Theorem 2 *The Laplace transform of the response time density in an M/M/1 queue with FCFS queueing discipline and non-preemptive arrivals to a random position in the queue is*

$$\frac{(1-\rho)\mu}{s+\mu} - (1-\rho)\int_0^\rho \frac{(z-r_2)^{\beta-1}}{(z-r_1)^{\beta+2}}\int_{r_1}^z \frac{v-r_1}{v}\left(\frac{v-r_1}{v-r_2}\right)^\beta \left[\frac{\lambda}{(s+\mu)(1-v)v}+\frac{1}{(1-v)^2}\right] dv\,dz - \frac{\mu}{(s+\mu)(1-z)z}$$

Proof

Conditioning on the queue length faced on arrival by the tagged customer, the required Laplace transform is

$$\begin{aligned}
& (1-\rho)L_{00}(s) + (1-\rho)\sum_{q=1}^{\infty}\rho^q\sum_{n=1}^{q}(1/q)L_{q-n,n}(s) \\
= \;& (1-\rho)L_{00}(s) + (1-\rho)\sum_{n=1}^{\infty}\sum_{q=n}^{\infty}\rho^q(1/q)L_{q-n,n}(s) \\
= \;& (1-\rho)L_{00}(s) + (1-\rho)\sum_{n=1}^{\infty}\sum_{q=0}^{\infty}\frac{\rho^{q+n}}{q+n}L_{qn}(s) \\
= \;& (1-\rho)L_{00}(s) + (1-\rho)\int_0^\rho \frac{G(z,z)-G(z,0)}{z}\mathrm{d}z
\end{aligned}$$

and the result follows. 2

Finally, it is interesting to state the result for the Laplace transform of the sojourn time density in the eventuality that the tagged customer arrives at the *end* of the queue.

Theorem 3 *For customers that arrive at the end of the queue which later customers join at random positions, the Laplace transform of sojourn time density is*

$$(1-\rho)G(0,\rho)$$

The proof is trivial.

analyticity of $G(x,y)$ at $y = r_1 < 1$. Now hold $x = y - u^{-1}(y-r_1)^{\beta+1}(y-r_2)^{-\beta}$ for any u such that $|x| < 1$. Then, as $y \to r_1$, the argument of Φ remains at u. Thus, for analyticity of $G(x,y)$, we require $\Phi(u) = 0$ for all u such that x stays in the unit disk when y is in the unit disk. Hence the function Φ is everywhere zero and we have:

$$\mu(y-r_1)^{\beta+2}(y-r_2)^{1-\beta}G = -y\int_{r_1}^{y}\frac{h(v)}{v}(v-r_1)\left(\frac{v-r_1}{v-r_2}\right)^{\beta}\mathrm{d}v$$

as required. 2

Now consider the limit of $G(x,y)$ as $y \to 0$. The Maclaurin expansion of the integrand at $v = 0$ is given by the following expansions:

$$\begin{aligned}(v-r_1)^{\beta+1}(v-r_2)^{-\beta} &= -r_1\left(\frac{r_1}{r_2}\right)^{\beta}(1-(\beta+1)v/r_1)(1+\beta v/r_2)) + o(v)\\ &= -r_1\left(\frac{r_1}{r_2}\right)^{\beta}(1-(r_1+r_2)v/(r_1r_2)) + o(v)\\ &= -r_1\left(\frac{r_1}{r_2}\right)^{\beta}(1-(s+\lambda+\mu)v/\lambda) + o(v)\end{aligned}$$

and

$$h(v) = \frac{\lambda\mu}{(s+\mu)(1-x(0))}\frac{1}{v} + \frac{\lambda\mu}{(s+\mu)(1-x(0))^2}x'(0) + \frac{\mu}{(1-x(0))^2} + O(v)$$

where $x(v) = v - C(v-r_1)^{\beta+1}(v-r_2)^{-\beta}$ so that, differentiating at $v = 0$:

$$\begin{aligned}x'(0) &= 1 - C(\beta+1)\left(\frac{r_1}{r_2}\right)^{\beta} + C\beta\left(\frac{r_1}{r_2}\right)^{\beta+1}\\ &= 1-(\beta+1)x(0)/r_1 + \beta x(0)/r_2\\ &= 1 - x(0)(s+\lambda+\mu)/\lambda\end{aligned}$$

Substituting into the expansion for $h(v)$ and simplifying then yields:

$$h(v) = \frac{\lambda\mu}{(s+\mu)(1-x(0))}\frac{1}{v} + \frac{\mu(s+\lambda+\mu)}{(s+\mu)(1-x(0))} + O(v)$$

Thus, the first two terms of the Maclaurin expansion of the integrand, multiplied by v, near $v = 0$ are given by:

$$\begin{aligned}&-\frac{\mu r_1}{(s+\mu)(1-x(0))}\left(\frac{r_1}{r_2}\right)^{\beta}\left(\lambda\frac{1}{v} + (s+\lambda+\mu)\right)\left(1 - \frac{s+\lambda+\mu}{\lambda}v\right) + O(v)\\ = \quad &-\frac{\mu r_1}{(s+\mu)(1-x(0))}\left(\frac{r_1}{r_2}\right)^{\beta}\frac{\lambda}{v} + O(v)\end{aligned}$$

Hence, as $y \to 0$, y multiplied by the integral approaches $\frac{\mu r_1}{(s+\mu)(1-x)}\left(\frac{r_1}{r_2}\right)^{\beta}\lambda$, there being no singularities in the range of integration other than at zero. Therefore:

$$-\mu r_1^2 r_2\left(\frac{r_1}{r_2}\right)^{\beta}G(x,0) = -\frac{\mu r_1}{(s+\mu)(1-x)}\left(\frac{r_1}{r_2}\right)^{\beta}\lambda$$

Cancellation of terms and using $r_1 r_2 = \lambda/\mu$ then gives:

$$G(x,0) = \frac{\mu}{(s+\mu)(1-x)}$$

as required since we have already observed that $L_{m0}(s) = \mu/(s+\mu)$ for all $m \geq 0$.

We are now in a position to state our result on the response time distribution in the queue with arrivals to a random position.

Integrating, we now obtain:

$$\log u = \frac{1}{2}\log(\mu y^2 - (s+\lambda+\mu)y + \lambda) - \gamma \log \frac{y-r_2}{y-r_1} + C'$$

where C' is an arbitrary constant and

$$\gamma = \frac{s+\lambda+\mu}{2\mu(r_2-r_1)} = \frac{r_1+r_2}{2(r_2-r_1)} = \frac{\alpha+\beta}{2} = \beta + 1/2$$

Consequently we have, for one solution of the auxiliary equations,

$$y - x = C(y-r_1)\left(\frac{y-r_1}{y-r_2}\right)^{\beta}$$

where C is an arbitrary constant.

To find an independent solution, we solve the equation:

$$\frac{\mathrm{d}y}{\mu(y-r_1)(y-r_2)} = \frac{\mathrm{d}G}{(-2\mu y + (s+\lambda+\mu) + \lambda/y)G - h(y)}$$

where $h(y) = H(x(y), y)$ with

$$x(y) = y - C(y-r_1)^{\beta+1}(y-r_2)^{-\beta}$$

We rewrite this equation as:

$$\frac{\mathrm{d}G}{\mathrm{d}y} + \left(\frac{2\mu y - (s+\lambda+\mu) - \lambda/y}{\mu(y-r_1)(y-r_2)}\right) G = \frac{-h(y)}{\mu(y-r_1)(y-r_2)}$$

The coefficient of G may be written in the form:

$$\frac{2\mu y - (s+\lambda+\mu)}{\mu y^2 - (s+\lambda+\mu)y + \lambda} - \frac{\lambda}{\mu}\frac{1}{y(y-r_1)(y-r_2)}$$

Expanding the last term in partial fractions yields:

$$\frac{1}{y(y-r_1)(y-r_2)} = \frac{1}{r_1 r_2}\left\{\frac{1}{y} + \frac{\beta}{y-r_2} - \frac{\alpha}{y-r_1}\right\}$$

where $r_1 r_2 = \lambda/\mu = \rho$ and we recall the definitions from section 2 $\alpha = r_2/(r_2 - r_1)$ and $\beta = r_1/(r_2 - r_1) = \alpha - 1$.

Thus, the integrating factor of the equation is $\mu(y-r_1)^{2+\beta}(y-r_2)^{1-\beta}y^{-1}$ and we obtain the independent solution of the auxiliary equations:

$$\mu(y-r_1)^{\beta+2}(y-r_2)^{1-\beta}G + y\int_{r_1}^{y}\frac{h(v)}{v}(v-r_1)\left(\frac{v-r_1}{v-r_2}\right)^{\beta}\mathrm{d}v = Dy$$

for arbitrary constant D. The lower limit of the definite integral is arbitrary in the interval $[-1, 1]$ in which G is defined and we choose r_1 (which is positive and less than unity for $s > 0$) for reasons which will become apparent.

The general solution of the partial differential equation may therefore be written:

$$\mu(y-r_1)^{\beta+2}(y-r_2)^{1-\beta}G + y\int_{r_1}^{y}\frac{h(v)}{v}(v-r_1)\left(\frac{v-r_1}{v-r_2}\right)^{\beta}\mathrm{d}v =$$
$$y\Phi\left(\frac{y-r_1}{y-x}\left[\frac{y-r_1}{y-r_2}\right]^{\beta}\right)$$

where Φ is an arbitrary function which we will determine as follows.

Since $G(x, y)$ is analytic in the unit x- and y- disks, let us consider the limit $y \to r_1$ which we know lies in the unit disk for $s > 0$. The coefficient of G vanishes as does the integral, the integrand not having a singularity at $v = r_1$. Thus the right hand side, $\Phi\left(\frac{y-r_1}{y-x}\left[\frac{y-r_1}{y-r_2}\right]^{\beta}\right)$ must also vanish to preserve

Proof

We obtain, for $n > 0$,:

$$\begin{aligned} F_{mn}(t+h) &= (1-(\lambda+\mu)\,h)\,F_{mn}(t) \\ &\quad + \lambda h \frac{m+1}{m+n+1} F_{m+1,n}(t) \\ &\quad + \lambda h \frac{n}{m+n+1} F_{m,n+1}(t) \\ &\quad + \mu h F_{m,n-1}(t) \end{aligned}$$

with the base case to the recurrence $F_{m,0}(t) = 1 - e^{-\mu t}$ for all $m \geq 0$ since, if there are no customers ahead at any time t, the tagged customer commences service immediately and cannot be preempted. Rearranging, we obtain the differential equation in F:

$$\begin{aligned} (m+n+1)F'_{mn}(t) &= -(\lambda+\mu)(m+n+1)F_{mn}(t) \\ &\quad + \lambda(m+1)F_{m+1,n}(t) + \lambda n F_{m,n+1} \\ &\quad + \mu(m+n+1)F_{m,n-1}(t) \end{aligned}$$

Similarly, we find that the same equation holds also for $n = 0$ provided we define $F_{m,-1}(t) = 1$ for all $t \geq 0$. (For $t = 0$ the definition merely states that the exponential service time density has value μ at the origin.) Now let the density function $f_{mn}(t) \equiv F'_{mn}(t)$ have Laplace Transform $L_{mn}(s)$. Then, using the result that $f'_n(t)$ has Laplace Transform $sL_{mn}(s) - f_{mn}(0) = sL_n(s)$ for $n > 0$, we can differentiate the above recurrence and take Laplace transforms as follows:

$$\begin{aligned} &(m+n+1)(s+\lambda+\mu)L_{mn} - (m+1)\mu\delta_{n0} = \\ &\quad \lambda(m+1)L_{m+1,n} + \lambda n L_{m,n+1} + \mu(m+n+1)L_{m,n-1} \end{aligned}$$

for $m, n \geq 0$, where δ is the Dirac delta function defined by $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$.

Multiplying the above equation by $x^m y^n$ and summing over all $m, n \geq 0$ yields the following linear first-order partial differential equation:

$$\begin{aligned} &(\mu xy - (s+\lambda+\mu)x + \lambda)\frac{\partial G}{\partial x} + (\mu y^2 - (s+\lambda+\mu)y + \lambda)\frac{\partial G}{\partial y} = \\ &\quad (-2\mu y + (s+\lambda+\mu) + \lambda/y)G - H(x,y) \end{aligned}$$

where

$$H(x,y) = \frac{\lambda\mu}{(s+\mu)(1-x)y} + \frac{\mu}{(1-x)^2}$$

Solving by Lagrange's method, we seek two independent solutions of the following auxiliary (ordinary differential) equations:

$$\begin{aligned} \frac{\mathrm{d}x}{\mu xy - (s+\lambda+\mu)x + \lambda} &= \frac{\mathrm{d}y}{\mu y^2 - (s+\lambda+\mu)y + \lambda} \\ &= \frac{\mathrm{d}G}{(-2\mu y + (s+\lambda+\mu) + \lambda/y)G - H(x,y)} \end{aligned}$$

Letting $u = y - x$, subtracting the numerators and denominators of the first two ratios, we obtain the differential equation:

$$\begin{aligned} \frac{\mathrm{d}u}{u} &= \frac{(\mu y - (s+\lambda+\mu))\mathrm{d}y}{\mu y^2 - (s+\lambda+\mu)y + \lambda} \\ &= \frac{1}{2}\left(\frac{(2\mu y - (s+\lambda+\mu)}{\mu y^2 - (s+\lambda+\mu)y + \lambda}\right)\mathrm{d}y - \frac{s+\lambda+\mu}{2\mu(y-r_1)(y-r_2)}\mathrm{d}y \end{aligned}$$

where r_1 and $r_2, r_1 \leq r_2$, are the roots of the equation $\mu z^2 - (s+\lambda+\mu)z + \lambda = 0$, exactly the same as in section 2. Now,

$$\frac{1}{(y-r_1)(y-r_2)} = \frac{1}{r_2 - r_1}\left(\frac{1}{y-r_2} - \frac{1}{y-r_1}\right)$$

where $\alpha = \frac{r_2}{r_2 - r_1}$ and $\beta = \frac{r_1}{r_2 - r_1}$. The solution for $G(z)$ is therefore:

$$\frac{G(z)(z-r_1)^\alpha}{z(-z+r_2)^\beta} = L_0(s) \int_{r_1}^{z} \left[\frac{\rho(v-r_1)^{\alpha-1}}{v^2(-v+r_2)^{\beta+1}} + \frac{(v-r_1)^{\alpha-1}}{(1-v)(-v+r_2)^{\beta+1}} \right] \mathrm{d}v$$

where the lower limit of integration is set to r_1 since the left hand side of the equation is zero at $z = r_1 < 1$ which is in the region of analyticity for $G(z)$. Noting that $\alpha = \beta + 1$ this may be written

$$\frac{G(z)(z-r_1)^\alpha}{z(-z+r_2)^\beta} = L_0(s) \int_{r_1}^{z} \frac{(v-r_1)^\beta}{(-v+r_2)^{\beta+1}} \left[\frac{\rho}{v^2} + \frac{1}{1-v} \right] \mathrm{d}v$$

so that

$$G(z) = \left(\frac{z}{z-r_1} \right) \left(\frac{-z+r_2}{z-r_1} \right)^\beta \left(\frac{\mu}{\mu+s} \right) \int_{r_1}^{z} \frac{(v-r_1)^\beta}{(-v+r_2)^{\beta+1}} \left[\frac{\rho}{v^2} + \frac{1}{1-v} \right] \mathrm{d}v$$

It now remains only to decondition the Laplace transform, $L_n(s)$ by considering the steady-state probability that $N = n$, for all n, when the tagged customer arrives. We thus obtain the unconditional Laplace transform, $L(s)$, as

$$\begin{aligned} L(s) &= \sum_{n=0}^{\infty} Pr\{\ N = n\ \} L_n(s) \\ &= (1-\rho) \sum_{n=0}^{\infty} \rho^n L_n(s) \\ &= (1-\rho)\, G(\rho) \end{aligned}$$

and the result follows. 2

3 A queue with random arrivals and fixed order of service

We now consider the same queue except that arrivals join at a random position and are served in order of the queue.

Again, we let T denote the response time random variable of a task, but now define for $m, n \geq 0$,

$$F_{mn}(t) = Pr\{\ T \leq t \mid M = m, N = n\ \}$$

where the random variables M and N are the numbers of tasks in the queue behind and ahead of, respectively, the tagged task just before its arrival instant. We again consider the intervals $[0, h)$ and $[h, t+h)$, where time 0 is the arrival instant of the tagged task, and the possible state transitions that can occur in the former interval. In contrast to the previous case, the position at which a customer joins the queue is now crucial.

Let the density function $f_{mn}(t) = F'_{mn}(t)$ have Laplace transform $L_{mn}(s)$ and define the generating function

$$G(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} L_{mn}(s) x^m y^n$$

Then we have:

Proposition 1

$$G(x,y) = -\frac{(y-r_2)^{\beta-1}}{(y-r_1)^{\beta+2}} y \int_{r_1}^{y} \frac{(v-r_1)^{\beta+1}}{(v-r_2)^\beta} \left[\frac{\lambda}{(s+\mu)(1-\eta(v))v^2} + \frac{1}{(1-\eta(v))^2 v} \right] dv$$

where

$$\eta(v) = v - (y-x) \left(\frac{v-r_1}{y-r_1} \right)^{\beta+1} \left(\frac{y-r_2}{v-r_2} \right)^\beta$$

with the base case

$$F_0(t) = 1 - e^{-\mu t}$$

from the assumption about exponentially distributed service times. Rearranging and taking the limit $h \to 0$, we obtain:

$$\begin{aligned} F_n'(t) &= \lambda F_{n+1}(t) \\ &\quad + \frac{\mu}{n} F_0(t) + \frac{\mu(n-1)}{n} F_{n-1}(t) \\ &\quad - (\lambda + \mu) F_n(t) \end{aligned}$$

Differentiating again and writing the density function $f_n(t) = \frac{\mathrm{d}}{\mathrm{d}t} F_n(t)$ for $n \geq 0$, we obtain for $n > 0$:

$$f_n'(t) = \lambda f_{n+1}(t) + \frac{\mu}{n} f_0(t) + \frac{\mu(n-1)}{n} f_{n-1}(t) - (\lambda + \mu) f_n(t)$$

Now let $f_n(t)$ have Laplace Transform $L_n(s)$. Then, by a simple integration by parts, we find that $f_n'(t)$ has Laplace Transform $sL_n(s) - f_n(0) \;=\; sL_n(s)$ for $n > 0$, $f_n(t)$ being a mixture of at least two convolutions of the exponential density with parameter μ. (However, observe that $f_0(0) = \mu$.) In terms of the Laplace transform, the recurrence now becomes:

$$n(s + \lambda + \mu)\, L_n(s) = n\lambda\, L_{n+1}(s) + \mu\, L_0(s) + \mu(n-1)\, L_{n-1}(s)$$

To solve this recurrence we introduce the generating function

$$G(z) = \sum_{n=0}^{\infty} z^n L_n(s)$$

Multiplying the above equation by z^n and summing from $n = 1$ to ∞ we obtain:

$$\begin{aligned} (s + \mu + \lambda) z G'(z) &= \lambda G'(z) - \lambda z^{-1}(G(z) - L_0(s)) \\ &\quad + \frac{\mu L_0(s) z}{1 - z} + \mu z^2 G'(z) \end{aligned}$$

This yields the following differential equation:

$$\begin{aligned} G'(z) &= \frac{\rho G(z)}{z(z - r_1)(z - r_2)} \\ &\quad - L_0(s) \left[\frac{\rho}{z(z - r_1)(z - r_2)} + \frac{z}{(1 - z)(z - r_1)(z - r_2)} \right] \end{aligned}$$

where r_1 and $r_2, r_1 \leq r_2$, are the roots of the quadratic equation

$$\mu v^2 - (s + \lambda + \mu) v + \lambda = 0$$

We solve the differential equation by determining the integrating factor:

$$I = exp\left(- \int \frac{\rho}{z(z - r_1)(z - r_2)} dz \right)$$

Splitting the integrand into partial fractions, we have

$$\begin{aligned} \log I &= -\rho \int \left[\frac{1}{z\rho} - \frac{1}{r_1(r_2 - r_1)(z - r_1)} + \frac{1}{r_2(r_2 - r_1)(z - r_2)} \right] \\ &= -\log z + \frac{r_2}{r_2 - r_1} \log(z - r_1) - \frac{r_1}{r_2 - r_1} \log(z - r_2) \end{aligned}$$

noting that $\rho = r_1 r_2$. Thus

$$I = \frac{(z - r_1)^{\alpha}}{z(-z + r_2)^{\beta}}$$

1 Introduction

Queues with various queueing disciplines have been studied extensively for many years but there are few results on queues in which random choices are made by the server and/or arrivals. Such variants are interesting on the one hand and reflect certain real world situations on the other. For example, some complex transaction processing schedulers in database systems are highly complex and non-deterministic in nature. Requests may be attached to processes which are scheduled for service according to a wide range of criteria, resulting in an unpredictable order of service. A random selection order can approximate such a system well. A degree of control can be added to such a system by the use of monitors which will impose an order on tasks' processing, but complex priority mechanisms still allow "queue jumping" by many tasks into unknown positions in the queue which one might model as if random.

We investigate the simplest case–the M/M/1 queue at equilibrium–first with random queueing discipline, under which the next customer to serve is chosen from the queue with equal probability and then with random arrivals, i.e. arrivals which join the queue in a randomly selected position. These two variants of the M/M/1 queue are considered in terms of response time in the next two sections—notice that queue length distribution is the same as for FCFS, or any other work conserving queueing discipline. Consequently, by Little's result, the queues we consider must both have mean sojourn time equal to that of the exponential distribution well known for the M/M/1 queue with FCFS discipline.

2 A queue with random selection

We first consider a single-server queue with Poisson arrivals at rate λ, and negative-exponential service times with mean $1/\mu$. The queueing discipline is random, i.e. the next customer selected for service is chosen from the queue with equal probability. It therefore matters not at what position an arriving task joins the queue, provided pre-emption is disallowed. We identify a special, "tagged" task whose progression through the queue we shall monitor as the basis of our analysis. The general method is similar to that used in Chapter 9 of [?] in the context of a Processor Sharing queue and [?].

Let T denote the response time random variable of a task, i.e. the sum of the task's queueing time and service time, and define

$$F_n(t) = Pr\{\ T \leq t \mid N = n\ \}$$

where the random variable N is the number of tasks in the queue at the arrival instant of the tagged task. Consider now the intervals $[0,h)$ and $[h,t+h)$ where time 0 is the arrival instant of the tagged task. During the initial interval $[0,h)$, as $h \to 0$, there are two possible events: firstly, an arrival with probability $\lambda h + o(h)$, and secondly a service completion with probability μh. In the case of a service completion the next customer to be served is selected randomly from the queue. Note that the analysis is insensitive to the positions at which customers join the queue, as anticipated above. This approach leads to the following result.

Theorem 1 *Response time in the M/M/1 queue with random selection has probability density function with Laplace transform:*

$$L(s) = \frac{(1-\rho)\lambda}{(\rho - r_1)(\mu + s)} \left(\frac{r_2 - \rho}{\rho - r_1}\right)^{\beta} \int_{r_1}^{\rho} \frac{(v - r_1)^{\beta}}{(-v + r_2)^{\beta+1}} \left[\frac{\rho}{v^2} + \frac{1}{1-v}\right] dv$$

where

$$r_1, r_2 = \frac{s + \mu + \lambda \mp \sqrt{(s+\mu+\lambda)^2 - 4\lambda\mu}}{2\mu}$$

Proof

Since the forward recurrence time of an exponential random variable is an identical exponential random variable, it is as if the tagged arrival occurred at time h in the state reached after any (or none) of the possible aforementioned transitions. Thus, by the Law of Total Probability, we have

$$\begin{aligned} F_n(t+h) \quad = \quad & (1 - (\lambda + \mu)\, h)\, F_n(t) \\ & + \lambda h F_{n+1}(t) \\ & + \mu h \left(\frac{1}{n} F_0(t) + \frac{n-1}{n} F_{n-1}(t) \right) \end{aligned}$$

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Response Times in Two Random Queues

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Abstract The effect of two notions of random choice in an M/M/1 queue is investigated in terms of response time distribution. In the first queue, the next customer to be served is chosen at random whereas in the second, arriving customers join the queue at an random position (without preemption) but are served in order. We analyse the differences between these queues in terms of their response (or sojourn) time distributions, the Laplace-Stieltjes transforms of which we derive as analytical formulae.

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