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Quantitative
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Analytical Computation of Lyapunov Exponents in Stochastic Event Graphs

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Abstract We consider a simple stochastic model with synchronization, representing for instance the evolution of an elementary event graph. We develop a computational technique that leads in some cases to a the complete, exact solution for both transient and stationary quantities of the model. The technique consists in translating the $(\max, +)$ linear system which describes the evolution of the event graph into a (standard) linear recurrence for the joint transforms of the state variables. These exact results can be used as a basis for evaluating the relative merits of several bounding schemes.

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1 Introduction

The study reported here was initially motivated by the paper of Baccelli and Konstantopoulos [3]. In this paper, the authors give a methodolgy to derive bounds for the *cycle time* in a class of stochastic discrete event systems described by evolution equations involving addition and maximization. Our aim is to obtain the *exact* solution for models in this class, and to use the solution as a testbed for evaluating the accuracy of these bounds. We therefore study the simplest model in this class, represented in figure 1. It may be interpreted as the infinite and periodic task graph describing the execution of an elementary cyclic parallel program on two processors. It may also represent the evolution of a stochastic Petri Net (with a particular convention concerning the handling of tokens).

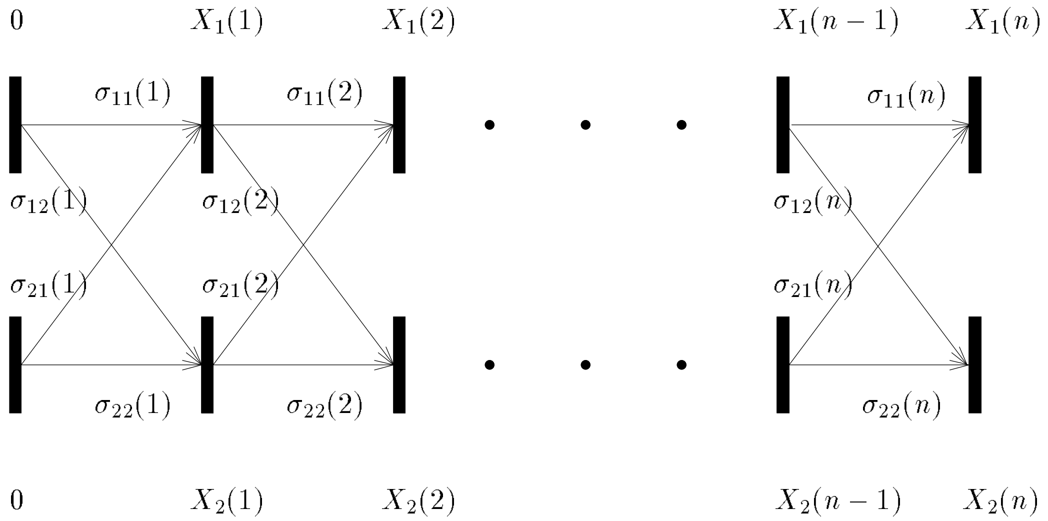

Figure 1: Representation of the stochastic recurrence

Let $\{\sigma_{ij}(n)\}_1^\infty$, $i,j \in \{1,2\}$ be four independent sequences of i.i.d random variables (RVs), hereafter called "input sequences". We are interested in solving the stochastic recurrence:

$$\begin{aligned} X_1(n) &= \max\{X_1(n-1)+\sigma_{11}(n), X_2(n-1)+\sigma_{21}(n)\} \\ X_2(n) &= \max\{X_1(n-1)+\sigma_{12}(n), X_2(n-1)+\sigma_{22}(n)\} \end{aligned} \tag{1}$$

with initial conditions $X_1(0) = X_2(0) = 0$.

This system is the simplest non-trivial linear recurrence in the semi-field $(\max, +)$. The general theory of such equations is developped in [1], and it is known that under natural conditions on the input sequences, the RVs $X_1(n)/n$ and $X_2(n)/n$ both converge almost sureley to a constant, γ, often referred to as the "Lyapunov Exponent (anciently, "Kingman constant") of the system. This constant may be interpreted in terms of *cycle times* and *throughput*. It is an essential performance measure of the discrete event system modelled by (1), and as such, its computation or estimation has recently received much attention in the literature.

In [10], a method is proposed to obtain the cycle time of the system. It is based on the fact that the variables $\{X_1(n) - X_2(n)\}_n$ form a Markov Chain. The cycle time γ can be expressed in terms of its stationnary distribution. In the case where the input sequence are not discrete, this involves the solution of integral equations. In [10], computations are carried out when all four sequences have the same distribution (the "totally symmetric case"), and when this distribution is exponential, Bernoulli or uniform.

The investigation of the "transients" of the system is interesting from many points of view. First, it allows to appreciate the speed of the convergence of the distributions of $X_i(n)/n$ $(i = 1, 2)$ and $X_1(n) - X_2(n)$ to their respective limits. Secondly, one may use the successive distributions of $X_1(n)$ to compute bounds, either by using known "oversynchronization" and "undersynchronization" techniques for task graphs [2] with the representation of figure 1, or following the method exposed in [3]. The comparison of these different approximation schemes, based on exact results obtained in this paper, is currently under development.

In a first part of this paper, we revisit the method of Reising *et al.* for discrete distributions, and show how the joint generating functions of the state vector can be computed recursively by a linear recurrence (theorem 2.1). As a corrolary, we obtain two distinct ways of computing the Lyapunov exponent of the system.

In a second section, we propose a method to compute the successive joint distributions of the couple $(X_1(n), X_2(n))$ in the case where the distributions are continuous. We prove that the joint Laplace transforms are given by a linear recurrence (theorem 3.1).

The rest of the paper gives examples of practical computations in simple cases. We study the case of exponential random variables in the "symmetric" (section 4.1) and "semi-symmetric" (section 4.2) cases. In these cases, the linear recurrence can be expressed as a simpler recurrence on a small number of polynomials (lemma 4.1 and (28). This method is then developped for discrete distributions with infinite support, for which the case of Bernoulli RV's with arbitrary parameters is solved (section 5).

2 The Linear Recurrence in the Discrete Case

Assume in this section that $\sigma_{1,1}, \sigma_{1,2}, \sigma_{2,1}$ and $\sigma_{2,2}$ have all a discrete distribution. In this case, we shall refine the technique proposed by Resing *et al.* in [10] and obtain two ways to compute the cycle time γ and the asymptotic variance σ of the system.

Let, for all $n \geq 0$,

$$Z(n) \;=\; X_1(n) - X_2(n), \qquad \eta(n) = \min\{X_1(n), X_2(n)\}\ .$$

It is known [10] that under the independence assumption of the input sequences, $\mathcal{Z} = \{Z(n)\}_n$ is an homogeneous Markov chain with state space in $\mathbb{Z}$. If the distributions of the $\sigma_{i,j}$ are all bounded, the state space of this Markov chain is bounded as well.

It is easy to show that:

$$\begin{cases} X_1(n+1) &=& \eta(n) + \max\{Z(n)^+ + \sigma_{11}(n+1), Z(n)^- + \sigma_{21}(n+1)\} \\ X_2(n+1) &=& \eta(n) + \max\{Z(n)^+ + \sigma_{12}(n+1), Z(n)^- + \sigma_{22}(n+1)\}\ . \end{cases} \tag{2}$$

and

$$\forall n \geq 0, \qquad \eta(n+1) \;=\; \eta(n) + \quad \min \quad \{Z(n)^- + \max(\sigma_{11}(n+1), \sigma_{12}(n+1)),$$

$$Z(n)^+ + \max(\sigma_{21}(n+1), \sigma_{22}(n+1))\} \;, \qquad (3)$$

where, by notation: $x^+ = max(x,0)$ and $x^- = -min(x,0)$.

Finally, let $\Phi_n(s,t) = \mathbb{E}(s^{X_1(n)} t^{X_2(n)})$.

Theorem 2.1 *The sequence Φ_n is given by:*

$$\Phi_n(s,t) \;=\; \sum_{k\in\mathbb{Z}} P_n^{(k)}(st) s^{k^+} t^{k^-} \;, \qquad (4)$$

where $P_n^{(k)}(x) = \mathbb{E}(x^{\eta(n)} \mathbf{1}_{\{Z(n)=k\}})$. Moreover, the sequence of vectors $\mathbf{P}_n(x) = (P_n^{(k)}(x))_{k\in\mathbb{Z}}$ satisfies the linear recurrence:

$$\mathbf{P}_{n+1}(x) \;=\; \mathbf{M}(x)\, \mathbf{P}_n(x) \;, \qquad (5)$$

with

$$M_{jk}(x) \;=\; \mathbb{E}(x^{\min\{k^- + \max(\sigma_{11}(n),\sigma_{12}(n)), k^+ + \max(\sigma_{21}(n),\sigma_{22}(n))\}} \mathbf{1}_{\{Z(n)=j\}} | Z(n-1) = k) \;. \qquad (6)$$

Note that the definition of the matrix $\mathbf{M}(x)$ in (6) does not depend on $n \geq 1$ because the chain $\mathcal{Z}$ is homogeneous.

Proof Decomposing Φ_n according to the value of $Z(n)$ and using the variable $\eta(n)$, one obtains:

$$\begin{aligned}
\Phi_n(s,t) &= \sum_{k\in\mathbb{Z}} \mathbb{E}(s^{\eta(n)+k^+} t^{\eta(n)+k^-} \mathbf{1}_{\{Z(n)=k\}}) \\
&= \sum_{k\in\mathbb{Z}} s^{k^+} t^{k^-} \mathbb{E}((st)^{\eta(n)} \mathbf{1}_{\{Z(n)=k\}}) \;.
\end{aligned}$$

This proves (4). Assume now that $n \geq 1$. Conditioning now on $Z(n-1)$, and making use of (3) and the Markov property, one has:

$$\begin{aligned}
P_n^{(j)}(x) &= \sum_{i\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \mathbb{E}(x^{\eta(n-1)+\min\{k^- + \max(\sigma_{11}(n),\sigma_{12}(n)), k^+ + \max(\sigma_{21}(n),\sigma_{22}(n))\}} \mathbf{1}_{\{Z(n+1)=j\}} \\
&\qquad\qquad |Z(n-1) = k, \eta(n-1) = i)\; \mathbb{P}(Z(n-1) = k, \eta(n-1) = i) \\
&= \sum_{i\in\mathbb{N}} \sum_{k\in\mathbb{Z}} x^i M_{jk} \mathbb{P}(Z(n-1) = k, \eta(n-1) = i) \\
&= \sum_{k\in\mathbb{Z}} M_{jk} \mathbb{E}(x^{\eta(n-1)} | Z(n-1) = k) \mathbb{P}(Z(n-1) = k) \;,
\end{aligned}$$

which proves (5). □

One immediately notices that the matrix $\mathbf{M}(1) = \boldsymbol{\Pi}$ is the transition matrix of the Markov chain $\mathcal{Z}$ and that $\pi_n = \mathbf{P}(1)$ is the probability distribution vector of $Z(n)$.

The representation of Theorem 2.1 allows to compute transient and stationary quantities of the system in two (seemingly) distinct ways: one using the stationary distribution of $\mathcal{Z}$ (Theorem 2.2), and one using uniquely the characteristic polynomial of $\mathbf{M}(x)$ (Theorem 2.3).

Let $d(n) = X_1(n) - X_1(n-1)$. We are going to analyze some properties of the sequence $\{d(n)\}_n$. The properties of the variables $X_2(n) - X_2(n-1)$ will follow easily.

Theorem 2.2

i/ We have:

$$\mathbb{E}d(n) \;=\; \mathbf{h}\;\pi_{n-1} \;=\; \mathbf{h}\;\boldsymbol{\Pi}^{n-1}\pi_0 \;. \tag{7}$$

where $\mathbf{h}$ *is the vector:*

$$\mathbf{h} \;=\; \mathbf{e}\mathbf{M}'(1) \;+\; \mathbf{f}(\boldsymbol{\Pi} - \mathbf{I}) \;,$$

with $\mathbf{e} = (1)_{k\in\mathbb{Z}}$ *and* $\mathbf{f} = (k^+)_{k\in\mathbb{Z}}$.

ii/ The covariances of the sequence $\{d(n)\}$ *are given by:*

$$n \geq 2 \qquad \mathbb{E}(d(n)d(1)) \;=\; \mathbf{h}\boldsymbol{\Pi}^{n-2}(\mathbf{M}'(1) + \boldsymbol{\Pi}\mathbf{F}\boldsymbol{\Pi} - \mathbf{F})\pi_0 \tag{8}$$

$$\mathbb{E}(d(1)^2) \;=\; E(d(1)) \;+\; (\mathbf{e}\mathbf{M}''(1) + 2\mathbf{f}\mathbf{M}'(1) - 2\mathbf{h}\mathbf{F})\pi_0 \;, \tag{9}$$

where $\mathbf{F}$ *is the diagonal matrix with* $\mathbf{F}_{ii} = i^+$.

iii/ If $\mathcal{Z}$ *is stationary and ergodic with stationary distribution* π*, one has:*

$$\lim_{n\to\infty} \mathbb{E}d(n) \;=\; \gamma \;=\; \mathbf{e}\mathbf{M}'(1)\pi \;.$$

and, in distribution:

$$\lim_{n\to\infty} \frac{X_1(n) - n\gamma}{\sqrt{n}} \;=\; \mathcal{N}(0,\sigma)$$

with

$$\sigma^2 \;=\; \gamma - 3\gamma^2 \;+\; \mathbf{e}[\mathbf{M}''(1) + 2\mathbf{M}'(1)(\mathbf{I} - \boldsymbol{\Pi} - \pi\mathbf{e})^{-1}\mathbf{M}'(1)]\;\pi \;. \tag{10}$$

Proof From Theorem 2.1, we have:

$$\mathbb{E}(s^{X_1(n)}) \;=\; \mathbf{f}(s)\mathbf{p}_n(s) \;, \tag{11}$$

where $\mathbf{f}(s) = (s^{k^+})_{k\in\mathbb{Z}}$. The vectors $\mathbf{f}(s)$ and $\mathbf{p}_n(s)$ are obviously differentiable with respect to s. Moreover, $\mathbf{f}(1) = \mathbf{e}$ and $\mathbf{f}'(1) = \mathbf{f}$. Therefore:

$$EX_1(n) \;=\; \mathbf{f}\pi_n \;+\; \mathbf{e}\mathbf{p}_n'(1) \;.$$

Using the relation: $\mathbf{p}_n(s) = \mathbf{M}(s)\mathbf{p}_{n-1}(s)$, we have:

$$\mathbf{p}_n'(1) \;=\; \mathbf{M}'(1)\pi_{n-1} \;+\; \boldsymbol{\Pi}\mathbf{p}_{n-1}'(1) \;. \tag{12}$$

If follows that for $n \geq 1$:

$$EX_1(n) - EX_1(n-1) \;=\; \mathbf{f}(\boldsymbol{\Pi}\pi_{n-1} - \pi_{n-1}) \;+\; \mathbf{e}\mathbf{M}'(1)\pi_{n-1} + \mathbf{e}(\boldsymbol{\Pi} - I)\mathbf{p}_{n-1}'(1) \;.$$

But $\mathbf{e}\boldsymbol{\Pi} = \mathbf{e}$, and this relation rewrites as (7). This proves i/.

The covariance of $d(1)$ and $d(n)$ is:

$$\mathbb{E}(d(1)d(n)) \;=\; \mathbb{E}(d(n)X_1(1)) \;-\; \mathbb{E}(d(n)X_1(0)) \;.$$

For all $n \geq 2$, one has:

$$\mathbb{E}(d(n)X_1(1)) \;=\; \mathbf{h}\boldsymbol{\Pi}^{n-2}\mathbb{E}(\pi_1 X_1(1)) \; .$$

But the k-th component of $\mathbb{E}(\pi_1 X_1(1))$ is $\mathbb{E}(\mathbf{1}_{\{Z(1)=k\}}X_1(1)) = {P^{(k)}}'_1(1) + k^+ P_1^{(k)}(1)$, from the definition of $P_n^{(k)}(x)$. Therefore,

$$\mathbb{E}(\pi_1 X_1(1)) \;=\; \mathbf{p}'_n(1) \;+\; \mathbf{F}\pi_1,$$

and likewise,

$$\mathbb{E}(\pi_1 X_1(0)) \;=\; \mathbf{p}'_n(0) \;+\; \mathbf{F}\pi_0.$$

Computing the difference and using $\mathbf{p}'_1(1) = \mathbf{M}'(1)\pi_0 + \boldsymbol{\Pi}\mathbf{p}'_0(1)$, one obtains (8).

Let us compute the variance of $d(1)$. Differentiating (11) twice, we have:

$$E[X_1(n)(X_1(n)-1)] \;=\; \mathbf{g}\pi_n \;+\; 2\mathbf{f}\mathbf{p}'_n(1) \;+\; \mathbf{e}\mathbf{p}''_n(1) \; ,$$

with $\mathbf{g} = \mathbf{f}''(1) = \mathbf{0}$. Moreover, $\mathbf{p}''_n(1) = \mathbf{M}'(1)\pi_n + 2\mathbf{M}'(1)\mathbf{p}'_{n-1}(1) + \boldsymbol{\Pi}\mathbf{p}''_{n-1}(1)$. Then, writing:

$$E[d(1)^2] \;=\; E[X_1(1)^2] - E[X_1(0)^2] - 2E[d(1)X_1(0)] \; ,$$

and using previous calculations, one obtains:

$$\begin{aligned} E[d(1)^2] &= Ed(1) \;+\; 2\mathbf{f}(\mathbf{M}'(1)\pi_0 + (\boldsymbol{\Pi}-\mathbf{I})\mathbf{p}'_0(1)) \;+\; \mathbf{e}(\boldsymbol{\Pi}-\mathbf{I})\mathbf{p}''_0(1) \;-\; 2\mathbf{h}(\mathbf{F}\pi_0 + \mathbf{p}'_0(1)) \\ &= Ed(1) \;+\; 2(\mathbf{f}\mathbf{M}'(1) - \mathbf{h}\mathbf{F})\pi_0 \;+\; 2(\mathbf{f}(\boldsymbol{\Pi}-\mathbf{I}) - \mathbf{h})\mathbf{p}'_0(1) \; . \end{aligned}$$

This rewrites as (9).

If $\mathcal{Z}$ has a stationary distribution π, then for all non-zero vector π_0, we have: $\lim_{n\to\infty} \boldsymbol{\Pi}^n\pi_0 = \pi$ (see *e.g.* [11]). Therefore, $\lim_{n\to\infty}\mathbb{E}(d(n)) = \mathbf{h}\pi = \mathbf{e}\mathbf{M}'(1)\pi$ (because $\boldsymbol{\Pi}\pi = \pi$). The validity of the central limit theorem for $X_1(n)$ has been proved in [10]. In this paper, the asymptotic variance is given by the formula:

$$\sigma^2 \;=\; \mathbb{E}_\pi(d(1)^2) + 2\sum_{l=2}^{\infty}\mathbb{E}_\pi[(d(1)-\gamma)(d(l)-\gamma)] \; ,$$

where $\mathbb{E}_\pi$ denotes the expectation given that the distribution of $Z(0)$ is given by π (that is, $\mathcal{Z}$ is stationary). Assume first that $n \geq 2$. From (8), and remembering that $\lim_{n\to\infty}\boldsymbol{\Pi}^n = \pi\mathbf{e}$, one writes:

$$\begin{aligned} &\mathbb{E}_\pi[(d(1)-\gamma)(d(l)-\gamma)] = \mathbb{E}_\pi(d(1)d(n)) - \gamma^2 \\ &\quad= \mathbf{h}[\boldsymbol{\Pi}^{n-2} - \pi\mathbf{e} + \pi\mathbf{e}][\mathbf{M}'(1) + (\mathbf{I}-\boldsymbol{\Pi})\mathbf{F}]\pi - \gamma^2 \\ &\quad= \mathbf{h}[\boldsymbol{\Pi}^{n-2} - \pi\mathbf{e}]\mathbf{M}'(1)\pi + \mathbf{h}[\boldsymbol{\Pi}^{n-2} - \pi\mathbf{e}](\mathbf{I}-\boldsymbol{\Pi})\mathbf{F}\pi + \gamma^2 - \gamma^2 \\ &\quad= \mathbf{h}[\boldsymbol{\Pi}^{n-2} - \pi\mathbf{e}]\mathbf{M}'(1)\pi + \mathbf{h}[\boldsymbol{\Pi}^{n-2} - \boldsymbol{\Pi}^{n-1}]\mathbf{F}\pi \\ &\quad= \mathbf{e}\mathbf{M}'(1)[\boldsymbol{\Pi}^{n-2} - \pi\mathbf{e}]\mathbf{M}'(1)\pi + \mathbf{f}[\boldsymbol{\Pi}^{n-1} - \boldsymbol{\Pi}^{n-2}]\mathbf{F} + \mathbf{h}[\boldsymbol{\Pi}^{n-2} - \boldsymbol{\Pi}^{n-1}]\mathbf{F}\pi \; . \end{aligned}$$

It follows that:

$$\sum_{l=2}^{\infty}\mathbb{E}_\pi[(d(1)-\gamma)(d(l)-\gamma)] \;=\; \left(\mathbf{e}\mathbf{M}'(1)(\sum_{n=0}^{\infty}[\boldsymbol{\Pi}^n - \pi\mathbf{e}])\mathbf{M}'(1) + \mathbf{f}(\pi\mathbf{e}-\mathbf{I})\mathbf{F} + \mathbf{h}(\mathbf{I}-\pi\mathbf{e})\mathbf{F}\right)\pi \; .$$

To compute the series, notice that if $n > 0$, $\boldsymbol{\Pi}^n - \pi\mathbf{e} = (\boldsymbol{\Pi} - \pi\mathbf{e})^n$. Therefore, $\sum_{n=0}^{\infty}[\boldsymbol{\Pi}^n - \pi\mathbf{e}] = (\mathbf{I} - \boldsymbol{\Pi} - \pi\mathbf{e})^{-1} - \pi\mathbf{e}$. The above expression then reduces to:

$$\sum_{l=2}^{\infty} \mathbb{E}_\pi[(d(1)-\gamma)(d(l)-\gamma)] \ = \mathbf{e}\mathbf{M}'(1)(\mathbf{I} - \boldsymbol{\Pi} - \pi\mathbf{e})^{-1}\mathbf{M}'(1)\pi + \mathbf{h}(\mathbf{I} - \pi\mathbf{e})\mathbf{F}\pi - \gamma^2 \ .$$

Together with (9), one obtains (10). □

Theorem 2.3 *Let $P(z,x) = Det(\mathbf{M}(x) - z\mathbf{I})$ be the caracteristic polynomial of $\mathbf{M}(x)$. Let:*

$$P_x = \frac{\partial P}{\partial x}(1,1), \quad P_z = \frac{\partial P}{\partial z}(1,1), \quad P_{xx} = \frac{\partial^2 P}{\partial x^2}(1,1), \quad P_{xz} = \frac{\partial^2 P}{\partial x \partial z}(1,1), \ \textit{and} \ P_{zz} = \frac{\partial^2 P}{\partial z^2}(1,1).$$

Then:

$$\gamma \ = \ - \ \frac{P_x}{P_z} \ , \tag{13}$$

and

$$\sigma \ = \ \gamma \ - \ \gamma^2 - \ \frac{1}{P_z}\left(P_{xx} + 2\gamma P_{xz} + \gamma^2 P_{zz}\right) \ . \tag{14}$$

Proof Using, for instance, the generating matrix $\sum_{n=0}^{\infty} t^n \mathbf{M}(x)^n$, one can show that for all $n \geq 0$ and in a neighborhood V_1 of $s = 1$, one has (see [11, p. 19]):

$$X_1^*(n)(s) \ = \ \sum_{l=0}^{L} \sum_{j=1}^{m_i - 1} n^j \alpha_{lj}(s) \lambda_l^n(s) \ ,$$

where $\lambda_1, \ldots, \lambda_L$ are the L distinct eigenvalues of $\mathbf{M}(s)$, m_l the multiplicity of λ_l and the α_{lj} are functions independent of n, analytic in V_1.

The Perron-Frobenius eigenvalue of $\mathbf{M}(s)$, say $\lambda_1(s)$, is real, isolated ($m_1 = 1$) and less than 1, because $\mathbf{M}(s)$ is substochastic for $s \in [0,1]$. Moreover, $\lambda_1(1) = 1$ and there exists $\rho < 1$ such that

$$\forall s \in V_1, \ \forall l, 2 \leq l \leq L, \qquad |\lambda_l(s)| < \rho \ .$$

Consequently,

$$X_1^*(n)(s) \ = \ \lambda_1^n(s) \ (1 + o(\rho^n)) \ .$$

A straightforward asymptotic expansion in $X_1^*(n)(e^{-s/\sqrt{n}})$ allows to conclude that:

$$\frac{X_1(n) - n\gamma}{\sigma\sqrt{n}} \rightarrow \mathcal{N}(0,1),$$

with

$$\gamma = \lambda_1'(1) \quad \text{and} \quad \sigma^2 = \lambda_1''(1) + \gamma - \gamma^2 \ .$$

The derivatives of $\lambda_1(s)$ at $s = 1$ can hopefully be computed without an explicit expression for this function. It suffices to note that $P(z,x) = \prod_{l=1}^{L}(z - \lambda_l(s))^{m_l}$, so that:

$$P_z \ = \ \sum_{l=1}^{L} \prod_{j \neq l} m_l (z - \lambda_l(s))^{m_l - 1} \ |_{s=1} \ = \ \prod_{j \neq 1}(z - \lambda_j(1))$$

and

$$P_{xz} \;=\; -\sum_{l=1}^{L} \lambda_l'(s) \prod_{j\neq l} m_l(z-\lambda_l(s))^{m_l-1} \,|_{s=1} \;=\; -\lambda_1'(1) P_z \ .$$

The expression for $\lambda_1'(1)$ follows. A similar calculation yields $\lambda_1''(1)$. □

3 The continuous case

For the moment, we do not assume anything particular on the distribution of the input sequences. For $i,j \in 1,2$, let us denote $S_{ij}(x) = \mathbb{P}(\sigma_{ij}(n) \leq x)$. Let, for all $n \geq 0$:

$$F_n(x,y) = P(X_1(n) \leq x,\ X_2(n) \leq y),$$

and

$$\Phi_n(s,t) = E(e^{-sX_1(n)-tX_2(n)}) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-sx-ty} dF_n(x,y) \ .$$

This Laplace Transform is defined *a priori* on the domain $i\mathbb{R} \times i\mathbb{R}$. If the random variables $X_1(n)$ and $X_2(n)$ are known to be positive (the most common case), the domain can be extended to $\{\Re(s) \geq 0\} \times \{\Re(t) \geq 0\}$. It is well known that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} e^{-sx-ty} F_n(x,y) dx dy = \frac{1}{st} \Phi_n(s,t) \ . \tag{15}$$

The initial values of these functions are naturally $F_0(x,y) = \mathbf{1}_{\{x,y \geq 0\}}$ and $\Phi_0(s,t) = 1$.

From (1), one has:

$$\begin{aligned}
F_{n+1}(x,y) &= \mathbb{P}(X_1(n) + \sigma_{11}(n+1) \leq x, X_2(n) + \sigma_{21}(n+1) \leq x, \\
&\qquad X_1(n) + \sigma_{12}(n+1) \leq y, X_2(n) + \sigma_{22}(n+1) \leq y) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{P}(\sigma_{11}(n+1) \leq x-u, \sigma_{21}(n+1) \leq x-v, \\
&\qquad\qquad \sigma_{12}(n+1) \leq y-u, \sigma_{22}(n+1) \leq y-v) dF_n(u,v) \\
&= \int_0^{\inf(x,y)} \int_0^{\inf(x,y)} S(x,y,u,v) dF_n(u,v) \ ,
\end{aligned} \tag{16}$$

where $S(x,y,u,v) = S_{11}(x-u)S_{21}(x-v)S_{12}(y-u)S_{22}(y-v)$.

Obeying to a standard reflex in such a context, we turn to Laplace transforms, hoping to obtain a more appealing recurrence relation. From (15) and (16), we have:

$$\begin{aligned}
&\frac{1}{st}\Phi_{n+1}(s,t) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-sx-ty} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_{\{u \leq \inf(x,y)\}} \mathbf{1}_{\{v \leq \inf(x,y)\}} S(x,y,u,v) dF_n(u,v) dx dy \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-sx-ty} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\mathbf{1}_{\{u \leq v \leq \inf(x,y)\}} + \mathbf{1}_{\{u < v \leq \inf(x,y)\}} \right) S(x,y,u,v) dF_n(u,v) dx dy \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_{\{u > v\}} \int_u^{\infty} \int_u^{\infty} e^{-sx-ty} S(x,y,u,v) dx dy dF_n(u,v)
\end{aligned}$$

$$
\begin{aligned}
& + \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{\{u\leq v\}} \int_v^{\infty}\int_v^{\infty} e^{-sx-ty} S(x,y,u,v)dxdydF_n(u,v) \\
= & \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{\{u>v\}} e^{-u(s+t)} \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-sx-ty} S(x+u,y+u,u,v)dxdydF_n(u,v) \\
& + \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{\{u\leq v\}} e^{-v(s+t)} \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-sx-ty} S(x+v,y+v,u,v)dxdydF_n(u,v) \qquad (17)
\end{aligned}
$$

Defining the functions:

$$
\begin{aligned}
A(s,t) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{\{u>v\}} e^{-u(s+t)} \int_{\mathbb{R}}\int_{\mathbb{R}} S(x+u,y+u,u,v)e^{-sx-ty}dxdydF_n(u,v) \\
B(s,t) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{\{v\geq u\}} e^{-v(s+t)} \int_{\mathbb{R}}\int_{\mathbb{R}} S(x+v,y+v,u,v)e^{-sx-ty}dxdydF_n(u,v) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{\{u\geq v\}} e^{-u(s+t)} \int_{\mathbb{R}}\int_{\mathbb{R}} S(x+u,y+u,v,u)e^{-sx-ty}dxdydF_n(v,u) \ ,
\end{aligned}
$$

equation (17) reduces to:

$$
\frac{1}{st}\Phi_{n+1}(s,t) \ = \ A_n(s,t) + B_n(s,t) \ . \qquad (18)
$$

Note that the definitions of A and B are slightly asymetric. This is due to the fact that F_n may in all generality have a mass on the diagonal $u = v$. A more symmetric approach would be to isolate this diagonal and introduce three functions, as will be done in the discrete case (see (37) in section 5). In the cases we shall analyze in the following, this problem does not arise, and the decomposition (18) will be sufficient.

The rest of this section is devoted to the proof of the following theorem:

Theorem 3.1 *There exists a function $K(z,s,t)$ such that, for all $n \geq 1$:*

$$
\Phi_{n+1}(s,t) \ = \ \frac{1}{2i\pi}\int_{i\mathbb{R}} K(z,s,t)\ \Phi_n(s+t+z,-z)\ dz \ .
$$

The function K is a 4-linear functional of the Laplace Transforms σ^*_{ij}, $1 \leq i,j \leq 2$.

The inner integral of A and B equals:

$$
\int_{\mathbb{R}} S_{11}(x)S_{21}(x+u-v)e^{-sx}dx \ \int_{\mathbb{R}} S_{12}(y)S_{22}(y+u-v)e^{-ty}dy \ . \qquad (19)
$$

Each of these terms is a Laplace transform. The first one is the Laplace transform of the r.v. $\max\{\sigma_{11}, \sigma_{21}+u-v\}$. Denote this r.v. by S, and let S^* be its Laplace transform. According to [8, p. 128], we have:

$$
\begin{aligned}
S^*(s) \ = \ & -\lim_{z\to 0,\Re(z)>0} [\frac{1}{2i\pi}\int_{i\mathbb{R}} \sigma^*_{11}(s+y)\sigma^*_{21}(-y)e^{y(u-v)}\frac{dy}{y-z} \\
& + \frac{1}{2i\pi}\int_{i\mathbb{R}} \sigma^*_{11}(-y)\sigma^*_{21}(s+y)e^{-(s+y)(u-v)}\frac{dy}{y-z}] \ .
\end{aligned}
$$

If follows that equation (19) can be rewritten as

$$
\lim_{z_1,z_2\to 0,\Re(z_1),\Re(z_2)>0} (\frac{1}{2i\pi})^2 \int_{i\mathbb{R}}\int_{i\mathbb{R}} H(s,t,y_1,y_2)e^{y_1(u-v)}e^{y_2(u-v)}\frac{dy_1}{y_1-z_1}\frac{dy_2}{y_2-z_2} \ ,
$$

with

$$
\begin{aligned}
H(s,t,y_1,y_2) &= \sigma_{11}^*(s+y_1)\sigma_{21}^*(-y_1)\sigma_{12}^*(s+y_2)\sigma_{22}^*(-y_2)e^{(y_1+y_2)(u-v)} \\
&+\sigma_{11}^*(s+y_1)\sigma_{21}^*(-y_1)\sigma_{12}^*(-y_2)\sigma_{22}^*(s+y_2)e^{(y_1-y_2-t)(u-v)} \\
&+\sigma_{11}^*(-y_1)\sigma_{21}^*(s+y_1)\sigma_{12}^*(s+y_2)\sigma_{22}^*(-y_2)e^{(y_2-y_1-s)(u-v)} \\
&+\sigma_{11}^*(-y_1)\sigma_{21}^*(s+y_1)\sigma_{12}^*(-y_2)\sigma_{22}^*(s+y_2)e^{-(y_1+y_2+s+t)(u-v)} \ .
\end{aligned}
$$

Integrating the exponential term of the first term on the domain $\{u > v\}$ against $e^{-(s+t)u}dF_n(u,v)$, we obtain

$$
J_n(s+t-y_1-y_2, y_1+y_2) = -\frac{1}{2i\pi}\int_{i\mathbb{R}} \Phi_n(s+t+z,-z)\frac{dz}{z+y_1+y_2} \ .
$$

Repeating the process for the three other terms, we get:

$$
\begin{aligned}
A_n(s,t) = & \lim_{z_1,z_2\to 0, \Re(z_1),\Re(z_2)>0} \left(\frac{1}{2i\pi}\right)^2 \int_{i\mathbb{R}}\int_{i\mathbb{R}} \frac{1}{2i\pi}\int_{i\mathbb{R}} \Bigg\{ \\
& \sigma_{11}^*(s+y_1)\sigma_{21}^*(-y_1)\sigma_{12}^*(s+y_2)\sigma_{22}^*(-y_2)\frac{1}{z+y_1+y_2} \\
& +\sigma_{11}^*(s+y_1)\sigma_{21}^*(-y_1)\sigma_{12}^*(-y_2)\sigma_{22}^*(s+y_2)\frac{1}{z+y_1-y_2-t} \\
& +\sigma_{11}^*(-y_1)\sigma_{21}^*(s+y_1)\sigma_{12}^*(s+y_2)\sigma_{22}^*(-y_2)\frac{1}{z+y_2-y_1-s} \\
& +\sigma_{11}^*(-y_1)\sigma_{21}^*(s+y_1)\sigma_{12}^*(-y_2)\sigma_{22}^*(s+y_2)\frac{1}{z-y_1-y_2-s-t}\Bigg\} \\
& \Phi_n(s+t+z,-z)dz\frac{dy_1}{y_1-z_1}\frac{dy_2}{y_2-z_2} \ .
\end{aligned} \tag{20}
$$

Likewise, the function $B_n(s,t)$ has a similar form. This proves the theorem.

4 Application to Continuous Distributions

Assume now that the four input sequences are made of i.i.d. exponential variables, and let a, b, c and d be the parameters of the variables $\sigma_{11}, \sigma_{12}, \sigma_{21}$ and σ_{22} respectively. This amounts to say that:

$$
S(x,y,u,v) = (1-e^{a(u-x)})(1-e^{b(u-y)})(1-e^{c(v-x)})(1-e^{d(v-y)}).
$$

If $a = d$ and $b = c$, then it is easy to convince oneself that the variables $X_1(n)$ and $X_2(n)$ are exchangeable for any n, that is to say the function $F_n(x,y)$ is symmetric in (x,y), and the function $\Phi_n(s,t)$ is symmetric in (s,t). In that case, $A(s,t) = B(t,s)$. We shall call this case "semi-symmetric" hereafter.

Further, we have from (19):

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} S(x+u, y+u, u, v)e^{-sx-ty}dxdy \\
&= \int_0^\infty (1-e^{-ax})(1-e^{b(v-u-x)})e^{-sx}dx \int_0^\infty (1-e^{-by})(1-e^{a(v-u-y)})e^{-ty}dy \\
&= \left(\frac{a}{s(s+a)} - \frac{ae^{b(v-u)}}{(s+b)(s+a+b)}\right)\left(\frac{b}{t(t+b)} - \frac{be^{a(v-u)}}{(t+a)(t+a+b)}\right) \ .
\end{aligned}
$$

The computation of Φ_{n+1} is therefore reduced to that of:

$$\begin{aligned} A_n(s,t) &= 2ab \int_0^\infty \int_0^u e^{-u(s+t)} \left(\frac{1}{s+a} - \frac{se^{b(v-u)}}{(s+b)(s+a+b)} \right) \\ &\qquad \left(\frac{1}{t+b} - \frac{te^{a(v-u)}}{(t+a)(t+a+b)} \right) dF_n(u,v) \ . \end{aligned} \tag{21}$$

4.1 The totally symmetric case

We assume in this section that $a = b$. We will actually take $a = 1$. It can be easily checked that $A(s,t)$ is now symmetric in (s,t) so that the expression for Φ_{n+1} reduces even more to yield:

$$\begin{aligned} \Phi_{n+1}(s,t) &= \frac{2}{(s+1)(t+1)} \int_0^\infty \int_0^u e^{-u(s+t)} \left(1 - \frac{se^{v-u}}{s+2} \right) \left(1 - \frac{te^{v-u}}{t+2} \right) dF_n(u,v) \\ &= \frac{2}{(s+1)(t+1)} (J_n(s+t,0) - (\frac{s}{s+2} + \frac{t}{t+2}) J_n(s+t+1,-1) \\ &\qquad + \frac{st}{(s+2)(t+2)} J_n(s+t+2,-2)) \ , \end{aligned}$$

where

$$J_n(s,t) = \int_{\mathbb{R}} \int_0^x e^{-sx-ty} dF_n(x,y) \ . \tag{22}$$

In order to obtain a functional recurrence for the sequence Φ_n, there remains to express the function J_n in terms of Φ_n. This can miraculously be done using theorem A.2 in appendix A. One obtains:

$$J_n(s,t) = -\frac{1}{2i\pi} \int_{i\mathbb{R}} \Phi_n(s+t+z,-z) \frac{dz}{z+t} \ .$$

This leads to:

$$\begin{aligned} \Phi_{n+1}(s,t) &= -\frac{2}{(s+1)(t+1)(s+2)(t+2)} \\ \frac{1}{2i\pi} \int_{i\mathbb{R}} & \left((s+2)(t+2)\frac{1}{z} - 2(st+s+t)\frac{1}{z-1} + st\frac{1}{z-2} \right) \Phi_n(s+t+z,-z) dz \ . \end{aligned} \tag{23}$$

The presence of a "$1/z$" in the integral causes actually no problem. This will be explained in a later version of this report.

When $n = 0$, this equation becomes:

$$\begin{aligned} \Phi_1(s,t) &= -\frac{2}{(s+1)(t+1)(s+2)(t+2)} \\ &\quad \frac{1}{2i\pi} \int_{i\mathbb{R}} \left((s+2)(t+2)\frac{1}{z} - 2(st+s+t)\frac{1}{z-1} + st\frac{1}{z-2} \right) dz \\ &= \frac{2}{(s+1)(t+1)(s+2)(t+2)} \frac{1}{2} ((s+2)(t+2) - 2(st+s+t) + st) \\ &= \frac{4}{(s+1)(t+1)(s+2)(t+2)} \ , \end{aligned} \tag{24}$$

by virtue of lemma A.3 (the problem with $z = 0$ will be solved later). This computation was of course hardly necessary: one obtains directly from (16) the value $F_1(x, y) = (1 - e^{-x})^2(1 - e^{-y})^2$.

The computation of Φ_2 is much more interesting, as it gives the intuition for the general result. We shall omit it for the sake of brevity, and state instead the principal lemma.

Lemma 4.1 *For all $n \geq 1$, the function Φ_n has the following form:*

$$\Phi_n(s,t) \; = \; \frac{4^n}{(s+1)(s+2)(t+1)(t+2)} \, \frac{\mathcal{N}(s,t)}{\mathcal{D}(s+t)} \; ,$$

where:

$$\begin{aligned} \mathcal{D}(x) &= [(x+1)(x+2)^2(x+3)^2(x+4)^2]^{n-1} \\ \mathcal{N}(s,t) &= P_n(s+t) \; + \; st \; Q_n(s+t) \; , \end{aligned}$$

and P_n, Q_n are polynomials. The degree of P_n is $3(n-1)$, and the degree of Q_n is $3n-5$ for $n > 1$.

Proof

The proof is by induction. The case $n = 1$ is already solved in view of (24): take $P_1(x) = 1$ and $Q_1(x) = 0$. The degree of P_1 is zero.

Assume the lemma holds for some n. Then, by (23), we have, denoting $x = s + t$:

$$\begin{aligned} \Phi_{n+1}(s,t) \; &= \; -\frac{2}{(s+1)(t+1)(s+2)(t+2)} \\ \frac{1}{2i\pi}\int_{i\mathbb{R}} & \left((s+2)(t+2)\frac{1}{z} - 2(st+x)\frac{1}{z-1} + st\frac{1}{z-2}\right) \\ & \frac{P_n(x) - z(x+z)Q_n(x)}{\mathcal{D}_n(x)(x+z+1)(x+z+2)(z-1)(z-2)}dz \; . \end{aligned} \tag{25}$$

All is needed is therefore a way to compute the integral:

$$I(\zeta) = \frac{1}{2i\pi}\int_{i\mathbb{R}} \frac{P_n(x) - z(x+z)Q_n(x)}{(x+z+1)(x+z+2)(z-1)(z-2)} \, \frac{dz}{z-\zeta}$$

for a fixed, pure imaginary number x and a positive number ζ.

The integrand of $I(\zeta)$ is a rational function in z, and the degree of the numerator (=2) is less than that of the denominator (=5). A decomposition in elementary fractions yields:

$$\begin{aligned} & \frac{P_n(x) - z(x+z)Q_n(x)}{(x+z+1)(x+z+2)(z-1)(z-2)(z-\zeta)} \\ & \quad \frac{\alpha(x,\zeta)}{x+z+1} + \frac{\beta(x,\zeta)}{x+z+2} + \frac{\gamma(x,\zeta)}{z-1} + \frac{\delta(x,\zeta)}{z-2} + \frac{\varepsilon(x,\zeta)}{z-\zeta} \; . \end{aligned} \tag{26}$$

With the use of lemma A.3, integration of this equation yields:

$$\begin{aligned} I(\zeta) \; &= \; \alpha(x,\zeta)sgn(-1-x) + \beta(x,\zeta)sgn(-2-x) + \gamma(x,\zeta)sgn(1) + \delta(x,\zeta)sgn(2) \\ & +\varepsilon(x,\zeta)sgn(\zeta) \; = \; \frac{1}{2}(\alpha + \beta - \gamma - \delta - \varepsilon) \; . \end{aligned}$$

On the other hand, it is clear from the decomposition (26) that $\alpha+\beta+\gamma+\delta+\varepsilon=0$ (multiply by z and let z go to infinity). Therefore, $I(\zeta)=\alpha(x,\zeta)+\beta(x,\zeta)$. The values of $\alpha(x,\zeta)$ and $\beta(x,\zeta)$ are respectively:

$$\begin{aligned}\alpha(x,\zeta) &= -\frac{P_n(x)-(x+1)Q_n(x)}{(x+2)(x+3)(x+1+\zeta)}\\ \beta(x,\zeta) &= -\frac{P_n(x)-2(x+2)Q_n(x)}{(x+3)(x+4)(x+2+\zeta)}\ .\end{aligned}$$

Plugging these expressions in (25) and reducing to the same denominator (the help of the MAPLE software was valuable here), one obtains:

$$\begin{aligned}\Phi_{n+1}(s,t) &= -\frac{2}{(s+1)(t+1)(s+2)(t+2)}\\ &\frac{2}{D_n(x)}\frac{A(x)P_n(x)+B(x)Q_n(x)+st(C(x)P_n(x)+D(x)Q_n(x))}{(x+1)(x+2)^2(x+3)^2(x+4)^2}\ ,\end{aligned}\tag{27}$$

where:

$$\begin{aligned}A(x) &= 2(x+4)(5x^2+20x+18)\\ B(x) &= x(x+1)^2(x+4)(3x+8)\\ C(x) &= 5x+14\\ D(x) &= (x+1)(x^2-8)\ .\end{aligned}$$

The lemma is therefore proved, and the polynomials P_{n+1} and Q_{n+1} are given by the recurrence:

$$\begin{pmatrix} P_{n+1}(x)\\ Q_{n+1}(x)\end{pmatrix}=\begin{pmatrix} A(x) & B(x)\\ C(x) & D(x)\end{pmatrix}\begin{pmatrix} P_n(x)\\ Q_n(x)\end{pmatrix}\ .\tag{28}$$

One easily checks that the degree of P_{n+1} is $\max\{3+3(n-1),4+3n-5\}=3n$ and that of Q_{n+1} is 2 if $n=1$ and $\max\{3(n-1)+1,3n-5+3\}=3n-2$ if $n>1$.

The recurrence (28) may be solved by elementary means.

Lemma 4.2 *For all $n\geq 1$, the polynomials P_n and Q_n are given by:*

$$\begin{aligned}P_n(x) &= \frac{1}{\lambda_1(x)-\lambda_2(x)}\left[\lambda_1^n(x)-\lambda_2^n(x)-D(x)\left(\lambda_1^{n-1}(x)-\lambda_2^{n-1}(x)\right)\right]\\ Q_n(x) &= \frac{-C(x)}{\lambda_1(x)-\lambda_2(x)}\left[\lambda_1^{n-1}(x)-\lambda_2^{n-1}(x)\right](*)\ ,\end{aligned}$$

where

$$\lambda_i(x)=\frac{1}{2}\left(A(x)+D(x)\pm\sqrt{(A(x)-D(x))^2+4B(x)D(x)}\right)\ .$$

Proof Let $Y_n(x)$ be the column vector whose entries are $P_n(x)$ and $Q_n(x)$, and let $M(x)$ denote the 2×2 matrix which appears in (27). We then have: $Y_{n+1}(x)=M(x)Y_n(x)$ for $n\geq 1$, with

$Y_1(x) = (1,0)^t$. If $\mathcal{Y}(z,x) = \sum_{n=1}^{\infty} Y_n(x) z^{n-1}$, then $\mathcal{Y}$ obeys the matrix equation: $\mathcal{Y}(z,x)(I - zM(x)) = Y_1(x)$. Inversion of this equation (at points where $I - zM(x)$ is not singular) yields:

$$\mathcal{Y}(z,x) \;=\; \frac{1}{P(z,x)} \begin{pmatrix} 1 - zD(x) & -zB(x) \\ -zC(x) & 1 - zA(x) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{29}$$

where $P(z,x) = det(I - zM(x)) = z^2(A(x)D(x) - B(x)C(x)) - z(A(x)+D(x)) + 1 = (1 - z\lambda_1(x))(1 - z\lambda_2(x))$. An expansion in series of z of the right-hand-side of (29), followed by the identification of the coefficients proves the lemma. For readers not fully convinced by arguments based on formal series, the validity of these computations will be proved in a later version of this report. □

We are now in position to state a number of properties of the solution of the system (1):

Proposition 4.3 *Let $\{(X_1(n), X_2(n))\}_0^\infty$ be the solution of (1). We have:*

$$EX_1(n) = EX_2(n) = \frac{407}{228}\, n \;+\; \frac{344}{1083} \;+\; \frac{126}{361}\frac{(-1)^n}{18^n}\,. \tag{30}$$

In particular, the Lyapunov exponent of the system is:

$$\gamma = \lim_{n\to\infty} \frac{X_1(n)}{n} \;=\; \frac{407}{228}\,. \tag{31}$$

Let $\Delta(n) = X_1(n) - X_2(n)$. Then:

$$\begin{aligned} P(\Delta(n) \le x) \;&=\; \frac{1}{19}\left(\frac{23}{2}e^x - 2e^{2x} + \frac{7}{6}\frac{(-1)^{n-1}}{18^{n-1}}(e^x - e^{2x})\right)\mathbf{1}_{\{x<0\}} \\ &\quad + \frac{1}{19}\left(19 - \frac{23}{2}e^{-x} + 2e^{-2x} - \frac{7}{6}\frac{(-1)^{n-1}}{18^{n-1}}(e^{-x} - e^{-2x})\right)\mathbf{1}_{\{x\ge 0\}}\,. \end{aligned}$$

In particular, $\Delta(n)$ converges (geometrically fast) in law to a RV $\Delta(\infty)$ with distribution ([10]):

$$P(\Delta(\infty) \le x) \;=\; \frac{1}{19}\left(\frac{23}{2}e^x - 2e^{2x}\right)\mathbf{1}_{\{x<0\}} \;+\; \frac{1}{19}\left(19 - \frac{23}{2}e^{-x} + 2e^{-2x}\right)\mathbf{1}_{\{x\ge 0\}}\,.$$

Proof The Laplace transform of $X_1(n)$ is:

$$\begin{aligned} X_n^*(s) = \Phi_n(s,0) \;&=\; \frac{4^n}{2(s+1)(s+2)}\,\frac{\mathcal{N}(s,0)}{\mathcal{D}(s)} \\ &=\; \frac{4^n}{2(s+1)(s+2)}\,\frac{P_n(s)}{[(s+1)(s+2)^2(s+3)^2(s+4)^2]^{n-1}}\,. \end{aligned}$$

Formula (30) follows by differentiation, with or without the help of MAPLE.

Likewise, the laplace transform of $\Delta(n)$ is given by:

$$\begin{aligned} \Delta_n^*(s) = \Phi_n(s,-s) \;&=\; \frac{4^n}{(1+s)(2+s)(1-s)(2-s)}\,\frac{P_n(0) - s^2Q_n(0)}{\mathcal{D}(0)} \\ &=\; \frac{1}{19}\,\frac{76 - 7s^2(1 - (-18)^{1-n})}{(1+s)(2+s)(1-s)(2-s)}\,. \end{aligned}$$

Inverting this Laplace transform yields the distribution of $\Delta(n)$. □

Other properties of the couple $(X_1(n), X_2(n))$ can be derived from the value of Φ_n, such as their higher moments and their covariance.

4.2 The semi-symmetric case

We now turn to the solution of the semi-symmetric case. Il follows the same steps as in the totally symmetric case, and many details will be ommited. It should be emphasized here that the use of an algebraic manipulation package (in our case, MAPLE) seems essential to be able to complete these calculations.

At the end of section 3, we were left with the formula: $\Phi_{n+1}(s,t) = st(A_n(s,t) + A_n(t,s))$, A_n being given by (21). Expanding the product and gives:

$$\begin{array}{rcll} A_n(s,t) & = & ab[(s+a)(s+b)(t+a)(t+b)(s+a+b)(t+a+b)]^{-1} & \\ & & (\ (s+b)(t+a)(s+a+b)(t+a+b) & J_n(s+t,0) \\ & & -t(s+b)(t+b)(s+a+b) & J_n(s+t+a,-a) \\ & & -s(s+a)(t+a)(t+a+b) & J_n(s+t+b,-b) \\ & & +(s+a)(t+b) & J_n(s+t+a+b,-a-b)\ )\ , \end{array} \tag{32}$$

where J_n is still given by (22). As in section 4.1, the computation of Φ_2 allows to guess what the general structure of Φ_n is. Again, we skip it to state directly:

Lemma 4.4 *For all $n \geq 1$, the function Φ_n has the following form:*

$$\Phi_n(s,t) \ = \ \frac{(ab)^{n+1}}{(s+a)(s+b)(s+a+b)(t+a)(t+b)(t+a+b)} \, \frac{\mathcal{N}_n(s,t)}{\mathcal{D}(s+t)^{n-1}}\ ,$$

where:

$$\begin{array}{rcl} \mathcal{D}(x) & = & (a+x)(b+x)(a+b+x)^2(2a+x)^2(2b+x)^2 \\ & & \qquad (2b+a+x)^2(2a+b+x)^2(2a+2b+x)^2 \\ \mathcal{N}_n(s,t) & = & P_n(s+t) \ + \ st\ Q_n(s+t) \ + \ (st)^2\ R_n(s+t), \end{array}$$

and P_n, Q_n and R_n are polynomials.

Proof □

The proof is similar to that of lemma 4.1. The case $n = 1$ is easily proved since it is clear that $F_1(x,y) = (1-e^{-ax})(1-e^{-bx})(1-e^{-ay})(1-e^{-by})$, the Laplace Transform of which is $\Phi_1(s,t) = a^2b^2(a+b+2s)(a+b+2t)/(s+a)(s+b)(s+a+b)(t+a)(t+b)(t+a+b)$. We have therefore $P_1(x) = 2x(a+b)+(a+b)^2$, $Q_1(x) = 4$ and $R_1 = 0$.

For the general case, one computes the integral

$$I(\zeta) = \frac{1}{2i\pi}\int_{i\mathbb{R}} \frac{P_n(x) - z(x+z)Q_n(x) + z^2(x+z)^2R_n(x)}{(x+z+a)(x+z+b)(x+z+a+b)(a-z)(b-z)(a+b-z)}\ \frac{dz}{z-\zeta}$$

by expanding the integrand in elementary fractions of z (the degree of the numerator is still less than that of the denominator). The use of the induction hypothesis and algebraic manipulations show that the vector of polynomials $Y_n = (P_n(x), Q_n(x), R_n(x))^t$ is given by a linear recurrence $Y_{n+1} = M(x)Y_n(x)$, as in (27). A MAPLE program implementing this recurrence and computing the Lyapunov exponent of the system is given in appendix B.

Repeating the analysis of the totally symmetric case, we define the vector $Y_n(x) = (P_n(x), Q_n(x), R_n(x))^t$ and the generating function $\mathcal{Y}(z,x) = \sum_{n=1}^{\infty} Y_n(x) z^{n-1}$, which satisfies:

$$\mathcal{Y}(z,x) \;=\; \frac{[I - zM(x)\tilde{]}Y_1(x)}{(1-z\lambda_1(x))(1-z\lambda_2(x))(1-z\lambda_3(x))}\;, \tag{33}$$

where $[A\tilde{]}$ denotes the comatrix of some matrix A, and $(1-z\lambda_1(x))(1-z\lambda_2(x))(1-z\lambda_3(x))$ is the determinant of $I - zM(x)$. It follows that $Y_n(x)$ can be expressed in terms of the function

$$\begin{aligned}\phi_n(x) \;&=\; [z^{n-1}]\frac{1}{(1-z\lambda_1(x))(1-z\lambda_2(x))(1-z\lambda_3(x))}\\ &=\; \frac{\lambda_1(x)^{n+1}(\lambda_3(x)-\lambda_3(x)) + \lambda_2(x)^{n+1}(\lambda_1(x)-\lambda_3(x)) + \lambda_3(x)^{n+1}(\lambda_2(x)-\lambda_1(x))}{(\lambda_1(x)-\lambda_2(x))(\lambda_2(x)-\lambda_3(x))(\lambda_3(x)-\lambda_1(x))}\;.\end{aligned}$$

We skip the painful details as the explicit expansion of P_n, Q_n and R_n (and therefore Φ_n) is too complicated to learn us anything. Note however that if the parameters a and b are given a particular numerical value, computations with MAPLE become again feasible, and the results are compact enough.

We instead concentrate on the computation of the Lyapunov exponent of the system. This is a simpler problem owing to the fact that the Laplace transform of $X_1(n)$ is simply:

$$X_1^*(n)(s) \;=\; \frac{(ab)^n}{(a+b)(a+s)(b+s)(a+b+s)}\frac{1}{(a+s)^{n-1}(b+s)^{n-1}}\frac{P_n(s)}{[\mathcal{D}(x)]^{n-1}},$$

where $\mathcal{D}$ is given in Lemma 4.4. The expected value of $X_1(n)$ is therefore:

$$\begin{aligned}EX_1(n) \;&=\; -\frac{P_n'(0)}{P_n(0)}\\ &\quad +(n-1)\left(\frac{1}{a}+\frac{1}{b}+2\left(\frac{1}{a+b}+\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2b+a}+\frac{1}{2a+b}+\frac{1}{2(a+b)}\right)\right)\\ &\quad +\frac{1}{a}+\frac{1}{b}+\frac{1}{a+b}\;.\end{aligned} \tag{34}$$

The value of $P_n(0)$ is easily shown to be $P_1(0)\lambda_1(0)^{n-1}$, where $\lambda_1(0) = M_{1,1}(0) = 64a^2b^2(a+2b)^2(2a+b)^2(a+b)^4$: this is simply due to the fact that $M_{1,2}(0) = M_{1,3}(0) = 0$. This eigenvalue is actually the *dominant* eigenvalue of $M(0)$, that is, the one with largest modulus (which is not shown here).

Let $e_1 = (1,0,0)$. Then we have:

$$e_1\mathcal{Y}(z,0) = \sum_{n=1}^{\infty} z^{n-1}P_n(0) = \frac{P_1(0)}{1-\lambda_1(0)z} = \frac{e_1[I-zM(0)\tilde{]}Y_1(0)}{(1-z\lambda_1(0))(1-z\lambda_2(0))(1-z\lambda_3(0))}$$

so that:

$$e_1[I-zM(0)\tilde{]}Y_1(0) = P_1(0)(1-z\lambda_2(0))(1-z\lambda_3(0))\;. \tag{35}$$

Differentiating equation (33) with respect to x, we have:

$$\sum_{n=1}^{\infty} P_n'(0)z^{n-1} = \frac{\partial\mathcal{Y}}{\partial x}(z,0) = \frac{z\lambda_1'(0)\; e_1[I-zM(0)\tilde{]}Y_1(0)}{(1-z\lambda_1(0))^2(1-z\lambda_2(0))(1-z\lambda_3(0))} + O\left(1/(1-z\lambda_1(0))\right)\;.$$

The right-hand side of this formula is a rational function of z which first singularity is at $z = 1/\lambda_1(0)$ (because $\lambda_1(0)$ is the dominant eigenvalue of $M(0)$). This singularity is a pole of order 2. It follows (see for instance [6]) that:

$$P'_n(0) = [z^{n-1}]\frac{\partial \mathcal{Y}}{\partial x}(z,0) = (n-1)\lambda_1(0)^{n-2}\frac{\lambda'_1(0)\ e_1[I-(\lambda_1(0))^{-1}M(0)]\tilde{}Y_1(0)}{(1-\lambda_2(0)/\lambda_1(0))(1-\lambda_3(0)/\lambda_1(0))}\ (1+o(1))\ .$$

Using (35), we obtain:

$$\frac{P'_n(0)}{P_n(0)} = n\ \frac{\lambda'_1(0)}{\lambda_1(0)}\ (1+o(1))\ .$$

There remains to evaluate $\lambda'_1(0)$, which we do as follows: let $\Pi(z,x) = (z-\lambda_1(x))(z-\lambda_2(x))(z-\lambda_3(x))$ be the characteristic polynomial of $M(x)$. On the one hand, we have:

$$\frac{\partial \Pi(z,x)}{\partial x} = \Pi(z,x)\sum_{i=1}^{3}\frac{\lambda'_i(x)}{z-\lambda_i(x)}\ .$$

Letting $z \to \lambda_1(0)$ and $x \to 0$ in the above expression, one gets (all the necessary differentiability and continuity conditions are trivially satisfied):

$$\lambda'_1(0) = -\frac{\partial \Pi}{\partial x}(\lambda_1(0),0)\frac{1}{(\lambda_1(0)-\lambda_2(0))(\lambda_1(0)-\lambda_3(0))}\ .$$

On the other hand, we have:

$$\frac{\partial \Pi}{\partial z}(\lambda_1(0),0) = (\lambda_1(0)-\lambda_2(0))(\lambda_1(0)-\lambda_3(0))\ .$$

Finally, we obtain

$$\lambda'_1(0) = -\frac{\partial \Pi}{\partial x}(\lambda_1(0),0)\left(\frac{\partial \Pi}{\partial z}(\lambda_1(0),0)\right)^{-1}\ . \tag{36}$$

In order that these computations be perfectly valid, there remains to be proved that the eigenvalue $\lambda_1(x)$ is indeed the dominant one for all valued of x. A way to obtain this result would be to show that all three $\lambda_i(x)$, $i = 1,2,3$ are never equal in modulus (actually, all seem to be real). As $\lambda_1(x)$ is dominant at $x = 0$ (which is also to be proved), it is then dominant everywhere.

The MAPLE program based on equations (34) and (36), given in appendix B. Its produces (among other things) the following result:

Lemma 4.5 *Let $r = a/b$. The Lyapunov exponent of the system is given by:*

$$\gamma = \frac{1}{16a(a+1)}\ \frac{N(r)}{D(r)},$$

where

$$\begin{aligned} N(r) &= 160r^{10} + 1776r^9 + 8220r^8 + 21378r^7 + 35595r^6 + 41566r^5 \\ &\quad +35595r^4 + 21378r^3 + 8220r^2 + 1776r + 160 \\ D(r) &= 8r^8 + 80r^7 + 321r^6 + 690r^5 + 880r^4 + 690r^3 + 321r^2 + 80r + 8\ . \end{aligned}$$

A plot of this function is given in figure 2 (for $a = 1$). Note that $D(r)$ and $N(r)/r$ are symmetric polynomials in r, which reflects the fact that γ is symmetric in (a, b). The limit of γ when $r \to \infty$ is $5/4$. The case $a = \infty$ actually admits a simpler solution, which we detail ar the end of this section.

To conclude the semi-symmetric case, we give a description of the behavior of the RV $\Delta(n)$:

Proposition 4.6 *The distribution of $\Delta(n)$ converges geometrically fast to that of $\Delta(\infty)$.*

Proof The Laplace transform of $\Delta(n)$ is:

$$\begin{aligned}
\Delta^*(n)(u) = \Phi_n(u, -u) &= \frac{a^2b^2(a+b)^2}{(a^2-u^2)(b^2-u^2)(a+b)^2-u^2)} \, \frac{P_n(0) - u^2Q_n(0) + u^2R_n(0)}{(ab)^{n-1}\lambda_1(0)} \\
&= \frac{a^2b^2(a+b)^2}{(a^2-u^2)(b^2-u^2)(a+b)^2-u^2)} \left(1 - u^2\frac{Q_n(0)}{P_n(0)} + u^4\frac{R_n(0)}{P_n(0)}\right) .
\end{aligned}$$

With the expression of $P_n(0), Q_n(0)$ and $R_n(0)$ in terms of the function ϕ_n, and knowing that $\lambda_1(0)$ is the dominant eigenvalue, one obtains that the last term satisfies:

$$\begin{aligned}
(\ldots) &= 1 - u^2\frac{\lambda_1(0)\left[Q_2(0) - (\lambda_2(0) + \lambda_3(0))Q_1(0) - u^2R_2(0)\right]}{P_1(0)(\lambda_1(0) - \lambda_2(0))(\lambda_1(0) - \lambda_3(0))} \\
&\quad + O(\rho^n) ,
\end{aligned}$$

where $\rho = \max\{|\lambda_2(0)|, |\lambda_3(0)|\}/|\lambda_1(0)$. This proves the theorem. The limiting distribution can be computed from the expression above. It does not seem to possess a simple expression, so we skip it.

□

The case $a = \infty$

This corresponds to the interesting case where, in a two-processor system, computation times are negligible compared to interprocessor communication delays.

Proposition 4.7 *For any $n \geq 1$, the Laplace transform of the couple $(X_1(n), X_2(n))$ is given by:*

$$\Phi_n(s,t) = \frac{b^{2n}}{(b+s)(b+t)} \left(\frac{3(s+t)+4b}{(s+t+b)(s+t+2b)^2}\right)^{n-1} .$$

One has:

$$EX_1(n) = EX_2(n) = \frac{5n-1}{4b} .$$

The Lapunov exponent of the system is $5/4$. For any $n \geq 1$, the distribution of $\Delta(n)$ is independent of n and is:

$$P(\Delta(n) \leq x) = 1/2\ e^{-bx}\mathbf{1}_{\{x<0\}} + (1-e^{-bx})/2\ \mathbf{1}_{\{x \geq 0\}} .$$

Proof

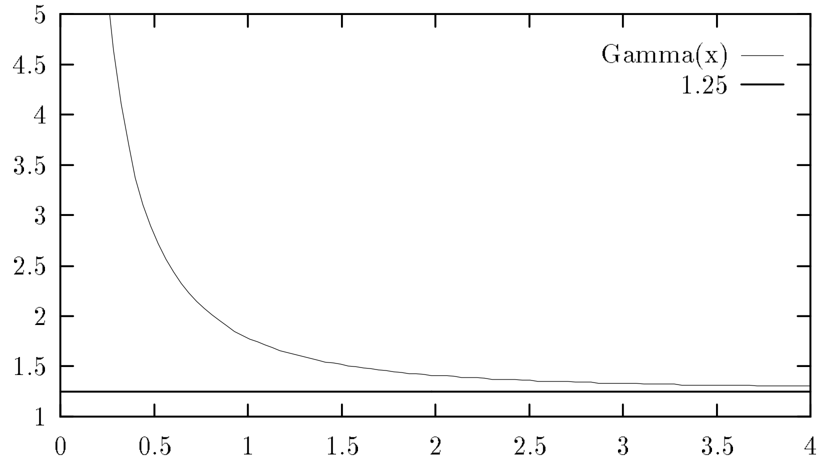

Figure 2: The curve $r \mapsto \gamma$ (*)

It is easy to see that the recurrence on Φ_n is now reduced to:

$$\begin{aligned}\Phi_{n+1}(s,t) &= \left(\frac{b}{b+s}+\frac{b}{b+t}\right) J_n(s+t,0) \ + \ \frac{b(s+t)}{(b+s)(b+t)} J_n(s+t+b,-b) \\ &= \frac{b}{(b+s)(b+t)}\left((2a+s+t)J_n(s+t,0)-(s+t)J_n(s+t+b,-b)\right) .\end{aligned}$$

It then follows that Φ_n is of the form: $\Phi_n(s,t) = b^{2n}P_n(s+t)/[(b+s+t)(2b+s+t)^2]^{n-1}$, and that $P_{n+1}(x) = (3x+4b)P_n(x)$. The result follows. □

4.3 Extensions

We indicate a number of possibilities to extend the computations to other models/assumptions.

- General Exponential case. The symmetry between A_n and B_n disappears, and $\Phi_n(s,t)$ is not symmetric anymore in (s,t). However, it is still a rational function, and it is likely that a polynomial recurrence may be used, based on the decomposition of the numerator of Φ_n isolating s and t (as in the general Bernoulli case, see section 5).

- Other distributions. The possiblity to acheive the computations is strongly connected to the relative simplicity of the inner integral that appears in the definition of A_n. In the (totally symmetric) Erlang-2 case, polynomial terms appear, which lead to an expression of Φ_{n+1} in terms of the function J_n *and* of its derivatives. However, the derivatives of J_n can be computed from the very formulas of Appendix A without further complications.

- Higher dimensions. It is of course very limitating to be able to analyze only two dimensional systems. In principle, the technique described here may be applied to systems with more than two state variables, through a quite straightforward induction. However, the iterations of the formulas of Appendix A to compute functions like $E(exp(-s_1X_1 - s_2X_2 - \ldots s_nX_n)\mathbf{1}\{X_1 > X_2 > \ldots X_n\})$ becomes quiclky problematic. A lot of guessing is involved in order to obtain the general form of the solution.

- (min, +)-systems. The theory above does not assume *a priori* that random variables should be positive. Hence, applying the technique to negative RV's and changing signs, one may compute the performance of simple liniear systems in the $(min, +)$ dioid.

- (min, max, +) systems. The approach of section 2 has been applied successfully to dynamic systems which evolution equations contain the min, max and + operators [9].

- Other models with synchronizations. The technique developped above seems to allow the analysis (both in transient and stationary regime) of some basic two-node queueing systems with synchronization whose evolution equations involve only addition and maximization (or minimization, *cf.* a previous remark). Such systems include the Fork/Join queue, with evolution equations (on the waiting time):

$$W_n^1 = [W_{n-1}^1 + \sigma_{n-1}^1 - \tau_n]^+$$
$$W_n^2 = [W_{n-1}^2 + \sigma_{n-1}^2 - \tau_n]^+,$$

  (solved in [7] in steady state with complicated computations) and the round-robin routing model, with evolution equations (for the workload at instants of arrival in the system):

$$W_n^1 = [W_{n-1}^1 + u_n^1\sigma_{n-1} - \tau_n]^+$$
$$W_n^2 = [W_{n-1}^2 + u_n^2\sigma_{n-1} - \tau_n]^+,$$

  where $u_n^1 = 1$ iff n is odd, and $u_n^1 + u_n^2 = 1$. For the last model, qualitative results are known (mainly, stochastic comparisons with the Bernoulli routing shceme) but hardly any quantitative ones. Moreover, no results are known if resequencing is added before exiting the system.

5 Return to the discrete case

The case of lattice distributions can be handled in principle with the approach of the previous section. There are however a few particularities in the analysis of discrete distributions. The analysis is similar to the previous one, with Laplace transforms repaced by generating functions, and the contour $i\mathbb{R}$ replaced by the unit circle.

Section 2 described how the use of a Markov chain underlying the evolution of the system allows the solution of the problem by linear algebra techniques. However, if the support of the distributions involved is infinite, the matrices involved are also infinite, and the proactical solution of the porblem is not evident. The approach which follows, although applied to a case of finite distributions, allows to handle more difficult cases, such as geometric distributions.

As before, let $F_n(x,y) = \mathbb{P}\{X_1(n) \leq x, X_2(n) \leq y\}$, and let now $f_n(x,y) = \mathbb{P}\{X_1(n) = x, X_2(n) = y\}$. Set: $\Phi_n(s,t) = \mathbb{E}(s^{X_1(n)} t^{X_2(n)})$. This generating function is defined (at least) on the set $\{|s| = |t| = 1\}$. We have:

$$\sum_x \sum_y s^x t^y F_n(x,y) = \frac{1}{1-s}\,\frac{1}{1-t}\;\Phi_n(s,t)\ .$$

The recurrence on the distributions, viz.:

$$F_{n+1}(x,y) \;=\; \int_0^{\inf(x,y)} \int_0^{\inf(x,y)} S(x,y,u,v) dF_n(u,v)\ ,$$

translates now into:

$$\Phi_{n+1}(s,t) \;=\; A_n(s,t) + B_n(s,t) + C_n(s,t)\ ,$$

with:

$$\begin{array}{rcl} A_n(s,t) & = & \sum\sum s^x t^y \mathbf{1}_{\{x>y\}} f_n(x,y) \\ B_n(s,t) & = & \sum\sum s^x t^y \mathbf{1}_{\{x<y\}} f_n(x,y) \\ C_n(s,t) & = & \sum\sum s^x t^y \mathbf{1}_{\{x=y\}} f_n(x,y)\ . \end{array} \tag{37}$$

The functions A_n, B_n and C_n are computed from Φ_n *via* contour integrals, as in the continuous case. The relevant formula will appear in a later version of this document.

5.1 The totally symmetric Bernoulli case

In this section, it is assumed that the four input sequences are made of Bernoulli variables with parameter $a = P(\sigma = 1)$. For convenience, let $\bar{a} = 1 - a$.

Proposition 5.1 *The lyapunov exponent of the system is given by:*

$$\gamma(a) \;=\; \frac{a(a-2)(2a^3 - a^2 - 2a + 2)}{1 + 2a(a-1)(a^2 - 3a + 1)} \;=\; 1 - \frac{(2a+1)(1-a)^4}{1 + 2a(a-1)(a^2 - 3a + 1)}\ . \tag{38}$$

In particular, $\gamma(1/2) = 6/7$ *([10]). The random variable* $\Delta(n)$ *converges exponentially fast to a* $\Delta(\infty)$ *given by:*

$$\mathbb{P}(\Delta(\infty) = 0) = \frac{1 - 2a\bar{a}}{1 + 2a\bar{a}(a^2 - 3a + 1)}, \qquad \mathbb{P}(\Delta(\infty) = \pm 1) = \frac{a(2-a)\bar{a}^2}{1 + 2a\bar{a}(a^2 - 3a + 1)}.$$

Proof The function Φ_n can be written in the form $\Phi_n(s,t) = (s+t)P_n(st) + Q_n(st)$, where P_n and Q_n are polynomials. Theses polynomials obey the linear recurrence:

$$\begin{pmatrix} P_{n+1}(x) \\ Q_{n+1}(x) \end{pmatrix} = \begin{pmatrix} 2a(1-a)x & a(2-a)(1-a)^2 \\ 2x((1-a)^2 + a^2 x) & (1-a)^4 + a^2(2-a)^2 x \end{pmatrix} \begin{pmatrix} P_n(x) \\ Q_n(x) \end{pmatrix}\ , \tag{39}$$

with initial conditions $P_0(x) = 0$ and $Q_0(x) = 1$. The solution of this recurrence using the technique of section 4.1 leads, thanks to MAPLE, to:

$$EX_1(n) = \gamma n + \beta\left(1 - (2a(a-1)(a^2 - 3a + 1))^n\right)\ ,$$

with γ given in (38), and

$$\beta = \frac{a(a-1)^2(a-2)(2a^2-2a+1)}{(1+2a(a-1)(a^2-3a+1))^2} \ .$$

Formula (38) follows because the term $2a(a-1)(a^2-3a+1)$ is always less than 1 in modulus. Note that $EX_1(n)$ is actually a polynomial in a although it may not seem at first glance. Observe also that the term $2a(a-1)(a^2-3a+1)$ changes sign at $a_0 = (3-\sqrt{5})/2$. The generating function of the RV $\Delta(n)$ is given by $\Phi_n(s, \frac{1}{s}) = P_n(1)(s+\frac{1}{s}) + Q_n(1)$. Given the expressions for P_n and Q_n, it turns out that:

$$\begin{aligned} P_n(1) &= \frac{a(2-a)(1-a)^2}{1+2a(a-1)(a^2-3a+1)}\Big(1-(2a(a-1)(a^2-3a+1))^n\Big) \\ Q_n(1) &= \frac{(2a^2-2a+1)+2a(2-a)(1-a)^2(2a(a-1)(a^2-3a+1))^n}{1+2a(a-1)(a^2-3a+1)}, \end{aligned}$$

from which the result follows. Note that $\Delta(n)$ does not depend on $n \geq 1$ when $a = a_0$. Finally, if $a = 1/2$, one has:

$$\mathbb{P}(\Delta(n) = \pm 1) = \frac{3}{14}(1-8^{-n}), \qquad \mathbb{P}(\Delta(n) = 0) = \frac{4}{7} + \frac{3}{7}8^{-n} \ .$$

□

5.2 The general Bernoulli Case

Let a, b, c and d denote the Bernoulli parameters of the four input sequences. It will actually more convenient to use $\bar{a} = 1-a, \bar{b} = 1-b, \bar{c} = 1-c$ and $\bar{d} = 1-d$.

The generating function Φ_n satisfies the recurrence:

$$\begin{aligned} \Phi_{n+1}(s,t) &= (\bar{a}+as)(\bar{b}+bt) && J_n(st,1) \\ &+ (\bar{a}\bar{c}s+(1-\bar{a}\bar{c}))(\bar{b}\bar{d}t+(1-\bar{b}\bar{d})) && K_n(st) \\ &+ (\bar{c}+cs)(\bar{d}+dt) && L_n(1,st) \ , \end{aligned} \tag{40}$$

with initial condition $\Phi_0(s,t) = 1$. Computing the first values of Φ_n, one quiclky realizes that this function must have the form:

$$\Phi_n(s,t) \ = \ P_n(st) \ + \ sQ_n(st) \ + \ tR_n(st) \ ,$$

where as usual, P_n, Q_n and $R-n$ are polynomials. Moreover, the computations of J_n, K_n and L_n gives simply:

$$K_n(x,1) = P_n(x), \qquad J_n(x,1) = Q_n(x), \qquad K_n(x) = R_n(x) \ .$$

Therefore, (40) translates into the linear recurrence for the vector $Y_n(x) = (P_n(x), Q_n(x), R_n(x))$: $Y_{n+1}(x) = M(x)Y_n(x)$ with

$$\begin{pmatrix} \bar{a}\bar{b}\bar{c}\bar{d} + x(1-\bar{a})(1-\bar{b}\bar{d}) & x(abx+\bar{a}\bar{b}) & x(cdx+\bar{c}\bar{d}) \\ \bar{b}\bar{d}(1-\bar{a}\bar{c}) & a\bar{b}x & c\bar{d}x \\ \bar{a}\bar{c}(1-\bar{b}\bar{d}) & b\bar{a}x & d\bar{c}x \end{pmatrix},$$

with $Y_0(x) = Y_0 = (1,0,0)^t$. Is it still necessary to tell how to solve this recurrence? Let us rather derive the Lyapunov exponent, following the reasoning of section 4.2. With $\mathcal{Y}(z,x) = \sum_{n=0}^{\infty} Y_n(x) z^n$, we have:

$$\mathcal{Y}(z,x) = \frac{[I - zM(x)\tilde{]}Y_0}{(1 - z\lambda_1(x))(1 - z\lambda_2(x))(1 - z\lambda_3(x))} .$$

Let $u = (1,1,1)$. From the fact that $\Phi_n(1,1) = P_n(1) + Q_n(1) + R_n(1) = 1$, and because the matrix $M(0)^t$ is stochastic, we have:

$$u\mathcal{Y}(z,0) = \frac{1}{1-z} = \frac{u[I - zM(0)\tilde{]}Y_0}{(1-z)(1 - z\lambda_2(0))(1 - z\lambda_3(0))} ,$$

so that $u[I - zM(0)\tilde{]}Y_0 = (1 - z\lambda_2(0))(1 - z\lambda_3(0))$. Now, $EX_1(n) = d\Phi(s,1)/ds|_{s=1} = P_n'(1) + Q_n'(1) + R_n'(1) + Q_n(1)$. Clearly, $Q_n(1) = o(n)$, so we just have to look at:

$$\begin{aligned}
&(P_n + Q_n + R_n)'(1) \\
&= [z^n]u\frac{\partial \mathcal{Y}}{\partial x}(z,1) = [z^n]\frac{z\lambda_1'(1)\ u[I - zM(1)\tilde{]}Y_0}{(1-z)^2(1 - z\lambda_2(1))(1 - z\lambda_3(1))} + O\left(1/(1-z)\right) \\
&= n\lambda_1'(1)(1 + o(n)) .
\end{aligned}$$

We can use (36) to compute $\lambda_1'(1)$. A MAPLE program based on these formulas eventually gives:

Proposition 5.2 *The Lyapunov exponent of the system is given by:*

$$\gamma = 1 - \frac{\delta}{\bar{a}}\bar{d}(\bar{b}\bar{c}(1-\bar{a})(1-\bar{d}) - (1 - (1-\bar{b})(1-\bar{c}))^2) ,$$

with

$$\begin{aligned}
\delta = {} & 1 - 2\bar{a}\bar{b}\bar{c} + 2\bar{a}\bar{c}\bar{d} - 2\bar{a}\bar{b}\bar{c}\bar{d} - \bar{a}\bar{d} - 2\bar{b}\bar{c}\bar{d} + \bar{a}\bar{c} + \bar{b}\bar{d} + \\
& \bar{a}\bar{b} + \bar{c}\bar{d} - \bar{c} - \bar{b} + \bar{c}\bar{b} + \bar{a}^2\bar{c}\bar{b} - \bar{a}\bar{c}^2\bar{d} + 2\bar{a}\overline{db} + \overline{db}^2\bar{a}\bar{c} - \bar{d}^2\bar{b}\bar{a}\bar{c} + \bar{a}\bar{c}^2\bar{b}\bar{d} - \bar{a}^2 \\
& \bar{c}\overline{db} - \bar{b}^2\bar{d}\bar{a} + \bar{b}\bar{d}^2\bar{c} .
\end{aligned}$$

Finally, the generating function of the RV $\Delta(n)$ is: $\Delta_n^*(u) = P_n(1) + uQ_n(1) + u^{-1}Q_n(1)$, which is to say that this distribution is (as expected) concentrated on $\{-1,0,1\}$. The value of the masses are extracted form the generating vector $\mathcal{Y}(z,1)$ and are:

$$Y_n(1) = [z^n]\frac{[I - zM(1)\tilde{]}Y_0}{(1-z)(1 - z\lambda_2(1))(1 - z\lambda_3(1))} .$$

The actual computations are left to the reader. When $n \to \infty$, this vector converges exponentially fast to

$$Y_\infty(1) = \frac{[I - M(1)\tilde{]}Y_0}{(1 - \lambda_2(1))(1 - \lambda_3(1))} .$$

The rate of convergence is given by $|\lambda_2(0)|$, the modulus of the second eigenvalue of $M(0)$.

A Joint distributions and Laplace Transforms

This section is essentially an excerpt of [8], adapted to the computations of the paper. Such formulas also appear in the works of J.W. Cohen [4]. The technique relies on the following lemma of the complex variable calculus.

Lemma A.1 *Let L be a smooth contour of $\mathbb{C}$, which separates $\mathbb{C}$ in two domains L^+ (to the left) and L^- (to the right). Let ϕ be a function defined on L satisfying Hölder's condition. Assume a function Φ, defined on $\mathbb{C} \setminus L$ is analytic on L^+ and L^- and admits a limit when $z \to t \in L$ both in L^+ and L^- and is such that:*

$$\forall t \in L, \qquad \lim_{z \to t, z \in L^+} \Phi(z) \; - \lim_{z \to t, z \in L^-} \Phi(z) \; = \; \phi(t) \; .$$

If furthermore $\Phi(z)$ vanishes when $|z| \to \infty$, then

$$\forall z \in \mathbb{C}, \qquad \Phi(z) \; = \; \frac{1}{2i\pi} \int_L \frac{\phi(t)}{t-z} dt \; .$$

The proof of this result may be found in [5] in the case of a closed contour (*). The proof for an smooth open contour can be found in [8]. Note that the contour has to be smooth at infinity, that is be sufficiently regular and tend to infinity in opposite directions.

Let A and B be two real RVs, not necessarily positive nor independent. Assume one knows their joint distribution by its Laplace transform:

$$G(x,y) \; = \; \mathbb{E}(e^{-xA-yB}) \qquad \forall \; \Re(x) = 0, \; \Re(y) = 0 \; .$$

Note that this fonction is defined only on $i\mathbb{R} \times i\mathbb{R}$ since A and B may be negative.

Let $J(s,t) = E(e^{-sA-tB} \mathbf{1}_{\{A \geq B\}})$. This function is defined on the domain $\{(s,t) | \Re(s+t) = 0, \Re(s \geq 0)\}$. It can be computed from G as follows:

Lemma A.2 *If for any $x \in i\mathbb{R}$ the function $y \mapsto G(x+y,-y)$ admits a limit when $|y| \to \infty$, then:*

$$J(s,t) \; = \; -\frac{1}{2i\pi} \int_{i\mathbb{R}} G(s+t+z,-z) \frac{dz}{z+t} \; .$$

Proof Fix $x \in i\mathbb{R}$. Define the function Φ_x on the domain $\mathbb{C} \setminus i\mathbb{R}$ by:

$$\Phi_x(y) \; = \; \begin{cases} - \quad \mathbb{E}(e^{-xA} e^{-y(A-B)} \; \mathbf{1}_{\{A-B \geq 0\}}) & \text{if } \Re(y) > 0 \\ \quad \mathbb{E}(e^{-xA} e^{-y(A-B)} \; \mathbf{1}_{\{A-B < 0\}}) & \text{if } \Re(y) < 0 \; . \end{cases} \; .$$

On easily checks that

$$\lim_{z \to y, \Re(z) < 0} \Phi_x(z) \; - \lim_{z \to y, \Re(z) > 0} \Phi_x(z) \; = \; G(x+y,-y) \quad \forall \; y \in i\mathbb{R}.$$

If $A - B$ possesses a distribution without mass at 0, Φ_x vanishes at infinity in all directions other than $i\mathbb{R}$. If moreover the mapping $y \mapsto H(x,y)$ satisfies Hölder's condition on $i\mathbb{R}$ and goes to zero for $|y| \to \infty$, we can apply lemma A.1 to obtain:

$$\Phi_x(z) \; = \; \frac{1}{2i\pi} \int_{i\mathbb{R}} G(x+y,-y) \frac{dy}{y-z} \quad \forall \; z \in \mathbb{C} \; - i\mathbb{R} \; .$$

The lemma is proved in this case. When $A - B$ has a mass in zero, there is a trick, which will be exposed here later. See [8] for the moment. $\square$

Define the "sign" function on $\mathbb{C} \setminus i\mathbb{R}$ by:

$$sgn(\zeta) \;=\; \frac{1}{2i\pi} \int_{i\mathbb{R}} \frac{dz}{z - \zeta} \;.$$

Lemma A.3 *The value of the sign function is:*

$$sgn(\zeta) \;=\; \begin{cases} 1/2 & \textit{if } \Re(\zeta) < 0 \\ -1/2 & \textit{if } \Re(\zeta) > 0. \end{cases}$$

Formulæ for the discrete case are forthcomming. They are very similar but involve the unit circle as integration contour.

B A MAPLE program for the semi-symmetric case

```
# MAPLE program to compute the Lyapunov exponent of the exponential,
# semi-symmetric system
#
Fract := 1/(a+s)/(b+s)/(a+b+s)/(b+t)/(a+t)/(b+a+t);

Intgd := subs( { s=x+y, t=-y }, Fract) / (y-z);

t1 := limit( Intgd*(x+y+a), y=-a-x );
t2 := limit( Intgd*(x+y+b), y=-b-x );
t3 := limit( Intgd*(x+y+a+b), y=-a-b-x );

J:=-subs({x=u+v,z=-v}, Na*t1+Nb*t2+Nab*t3);

J0:=subs({u=x,v=0},J);
Ja:=subs({u=x+a,v=-a},J);
Jb:=subs({u=x+b,v=-b},J);
Jab:=subs({u=x+b+a,v=-b-a},J);

M0 := (x*(a+b)+y+(a+b)^2)*(2*y+x*(a+b)+2*a*b);
Ma := -(x*b+y+b**2)*(2*y+x*(a+b));
Mb := -(x*a+y+a**2)*(2*y+x*(a+b));
Mab := y*(2*y+x*(a+b)+2*a*b);

NN := PP + y*QQ + y^2*RR;
Na := subs( y=-a*(x+a), NN );
Nb := subs( y=-b*(x+b), NN );
Nab := subs( y=-(a+b)*(x+(a+b)), NN );

NewP := (J0*M0 + Ja*Ma + Jb*Mb + Jab*Mab);
```

```
NumNewP := collect(numer(normal(NewP)),y):
PP1 := collect(coeff(NumNewP,y,0),{PP,QQ,RR}):
QQ1 := collect(coeff(NumNewP,y,1),{PP,QQ,RR}):
RR1 := collect(coeff(NumNewP,y,2),{PP,QQ,RR}):

B_11 := coeff(PP1,PP,1): B_12 := coeff(PP1,QQ,1): B_13 := coeff(PP1,RR,1):
B_21 := coeff(QQ1,PP,1): B_22 := coeff(QQ1,QQ,1): B_23 := coeff(QQ1,RR,1):
B_31 := coeff(RR1,PP,1): B_32 := coeff(RR1,QQ,1): B_33 := coeff(RR1,RR,1):

B := matrix(3,3,[B_11,B_12,B_13,B_21,B_22,B_23,B_31,B_32,B_33]):

#
# A procedure to compute recursively P_n(x)
#
Pi := proc(n)
if n=1  then 2*x*(a+b)+(a+b)^2;
else Pi(n-1)*B_11+Qi(n-1)*B_12+Ri(n-1)*B_13;
fi;
end;

#
# The computation of the Lyapunov exponent, using the characteristic
# polynomial of the matrix.
#
PolCar := collect(charpoly(B,z),z);

l10 := factor( subs( x=0 , B[1,1] ) ):

corr := 2/a+2/b+2*(1/(a+b)+1/(2*a+b)+1/(a+2*b)+1/2/(a+b));

Chi := -simplify( subs({x=0,z=l10}, diff(PolCar,x)) / diff(PolCar,z) ) ):

gamma := -simplify( Chi / l10 - corr ):
```

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