the following two assumptions are made:

Assumption 1 (A_1) *The total firing vectors* Z *and* Y *are finite if* R *is finite.*

This assumption is a property of Π (and the topology of the net of course) since the total firing vector does not depend on the timing variables σ^t of the system. This assumption states that if the number of arrivals is finite, then the system reaches a deadlock after a finite number of firings.

Assumption 2 (A_2) *If the network reaches a deadlock, then this deadlock is the reference deadlock.*

We can give a characterization of this property in terms of the vector $X = (Z, Y)$. If the network reaches a deadlock, then X is finite. If this is the reference deadlock, then in any MGC G, all the transitions fire the same number of times, $\forall t_1, t_2 \in G,\ X^{t_1} = X^{t_2}$. Conversely if X is finite and for all MGC G,

---

[4] The function Π^t, $t \in \mathcal{T}$ is equitable if $\Pi^t(m)$ goes to infinity when m goes to infinity

$\forall t_1, t_2 \in G,\ X^{t_1} = X^{t_2}$, then the network reaches the reference deadlock. Note that for any deadlock, the number of tokens in each routing place is necessarily zero. In order to see this, introduce F the $|\mathcal{B}| \times |\mathcal{B}|$-matrix defined by $F(t, t') = 1$ if $t \in \mathcal{B}$ and $t' \in ({}^{\bullet}t)^{\bullet}$. Since for all finite vectors Z

$$F \times \Pi(Z) = Z,$$

then the solution of (8)-(9) satisfies

$$F \times Z = P \times Z + Q \times Y + R, \tag{10}$$

which implies that the total number of tokens that ever leave place p is indeed equal to the sum of the initial number of tokens in p plus the total number that ever enter p.

5.2 Restriction of the Arrival Process

Let $R^t_{[m,n]}$ be the counting measure of $N_{[m,n]}$ on place ${}^{\bullet}t$:

$$R^t_{[m,n]} = \sum_{k \in \mathbb{Z}} I_{[m,n]}(k)(t), \tag{11}$$

where $I_{[m,n]}(k)(t) = 1$ iff $k \in \{T_l\}_{n \leq l \leq m}$ *and* the exogeneous token is sent to place ${}^{\bullet}t$.

Assumption (A_1) allows us to say that the network with the input process $\left(R_{[0,n]}(k)\right)_{k \in \mathbb{N}}$ reaches a deadlock. We denote by $Z_{[0,n]}$ and $Y_{[0,n]}$ the total firing vectors for this system. With $\Pi_{[0,\infty]} \equiv \Pi$, they verify:

$$\begin{aligned} Y_{[0,n]} &= A \otimes Y_{[0,n]} \oplus B \otimes Z_{[0,n]} \\ Z_{[0,n]} &= \Pi_{[0,\infty]} \left(P \times Z_{[0,n]} + Q \times Y_{[0,n]} + R_{[0,n]}\right) . \end{aligned}$$

If $t \in \mathcal{B}$, we denote by $U^{\bullet t}_{[0,n]}$ the total number of tokens that enter place ${}^{\bullet}t$. Finally, let $U_{[0,n]} = \{U^{\bullet t}_{[0,n]},\ t \in \mathcal{B}\}$. By definition, we have

$$U_{[0,n]} = P \times Z_{[0,n]} + Q \times Y_{[0,n]} + R_{[0,n]}. \tag{12}$$

Since $Z_{[0,n]}$ and $Y_{[0,n]}$ are finite, $U_{[0,n]}$ is also finite.

Now, we introduce the system generated by the restricted input process $(R_{[m,n]}(k))_{k \in \mathbb{N}}$. We connect this system with the original one by taking

$$\nu^p_{[m,\infty]}(k) \quad \equiv \quad \nu^p(k + U^{\bullet t}_{[0,m-1]}), \tag{13}$$

for all $t \in \mathcal{T}$ and for all $p \in \mathcal{R}$. This definition, which states that routing sequences are 'marks' of the arrival point process, is essential for the following. With this, the function $\Pi_{[m,\infty]}$ is defined by

$$\Pi^t_{[m,\infty]}(k) \equiv \sum_{l=1}^{k} \mathbf{1}_{\nu^{\bullet t}_{[m,\infty]}(l) = t} = \Pi^t(k + U^{\bullet t}_{[0,m-1]}) - \Pi^t(U^{\bullet t}_{[0,m-1]}) .$$

Finally we define the vectors $Y_{[m,n]}(k)$ and $Z_{[m,n]}(k)$, which verify the equations

$$\begin{aligned} Y_{[m,n]}(k) &= \bigoplus_{l=1}^{M} \big(A_l \otimes Y_{[m,n]}(k-l) \oplus B_l \otimes Z_{[m,n]}(k-l)\big) \\ Z_{[m,n]}(k) &= \Pi_{[m,\infty]}\bigg(\sum_{l=1}^{M} \big(P_l Z_{[m,n]}(k-l) + Q_l Y_{[m,n]}(k-l)\big) + R_{[m,n]}(k)\bigg), \end{aligned}$$

with the initial conditions : $\forall k < T_m$, $Y_{[m,n]}(k+1) = 0$ and $Z_{[m,n]}(k) = 0$.

5.3 The Separability and Monotonicity Properties

In this section, we prove that under the two assumptions A_1 and A_2, the variables $X_{[m,n]}(k)$ satisfy a set of monotonicity and conservation properties (which are known as the 'monotone-separable properties' - see Appendix 1) which form the basis for analyzing stability. We need a preliminary lemma.

Lemma 7 *If* $T_m > X_{[0,m-1]} + M$*, then for all* $k \geq T_m$*,* $X_{[0,n]}(k) = X_{[0,m-1]} + X_{[m,n]}(k)$*.*

Proof The proof holds by induction on k. For $k = T_m$, $X_{[1,m-1]} = X_{[1,m-1]}(k-l)$ $\forall l > 1$ yields

$$\begin{aligned} Y_{[0,n]}(k) &= Y_{[0,m-1]} \\ Z_{[0,n]}(k) &= \Pi\left((PZ_{[0,m-1]} + QY_{[0,m-1]} + R_{[0,m-1]} + I(T_m)\right). \end{aligned}$$

So, by definition of $\Pi_{[m,\infty]}$:

$$\begin{aligned} Y_{[0,n]}(k) &= Y_{[0,m-1]} + Y_{[m,n]}(k) \\ Z_{[0,n]}(k) &= Z_{[0,m-1]} + Z_{[m,n]}(k). \end{aligned}$$

For the case $k > T_m$, the induction property yields

$$\begin{aligned} Y_{[0,n]}(k) &= \bigoplus_{l=1}^{M}(A_l \otimes (Y_{[0,m-1]} + Y_{[m,n]}(k-l)) \oplus B_l \otimes (Z_{[0,m-1]} + Z_{[m,n]}(k-l))) \\ Z_{[0,n]}(k) &= \Pi\bigg(\sum_{l=1}^{M}\big(P_l(Z_{[0,m-1]} + Z_{[m,n]}(k-l)) \\ &\quad + Q_l(Y_{[0,m-1]} + Y_{[m,n]}(k-l))\big) + R_{[0,m-1]} + R_{[m,n]}(k)\bigg). \end{aligned}$$

Using the characterization of assumption (A_2) for $Y_{[0,m-1]}$,

$$\begin{aligned} Y_{[0,n]}(k) &= Y_{[0,m-1]} + \bigoplus_{l=1}^{M} \big(A_l \otimes Y_{[m,n]}(k-l) \oplus B_l \otimes Z_{[m,n]}(k-l)\big) \\ Z_{[0,n]}(k) &= \Pi(\sum_{l=1}^{M}(P_l Z_{[m,n]}(k-l) + Q_l Y_{[m,n]}(k-l)) + U_{[0,m-1]} + R_{[m,n]}(k)). \end{aligned}$$

This yields

$$\begin{aligned} Y_{[0,n]}(k) &= Y_{[0,m-1]} + Y_{[m,n]}(k) \\ Z_{[0,n]}(k) &= Z_{[0,m-1]} + Z_{[m,n]}(k)\,. \end{aligned}$$

■

Corollary 1 $X_{[0,n]} = X_{[0,m-1]} + X_{[m,n]}$.

Proof This is an immediate corollary of the previous lemma considering the fact that $X_{[0,n]}$ does not depend on T_m. ■

Lemma 8 (External Monotonicity) *If the two arrival counting measures R and R' satisfy the inequality $R'_{[m,n]}(k) \geq R_{[m,n]}(k)$ for all k, then $X'_{[m,n]}(k) \geq X_{[m,n]}(k)$ for all k.*

Proof The vector $X_{[m,n]}(k)$ is an increasing function of $\left(X_{[m,n]}(k-l)\right)_{l=1\cdots M}$ and of $R_{[m,n]}(k)$. The proof follows by a straightforward induction. ■

Lemma 9 (Conservation) *$X_{[m,n]}$ is finite and independent of the arrival times.*

Proof Corollary 1 says that $X_{[m,n]} = X_{[0,n]} - X_{[0,m-1]}$. Therefore, $X_{[m,n]}$ is finite and independent of the arrival times. ■

Let $\mathcal{X}_{[m,n]}$ be the *maximal dater* of the network for the finite input process $R_{[m,n]}$. By maximal dater, we mean here the time when the net enters the reference deadlock (see Appendix 2).

Lemma 10 (Separability) *Suppose that $T_r > \mathcal{X}_{[m,n]} - T_n + M$. Then for all $m \leq n$,*
if $k < T_r$, then $X_{[m,n]}(k) = X_{[m,r-1]}(k)$,
if $k \geq T_r$, then $X_{[m,n]}(k) = X_{[m,r-1]} + X_{[r,n]}(k)$.

Proof The case $t < T_r$ is trivial. For the case $t \geq T_r$, the proof holds by induction on k. It is very similar to the proof of lemma 7 and is not reported here. ■

Lemma 11 (Homogeneity) *Let $R'_{[m,n]}$ be the arrival process shifted by a constant C, $R'_{[m,n]}(k) = R_{[m,n]}(k+C)$. Then, $X'_{[m,n]}(k) = X_{[m,n]}(k+C)$.*

Proof This holds by immediate induction on k. ■

So, under assumptions A_1 and A_2, the counters $X^t(k)$ of a FCNet satisfy the conditions of the monotone-separable framework given in the Appendix 1.

Let

$$\begin{aligned} \mathcal{W}_{[m,n]} &\equiv \mathcal{X}_{[m,n]} - T_n, \\ W^t_{[m,n]}(k) &\equiv X^t_{[m,n]} - X^t_{[m,n]}(T_n + k), \quad k \geq 0\,. \end{aligned}$$

The variable $\mathcal{W}_{[m,n]}$ gives the residual time to deadlock, whereas $W^t_{[m,n]}(k)$ gives the residual number of firings of transition t after time k; both definitions assume that the arrival process is stopped at time T_n.

Since the counters $X^t(k)$ satisfy the conditions of the monotone-separable framework, the following theorem holds (see [4]):

Theorem 3 *For all n, $\mathcal{W}_{[m-1,n]} \geq \mathcal{W}_{[m,n]}$; for all $k \geq 0$, t and n, $W^t_{[m-1,n]}(k) \geq W^t_{[m,n]}(k)$, so that*

$$\exists \lim_{m\to-\infty} \uparrow \mathcal{W}_{[m,n]} \equiv \mathcal{W}_{[-\infty,n]}, \quad \exists \lim_{m\to-\infty} \uparrow W^i_{[m,n]}(t) \equiv W^i_{[-\infty,n]}(t), \tag{14}$$

where the notation $\lim_{m\to-\infty} \uparrow x(m)$ indicates that $x(m)$ is a non-decreasing function of m.

5.4 Stochastic Assumptions

All the random variables defined in what follows are assumed to be carried by some probability space $(\Omega, \mathcal{F}, P, \theta)$, where θ is a shift on Ω which is P-ergodic and such that $P[f \circ \theta] = P[f]$ for all measurable functions $f : \Omega \to \mathbb{R}^+$ (see [2] for more details on this formalism).

We assume that the point process associated with the counting measure $R_{[-\infty,+\infty]}(k)$ is θ-stationary and ergodic, and that it has a finite intensity. When taking $\{T_0 = 0\}$, this θ-stationarity assumption here means that

$$R_{[n,\infty]}(T_n + k) = R_{[0,\infty]}(k) \circ \theta^n \tag{15}$$

for all $k \in \mathbb{R}$ and n. Consider the $\mathcal{T}$-valued sequences $\nu_{[0,\infty]} = \{\nu^p_{[0,\infty]}(k)\}$, $p \in \mathcal{R}, k \in \mathbb{N}\}$ describing the routing decisions; we also assume that the following compatibility relation holds for all n.

$$\nu_{[n,\infty]} = \nu_{[0,\infty]} \circ \theta^n\,. \tag{16}$$

If $\mu = \{\mu(k)\}_{k\geq 0}$ denotes a sequence, for all integers $V \geq 0$, we will denote $\tau_V\mu$ the shifted sequence $\{\mu(V + k)\}_{k\geq 0}$. If $U = (U^p,\ p \in \mathcal{R})$ is a vector of non-negative integers, we will denote $\tau_U\nu$ the sequences $(\tau_{U^p}\nu^p,\ p \in \mathcal{R})$. Equation (13) and the above relation imply that

$$\nu_{[0,\infty]} \circ \theta^n = \tau_{U_{[0,n-1]}}\nu_{[0,\infty]} \tag{17}$$

where $U^{\bullet t}_{[0,n]}$ is the function defined in Equation (12).

It should be clear that under the above assumptions, the functions $\Pi^t_{[n,\infty]}$ satisfy the compatibility property

$$\Pi_{[n,\infty]} = \Pi_{[0,\infty]} \circ \theta^n, \tag{18}$$

so that the following compatibility relations also hold:

$$\begin{aligned} \mathcal{W}_{[n,n+k]} &= \mathcal{W}_{[0,k]} \circ \theta^n, \\ W^t_{[n,n+l]}(k) &= W^t_{[0,l]}(k) \circ \theta^n, \quad k \geq 0\,. \end{aligned}$$

Since all the firing times are positive, one can easily check that condition (19) below

$$\left\{\mathcal{W}_{[-n,0]} \to_{n\to\infty} \infty\right\} \overset{\text{a.s.}}{=} \left\{\exists i / W^i_{[-n,0]} \to_{n\to\infty} \infty\right\} \tag{19}$$

is satisfied.

Therefore, the following theorem holds (see [4]):

Theorem 4 *Under the above statistical assumptions, the following a.s. limit takes place for all c-dilation $c.N$ of N, with $c \geq 0$:*

$$\exists \lim_{n\to\infty} \frac{\mathcal{W}_{[0,n]}(c.N)}{n} = \Gamma(c) \tag{20}$$

where $\Gamma(c)$ is a constant. If the intensity λ of the input point process is such that $\lambda\Gamma(0) < 1$, then the limits $\mathcal{W}_{[-\infty,n]}(1.N) \equiv \mathcal{W}_{[-\infty,n]}$ and $W^t_{[-\infty,n]}(k)$ are a.s. finite for all n, k and t.

So, under the condition $\lambda\Gamma(0) < 1$, the residual processes $W^t_{[-\infty,n]}(.)$ admit a finite stationary regime which is given by the relation

$$W^t_{[-\infty,n]}(.) = W^t_{[-\infty,0]}(.) \circ \theta^n. \tag{21}$$

More precisely

Whenever $\lambda\Gamma(0) < 1$, if the initial condition is the reference deadlock, the stochastic process $W^t_{[0,n]}(.)$ converges in law to the stochastic process $W^t_{[-\infty,0]}(.)$, for all $t \in \mathcal{T}$.

This follows from the relation

$$W^t_{[0,n]}(.) = W^t_{[-n,0]}(.) \circ \theta^n$$

and from the fact that

$$= W^t_{[-n,0]}(.) \uparrow= W^t_{[-\infty,0]}(.) < \infty$$

which imply that

$$P[W^t_{[0,n]}(k) > x] = P[W^t_{[-n,0]}(k) > x] \leq P[W^t_{[-\infty,0]}(k) > x],$$

for all $x \in \mathbb{R}^+$ and $k \in \mathbb{N}$. In other words, we have proved that the residual processes $W^t_{[-n,0]}(.)$ were indeed bounded in probability (in fact, we proved more since we showed that they converge in law to finite stationary processes).

It is easy to construct the stationary marking process out of the stationary residual processes. The number of tokens $M^p_{[m,n]}(k)$ in place p at time $k + T_n$, for the input process $N_{[m,n]}$, is given by the relation

$$N^p_{[m,n]}(k) = M^p + \sum_{t \in p^\bullet} W^t_{[m,n]}(k) - \sum_{t' \in {}^\bullet p} W^{t'}_{[m,n]}(k), \tag{22}$$

where M^p denotes the marking of p in the reference deadlock. So, if $\lambda\Gamma(0) < 1$, we have constructed a stationary (θ-compatible) version of the marking process in place p, say $M^p(.)$, by taking

$$M^p(T_n + k) = M^p + \sum_{t \in p^\bullet} W^t_{[-\infty,n]}(k) - \sum_{t' \in {}^\bullet p} W^{t'}_{[-\infty,n]}(k), \tag{23}$$

for $k \in [0, T_{n+1} - T_n)$.

Remark Note that since $\nu^p_{[0,\infty]}(k) = \nu^p_{[n,n]}(l)$ for $l = k - U^p_{[0,n-1]} \leq U^p_{[n,n]}$, the infinite sequences $\nu_{[0,\infty]}$ are fully determined by the finite sub-sequences

$$(\nu^p_{[n,n]}(k),\ p \in \mathcal{R},\ 0 \leq k \leq U^p_{[n,n]})_{n \geq 0}$$

Thus, with our framework, for all nodes p, the whole routing sequence $\nu^p_{[0,\infty]}(k)$ is simply the concatenation of the routing sequences $\nu^p_{[n,n]}(l)$, $1 \leq l \leq U^p_{[n,n]}$.

6 More on the Stability of SI-FCNets

Throughout this section, we will assume that the free choice net under consideration is SI, and we will denote $t^{(i)}$ the unique transition of $\mathcal{B}$ which is an input of G_i.

6.1 Total Number of Events

If F is a SI-FCN, the equations for the total number of events can be put in a simplified form. Let

$$A^* = \bigoplus_{k=0}^{\infty} A^k.$$

We get

$$\begin{aligned} Y &= A^* \otimes B \otimes Z = C \otimes Z \qquad (24) \\ Z &= \Pi(P \times Z + Q \times (C \otimes Z) + R). \qquad (25) \end{aligned}$$

Since the network is SI, the matrix C has at most one finite element per line. Therefore, we can define C', R' by $C \otimes Z + R = C' \times Z + R'$ and P' by $P' = P + Q \times C'$. In this case, the total number of firings verify the equations:

$$\begin{aligned} Z &= \Pi(P' \times Z + R') \qquad (26) \\ Y &= C \otimes Z. \qquad (27) \end{aligned}$$

A MGC G_i is *input-connected* if for each transition in G_i, there is a path from $t^{(i)}$ to t.

This translates into the following algebraic criterion: all MGC's are input-connected if and only if the matrix $C = A^* \otimes B$ has no line composed only of $\varepsilon = \infty$s. Since this matrix has at most one non-ε element per line (from the SI assumption), all MGC's are input-connected if and only if C has exactly one non-ε element per line.

6.2 Reference Deadlock for SI-FCNets

Lemma 12 *Let F be a SI-FCN with all its MGC's input-connected. If F can reach a deadlock, then this deadlock is unique.*

Proof If a routing place p contains a token, then one of the transitions in $p^\bullet$ is enabled, thus this marking is not a deadlock. Let t be a transition in G_i, let us follow the longest path in G_i without tokens. This path leads to a transition which is enabled except if it is the input transition. Now, a marking verifying these conditions is necessarily unique. ■

Therefore, for SI-FCN with input-connected components, assumption (A_2) is redundant for a reference deadlock to exist.

Lemma 13 *Let F be a SI-FCN with all its MGC's input-connected. If the equation*

$$Z = \Pi(P' \times Z + R') \qquad (28)$$

has a unique finite solution for all finite R', then F has a reference deadlock.

Proof In view of the preceding lemma, it is sufficient to check A_1 in order to prove the existence of a reference deadlock. From Equation (26), since the MGC's are all input-connected, C is such that $Z < \infty$ implies $Y < \infty$. So, if $Z < \infty$, the system enters a deadlock, which is uniquely defined in view of the preceding lemma. ■

Lemma 14 *Assume furthermore that the routing sequences* $\{\nu^p(i)\}$ *are i.i.d. in* i *and mutually independent in* p. *Let* Ψ *be the* $|\mathcal{B}| \times |\mathcal{B}|$*-diagonal (conventional algebra) matrix with* t*-th diagonal term equal to* $P[\nu^{\bullet t}(1) = t]$. *Then condition* A_1 *is satisfied whenever the Perron-Frobenius eigenvalue of the positive matrix* $\Psi P'$ *is strictly less than* 1.

Proof Taking expectations on both sides of (28), we obtain (on line t)

$$\begin{aligned} E[Z^t] &= E[\sum_{i=1}^{(P'Z+R')^t} 1_{\nu^{\bullet t}(i)=t}] \\ &= \sum_{i=1}^{\infty} E[1_{(P'Z+R')^t \geq i} 1_{\nu^{\bullet t}(i)=t}] \\ &= \sum_{i=1}^{\infty} E[1_{(P'Z+R')^t > i-1} 1_{\nu^{\bullet t}(i)=t}]. \end{aligned}$$

But it is clear that the events $\{(P'Z + R')^t \leq i - 1\}$ and $\{\nu^{\bullet t}(i) = t\}$ are independent, and so

$$\begin{aligned} E[Z^t] &= \sum_{i=1}^{\infty} P[(P'Z + R')^t > i - 1] P[\nu^{\bullet t}(1) = t] \\ &= E[(P'Z + R')^t] P[\nu^{\bullet t}(1) = t] \\ &= (P'E[Z] + R')^t P[\nu^{\bullet t}(1) = t]. \end{aligned}$$

Finally

$$E[Z] = \Psi P' E[Z] + \Psi R', \tag{29}$$

and the fact that the Perron-Frobenius eigenvalue of $\Psi P'$ is strictly less than one implies that $E[Z]$ is finite, which in turns implies that Z is a.s. finite. ∎

A branching process interpretation of this result is fiven in Appendix 2.

6.3 Stability Region

The aim of this section is the computation of the constant $\Gamma(0)$ defined in Theorem 4, which allows us to characterize the stability region of the network, as we know.

Delayed Networks We now consider a more general framework with inputs both on routing places as above, but also on transitions. Inputs on transitions mean *delays*. Our assumption will be that each transition is possibly delayed by an exogeneous integer-valued input process, say $U^j(k)$ for transition j, with the interpretation that this transition cannot initiate more than $U^j(k)$ firings by time k. The evolution equations for such a network are then given by

$$\begin{aligned} Y(k) &= \left(\bigoplus_{l=1}^{M} (A_l \otimes Y(k-l) \oplus B_l \otimes Z(k-l)) \right) \oplus V(k) && (30) \\ Z(k) &= \Pi \left(\sum_{l=1}^{M} (P_l \times Z(k-l) + Q_l \times Y(k-l)) + R(k) \right) \oplus W(k), && (31) \end{aligned}$$

where $V(k)$ denotes the vector $U^j(k)$, $j \in \mathcal{A}$. and $W(k)$ the vector $U^j(k)$, $j \in \mathcal{B}$.

The case with no inputs on transitions is a particular case obtained when taking $U^j(k) = \infty$ for all j and k.

All the monotonicity properties extend to this more general framework. In addition, we obtain by an immediate induction that the solution of the above system satisfies the following property:

Lemma 15 *If the two delay processes U and U' satisfy the inequality $U(k) \geq U'(k)$ for all k, then $X(k) \geq X'(k)$ for all k.*

Total Number of Firings We have the following generalization for the equations of Lemma 6

$$\begin{aligned} Y &= A \otimes Y \oplus B \otimes Z \oplus V && (32) \\ Z &= \Pi (P \times Z + Q \times Y + R) \oplus W, && (33) \end{aligned}$$

with $V = V(\infty)$, $W = W(\infty)$. Note that the total number of events does only depend on the delay process through V and W. In particular, whenever $V = \infty$ and $W = \infty$, the total numbers of events in the delayed network and the non-delayed one are the same.

Computation of $\Gamma(0)$ The constant $\Gamma(0)$ is that associated with the network with arrival point process $0.N$. Thus, in the following, all exogeneous arrivals take place at time 0.

Consider a SI network, where all MGC's are input-connected. Assumption (A_1) is assumed to hold (so that (A_2) also holds). Let

$$X^{(i)}_{[m,n]} = X^{j}_{[m,n]}, \; j \in G_i, \tag{34}$$

denote the total number of firings of (any transition in) G_i, generated by $N_{[m,n]}$ (see the remark following Assumption (A_2)). Let

$$b^i = \gamma^i E[X^{(i)}_{[0,0]}], \tag{35}$$

where γ^i denotes the maximal Lyapunov exponent of the event-graph G_i (see [2], Chapter 7), and finally let

$$b = \max_{i=1,\ldots,I} b^i. \tag{36}$$

The parameter b^i will be referred to as the load factor of MGC G_i.

Lemma 16 *Under the foregoing assumptions,* $\Gamma(0) \geq b$.

Proof For all $i = 1, \ldots, I$, the total number of tokens to arrive in ${}^\bullet t^{(i)}$ and dedicated to G_i is $X^{(i)}_{[1,n]}$. This and external monotonicity (applied to G_i) imply that

$$\Gamma(0) \geq \lim_n \frac{X^{(i)}_{[1,n]}}{n} \frac{\widetilde{\mathcal{X}}^{(i)}_{[1,n]}}{X^{(i)}_{[1,n]}},$$

where $\widetilde{\mathcal{X}}^{(i)}_{[1,n]}$ denotes the maximal dater of the MGC G_i in isolation, with an initial number of tokens in its input place equal to $X^{(i)}_{[1,n]}$.

Using the additive property for the total number of events (see Corollary 1) and Birkoff's pointwise ergodic theorem, we obtain that

$$\lim_n \frac{X^{(i)}_{[1,n]}}{n} = E[X^{(i)}_{[0,0]}] \quad a.s.$$

From the first order ergodic theorem on marked graphs (see [2], Chapter 7)

$$\lim_n \frac{\widetilde{\mathcal{X}}^{(i)}_{[1,n]}}{X^{(i)}_{[1,n]}} = \gamma^i \quad a.s.$$

So $\Gamma(0) \geq b_i$, for all i. ■

The main result of this section is the following theorem:

Theorem 5 *Under the foregoing assumptions,* $\Gamma(0) = b$*, where* b *is defined in (35), (36).*

The proof of this theorem is based on an induction on the number I of MGC's (which will be assumed to be positive). So we start with:

Lemma 17 *Theorem 5 holds for* $I = 1$.

Proof In view of Lemma 16, it is enough to prove that $\Gamma(0) \leq b$. Assume that $E[\Pi^{t^{(1)}}(R^{t^{(1)}}_{[0,0]})] > 0$. Let c be a positive real number such that

$$c < \frac{E[\Pi^{t^{(1)}}(R^{t^{(1)}}_{[0,0]})]}{E[X^{(1)}_{[0,0]}]} \leq 1. \tag{37}$$

and let n be a positive integer.[5] Consider the delays

$$U^j(k) = \begin{cases} X^j_{[0,cn]} & \text{for } 0 \leq k < \mathbf{V}(n); \\ \infty & \text{for } \mathbf{V}(n) \leq k, \end{cases} \tag{38}$$

where $\mathbf{V}(n)$ is a real number to be defined later.

With these delays, *at most* $X^j_{[0,cn]}$ firings are allowed on transition j before time $\mathbf{V}(n)$. We now prove that if we take $\mathbf{V}(n) = \infty$, then the total number of events on transition j is *exactly* $X^j_{[0,cn]}$. For this, we compare the solutions $\widetilde{Y}$ and $\widetilde{Z}$ of Equations (32)-(33) for the delayed system, which read

$$\begin{aligned} \widetilde{Y} &= A \otimes \widetilde{Y} \oplus B \otimes \widetilde{Z} \oplus Y_{[0,cn]} \\ \widetilde{Z} &= \Pi\left(P \times \widetilde{Z} + Q \times \widetilde{Y} + R_{[0,n]}\right) \oplus Z_{[0,cn]}. \end{aligned}$$

and of the non-delayed system with input process $0.N_{[0,cn]}$, which can be rewritten as follows:

$$\begin{aligned} Y_{[0,cn]} &= A \otimes Y_{[0,cn]} \oplus B \otimes Z_{[0,cn]} \oplus Y_{[0,cn]} \\ Z_{[0,cn]} &= \Pi\left(P \times Z_{[0,cn]} + Q \times Y_{[0,cn]} + R_{[0,cn]}\right) \oplus Z_{[0,cn]} \end{aligned}$$

(the $\oplus$-addition of the extra terms $Y_{[0,cn]}$ and $Z_{[0,cn]}$ leaves the solution of the system unchanged).

Note that the two systems are quite different. For instance, if $\Pi^{t^{(1)}}(R^{t^{(1)}}_{[0,n]}) \geq Z^{t^{(1)}}_{[0,cn]}$, then the delayed system is such that transition $t^{(1)}$ never starves before the maximal dater of transition $t^{(1)}$, which is not the case in the non-delayed system with input process $0.N_{[0,cn]}$.

Since $R_{[0,n]} \geq R_{[0,cn]}$, we obtain from the monotonicity property of the solution Y, Z of the system

$$\begin{aligned} Y &= A \otimes Y \oplus B \otimes Z \oplus Y_{[0,cn]} \\ Z &= \Pi\left(P \times Z + Q \times Y + R\right) \oplus Z_{[0,cn]}. \end{aligned}$$

seen as a function of R, that $\widetilde{Y} \geq Y_{[0,cn]}$ and $\widetilde{Z} \geq Z_{[0,cn]}$. But from the very equation satisfied by $\widetilde{Y}$ and $\widetilde{Z}$, $\widetilde{Y} \leq Y_{[0,cn]}$ and $\widetilde{Z} \leq Z_{[0,cn]}$, and so $\widetilde{Y} = Y_{[0,cn]}$ and $\widetilde{Z} = Z_{[0,cn]}$ indeed.

The variables associated with the system delayed by U as above (with $\mathbf{V}(n) = \infty$) will be denoted by a tilde. In particular, we will denote $\widetilde{\mathcal{X}}_n$ the maximal dater in this system, namely

$$\widetilde{\mathcal{X}}_n = \max_j \widetilde{\mathcal{X}}^j(\widetilde{X}^j) = \max_j \widetilde{\mathcal{X}}^j(X^j_{[0,cn]}).$$

---

[5] In general, cn is not an integer. One should then read 'Integer Part (cn), instead of cn; we keep the notation cn for seek of simple notations

The event

$$\mathcal{E}_n = \left\{ \Pi^{t^{(1)}}(R^{t^{(1)}}_{[0,n]}) \geq X^{(1)}_{[0,cn]} \right\}$$

is such that $\mathcal{E}_n$ tends to Ω a.s. as n tends to ∞. This follows from (37) as a direct consequence of Birkoff's pointwise ergodic theorem. On this event, more than $X^{(1)}_{[0,cn]}$ tokens dedicated to $t^{(1)}$ are present in ${}^{\bullet}t^{(1)}$ at time 0, and so we have

$$\widetilde{\mathcal{X}}^j(X^j_{[0,cn]}) = \mathcal{X}^j_{[0,\infty]}(X^j_{[0,cn]}).$$

But in view of the first-order ergodic theorem on marked graphs ([2], Chapter 7), and of Birkoff's theorem,

$$\begin{aligned} \lim_n \frac{\max_{j\in G_1} \mathcal{X}^j_{[0,\infty]}(X^j_{[0,cn]})}{n} &= \lim_n \frac{\max_{j\in G_1} \mathcal{X}^j_{[0,\infty]}(X^j_{[0,cn]})}{X^j_{[0,cn]}} \lim_n \frac{X^j_{[0,cn]}}{n} \\ &= \gamma^1 c E[X^{(1)}_{[0,0]}] \quad a.s. \end{aligned}$$

Since no transition of $\mathcal{B}$ has and-convergence,

$$\max_{j\in\mathcal{B}} \widetilde{\mathcal{X}}^j(X^j_{[0,cn]}) \leq \max_{j\in G_1} \mathcal{X}^j_{[0,\infty]}(X^j_{[0,cn]}) + a_n,$$

where a_n/n tends to 0 as n goes to ∞. So

$$\lim_n \frac{\widetilde{\mathcal{X}}_n}{n} 1_{\mathcal{E}_n} = \gamma^1 c E[X^{(1)}_{[0,0]}] \quad a.s. \tag{39}$$

We now prove that

$$\lim_n \frac{\widetilde{\mathcal{X}}_n}{n} = \gamma^1 c E[X^{(1)}_{[0,0]}] \quad a.s. \tag{40}$$

Since

$$\lim_n \frac{\widetilde{\mathcal{X}}_n}{n} = \lim_n \frac{\widetilde{\mathcal{X}}_n}{n} 1_{\mathcal{E}_n} + \frac{\widetilde{\mathcal{X}}_n}{n} 1_{\overline{\mathcal{E}_n}},$$

it is enough to prove that

$$\lim_n \frac{\widetilde{\mathcal{X}}_n}{n} 1_{\overline{\mathcal{E}_n}} = 0.$$

From the monotonicity property of the solutions $Y(k)$, $Z(k)$ of (30)-(31), seen as a function of $R(k)$, we obtain that

$$\widetilde{\mathcal{X}}_n \leq \mathcal{X}_{[0,cn]},$$

so that

$$\lim_n \frac{\widetilde{\mathcal{X}}_n}{n} 1_{\overline{\mathcal{E}_n}} \leq \lim_n \frac{\mathcal{X}_{[0,cn]}}{n} \lim_n 1_{\overline{\mathcal{E}_n}} = c\Gamma_0 0 = 0 \quad a.s.$$

which concludes the proof of (40).

Let us now compute the number of tokens present in each place of the delayed network at time $\widetilde{\mathcal{X}}_n$. As it is proved above, transition j fires $X^j_{[0,cn]}$ times in the delayed network. So at time $\widetilde{\mathcal{X}}_n$, each serial place in $G_{(1)}$ has a marking equal to that of the reference deadlock. In view of (10), for each routing place $p = {}^\bullet t$, $t \in \mathcal{B}$

$$R^t_{[0,cn]} + \sum_{s \in {}^\bullet p} X^s_{[0,cn]} = \sum_{u \in p^\bullet} X^u_{[0,cn]}.$$

So the total number of tokens in place p at time $\widetilde{\mathcal{X}}_n$ is

$$R^t_{[0,n]} + \sum_{s \in {}^\bullet p} X^s_{[0,cn]} - \sum_{u \in p^\bullet} X^u_{[0,cn]} = R^t_{[cn+1,n]}.$$

We are now in a position to conclude the proof of the lemma. We now take $\mathbf{V}(n) = \widetilde{\mathcal{X}}_n$ in the delay function and we denote $\overline{X}$ and $\overline{\mathcal{X}}$ the variables of the delayed network obtained for this new delay process. From Lemma 15, we have

$$\mathcal{X}_{[0,n]} \leq \overline{X}_{[0,n]}. \tag{41}$$

In view of what precedes, from time $\mathbf{V}(n)$ on, the delayed network coincides with the network with arrival point process $\widetilde{\mathcal{X}}_n + 0.N_{[cn+1,n]}$. So in view of homogeneity

$$\overline{X}_{[0,n]} = \widetilde{\mathcal{X}}_n + \mathcal{X}_{[cn+1,n]}. \tag{42}$$

From (41) and (42),

$$\mathcal{X}_{[0,n]} \leq \widetilde{\mathcal{X}}_n + \mathcal{X}_{[cn+1,n]}. \tag{43}$$

Dividing both terms of this inequality by n and letting n go to ∞, when using (40), we obtain

$$\Gamma(0) \leq \gamma^1 c E[X^{(1)}_{[0,0]}] + (1-c)\Gamma(0),$$

which in turn implies that

$$\Gamma(0) \leq \gamma^1 E[X^{(1)}_{[0,0]}].$$

■

The induction requires the construction of two upper-bound networks.

First Upper-Bound Network Assume that the expectation of the number of exogeneous arrivals to MGC G_I is positive, that is

$$E[\Pi^{t^{(I)}}(R^{t^{(I)}}_{[0,0]})] > 0.$$

This is not a restriction, up to a change of numbering of MGC's. The following lemma is an immediate consequence of the first order ergodic theorems in [2], Chapter 7:

Lemma 18 *When multipling all firing times in MGC G_i by a common positive constant s, its Lyapunov exponent becomes $s\gamma^i$.*

This property and monotonicity with respect to service times (see [1]) imply that the constant $\Gamma(0)$ is bounded from above by that of the network with all firing times of transitions in G_i, $i \neq I$, multiplied by $b/\gamma^i \geq 1$, and all firing times of transitions in G_I multiplied by $(b+d)/\gamma^I \geq 1$, where d is some positive real number. In this *first upper-bound network*, all MGC's have a load factor equal to b, but for G_I, the load factor of which is $b+d$.

Thus, it is enough to prove that the constant associated with this upper-bound network is bounded from above by $b+d$ to conclude the proof of the theorem. Indeed, we will then have $\Gamma(0) \leq b+d$ for all $d > 0$, that is $\Gamma(0) \leq b$.

In order to keep light notations, we will assume from now on that our initial network satisfies the properties $b^I = b+d$, $b^i = b$, $i \neq I$ (or equivalently, we will use the same notations for the initial and the first upper-bound network).

Second Upper-Bound Network Let M be a positive integer, and c be a positive real number such that

$$Mc < \frac{E[\Pi^{t^{(I)}}(R^{t^{(I)}}_{[0,0]})]}{E[X^{(I)}_{[0,0]}]} \leq 1. \tag{44}$$

Let n be a positive integer.Let $\mathcal{E}_n$ be the event

$$\mathcal{E}_n = \left\{ \Pi^{t^{(I)}}(R^{t^{(I)}}_{[0,n]}) \geq X^{(I)}_{[0,Mcn]} \right\}.$$

In view of (44), we obtain from Birkoff's pointwise ergodic theorem that

$$\lim_n \mathcal{E}_n = \Omega \quad a.s. \tag{45}$$

We now define, by induction on $m = 1, \ldots, M+1$, a sequence $U_m(k)$ of delay processes, all defined on the event $\mathcal{E}_n$. The second upper-bound network is obtained by delaying the first upper-bound network with the delay function $U_{M+1}(k)$.

$\mathbf{U_1}$.

For all transitions $j \in G_I$, we take

$$U^j_1(k) = X^j_{[0,cn]} \quad \text{for all } k. \tag{46}$$

On $\mathcal{E}_n$, the number of events of $j \in G_I$ in the delayed network is exactly $X^{(I)}_{[0,cn]}$, and the associated daters, which will be denoted $\widetilde{\mathcal{X}}^j_{1,[0,n]}$, $j \in G_I$ are not affected by the behavior of the rest of the network. Let $\mathbf{V}_1$ be the maximal dater of G_I in this delayed network:

$$\mathbf{V}_1 = \max_{j \in G_I} \widetilde{\mathcal{X}}^j_{1,[0,n]}(X^j_{[0,cn]}) + \sigma^j. \tag{47}$$

We define the delays $U_1(k)$ for all transitions j which do not belong to G_I by

$$U^j_1(k) = \begin{cases} 0 & \text{for } 0 \leq k < \mathbf{V}_1; \\ X^j_{[0,cn]} & \text{for } \mathbf{V}_1 \leq k, \end{cases} \tag{48}$$

We denote $\widetilde{\mathcal{X}}^j_{1,[0,n]}$ the daters of this delayed network and $\widetilde{\mathcal{X}}^j_{1,[0,n]}$ the corresponding counters for all j. In particular, we denote $\widetilde{\mathcal{X}}_{1,[0,n]}$ the maximal dater.

We now prove that the total number of events $\widetilde{X}^j_{1,[0,n]}$ on transition j is such that

$$\widetilde{X}^j_{1,[0,n]} = X^j_{[0,cn]}, \tag{49}$$

for all transitions j. The proof is very similar to the one used in Lemma 17: the total numbers of events in the delayed system are given by the equations

$$\begin{aligned} \widetilde{Y}_{1,[0,n]} &= A \otimes \widetilde{Y}_{1,[0,n]} \oplus B \otimes \widetilde{Z}_{1,[0,n]} \oplus Y_{[0,cn]} \\ \widetilde{Z}_{1,[0,n]} &= \Pi \left(P \times \widetilde{Z}_{1,[0,n]} + Q \times \widetilde{Y}_{1,[0,n]} + R_{[0,n]} \right) \oplus Z_{[0,cn]}. \end{aligned}$$

Equation (49) is obtained by the same type of arguments as in Lemma 17, by comparing the solution of the last equation and the solution of the equation for the non-delayed system with input $0.N_{[0,nc]}$, which can be rewritten as

$$\begin{aligned} Y_{[0,cn]} &= A \otimes Y_{[0,cn]} \oplus B \otimes Z_{[0,cn]} \oplus Y_{[0,cn]} \\ Z_{[0,cn]} &= \Pi \left(P \times Z_{[0,cn]} + Q \times Y_{[0,cn]} + R_{[0,cn]} \right) \oplus Z_{[0,cn]}. \end{aligned}$$

So at time $\mathbf{V}_1$, for all transitions j which do not belong to G_I, no firing has yet taken place, and exactly $X^j_{[0,cn]}$ firings still have to take place.

In view of the first-order ergodic theorem on marked graphs, the additive property for total number of events and Birkoff's pointwise ergodic theorem, we obtain:

Lemma 19

$$\lim_n \frac{V_1}{n} = c\gamma^I E[X^{(I)}_{[0,0]}] = c(b+d) \quad a.s. \tag{50}$$

Lemma 20

$$\lim_n \frac{\widetilde{\mathcal{X}}_{1,[0,n]}}{n} = \max_{i \neq I} c\gamma^i E[X^{(i)}_{[0,0]}] = bc \quad a.s. \tag{51}$$

Proof Consider the subnetwork F' of the initial FC net F, which is composed of the set of transitions $\mathcal{T} - G_I$ and the set of places $\mathcal{P} - \mathcal{P}_I$. Network F' is a FC net with less than I MGC's, so that we are in a position to assume that Theorem 5 holds for this network (induction assumption). For the delays $U_1(k)$, F' is a FC net with all its exogeneous arrivals taking place at time $\mathbf{V}_1$, and with total number of events on transition j equal to $X^j_{[0,cn]}$ (Equation (49)). So, from the induction assumption,

$$\lim_n \frac{\widetilde{\mathcal{X}}_{1,[0,n]}}{nc} = \max_{i\neq I} \gamma^i E[X^{(i)}_{[0,0]}] = b \quad a.s.$$

■

$\mathbf{U_m}$, $\mathbf{1 < m \leq M}$.

Assume that the function $U_p(k)$, $p \leq m$ and the parameters $\mathbf{V}_p$, and $\widetilde{\mathcal{X}}_{p,[0,n]}$, $p \leq m$, are defined for some $m < M$, and such that

$$\lim_n \frac{V_p}{n} = cp(b+d) \quad a.s. \tag{52}$$

for all $p = 1, \ldots, m$ and

$$\lim_n \frac{\widetilde{\mathcal{X}}_{p,[0,n]}}{n} = c(p(b+d)+b) \quad a.s. \tag{53}$$

for all $p = 1, \ldots, m-1$, so that the event

$$\mathcal{F}_{m,n} = \mathcal{E}_n \bigcap \left\{ \widetilde{\mathcal{X}}_{p-1,[0,n]} < \mathbf{V}_p, \ \forall p \leq m \right\} \tag{54}$$

tends to Ω a.s. when n goes to ∞. This holds for $m = 1$ when taking $\widetilde{\mathcal{X}}_{0,[0,n]} - 0$.

We define the same quantities for $m+1$ on the event $\mathcal{F}_{m,n}$, as follows:

For $j \in G_I$, we take

$$U^j_{m+1}(k) = \begin{cases} U^j_m(k) & \text{for } 0 \leq k < \mathbf{V}_m; \\ X^j_{[0,(m+1)cn]} & \text{for } \mathbf{V}_m \leq k, \end{cases} \tag{55}$$

Again, since $\mathcal{E}_n$ holds, the number of events of $j \in G_I$ in this delayed network is exactly $X^{(I)}_{[0,(m+1)cn]}$, and the associated daters $\widetilde{\mathcal{X}}^j_{m+1,[0,n]}$, $j \in G_I$ are not affected by the behavior of the rest of the network. So we can define

$$\mathbf{V}_{m+1} = \max_{j \in G_I} \widetilde{\mathcal{X}}^j_{m+1,[0,n]}(X^j_{[0,(m+1)cn]}) + \sigma^j. \tag{56}$$

For $j \notin G_I$, we take

$$U^j_{m+1}(k) = \begin{cases} U^j_m(k) & \text{for } 0 \leq k < \mathbf{V}_{m+1}; \\ X^j_{[0,(m+1)cn]} & \text{for } \mathbf{V}_{m+1} \leq k, \end{cases} \tag{57}$$

Let $\widetilde{\mathcal{X}}^j_{m+1,[0,n]}$ and $\widetilde{\mathcal{X}}^j_{m+1,[0,n]}$ denote the corresponding daters and counters respectively, and in particular, let $\widetilde{\mathcal{X}}_{m+1,[0,n]}$ be the maximal dater in this delayed network.

By the same arguments as in the case $m = 1$, we prove that:

$$\lim_n \frac{V_{m+1}}{n} = (m+1)c\gamma^I E[X^{(I)}_{[0,0]}] = (m+1)c(b+d) \quad a.s.$$

Since $d > 0$, this, Equation (53) and Birkoff's Theorem allow to prove that $\lim_n \mathcal{F}_{m+1,n} = \Omega$ a.s.

In addition, on $\mathcal{F}_{m,n}$, the total number of events to take place on transition j from time $\mathbf{V}_{m+1}$ on is exactly $X^j_{[mcn+1,(m+1)cn]}$. This, plus arguments similar to those in the proof of Lemma 20 allow us to show that

$$\lim_n \frac{\widetilde{\mathcal{X}}_{m+1,[0,n]}}{nc} = m(b+d) + b \quad a.s. \tag{58}$$

$\mathbf{U_{M+1}}$.

The last delay functions are defined on the event

$$\mathcal{F}_n = \mathcal{F}_{M,n}.$$

Let

$$\mathbf{W_n} = \widetilde{\mathcal{X}}_{M,[0,n]} = \max_{j \notin G_I} \widetilde{\mathcal{X}}^j_{M,[0,n]}(X^j_{[0,Mcn]}). \tag{59}$$

For all j,we take

$$U^j_{M+1}(k) = \begin{cases} U^j_M(k) & \text{for } 0 \le k < \mathbf{W_n}; \\ X^j_{[0,n]} & \text{for } \mathbf{W_n} \le k, \end{cases} \tag{60}$$

By the same arguments as above, on the event $\mathcal{F}_{M,n}$

$$\lim_n \frac{\mathbf{W_n}}{nc} = M(b+d) + b \quad a.s. \tag{61}$$

Proof of Theorem 5 From Lemma 15, adding the delay process $U_{M+1}(k)$ to the first upper-bound network leads to a second upper-bound network, the variables of which will be denoted $\widetilde{X}^j_n$ and $\widetilde{\mathcal{X}}^j_n$ for sake of simplicity.

By the same argument as in Lemma 17, at time $\mathbf{W_n}$, the serial places all have the marking of the reference deadlock, whereas the routing place $p = {}^\bullet t$, $t \in \mathcal{B}$ has a marking equal to $R^t_{[Mcn+1,n]}$.

So we obtain the following relation in the same way as relation (41) in Lemma 17)

$$\mathcal{X}_{[0,n]} 1_{\mathcal{F}_n} \le (\mathbf{W_n} + \mathcal{X}_{[Mcn+1,n]}) 1_{\mathcal{F}_n}. \tag{62}$$

Dividing both sides by n and letting n go to ∞, we obtain from Equation 61 that

$$\Gamma(0) \leq Mc(b+d) + bc + (1 - Mc)\Gamma(0),$$

which in turn implies

$$\Gamma(0) \leq \frac{M(b+d)+b}{M}.$$

Since this holds for all d and since M can be taken arbitrarily large, we finally obtain that $\Gamma(0) \leq b$. ■

7 Generalization to Variable Firing Times

Variable Firing Times We now consider the case when firing times are still integer-valued and bounded, but variable with time. Let $\sigma^t(m)$ be the firing time of the m-th firing of transition t. Let $\zeta^t(k)$ be the minimum of M and the time which elapsed since the last time t has started firing before time k. If we consider the variables to be left-continuous, we have:

Lemma 21

$$\begin{aligned} Y(k) &= \bigoplus_{l=1}^{M} (A_l(k) \otimes Y(k-l) \oplus B_l(k) \otimes Z(k-l)) && (63) \\ Z(k) &= \Pi\left(\sum_{l=1}^{M} (P_l(k) \times Z(k-l) + Q_l(k) \times Y(k-l)) + R(k) \right), && (64) \end{aligned}$$

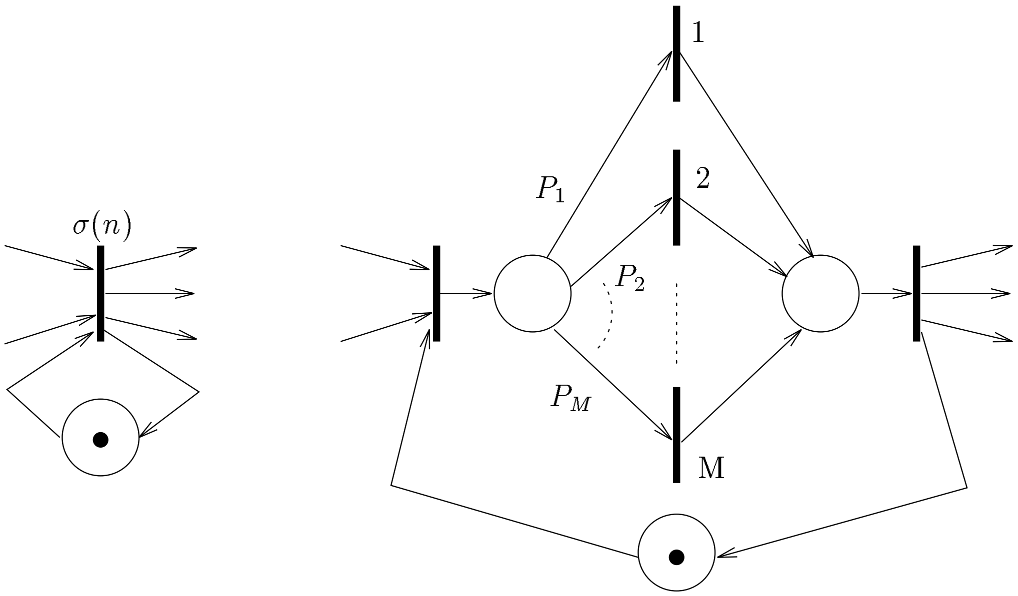
with $A_l(k)(t,t') = c$, *the number of tokens in the initial marking of the place between* t' *and* t *if* $\zeta^t(k) = l$, ∞ *otherwise (with a similar definition for* B*) and* $P_l(k)(t,t') = c$, *the number of tokens in the initial marking of the place between* t' *and* t *if* $\zeta^t(k) = l$, 0 *otherwise (with a similar definition for* Q*).*

A system with variable firing times also falls within the monotone separable framework and the same method as in the constant case can be applied (see [5]).

Finally, one can notice that a system with i.i.d. bounded stochastic firing times with integer values can be transformed into a system with constant firing times by replacing every timed transition with a random firing time by a subnetwork as in Figure 5.

Indeed, let $\sigma(n)$ be an i.i.d., integer-valued random process bounded by M. Then, the law of $\sigma(n)$ is determined by the sequence $P_1, \cdots, P_M$, with $\mathbf{P}(\phi(n) = i) = P_i$.

Note that if the original net was a SI-FCN, the net obtained after transformation is a MI-FCN. However this network falls within the separable and monotone framework.



Original transition
with a variable firing time

Transformation into a constant firing time
subnetwork

Figure 5: Transformation of a random firing time transition into a subnetwork composed with constant firing time transitions

Appendix

Appendix 1: Separable-Monotone Framework for Counters and Daters

We consider a discrete event system composed of a set of nodes, and submitted to an input point process $N = \{T_n\}_{n\in\mathbb{Z}}$. Let $c.N$ be the c-dilation of N, namely the point process with arrivals $c.N = \{cT_n\}_{n\in\mathbb{Z}}$. Let $N_{[m,n]}$ be the $[m,n]$-restriction of the point process N, namely the point process $\{T_l\}_{m\le l\le n}$. We will say that $N_{[m,n]} \le N'_{[m,n]}$ if $T_l \le T'_l$ for all $m \le l \le n$. In what follows, this point process will also be characterized through its counting measure $R_{[m,n]} : \mathbb{R} \to \mathbb{N}$, where $R_{[m,n]}(t)$ counts the number of points of $N_{[m,n]}$ which are less than t.

The discrete event system is characterized by two equivalent sets of variables:

- The daters: $\mathcal{X}^i_{[m,n]}(k) \in \mathbb{R}$ will denote the epoch of the k-th event on node i, when the system is submitted to $N_{[m,n]}$ (here, we take $k \in \mathbb{N}$ and $\mathcal{X}^i_{[m,n]}(k) = \infty$ if there are less than k events on node i).
- The counters: $X^i_{[m,n]}(t) \in \mathbb{N}$ will denote the number of events which take place on node i before time t (we will take this function left-continuous).

Note that counters and daters are related by

$$X^i_{[m,n]}(t) = \sum_{k\in\mathbb{N}} 1_{\{\mathcal{X}^i_{[m,n]}(k)\le t\}} \,. \tag{65}$$

The separable-monotone framework consists of the following set of assumptions:

External Monotonicity If $N_{[m,n]} \le N'_{[m,n]}$, then for all k and i (with obvious notations), $\mathcal{X}^i_{[m,n]}(k) \le \{\mathcal{X}'\}^i_{[m,n]}(k)$, which is equivalent to the property that for all t and i $X^i_{[m,n]}(t) \ge \{X'\}^i_{[m,n]}(t)$.

Conservation Let

$$X^i_{[m,n]} \equiv \lim_{t\to\infty} X^i_{[m,n]}(t) \,. \tag{66}$$

This limit exists since the function is non-decreasing. In words, $X^i_{[m,n]}$ counts the *total* number of events on node i for $N_{[m,n]}$. We assume that $X^i_{[m,n]}$ is *finite* and *independent* of the values taken by the variables T_l, $n \le l \le m$ (provided m, n and $\{T_l\}$ are finite of course). Of particular interest to us will be the *maximal dater* defined by:

$$\mathcal{X}_{[m,n]} = \max_i \mathcal{X}^i_{[m,n]}(X^i_{[m,n]}) \,. \tag{67}$$

Separability The separability assumption states that if $T_{l+1} \geq \mathcal{X}_{[m,l]} + M$, for some non-negative M, then

$$\begin{aligned} \mathcal{X}^i_{[m,n]}(k) &= \mathcal{X}^i_{[m,l]}(k), \quad k \leq X^i[m,l] \\ \mathcal{X}^i_{[m,n]}(k + X^i_{[m,l]}) &= \mathcal{X}^i_{[l+1,m]}(k), \quad k \geq 1 \end{aligned} \tag{68}$$

or equivalently

$$\begin{aligned} X^i_{[m,n]}(t) &= X^i_{[m,l]}(t), \quad t < T_{l+1} \\ X^i_{[m,n]}(t) &= X^i_{[m,l]} + X^i_{[l+1,m]}(t), \quad t \geq T_{l+1}. \end{aligned} \tag{69}$$

It is easy to check that the separation and the conservation properties imply that for all $m \leq l < n$, $X^i_{[m,n]} = X^i_{[m,l]} + X^i_{[l+1,n]}$ regardless of $\{T_l\}$.

Homogeneity The homogeneity assumption states that if $T'_l = T_l + c$, then $\{\mathcal{X}'\}^i_{[m,n]}(k) = \mathcal{X}^i_{[m,n]}(k) + c$ for all k and i or equivalently that $\{X'\}^i_{[m,n]}(t+c) = X^i_{[m,n]}(t)$ for all t and i.

Appendix 2: A Branching Process Interpretation

For all input transition t of a MGC, let O^t be the set

$$O^t = \{q \in \mathcal{R} \mid \exists t' \in G_i \text{ s.t. } q = t'^{\bullet}\},$$

where q is counted with multiplicity n if there are n arcs going from G_i to q. We will then say that q is an *offspring* of t with multiplicity n.

We now describe the dynamics of a pseudo marking process (this marking process is different from the one in the real system) on the set of places of $\mathcal{R}$, which is driven by the routing functions only. Fix an arbitrary priority order on the places of $\mathcal{R}$: p_j has priority over p_{j+1} etc. Assume that the jump of R at T_0 brings $m^{\bullet t}$ tokens to place ${}^{\bullet}t$, for all $t \in \mathcal{B}$. If $m^{p_1} > 0$, one token of p_1 is *moved* following the routing decision $t = \nu^{p_1}_{[0,0]}(1)$. Let n^{p_i} be the multiplicity of the offspring p_i of t (this multiplicity is zero if p_i does not belong to O^t). By definition, such a move leads to the new marking defined by $m^{p_i} + n^{p_i}$ for all $i > 1$ and $m^{p_1} + n^{p_1} - 1$ for p_1. If the new marking of p_1 is still positive, we move one token of p_1 as above, but according to the routing decision $\nu^{p_1}_{[0,0]}(2)$, which leads to a new marking; the procedure is repeated up to the time when no tokens are left in p_1 (this may never happen in which case this first step of the procedure never stops). We then move one token of type p_2 according to the routing decision $\nu^{p_2}_{[0,0]}(1)$, provided there is at least one token in place p_2 in the last obtained marking. This may possibly create new tokens of type p_1. The general rule is actually to move the token with highest priority at each step, according to the residual routing decisions. The procedure stops whenever there are no tokens left in the routing places.

Lemma 22 *The assumptions* (A_1) *(and therefore* (A_2)*) are satisfied if and only if the above procedure stops after an almost surely finite number of steps.*

The proof is omitted. It is based on a generalization of the Euler property for directed graphs called the Euler-Ordered property, which is introduced in [3]. Note that if the above stopping property holds for this specific ordering of the moves, it will hold for any other ordering.

In the particular case of i.i.d. routing decisions, independent on different nodes, one can naturally associate a multitype branching process with the set $\mathcal{R}$ by saying that an individual of type p has a set of offspring O^t with probability $P = P(\nu^p = t)$. Properties (A_1) (and (A_2)) will then a.s. hold whenever this multitype branching process is subcritical (namely whenever its population dies out a.s. for all finite initial conditions). This property boils down to checking that the maximal eigenvalue of the branching matrix is strictly less than 1 ([10]).

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