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Quantitative
Modeling
In
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Systems

G-Networks with Multiple Classes of Positive Customers, Negative Customers and Triggers

Erol Gelenbe (1), Ali Labed (2)
(1) Department of Electrical Engineering, Duke University
Durham, N.C. 27708-0291, USA
(2) EHEI, Université René Descartes, 45 rue des Saints-Pères
75270 PARIS CEDEX 06, France

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Abstract New queueing network models with "negative and positive" customers were introduced in [6], and shown to have product form. This model has undergone several generalizations to include triggers or signals which are special forms of customers whose role is to move other customers from some queue to another queue [7, 9, 12, 13]. In this paper we extend the basic model with triggers [12] to the case of multiple classes of customers and triggers. As in other multiple class queueing networks, a positive customer class is characterized by the routing probabilities and the service rate parameter at each service center while negative customers of different classes may have different "customer destruction *and movement triggering*" capabilities. In this paper all service time distributions are exponential and the service centers can be of the following types: FIFO

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(first-in-first-out), LIFO/PR (last-in-first-out with preemption), PS (processor sharing), with class dependent service rates.

1 Introduction

In recent work [6], a new class of queueing networks in which customers are either"negative" or "positive" was introduced. Positive customers enter a queue and receive service as ordinary queueing network customers. A negative customer will vanish if it arrives to an empty queue, and it will reduce by one the number of positive customers in queue otherwise. Negative customers do not receive service. Positive customers which leave a queue to enter another queue can become negative or remain positive. It has been shown [6] that networks of queues with a single class of positive and negative customers have a product form solution if the external positive or negative customer arrivals are Poisson, the service times of positive customers are exponential and independent, and if the movement of customers between queues is Markovian.

The single server queue with negative and positive customers has been examined in [7]. Stability conditions for these networks have been discussed in [9], while "triggers" which are specific customers which can order the re-routing of customers [12], and batch removal of customers by negative customers, have been introduced in [13, 14]. We call these generalized queueing networks "G-Networks" in order to distinguish them from the usual queueing network models. Additional primitives for these networks have also been introduced in [11]. G-Networks can be used to represent a variety of systems. The initial model in [6] was motivated by the analogy with neural networks [5]: each queue represents a neuron, and customers represent excitation (positive) or inhibition (negative) signals. Note that signals in biophysical neurons also take the form of random trains of impulses of constant size, much as customers traveling through a queueing network. Several other applications, including to some networking problems [15] have also been developed. Another possible application is to multiple resource systems: positive customers can be considered to be resource requests, while negative customers can correspond to decisions to cancel such requests. An application of G-networks to doubly redundant systems, where work is scheduled on two different processors and then cancelled at one of the processors if the work is successfully completed at the other, is detailed in [8].

The extension of the original model [6] to multiple classes has also been recently suggested by [10]. Other applications of G-networks are summarized in a recent survey article [16]. The current paper is a generalization of [17] where multiple classes of positive and negative customers are considered, while here we also include multiple classes of triggers. Thus in this paper we extend the model to G-Networks with *multiple classes of positive customers and one or more classes of negative customers and triggers.* Three types of service centers with their corresponding service disciplines are examined:

- Type 1 : first-in-first-out (FIFO),
- Type 2 : processor sharing (PS),
- Type 4 : last-in-first-out with preemptive resume priority (LIFO/PR).

Type 3 service centers [2] with an infinite number of servers *will not be covered by our results.* Furthermore, in this paper we deal only with exponentially distributed service times. The contribution of this paper is to establish the *product form stationary solution* for these multiple class G-Networks with Type 1, 2 and 4 service centers.

2 The model

We consider networks with an arbitrary number N of queues, an arbitrary number of positive customer classes K, and an arbitrary number of negative customer classes S. As in [6] we are only interested in open G-Networks since when the system is closed the total number of customers will decrease until there are no negative customers in the network.

In the sequel we will not distinguish between "negative customers" and "triggers" except in the most essential way. We will use the term "negative customer" to mean both. A trigger will then be a negative customer which moves another customer from one queue to the other, while in certain special cases it will move a customer from a queue to the outside of the network. In this latter case, it is equivalent to negative customers in our earlier work [6, 7, 9, 17] since pushing a customer from a queue to the outside of the network is equivalent to destroying it.

In our model, external arrival streams to the network are independent Poisson processes for positive customers of each class k or negative customers of each class c. We denote by $\Lambda_{i,k}$ the external arrival rate of *positive* customers of class k to queue i and by $\lambda_{i,m}$ be the external arrival rate of *negative* customers of class m to queue i. Only positive customers are served, and after service they may change class, service center and nature (positive to negative), or depart from the system.We assume that all service centers have exponential service time distributions. In the three types of service centers, each class of positive customers may have a distinct service rate $\mu_{i,k}$.

The movement of customers between queues, classes and nature (positive to negative) is represented by a Markov chain. A positive customer of class k which leaves queue i (after finishing service) goes to queue j as a positive customer of class l with probability $P^+[i,j]\ [k,l]$, or as a negative customer of class m with probability $P^-[i,j]\ [k,m]$. It may also depart from the network with probability $d[i,k]$. Obviously we have for all i,k:

$$\sum_{j=1}^{N}\sum_{l=1}^{R} P^+[i,j][k,l] \ + \ \sum_{j=1}^{N}\sum_{m=1}^{S} P^-[i,j][k,m] \ + \ d[i,k] = 1 \tag{1}$$

At its arrival in a non-empty queue, a negative customer selects a positive customer in the queue in accordance with the service discipline at that station. If the queue is empty, then the negative customer simply disappears. Once the positive customer is selected, the negative customer "tries" to *trigger the movement of* the selected customer. A negative customer, of some class m triggers the movement of the selected positive customer of some class k, at service center i with probability $K_{i,m,k}$. With probability $(1 - K_{i,m,k})$ it does not succeed. A negative customer disappears as soon as it tries to trigger the movement of its targeted customer. Recall that a negative customer is either exogenous, or is obtained by the transformation of a positive customer as it leaves a queue.

When the service center is of Type 1 (FIFO) we place the following constraint on the service rate and the *movement triggering* rate due to incoming negative customers:

$$\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} = c_i \tag{2}$$

Note that this constraint, together with (3) given below, have the effect of producing a *single positive customer class equivalent* for service centers with FIFO discipline. Note that because services are exponentially distributed, positive customers of a given class are indistinguishable for *movement triggering* because of the exponential service time assumption. The following constraints on the *movement triggering* probability are made:

- For all stations i of Type 1 and classes of negative customers m such that $\sum_{j=1}^{N}\sum_{l=1}^{R} P^{-}[j,i][l,m] > 0$

$$\text{for all classes of positive customers k and p}, K_{i,m,k} = K_{i,m,p} \tag{3}$$

  This constraint implies that a negative customer of some class m arriving from the network does not "distinguish" between the positive customer classes it tries to move, and that it treats them all in the same manner.

- For a Type 2 server, the probability that any one positive customer of the queue is selected by the arriving negative customer is $1/c$ if c is the total number of customers in the queue.

For Type 1 service centers, one may consider the following conditions which are simpler than (2) and (3):

$$\begin{aligned} \mu_{ik} &= \mu_{ip} \\ K_{i,m,k} &= K_{i,m,p} \end{aligned} \tag{4}$$

for all classes of positive customers k and p, and all classes of negative customers m.

2.1 State representation

We denote the state at time t of the queueing network by a vector $x(t) = (x_1(t), ..., x_N(t))$. Here $x_i(t)$ represents the state of service center i. The vector $x = (x_1, ..., x_N)$ will denote a particular value of the state and $|x_i|$ will be the total number of customers in queue i for state x.

For Type 1 and Type 4 servers, the instantaneous value of the state x_i of queue i is represented by the vector $(r_{i,j})$ whose length is the number of customers in the queue and whose jth element is the class index of the jth customer in the queue. Furthermore, the customers are ordered according to the service order (FIFO or LIFO); it is always the customer at the head of the list which is in service. We denote by $r_{i,1}$ the class number of the customer in service and by $r_{i,\infty}$ the class number of the last customer in the queue.

For a PS (Type 2) service station, the instantaneous value of the state x_i is represented by the vector $(x_{i,k})$ which is the number of customers of class k in queue i.

3 Main Results

Let $\Pi(x)$ denote the stationary probability distribution of the state of the network, if it exists. The following result establishes the product form solution of the network being considered.

Theorem 1 *Consider a G-network with the restrictions indicated above. If the system of non-linear equations*

$$\begin{aligned} q_{i,k} &= \frac{\Lambda_{i,k} + \Lambda_{i,k}^{+}}{\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}[\lambda_{i,m} + \lambda_{i,m}^{-}]} \\ \Lambda_{i,k}^{+} &= \sum_{j=1}^{N}\sum_{l=1}^{R} P^{+}[j,i][l,k]\mu_{j,l}q_{j,l} \\ &+ \sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{h=1}^{N}\sum_{m=1}^{S}\sum_{s=1}^{R} \mu_{j,l}q_{j,l}P^{-}[j,h][l,m]K_{h,m,s}q_{h,s}Q[h,i][s,k] \end{aligned} \tag{5}$$

$$+\sum_{j=1}^{N}\sum_{m=1}^{S}\sum_{s=1}^{R}\lambda_{j,m}K_{j,m,s}q_{j,s}Q[j,i][s,k] \quad (6)$$

$$\lambda^-_{i,m} = \sum_{j=1}^{N}\sum_{l=1}^{R}P^-[j,i][l,m]\mu_{j,l}q_{j,l} \quad (7)$$

has a solution such that

$$\textit{for each pair } i,k: \quad 0 < q_{i,k} \quad \textit{and} \quad \textit{for each station } i: \quad \sum_{k=1}^{R} q_{i,k} < 1$$

then the stationary distribution of the network state is

$$\Pi(x) = G \prod_{i=1}^{N} g_i(x_i) \quad (8)$$

where each $g_i(x_i)$ *depends on the type of service center* i. *The* $g_i(x_i)$ *in (5) have the following form :*

FIFO *If the service center is of Type 1, then*

$$g_i(x_i) = \prod_{n=1}^{|x_i|} q_{i,r_{i,n}} \quad (9)$$

PS *If the service center is of Type 2, then*

$$g_i(x_i) = |x_i|! \prod_{k=1}^{R} \frac{(q_{i,k})^{x_{i,k}}}{x_{i,k}!} \quad (10)$$

LIFO/PR *If the service center is of Type 4, then*

$$g_i(x_i) = \prod_{n=1}^{|x_i|} q_{i,r_{i,n}} \quad (11)$$

and G *is the normalization constant.*

Notice that $\Lambda^+_{i,k}$ may be written as

$$\Lambda^+_{i,k} = \sum_{j=1}^{N}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}P^+[j,i][l,k]$$
$$+ \sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} q_{j,l}Q[j,i][l,k]K_{j,m,l}[\lambda_{j,m}+\lambda^-_{j,m}] \quad (12)$$

Note that the conditions requiring that $q_{i,k} > 0$ and on that their sum over all classes at each center be less than 1 insure the existence of the normalizing constant G. The proof is based on simple algebraic manipulations of global balance equation as in [12, 17]. We begin with some technical lemmas.

Lemma 1 *The following flow equation is satisfied:*

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{k=1}^{R}\Lambda^{+}_{i,k}+\sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m}\\
&= \sum_{i=1}^{N}\sum_{k=1}^{R} q_{i,k}\mu_{i,k}(1-d[i,k])\\
&+ \sum_{i=1}^{N}\sum_{k=1}^{R}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} q_{j,l}Q[j,i][l,k]K_{j,m,l}[\lambda_{j,m}+\lambda^{-}_{j,m}]
\end{aligned}$$

Proof : Consider (12), then sum it for all the stations and all the classes and exchange the order of summations in the right-hand side of the equation :

$$\sum_{i=1}^{N}\sum_{k=1}^{R}\Lambda^{+}_{i,k}=\sum_{j=1}^{N}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}(\sum_{i=1}^{N}\sum_{k=1}^{R}P^{+}[j,i][l,k])+\sum_{i=1}^{N}\sum_{k=1}^{R}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} q_{j,l}Q[j,i][l,k]K_{j,m,l}[\lambda_{j,m}+\lambda^{-}_{j,m}]$$

Similarly, using equation (7)

$$\sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m}=\sum_{j=1}^{N}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}(\sum_{i=1}^{N}\sum_{m=1}^{S}P^{-}[j,i][l,m])$$

And,

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{k=1}^{R}\Lambda^{+}_{i,k}+\sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m}\\
&= \sum_{j=1}^{N}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}(\sum_{i=1}^{N}\sum_{k=1}^{R}P^{+}[j,i][l,k]+\sum_{i=1}^{N}\sum_{m=1}^{S}P^{-}[j,i][l,m])\\
&+ \sum_{i=1}^{N}\sum_{k=1}^{R}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} q_{j,l}Q[j,i][l,k]K_{j,m,l}[\lambda_{j,m}+\lambda^{-}_{j,m}]
\end{aligned}$$

According to the definition of the routing matrix P (equation 1), we have

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{k=1}^{R}\Lambda^{+}_{i,k}+\sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m}\\
&= \sum_{j=1}^{N}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}(1-d[j,l])\\
&+ \sum_{i=1}^{N}\sum_{k=1}^{R}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} q_{j,l}Q[j,i][l,k]K_{j,m,l}[\lambda_{j,m}+\lambda^{-}_{j,m}]
\end{aligned}$$

Thus the proof of the lemma is complete.

□

In order to clarify the manipulations of the global balance equations, we introduce the following intermediate results:

- Let the state dependent service rates for customers at service center j be denoted by $M_{j,l}(x_j)$ where x_j refers to the state of the service center and l is the class of the customer concerned. From the definition of the service rate $\mu_{j,l}$, we obtain for the three types of stations :

  FIFO and LIFO/PR $M_{j,l}(x_j) = \mu_{j,l} 1_{\{r_{j,1}=l\}}$,

  PS $M_{j,l}(x_j) = \mu_{j,l} \frac{x_{j,l}}{|x_j|}$.

- $N_{j,l}(x_j)$ is the *movement triggering* rate of class l positive customers due to external arrivals of all the classes of negative customers

  FIFO and LIFO/PR $N_{j,l}(x_j) = 1_{\{r_{j,1}=l\}} \sum_{m=1}^{S} K_{j,m,l} \lambda_{j,m}$

  PS $N_{j,l}(x_j) = \frac{x_{j,l}}{|x_j|} \sum_{m=1}^{S} K_{j,m,l} \lambda_{j,m}$.

- $A_{j,l}(x_j)$ is the condition which establishes that it is possible to reach state x_j by an arrival of a positive customer of class l

  FIFO $A_{j,l}(x_j) = 1_{\{r_{j,\infty}=l\}}$,

  LIFO/PR $A_{j,l}(x_j) = 1_{\{r_{j,1}=l\}}$,

  PS $A_{j,l}(x_j) = 1_{\{|x_{j,l}|>0\}}$.

- $Z_{j,l,m}(x_j)$ is the probability that a negative customer of class m, arriving from the network, will *trigger the movement of* a positive customer of class l.

  FIFO and LIFO/PR $Z_{j,l,m}(x_j) = 1_{\{r_{j,1}=l\}} K_{j,m,l}$

  PS $Z_{j,l,m}(x_j) = \frac{x_{j,l}}{|x_j|} K_{j,m,l}$.

- $Y_{j,m}(x_j)$ is the probability that a negative customer of class m which enters a non empty queue, will not *trigger the movement of* a positive customer.

  FIFO and LIFO/PR $Y_{j,m}(x_j) = \sum_{l=1}^{R} 1_{\{r_{j,1}=l\}}(1 - K_{j,m,l})$

  PS $Y_{j,m}(x_j) = \sum_{l=1}^{R} (1 - K_{j,m,l}) \frac{x_{j,l}}{|x_j|}$.

Denote by $(x_j + e_{j,l})$ the state of station j obtained by **adding to the server a positive customer of class** l. Let $(x_i - e_{i,k})$ be the state obtained by removing

from the end of the list a class k customer (if it exists, since otherwise $(x_i - e_{i,k})$ will not be defined).

Lemma 2 *For any Type 1, 2, or 4 service center, the following relations hold:*

$$M_{j,l}(x_j + e_{j,l}) \frac{g_j(x_j + e_{j,l})}{g_j(x_j)} = \mu_{j,l} q_{j,l} \tag{13}$$

$$N_{j,l}(x_j + e_{j,l}) \frac{g_j(x_j + e_{j,l})}{g_j(x_j)} = \sum_{m=1}^{S} (K_{j,m,l} \lambda_{j,m}) q_{j,l} \tag{14}$$

$$Z_{j,l,m}(x_j + e_{j,l}) \frac{g_j(x_j + e_{j,l})}{g_j(x_j)} = K_{j,m,l} q_{j,l} \tag{15}$$

The proof is purely algebraic.

□

Remark : As a consequence, we have from equations (12), (7) and (13):

$$
\begin{aligned}
\Lambda^+_{i,k} &= \sum_{j=1}^{N}\sum_{l=1}^{R} M_{j,l}(x_j+e_{j,l})\frac{g_j(x_j+e_{j,l})}{g_j(x_j)}P^+[j,i][l,k] \\
&+ \sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} q_{j,l}Q[j,i][l,k]K_{j,m,l}[\lambda_{j,m}+\lambda^-_{j,m}]
\end{aligned}
\tag{16}
$$

and

$$
\lambda^-_{i,m} = \sum_{j=1}^{N}\sum_{l=1}^{R} M_{j,l}(x_j+e_{j,l})\frac{g_j(x_j+e_{j,l})}{g_j(x_j)}P^-[j,i][l,m] \tag{17}
$$

Lemma 3 *Let i be any Type 1, 2, or 4 station, and let $\Delta_i(x_i)$ be:*

$$
\begin{aligned}
\Delta_i(x_i) = & \sum_{m=1}^{S} \lambda^-_{i,m}Y_{i,m}(x_i) \\
- & \sum_{k=1}^{R}(M_{i,k}(x_i)+N_{i,k}(x_i)) \\
+ & \sum_{k=1}^{R} A_{i,k}(x_i)(\Lambda_{i,k}+\Lambda^+_{i,k})\frac{g_i(x_i-e_{i,k})}{g_i(x_i)}
\end{aligned}
$$

Then for the three types of service centers, $1_{\{|x_i|>0\}}\Delta_i(x_i) = \sum_{m=1}^{S}\lambda^-_{i,m}1_{\{|x_i|>0\}}$.

The proof of Lemma 3 is in the appendix.

□

Let us now turn to the proof of the Theorem 1. Consider the global balance equation the networks considered is:

$$
\begin{aligned}
&\Pi(x)[\sum_{j=1}^{N}\sum_{l=1}^{R}(\Lambda_{j,l}+M_{j,l}(x_j)1_{\{|x_j|>0\}}+N_{j,l}(x_j)1_{\{|x_j|>0\}})] \\
= &\sum_{j=1}^{N}\sum_{l=1}^{R}\Pi(x-e_{j,l})\Lambda_{j,l}A_{j,l}(x_j)1_{\{|x_j|>0\}} \\
+ &\sum_{j=1}^{N}\sum_{l=1}^{R}\Pi(x+e_{j,l})N_{j,l}(x_j+e_{j,l})D[j,l] \\
+ &\sum_{j=1}^{N}\sum_{l=1}^{R}\Pi(x+e_{j,l})M_{j,l}(x_j+e_{j,l})d[j,l] \\
+ &\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} M_{j,l}(x_j+e_{j,l})\Pi(x+e_{j,l})P^-[j,i][l,m]Y_{i,m}(x_i)1_{\{|x_i|>0\}}
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} M_{j,l}(x_j+e_{j,l})\Pi(x+e_{j,l})P^{-}[j,i][l,m]1_{\{|x_i|=0\}} \\
+ \quad & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{R}\sum_{l=1}^{R} M_{j,l}(x_j+e_{j,l})\Pi(x-e_{i,k}+e_{j,l})P^{+}[j,i][l,k]A_{i,k}(x_i)1_{\{|x_i|>0\}} \\
+ \quad & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{R}\sum_{l=1}^{R} N_{j,l}(x_j+e_{j,l})\Pi(x-e_{i,k}+e_{j,l})Q[j,i][l,k]A_{i,k}(x_i)1_{\{|x_i|>0\}} \\
+ \quad & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{R}\sum_{l=1}^{R}\sum_{m=1}^{S} M_{j,l}(x_j+e_{j,l})\Pi(x+e_{i,k}+e_{j,l})P^{-}[j,i][l,m]Z_{i,k,m}(x_i+e_{i,k})D[i,k] \\
+ \quad & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{R}\sum_{l=1}^{R}\sum_{m=1}^{S}\sum_{h=1}^{N}\sum_{s=1}^{R}(M_{j,l}(x_j+e_{j,l})\Pi(x+e_{i,k}+e_{j,l}-e_{h,s})P^{-}[j,i][l,m] \\
& Z_{i,k,m}(x_i+e_{i,k})Q[i,h][k,s]A_{h,s}(x_h)1_{\{|x_h|>0\}})
\end{aligned}
$$

We divide both sides by $\Pi(x)$ and we assume that there is a product form solution. Then, we apply lemma 2.

$$
\begin{aligned}
&\sum_{j=1}^{N}\sum_{l=1}^{R}(\Lambda_{j,l} + M_{j,l}(x_j)1_{\{|x_j|>0\}} + N_{j,l}(x_j)1_{\{|x_j|>0\}}) \\
&= \sum_{j=1}^{N}\sum_{l=1}^{R} \frac{g_j(x_j - e_{j,l})}{g_j(x_j)}\Lambda_{j,l}A_{j,l}(x_j)1_{\{|x_j|>0\}} \\
&+ \sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} \lambda_{j,m}K_{j,m,l}q_{j,l}D[j,l] \quad + \quad \sum_{j=1}^{N}\sum_{l=1}^{R} \mu_{j,l}q_{j,l}d[j,l] \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} \mu_{j,l}q_{j,l}P^{-}[j,i][l,m]Y_{i,m}(x_i)1_{\{|x_i|>0\}} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} \mu_{j,l}q_{j,l}P^{-}[j,i][l,m]1_{\{|x_i|=0\}} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{R}\sum_{l=1}^{R} \mu_{j,l}q_{j,l}P^{+}[j,i][l,k]A_{i,k}(x_i)\frac{g_i(x_i - e_{i,k})}{g_i(x_i)}1_{\{|x_i|>0\}} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{R}\sum_{l=1}^{R}\sum_{m=1}^{S} \lambda_{j,m}K_{j,m,l}q_{j,l}Q[j,i][l,k]A_{i,k}(x_i)\frac{g_i(x_i - e_{i,k})}{g_i(x_i)}1_{\{|x_i|>0\}} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{R}\sum_{l=1}^{R}\sum_{m=1}^{S} \mu_{j,l}q_{j,l}P^{-}[j,i][l,m]K_{i,m,k}q_{i,k}D[i,k] \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N}\sum_{l=1}^{R}\sum_{k=1}^{R}\sum_{s=1}^{R}\sum_{m=1}^{S} \mu_{j,l}q_{j,l}P^{-}[j,i][l,m]K_{i,m,k}q_{i,k}Q[i,h][k,s]\frac{g_h(x_h - e_{h,s})}{g_h(x_h)}A_{h,s}(x_h)1_{\{|x_h|>0\}}
\end{aligned}
$$

We apply (7) to the fourth, fifth, eigth and ninth terms of the second member of the equation.

$$
\begin{aligned}
&\sum_{j=1}^{N}\sum_{l=1}^{R}(\Lambda_{j,l} + M_{j,l}(x_j)1_{\{|x_j|>0\}} + N_{j,l}(x_j)1_{\{|x_j|>0\}}) \\
&= \sum_{j=1}^{N}\sum_{l=1}^{R} \frac{g_j(x_j - e_{j,l})}{g_j(x_j)}\Lambda_{j,l}A_{j,l}(x_j)1_{\{|x_j|>0\}} \\
&+ \sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} \lambda_{j,m}K_{j,m,l}q_{j,l}D[j,l] \quad + \quad \sum_{j=1}^{N}\sum_{l=1}^{R} \mu_{j,l}q_{j,l}d[j,l] \\
&+ \sum_{i=1}^{N}\sum_{m=1}^{S} \lambda^{-}_{i,m}Y_{i,m}(x_i)1_{\{|x_i|>0\}} \\
&+ \sum_{i=1}^{N}\sum_{m=1}^{S} \lambda^{-}_{i,m}1_{\{|x_i|=0\}} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{R}\sum_{l=1}^{R} \mu_{j,l}q_{j,l}P^{+}[j,i][l,k]A_{i,k}(x_i)\frac{g_i(x_i - e_{i,k})}{g_i(x_i)}1_{\{|x_i|>0\}}
\end{aligned}
$$

$$+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{R}\sum_{l=1}^{R}\sum_{m=1}^{S} \lambda_{j,m} K_{j,m,l} q_{j,l} Q[j,i][l,k] A_{i,k}(x_i) \frac{g_i(x_i - e_{i,k})}{g_i(x_i)} 1_{\{|x_i|>0\}}$$
$$+ \sum_{i=1}^{N}\sum_{k=1}^{R}\sum_{m=1}^{S} \lambda^{-}_{i,m} K_{i,m,k} q_{i,k} D[i,k]$$
$$+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{k=1}^{R}\sum_{m=1}^{S} \lambda^{-}_{j,m} K_{j,m,l} q_{j,l} Q[j,i][l,k] \frac{g_i(x_i - e_{i,k})}{g_i(x_i)} A_{i,k}(x_i) 1_{\{|x_i|>0\}}$$

We group the first, sixth, seventh and ninth terms of the right side of the equation, and pass the two last terms of the first member to the second.

$$\begin{aligned}
&\sum_{j=1}^{N}\sum_{l=1}^{R}(\Lambda_{j,l}) \\
&= -\sum_{i=1}^{N}\sum_{k=1}^{R}(M_{i,k}(x_i) + N_{i,k}(x_i)) 1_{\{|x_i|>0\}} \\
&+ \sum_{i=1}^{N}\sum_{k=1}^{R} \frac{g_i(x_i - e_{i,k})}{g_i(x_i)} A_{i,k}(x_i) 1_{\{|x_i|>0\}} (\Lambda_{i,k} + \Lambda^{+}_{i,k}) \\
&+ \sum_{i=1}^{N}\sum_{m=1}^{S} \lambda^{-}_{i,m} Y_{i,m}(x_i) 1_{\{|x_i|>0\}} \\
&+ \sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} \lambda_{j,m} K_{j,m,l} q_{j,l} D[j,l] \quad + \quad \sum_{j=1}^{N}\sum_{l=1}^{R} \mu_{j,l} q_{j,l} d[j,l] \\
&+ \sum_{i=1}^{N}\sum_{m=1}^{S} \lambda^{-}_{i,m} 1_{\{|x_i|=0\}} \\
&+ \sum_{i=1}^{N}\sum_{k=1}^{R}\sum_{m=1}^{S} \lambda^{-}_{i,m} K_{i,m,k} q_{i,k} D[i,k]
\end{aligned}$$

We apply lemma 3 to the sum of the three first terms of the second equation.

$$
\begin{aligned}
&\sum_{j=1}^{N}\sum_{l=1}^{R}\Lambda_{j,l} \\
&= \sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m}1_{\{|x_i|>0\}} \\
&+ \sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S}\lambda_{j,m}K_{j,m,l}q_{j,l}D[j,l] \quad + \quad \sum_{j=1}^{N}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}d[j,l] \\
&+ \sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m}1_{\{|x_i|=0\}} \\
&+ \sum_{j=1}^{N}\sum_{k=1}^{R}\sum_{m=1}^{S}\lambda^{-}_{j,m}K_{j,m,k}q_{j,k}D[j,k]
\end{aligned}
$$

We group the first and fourth terms , and the second and fifth terms of the right side of the equation.

$$
\begin{aligned}
&\sum_{j=1}^{N}\sum_{l=1}^{R}\Lambda_{j,l} \\
&= \sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m} \\
&+ \sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S}q_{j,l}K_{j,m,l}(\lambda_{j,m}+\lambda^{-}_{j,m})D[j,l] \\
&+ \sum_{j=1}^{N}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}d[j,l]
\end{aligned}
$$

There are two manners in which the proof can be completed. One does not use Lemma 1 and is more concise. The second does use Lemma 1. Let us first proceed without using Lemma 1. We substitute the value of $D[j,l]$ and the value of $d[j,l]$

$$
\begin{aligned}
&\sum_{j=1}^{N}\sum_{l=1}^{R}\Lambda_{j,l} \\
&= \sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m} \;+\; \sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S}q_{j,l}K_{j,m,l}(\lambda_{j,m}+\lambda^{-}_{j,m}) \;+\; \sum_{j=1}^{N}\sum_{l=1}^{R}q_{j,l}\mu_{j,l} \\
&- \Big(\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{k=1}^{R}\sum_{m=1}^{S}q_{j,l}K_{j,m,l}Q[j,i][l,k](\lambda_{j,m}+\lambda^{-}_{j,m}) \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{k=1}^{R}\mu_{j,l}q_{j,l}P^{+}[j,i][l,k]\Big) \\
&- \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S}q_{j,l}\mu_{j,l}P^{-}[j,i][l,m]
\end{aligned}
$$

We substitute the value of qjl in the second and third terms and we group them.

$$\begin{aligned}
&\sum_{j=1}^{N}\sum_{l=1}^{R}\Lambda_{j,l} \\
&\quad = \sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m} \\
&\quad + \sum_{j=1}^{N}\sum_{l=1}^{R}\Lambda_{j,l} + \sum_{j=1}^{N}\sum_{l=1}^{R}\Lambda^{+}_{j,l} \\
&\quad - \sum_{j=1}^{N}\sum_{l=1}^{R}\Lambda^{+}_{j,l} - \sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m}
\end{aligned}$$

wich yields to the following equality which is obviously satisfied

$$\sum_{j=1}^{N}\sum_{l=1}^{R}\Lambda_{j,l} = \sum_{j=1}^{N}\sum_{l=1}^{R}\Lambda_{j,l}$$

This concludes the proof.

The second approach which does use Lemma 1 proceeds as follows.

We substitute the value of $D[j,l]$ and we add to both sides the quantity $\sum_{j=1}^{N}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}(1-d[j,l])$

$$\begin{aligned}
&\sum_{j=1}^{N}\sum_{l=1}^{R}\Lambda_{j,l} \\
+ &\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{R}\sum_{l=1}^{R}\sum_{m=1}^{S} q_{j,l}K_{j,m,l}Q[j,i][l,k](\lambda_{j,m}+\lambda^{-}_{j,m}) + \sum_{j=1}^{N}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}(1-d[j,l]) \\
= &\sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m} \\
+ &\sum_{j=1}^{N}\sum_{l=1}^{R}\sum_{m=1}^{S} q_{j,l}K_{j,m,l}(\lambda_{j,m}+\lambda^{-}_{j,m}) \\
+ &\sum_{j=1}^{N}\sum_{l=1}^{R}\mu_{j,l}q_{j,l}
\end{aligned}$$

we apply lemma 1 and factorize the last two terms of the second member

$$\begin{aligned}
&\sum_{j=1}^{N}\sum_{l=1}^{R}\Lambda_{j,l} + \sum_{j=1}^{N}\sum_{l=1}^{R}\Lambda^{+}_{j,l} + \sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m} \\
&\quad = \sum_{i=1}^{N}\sum_{m=1}^{S}\lambda^{-}_{i,m} + \sum_{j=1}^{N}\sum_{l=1}^{R} q_{j,l}(\mu_{j,l} + \sum_{m=1}^{S} K_{j,m,l}(\lambda_{j,m}+\lambda^{-}_{j,m}))
\end{aligned}$$

which , substituting the value of qjl , yields to the following equality which is obviously satisfied

$$\sum_{j=1}^{N}\sum_{l=1}^{R}\Lambda_{j,l} + \sum_{j=1}^{N}\sum_{l=1}^{R}\Lambda_{j,l}^{+} = \sum_{j=1}^{N}\sum_{l=1}^{R}\Lambda_{j,l} + \sum_{j=1}^{N}\sum_{l=1}^{R}\Lambda_{j,l}^{+}$$

This also concludes the proof.

□

As in the BCMP [2] theorem, we can also compute the steady state distribution of the number of customers of each class in each queue. Let y_i be the vector whose elements are $(y_{i,k})$ the number of customers of class k in station i. Let y be the vector of vectors (y_i). We omit the proof of the following result.

Theorem 2 *If the system of equations (5), (6) and (7) has a solution then, the steady state distribution $\pi(y)$ is given by*

$$\pi(y) = \prod_{i=1}^{N} h_i(y_i) \tag{18}$$

where the marginal probabilities $h_i(y_i)$ have the following form :

$$h_i(y_i) = (1 - \sum_{k=1}^{R} q_{i,k})|y_i|! \prod_{k=1}^{R} [(q_{i,k})^{y_{i,k}}/y_{i,k}!] \tag{19}$$

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4 Appendix

Proof of Lemma 3 : The proof consists in algebraic manipulations for the three types of stations.

LIFO/PR First consider an arbitrary LIFO station and recall the definition of Δ_i :

$$\begin{aligned}
1_{\{|x_i|>0\}}\Delta_i(x_i) &= 1_{\{|x_i|>0\}} \sum_{k=1}^{R} A_{i,k}(x_i)(\Lambda_{i,k} + \Lambda_{i,k}^{+}) \frac{g_i(x_i - e_{i,k})}{g_i(x_i)} \\
&- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} M_{i,k}(x_i) \quad - \quad 1_{\{|x_i|>0\}} \sum_{k=1}^{R} N_{i,k}(x_i) \\
&+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda_{i,m}^{-} Y_{i,m}(x_i)
\end{aligned}$$

Then, we substitute the values of $Y_{i,m}$, $M_{i,k}$, $N_{i,k}$ and $A_{i,k}$ for a LIFO station :

$$\begin{aligned}
1_{\{|x_i|>0\}}\Delta_i(x_i) &= 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} \ (\Lambda_{i,k} + \Lambda_{i,k}^{+})/q_{i,k} \\
&- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} \ \mu_{i,k} \\
&- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} \sum_{m=1}^{S} K_{i,m,k} \lambda_{i,m} \\
&+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda_{i,m}^{-} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} (1 - K_{i,m,k})
\end{aligned}$$

We use the value of $q_{i,k}$ from equation 5 to obtain after some cancelations of terms :

$$\begin{aligned}
1_{\{|x_i|>0\}}\Delta_i(x_i) &= 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} (\sum_{m=1}^{S} K_{i,m,k} \lambda_{i,m}^{-} + \sum_{m=1}^{S} \lambda_{i,m}^{-} (1 - K_{i,m,k}) \\
&= 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda_{i,m}^{-} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}
\end{aligned}$$

and as $1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} = 1_{\{|x_i|>0\}}$, we finally get the result :

$$1_{\{|x_i|>0\}}\Delta_i(x_i) = 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda_{i,m}^{-} \tag{20}$$

FIFO Consider now an arbitrary FIFO station :

$$\begin{aligned}1_{\{|x_i|>0\}}\Delta_i(x_i) &= 1_{\{|x_i|>0\}} \sum_{k=1}^{R} A_{i,k}(x_i)(\Lambda_{i,k}+\Lambda^{+}_{i,k})\frac{g_i(x_i-e_{i,k})}{g_i(x_i)} \\ &- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} M_{i,k}(x_i) \quad - \quad \sum_{k=1}^{R} 1_{\{|x_i|>0\}} N_{i,k}(x_i) \\ &+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda^{-}_{i,m} Y_{i,m}(x_i)\end{aligned}$$

Similarly, we substitute the values of $Y_{i,m}$, $M_{i,k}$, $N_{i,k}$, $A_{i,k}$ and $q_{i,k}$:

$$\begin{aligned}1_{\{|x_i|>0\}}\Delta_i(x_i) &= 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,\infty}=k\}} \ (\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} + \sum_{m=1}^{S} K_{i,m,k}\lambda^{-}_{i,m}) \\ &- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} \ \mu_{i,k} \quad - \quad 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} \\ &+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda^{-}_{i,m} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}(1-K_{i,m,k})\end{aligned}$$

We separate the last term into two parts, and regroup terms:

$$\begin{aligned}1_{\{|x_i|>0\}}\Delta_i(x_i) &= 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,\infty}=k\}} \ (\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} + \sum_{m=1}^{S} K_{i,m,k}\lambda^{-}_{i,m}) \\ &- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} \ (\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} + \sum_{m=1}^{S} K_{i,m,k}\lambda^{-}_{i,m}) \\ &+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda^{-}_{i,m} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}}\end{aligned}$$

Conditions (2) and (3) imply that the following relation must hold:

$$\begin{aligned}&\sum_{k=1}^{R} 1_{\{r_{i,\infty}=k\}} \ (\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} + \sum_{m=1}^{S} K_{i,m,k}\lambda^{-}_{i,m}) = \\ &\quad \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} \ (\mu_{i,k} + \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} + \sum_{m=1}^{S} K_{i,m,k}\lambda^{-}_{i,m})\end{aligned}$$

Thus, as $1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{r_{i,1}=k\}} = 1_{\{|x_i|>0\}}$, we finaly get the expected result :

$$1_{\{|x_i|>0\}}\Delta_i(x_i) = 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda^{-}_{i,m} \tag{21}$$

PS Consider now an arbitrary PS station :

$$\begin{aligned}
1_{\{|x_i|>0\}}\Delta_i(x_i) &= 1_{\{|x_i|>0\}} \sum_{k=1}^{R} A_{i,k}(x_i)(\Lambda_{i,k}+\Lambda_{i,k}^{+}) \frac{g_i(x_i - e_{i,k})}{g_i(x_i)} \\
&- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} M_{i,k}(x_i) \quad - \quad \sum_{k=1}^{R} 1_{\{|x_i|>0\}} N_{i,k}(x_i) \\
&+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \lambda_{i,m}^{-} Y_{i,m}(x_i)
\end{aligned}$$

As usual, we substitute the values of $Y_{i,m}$, $M_{i,k}$, $N_{i,k}$, $A_{i,k}$:

$$\begin{aligned}
1_{\{|x_i|>0\}}\Delta_i(x_i) &= 1_{\{|x_i|>0\}} \sum_{k=1}^{R} 1_{\{|x_{i,k}|>0\}} \frac{(\Lambda_{i,k}+\Lambda_{i,k}^{+})}{q_{i,k}} \frac{x_{i,k}}{|x_i|} \\
&- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} \mu_{i,k} \frac{x_{i,k}}{|x_i|} \\
&- 1_{\{|x_i|>0\}} \sum_{k=1}^{R} \frac{x_{i,k}}{|x_i|} \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m} \\
&+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \sum_{k=1}^{R} \lambda_{i,m}^{-} \frac{x_{i,k}}{|x_i|}(1-K_{i,m,k})
\end{aligned}$$

Then, we apply equation 5 to substitute $q_{i,k}$. After some cancelations of terms we obtain :

$$\begin{aligned}
1_{\{|x_i|>0\}}\Delta_i(x_i) &= 1_{\{|x_i|>0\}} \sum_{k=1}^{R} \frac{x_{i,k}}{|x_i|} \sum_{m=1}^{S} K_{i,m,k}\lambda_{i,m}^{-} \\
&+ 1_{\{|x_i|>0\}} \sum_{m=1}^{S} \sum_{k=1}^{R} \lambda_{i,m}^{-} \frac{x_{i,k}}{|x_i|}(1-K_{i,m,k})
\end{aligned}$$

Finally we have:

$$1_{\{|x_i|>0\}}\Delta_i(x_i) = 1_{\{|x_i|>0\}} \sum_{k=1}^{R} \frac{x_{i,k}}{|x_i|} \sum_{m=1}^{S} \lambda_{i,m}^{-} \qquad (22)$$

As $1_{\{|x_i|>0\}} \sum_{k=1}^{R} \frac{x_{i,k}}{|x_i|} = 1_{\{|x_i|>0\}}$, once again, we establish the relation we need. This concludes the proof of Lemma 3.

□