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Memorandum: SAI-RM-4

Date:- October, 1973

Subject: Lambda-definability and logical relations

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1. Introduction

In [4] we showed that in every model, D_∞, of the λ-calculus as constructed in [8] the strict ordering, $\prec$, is first-order definable using only application. Here we look at the, perhaps more pertinent, question of definability by pure λ-terms of such lattice-theoretic entities as $\perp$, $\top$, $\sqcup$, $\sqcap$ and Y, the least fixed-point operator.

The main method will be to construct certain, so-called, logical relations which are satisfied by all (constant vectors of) λ-definable elements and yet are not satisfied by [illegible], under discussion. The definition of logical is derived from a corresponding one of M. Gordon for the typed λ-calculus. This in turn generalised the idea of an invariant functional [2]. R. Milne [3] has independently developed analogues of the logical relations for use in equivalence proofs about programming languages.

It is not known whether logical relations also provide sufficient conditions for definability. In the second half of this memorandum we discuss this question for the typed case, obtaining necessary and sufficient conditions by using the more inclusive concept of an I-logical relation.

This memorandum is by no means self-contained. The reader should have some knowledge of both the typed and untyped λ K-calculi and be fairly familiar with Scott's models of the untyped λ K-calculus.

2. Pure definability in D_∞

A structure $\langle D,K,S,\cdot[\cdot]\rangle$ is called a (non-trivial) model of the λ-calculus if K and S have the usual properties and extensionality holds (and $|D| > 1$). Such structures give a denotational semantics for the λ K-calculus which we will use informally, confusing use and mention. Generally we will consider only the models D_∞, from [8], given by a Park retraction, $\psi_0 = \lambda f: D_1.f(t)$, where t is an isolated element of D_0. We will often use facts about such models, accompanied by a reference to the proof for the case $t=\bot$. The general proof is always similar. Also needed is the fact that if Y_λ is the paradoxical combinator, $\lambda f(\lambda x\ f(xx))(\lambda x\ f(xx))$, then, in D_∞, $Y_\lambda[f] = \bigsqcup_{n\geq 0} f^n[f[t] \sqsupseteq t \to t, \bot]$ $(f \in D_\infty)$ [5].

A relation $R \subseteq D^\kappa$, (κ an ordinal) on such a structure is logical iff:

$$\forall \vec{f} \in D^\kappa.(R(\vec{f}) \equiv (\forall \vec{x} \in D^\kappa.(R(\vec{x}) \to R(\vec{f}[\vec{x}])))).$$

Here κ is any ordinal and application of vectors is defined pointwise. An element $x \in D$ satisfies R iff $R(\hat{x})$ is true, where $\hat{x} \in D^\kappa$ is the constant vector such that $(\hat{x})_\lambda = x$ $(\lambda < \kappa)$.

An element $x \in D$ is λ-definable if x=M, for some closed λ-term M; it is λ-definable from $X \subseteq D$ iff there is a closed term M and $x_1 \ldots x_n$ in X such that $x = Mx_1 \ldots x_n$.

Theorem 1 1. Any closed λ-term satisfies any logical relation.
2. If x is λ-definable from $X \subseteq D$, and each element in X satisfies the logical relation R, then so does x.

Proof Clearly, if x and y satisfy a logical relation R, so does $x[y]$. So to finish the proof we need only show that K and S satisfy any such relation. Suppose R is logical. To show K satisfies R, assuming $R(\vec{x})$ we must show that $R(\hat{K}[\vec{x}])$. This, in turn, follows if $R(\hat{K}[\vec{x}][\vec{y}])$ when $R(\vec{y})$. But this holds as $\hat{K}[\vec{x}][\vec{y}] = \vec{x}$.

In/

In the same way we see that S satisfies R if $R(\hat{S}[\vec{x}][\vec{y}][\vec{z}])$ when $R(\vec{x})$, $R(\vec{y})$, and $R(\vec{z})$. But then we have successively, by the remark made at the beginning of the proof that $R(\vec{x}[\vec{z}])$, $R(\vec{y}[\vec{z}])$ and $R(x[\vec{z}][\vec{y}][\vec{z}]])$, concluding the proof.

Nothing is known about the converse of theorem 1. However it will be very useful for particular cases of undefinability. Here is a way of constructing logical relations $R \subseteq D_{\infty}^2$.

Suppose $R_0 \subseteq D_0^2$. Define $R_n \subseteq D_n^2$ by:

$$\forall f,g \in D_{n+1}.(R_{n+1}(f,g) \equiv \forall\, x,y \in D_n(R_n(x,y) \rightarrow R_n(fx,gy))).$$

Define $R_\infty \subseteq D_\infty^2$ by:

$$\forall\, d,e \in D_\infty(R_\infty(d,e) \equiv \forall n\; R_n(d_n,e_n)).$$

<u>Theorem 2</u> Suppose that $R_0(\top,\top)$, that $R_0(d,e)$ implies $R_1(\phi_0\, d, \phi_0\, e)$, for any d, e in D_0, and that R_0 is closed under unions of increasing sequences. Then:

1. R_∞ is logical.
2. R_∞ is closed under increasing sequences.
3. If R_0 is closed under $\sqcup$ ($\sqcap$) so is R_∞; if $R_0(\perp,\perp)$ ($R_0(\top_0,\top_0)$) then $R_\infty(\perp,\perp)$ ($R_\infty(\top,\top)$).

The construction also works for any $R_0 \subseteq D^K$ under the corresponding conditions, and the theorem analogous to theorem 2 can be proved; this extension will be assumed.

<u>Lemma 1.1</u> Suppose that $R_0(\top,\top)$ and $R_0(d,e)$ implies $R_1(\phi_0\, d, \phi_0\, e)$ for any d, e in D_0. Then,

$$\forall n\; \forall f,g \in D_n\; (R_n(f,g) \rightarrow R_{n+1}(\phi_n(f), \phi_n(g))) \text{ and}$$
$$\forall n\; \forall f,g \in D_{n+1}\; (R_{n+1}(f,g) \rightarrow R_n(\phi_n'(f), \phi_n'(g))).$$

1.2 If R_0 is closed under increasing sequences so is each R_n.

<u>Proof</u>/

Proof 1.1 By induction on n.

For n=0, note that if $R_1(f,g)$ then $R_0(ft,gt)$ from the definition of R_1 and the fact that $R_0(t,t)$.

For n+1, suppose $R_{n+1}(f,g)$ and suppose $R_{n+1}(f',g')$. By induction hypothesis $R_n(\psi_n f', \psi_n g')$. Therefore $R_n(f(\psi_n f'), g(\psi_n g'))$. and by the induction hypothesis, $R_{n+1}(\phi_n \circ f \circ \psi_n(f'), \phi_n \circ f \circ \psi_n(g'))$, which shows that $R_{n+2}(\phi_{n+1}f, \phi_{n+1}g)$. The other half is similar.

1.2 By induction on n. For n+1, let $\langle f^m, g^m\rangle_{m=0}^{\infty}$ be an (infinite) increasing sequence in R_{n+1} and suppose $R_n(x,y)$. Then $\langle f^m x, g^m y\rangle_{m=0}^{\infty}$ is an increasing sequence in R_n and so $\langle (\bigsqcup_m f^m)x, (\bigsqcup_m g^m)y\rangle$ is in R_n by induction hypothesis and the complete additivity of application in its first argument. This concludes the proof.

Proof of theorem 1 1 First suppose that $R_\infty(f,g)$ and $R_\infty(x,y)$. We will show that $R_\infty(fx,gy)$.

Now $(fx)_n = \bigsqcup_{m=n}^{\infty} \psi_{mn}(f_{m+1}x_m)$ and similarly for $(gy)_n$, [7]. Since $R_m(f_{m+1}x_m, g_{m+1}y_m)$ is true for any $m \geq n$, $R_n(\psi_{mn}(f_{m+1}x_m), \psi_{mn}(g_{m+1}y_m))$ follows by m - n applications of lemma 1.1, and then we see that $R_n((fx)_n, (gy)_n)$ by lemma 1.2 and the above formulae for $(fx)_n$ and $(gy)_n$.

Conversely, suppose that whenever $R_\infty(x,y)$ then $R_\infty(fx,gy)$ and yet for some n, $R_n(f_n, g_n)$ is false. By lemma 1.1 we can assume that $n>0$, and so for some $\langle x_{n-1}, y_{n-1}\rangle \in R_{n-1}$, $R_{n-1}(f_n x_{n-1}, g_n y_{n-1})$ is false. Let $x = \phi_{n-1\infty} x_{n-1}$ and define y similarly. By lemma 1.1, $R_\infty(x,y)$ is true and so therefore is $R_\infty(fx,gy)$ and, consequently, $R_{n-1}((fx)_{n-1}, (gy)_{n-1})$. But $(fx)_{n-1} = (f_n x_{n-1})$ and similarly for incorrect $(gy)_{n-1}$ (cf. the laws of application in [9]) and so $R_{n-1}(f_n x_{n-1}, g_n y_{n-1})$, a contradiction). (fx)

2 Suppose $\langle x^m, y^m\rangle_{m=0}^{\infty}$ is an (infinite) increasing sequence in R_∞.

Now/

Now, $(\bigsqcup_{m=0}^{\infty} x^m)_n = \bigsqcup_{n'=n}^{\infty} \psi_{n'n}(\bigsqcup_{m=0}^{\infty}(x^m)_{n'})$ and similarly for the y's, (cf. [7]).

Then one sees, successively that, $R_n((x^m)_n,(y^m)_n)$ for all m and n, $R_n(\bigsqcup_{m=0}^{\infty}(x^m)_n, \bigsqcup_{m=0}^{\infty}(y^m)_n)$, for all n, by lemma 1.2, $R_n(\psi_{n'n}(\bigsqcup_m (x^m)_n), \psi_{n'n}(\bigsqcup_m (y^m)_n)$ for $n' \geq n$, by lemma 1.1 and finally $R_n((\bigsqcup_m x^m)_n, (\bigsqcup_m y^m)_n)$ by lemma 1.2.

3 A straightforward inductive argument shows that if R_0 is closed under $\sqcup$ so is each R_n. Then, clearly, $R_\infty(\lambda x \lambda y(x_n \sqcup y_n), \lambda x \lambda y(x_n \sqcup y_n))$ and so R_∞ is closed under $\sqcup$ as $\sqcup = \bigsqcup_{n\geq 0}(\lambda x \lambda y(x_n \sqcup y_n))$ expresses $\sqcup$ as an increasing sequence in D_∞.

The argument for D_∞ is similar; it uses the fact that if f,g : X -> · where X and Y are continuous lattices then $(f \sqcap g)x=(fx) \sqcap (gx)$.

If $R_0(\perp,\perp)$ then $R_n(\perp,\perp)$, for any n, by lemma 1.1.

If $R_0(T_0,T_0)$ an easy inductive argument shows that $R_n(T_n,T_n)$ for all n, concluding the proof.

As an example, let $T_0=\{t\}$. Then by the assumed extension of) theorem 2, P_∞ is logical and so the 0th component of any closed λ-term is t. Therefore if $t \neq \perp$ neither $\perp$ nor, since $\perp \sqsubseteq T$, Y is not λ-definable; this is a result of Park [6].

The next example establishes all the definabilities among $\perp$, T, $\sqcup$, $\sqcap$ and Y for all possible values of t.

<u>Theorem 3</u> 1.1 If $t=\perp$, $Y=Y_\lambda$ and $\perp = Y_\lambda I$.

.2 In general, $\perp = YI$ and $Y = \lambda f(Y_\lambda(\lambda g \lambda x f(g\perp))I)$.

.3 If $t=T_0$, then $T=Y_\lambda K$.

.4 If $t=T_0$ and $D_0=\mathcal{D}=\{\perp,T\}$ then $\sqcap = Y_\lambda(\lambda g \lambda x \lambda y \lambda z\ g(xz)(yz))$.

2./

2. The only definabilities among $\perp$, T, $\sqcup$, $\sqcap$ and Y are those implied by 1.

<u>Proof</u> 1.1 This result is known - see [9].

.2. $\perp = YI$ is obvious.

Suppose $f \in D_\infty$ and let $\Gamma = \lambda g \lambda x \; f(g\perp)$. As $\Gamma t = \Gamma\perp = \lambda x f \perp$, we see that $Y_\lambda \Gamma = \bigsqcup_{n\geq 0} \Gamma^n(\lambda x f \perp)$. By induction on n, $\Gamma^n(\lambda x f \perp) = \lambda x f^{n+1}(\perp)$, giving $Y_\lambda \Gamma = \lambda x(Yf)$, and the result follows.

.3. As $Kt \sqsupseteq t$, $Y_\lambda K = \bigsqcup_{n\geq 0} K^n t$. As $K^n t \sqsupseteq T_n$ ($t = T_0$, here), for all n, $Y_\lambda K = T$.

.4. Let $\Gamma = \lambda g \lambda x \lambda y \lambda z g(xz)(yz)$. Since, in this D_∞, $x \sqsupseteq t$ iff $xt \sqsupseteq t$, one sees that $\Gamma t \sqsupseteq t$. Now, $t = \lambda x \lambda y \; x_0 \sqcap y_0$ is true in this lattice and then $\sqcap = Y_\lambda \Gamma$ follows by the usual inductive argument.

2. As $\perp$ and Y are interdefinable, only definabilities among $\perp$, T, $\sqcup$ and $\sqcap$ need be considered.

We must show that if $t \neq \perp$, then $\perp$ is not λ-definable from $\{T, \sqcup, \sqcap\}$; that if $t \neq T_0$, T is not λ-definable from $\{\perp, \sqcup, \sqcap\}$; that $\sqcup$ is not λ-definable from $\{\perp, T, \sqcap\}$ in all cases; and that if $t \neq T_0$ or $D_0 \neq \mathbb{O}$ then $\sqcap$ is not λ-definable from $\{\perp, T, \sqcup\}$.

To show that $\perp$ is not λ-definable from $\{T, \sqcup, \sqcap\}$, when $t \neq \perp$ let $R_0 = \{t, T_0\}$. The conditions of theorem 2 are easily checked and so R_∞ is logical. It also follows from theorem 2 that $\sqcup$, $\sqcap$ and T satisfy R_∞. Clearly $\perp$ does not. The conclusion then follows from theorem 1.2.

In the rest of the proof we shall first display an appropriate R_0 and leave the (admittedly tedious) details to the reader.

To show that if $t \neq T_0$, T is not λ-definable from $\{\perp, \sqcap, \sqcup\}$ take $R_0 = \{\perp, t\}$.

To/

To show that, in all cases, $\sqcup$ is not λ-definable from $\{\perp, T, \sqcap\}$, take $R_0=\{\langle T_0,T_0,T_0\rangle, \langle\perp,T_0,\perp\rangle, \langle T_0,\perp,\perp\rangle, \langle t,t,t\rangle, \langle\perp,t,\perp\rangle, \langle t,\perp,\perp\rangle, \langle\perp,\perp,\perp\rangle\}$. Note that $R_\infty(\perp,T_0,\perp)$ and $R_\infty(T_0,\perp,\perp)$ but not $R_\infty(T_0,T_0,\perp)$.

To show that if $t\neq T_0$ then $\sqcap$ is not λ-definable from $\{\perp,T,\sqcup\}$, take $R_0=\{\langle x,y,z\rangle \mid \{x,y\} \subseteq \{t,T_0\},\ x\neq t \text{ or } y\neq t,\ z \in \{\perp,t,T_0\}\} \cup \{\langle\perp,\perp,\perp\rangle\}$. Note that $R_\infty(t,T_0,\perp)$ and $R_\infty(T_0,t,\perp)$ but not $R_\infty(t,t,\perp)$.

To show that if $t=T_0$ and $D_0\neq \mathbb{O}$ then $\sqcap$ is not λ-definable from $\{\perp,T,\sqcup\}$, choose $u \in D_0$ distinct from $\perp$ and t and take $R_0=\{\langle t,u,u\rangle,\langle u,t,u\rangle,\langle t,t,u\rangle,\langle t,t,t\rangle\} \cup \{\langle x,y,\perp\rangle \mid x,y \in \{\perp,u,t\}\}$. Note that $R_\infty(t,u,u)$ and $R_\infty(u,t,u)$ but not $R_\infty(u,u,u)$. This concludes the proof.

It is interesting to note that when $t=T_0$ and $D_0=\mathbb{O}$ then a normal term can even equal an unsolvable term, for example, $I=Y_\lambda(\lambda f \lambda x \lambda y\ f(xy))$ (cf. I-J, when $t=\perp$ [9]).

Our method of constructing logical relations is by no means all-powerful. For example, we believe that if $t\neq T_0$ or $D_0\neq\mathbb{O}$ then ψ_0 is not λ-definable. Clearly, for the R_∞'s constructed so far, if $R_\infty(\vec{x})$ then $R_0(\lambda\alpha.((\vec{x})_\alpha)_0)$ and so $R_\infty(\lambda\alpha.((\vec{x})_\alpha)_0)$. Therefore $R_\infty(\hat{\psi}_0)$. On the other hand, suppose ψ_0 were λ-definable by a closed term M when $t=\perp$. Clearly (see [10]) M is not unsolvable, as $\psi_0\neq\perp$. So there are closed terms $M_1 \ldots M_k$ $(k\geq 0)$ such that $M\,M_1\ldots M_k=I$, but as mentioned above the 0th component of M_1 must be $\perp$ and so either $\perp=I$ or $\psi_0=I$, a contradiction. Perhaps an extension of Wadsworth's methods to the other D_∞'s would sort this out.

The last example concerns interdefinabilities among the members of $\{tt,ff,T,\sqcup,\sqcap,\supset\}$ in T_∞ [9] which is gotten by taking $t=\perp$ and D_0 to be the truth-value lattice displayed in figure 1.

T/

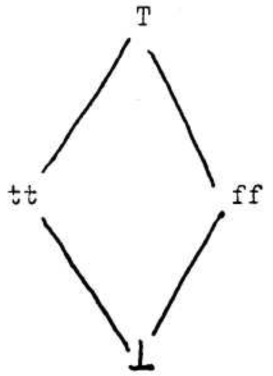

fig. 1

The conditional, $\supset$, is in $T_{\infty}^{3} \to T_{\infty}$, and is regarded as being in T_{∞} , in the usual way. It is defined by:

$$(z \supset x,y) = \begin{cases} x \sqcup y & (\text{if } z=T) \\ x & (\text{if } tt \sqsubseteq z \neq T) \\ y & (\text{if } ff \sqsubseteq z \neq T) \\ \perp & (\text{otherwise}) \end{cases}$$

It is known that T is λ -definable from $\{tt,ff,\sqcup\}$, $\sqcup$ is λ -definable from $\{\supset,T\}$ and $\sqcap$ can be defined from $\{tt,ff,\sqcup,\supset\}$. We will show that there are no more λ -definabilities of tt,ff,T, $\sqcup$ or $\sqcap$ other than those implied by the above ones; the situation for $\sqcap$ has only been partly clarified.

First, $\supset$ is not definable from $\{tt,ff,\sqcup,\sqcap,T\}$. Take $R_0=\{\langle\perp,\perp\rangle,\langle tt,ff\rangle,\langle ff,ff\rangle,\langle tt,tt\rangle,\langle tt,T\rangle,\langle T,T\rangle,\langle\perp,ff\rangle,\langle tt,\perp\rangle,\langle T,ff\rangle\}$ and note that $R_0(\supset tt\ ff\ tt, \supset ff\ ff\ tt)$ is false. Here and later theorem 2 is used implicitly.

tt is not definable from $\{ff,\sqcup,\supset,T\}$; take $R_0=\{\perp,ff,T\}$.
ff is not definable from $\{tt,\sqcup,\supset,T,\sqcap\}$; take $R_0=\{\perp,tt,T\}$.
T is not definable from any one of $\{tt,ff,\supset,\sqcap\}$, $\{tt,\sqcup,\supset,\sqcap\}$ or $\{ff,\sqcup,\supset,\sqcap\}$; take $R_0=\{\perp,tt,ff\}$, $\{\perp,tt\}$ or $\{\perp,ff\}$ respectively.

$\sqcup$ is not definable from either one of $\{tt,ff,T,\sqcap\}$ or $\{tt,ff,\sqcap,\supset\}$; take/

take $R_0=\{\langle tt,tt\rangle,\langle ff,ff\rangle,\langle\perp,\perp\rangle,\langle T,T\rangle,\langle tt,ff\rangle,\langle tt,\perp\rangle,\langle\perp,ff\rangle\}$ and note that $R_0(\sqcup\, tt\ tt, \sqcup\, tt\ ff)$ is false in the first case and take $R_0=\{\perp, tt, ff\}$ in the second case.

In the case of $\sqcap$, we would like to show that $\sqcap$ is not definable from any of the sets $\{tt, ff, \sqcup, T\}$, $\{tt, ff, \supset\}$, $\{ff, \sqcup, \supset, T\}$ or $\{tt, \sqcup, \supset, T\}$. For the first of these take $R_0=\{\langle tt,tt\rangle,\langle ff,ff\rangle,\langle T,T\rangle,\langle\perp,\perp\rangle,\langle tt,ff\rangle,\langle tt,T\rangle,\langle T,ff\rangle\}$ and note that $R_0(\sqcap\, ff\ tt, \sqcap\, ff\ ff)$ is false.

The trouble with the others is that if $R_\infty(\supset,\supset)$ then $R_\infty(\sqcap,\sqcap)$, for the R's considered here. For if $x,y \in T_0$ then $(x \supset (y \supset y, \perp), (y \supset \perp, y)) = x \sqcap y$, and so one can define from $\supset$ terms M_n $(n\geq 0)$ such that $M_n xy = x \sqcap y$ if x and y are in T_n. Therefore if $R_\infty(\supset,\supset)$ then $R_\infty(\lambda x \lambda y\ \ x_n \sqcap y_n)$ for any n and so $R_\infty(\sqcap,\sqcap)$.

On the other hand, $\sqcap$ is, in fact, not λ-definable from $\supset$. For suppose $\sqcap = M \supset$ for some closed λ-term M. If M is unsolvable then $\sqcap = \perp$, a contradiction; therefore M has the form $\lambda x_1 \ldots \lambda x_n . x_j\ M_1 \ldots M_k$ where $n>0$ and $1\leq j\leq n$. One can assume that $j\leq 3$ since one can always apply the identity $\sqcap = M \supset (M\supset)(M\supset)$. If $j=1$, then $\sqcap\ x_2 \ldots x_n = \supset M'_1 \ldots M'_k$ where $M'_r = (\lambda x_1 M_r) \supset$ $(1\leq r\leq k)$. Taking the $x_i = I$ and $F_0=\{\perp\}$ we see that $(M'_1)_0 = \perp$ and so $\sqcap\ I \ldots I = \perp$, a contradiction. If $j\neq 1$ then since $\sqcap\, xy = \sqcap\, yx$, for any x,y in T_∞, we have $xM'_1 \ldots M'_k = M \supset xy$ --- $= M \supset yx$ --- $= y\ M''_1 \ldots M''_k$ where the M'_i and M''_i are λ-definable from x,y, $\supset$ and ---. Since x,y are arbitrary members of T_∞, this is a contradiction.

Perhaps an extension of Wadsworth's ideas to LAMBDA [9], would settle these questions.

3. λ-definability in the full type hierarchy

For the sake of clarity, we will be a little more formal than in the last section.

The set of <u>type symbols</u> is the least set containing c and containing $(\sigma \rightarrow \tau)$ whenever it contains σ and τ; σ and τ are metavariables, possibly suffixed, ranging over type symbols and $(\sigma_1, \ldots, \sigma_n, \tau)$ abbreviates $(\sigma_1 \rightarrow (\sigma_2 \rightarrow \ldots (\sigma_n \rightarrow \tau) \ldots))$ $(n \geq 0)$.

The language of the typed λ-calculus has denumerably many <u>variables</u> α_i^{τ} $(i \geq 0)$ of each type τ. We will use α and β, with or without various decorations as metavariables over variables. The language has a set of <u>terms</u> which is given by:

1. α_i^{τ} is a term of type τ, $(i \geq 0)$,
2. if M and N are terms of type $(\sigma \rightarrow \tau)$ and σ respectively then (MN) is a term of type τ,
3. if M is a term of type τ then $(\lambda \alpha_i^{\sigma} . M)$ is a term of type $(\sigma \rightarrow \tau)$, $(i \geq 0)$:

M and N, possibly with suffices, will be used as metavariables over terms. The reader is assumed to know what a β, η-normal form of a term is and the elementary properties of normal forms; $M \approx N$ means that M and N have identical β-η-normal forms. By the Church-Rosser theorem this is an equivalence relation. Suppose that $K_{\sigma\tau}, = (\lambda \alpha_0^{\sigma}(\lambda \alpha_1^{\tau} \alpha_0^{\sigma}))$ and $S_{\sigma_1, \sigma_2, \sigma_3} = (\lambda \alpha_0^{(\sigma_1, \sigma_2, \sigma_3)}(\lambda \alpha_0^{(\sigma_1, \sigma_2)}(\lambda \alpha_0^{\sigma_1}((\alpha_0^{(\sigma_1, \sigma_2, \sigma_3)} \alpha_0^{\sigma_1})(\alpha_0^{(\sigma_1, \sigma_2)} \alpha_0^{\sigma_1})))))$. Then, as is well-known, the K's and S's generate all closed terms under application, to within $\approx$. The type subscripts in K and S will often be omitted, as will be as many other type symbols as is convenient; the resulting propositions are to be understood as being asserted for every consistent way of putting the symbols back in.

Our language also has a semantics based on the full type hierarchy $\{D_{\sigma}\}$ defined by:

D/

$D_{(\sigma\to\tau)} = (D_\sigma \to D_\tau)$ (the set of all functions from D_σ to D_τ), where D_ι is some given set.

The semantics is a function $[\![\ ]\!]$: Terms$\to$(Env$\to\bigcup_\sigma D_\sigma$) where Env, the set of environments, is the set of type respecting functions from the set of variables to $\bigcup D_\tau$, and is ranged over by ρ. Then, $[\![\ ]\!]$ is the unique function of that type such that: $\bigcup_\sigma D_\sigma$

1. $[\![\alpha^\tau_i]\!](\rho) = \rho(\alpha^\tau_i)$ $(i \geq 0)$
2. $[\![(MN)]\!](\rho) = [\![M]\!](\rho)([\![N]\!](\rho))$
3. $[\![(\lambda\alpha^\tau_i.M)]\!](\rho)(x) = [\![M]\!](\rho[x/\alpha^\tau_i])$ $(i \geq 0, x \in D_\tau)$

where $\rho[x/\alpha^\tau_i]$ is the environment ρ' such that

$$\rho'(\alpha^{\tau'}_{i'}) = \begin{cases} x & (\alpha^{\tau'}_{i'} = \alpha^\tau_i) \\ \rho(\alpha^{\tau'}_{i'}) & \text{(otherwise).} \end{cases}$$

Note that if M has type σ, $[\![M]\!](\rho) \in D_\sigma$. If M is closed then $[\![M]\!](\rho) = [\![M]\!](\rho')$ for any ρ and ρ' – so we often drop the reference to ρ for closed M. If $M \approx M'$ then $[\![M]\!](\rho) = [\![M']\!](\rho)$ for any ρ; we will give a converse later.

Suppose $\pi_\iota \in D_{(\iota\to\iota)}$ is a permutation. Permutations π_σ in any $D_{(\sigma\to\sigma)}$ can be defined by:

$$\pi_{(\sigma\to\tau)}(f) = \pi_\tau \circ f \circ \pi_\sigma^{-1} \quad (f \in D_{(\sigma\to\tau)}).$$

If M is closed term then $\pi([\![M]\!]) = [\![M]\!]$ (see [2]). However this does not characterise λ-definability. ??

{préservé par application et K et S vérifiant

For example ground equality, $=_\iota$, is permutation-invariant, but is certainly not λ-definable. Explicitly let O abbreviate (ι, ι, ι) and let tt and ff be $\lambda\alpha^\iota_0\lambda\alpha^\iota_1.\alpha^\iota_0$ and $\lambda\alpha^\iota_0\lambda\alpha^\iota_1.\alpha^\iota_1$ respectively. Then $=_\iota$ is defined by:

$$=_\iota xy \begin{cases} \text{tt (if } x = y) \\ \text{ff (if } x \neq y) \end{cases} \quad (x, y \in D_\iota)$$

But/

But the only λ-definable functionals of type $(\iota,\iota,0)$ are $\lambda\alpha_0^{\iota}\lambda\alpha_1^{\iota}\lambda\alpha_2^{\iota}\lambda\alpha_3^{\iota}\ \alpha_j^{\iota}$ for $0\leq j\leq 3$ none of which are $=_{\iota}$ if $|D_{\iota}|>1$.

M. Gordon proposed, as a possible remedy, that relations $R_{\iota}\subseteq D_{\iota}^2$ should be extended - not just permutations. Starting with such an R_{ι}, the R_{σ}'s are defined by:

$$R_{(\sigma\to\tau)}(f,g)\equiv\forall x,y\in D_{\sigma}.(R_{\sigma}(x,y)\to R_{\tau}(fx,gy)).$$

When R_{ι} is a permutation π_{ι}, $R_{\sigma}=\pi_{\sigma}$ for all σ. The definition generalises, in the obvious way, if one starts with $R_{\iota}\subseteq D^{\kappa}$, for any ordinal κ. If $R_{\sigma}\subseteq D_{\sigma}^{\kappa}$ is obtained from an R_{ι} in that way it is called κ-logical; $f\in D_{\sigma}$ satisfies it iff $R_{\sigma}(\hat{f})$ holds. With the obvious definitions of λ-definability and λ-definability from a set $X\subseteq\bigcup D_{\sigma}$, one shows that any λ-definable functional satisfies any κ-logical relation, of the right type and that if $\{R_{\sigma}\}$ is the system of relations obtained from some R_{ι}, and each member of X satisfies the appropriate R_{σ} and f is λ-definable from X, then x satisfies the appropriate R_{σ}. The proof is like that of theorem 1.1.

One can now see why $=_{\iota}$ is not λ-definable if $|D_{\iota}|>1$. Let 0,1 be distinct elements of D_{ι}. Let $R_{\iota}=\{\langle 0,0\rangle,\langle 0,1\rangle,\langle 1,0\rangle\}$. Then R(tt,ff) is false for R(1,0) and R(0,1) but not R(tt10,ff01). Therefore $R(=_{\iota},=_{\iota})$ is false for R(0,0) and R(0,1) but not $R(=_{\iota}(0)(0),=_{\iota}(0)(1))$.

As an example of non-relative definability, consider the universal quantifier $\forall_{\iota}$ of type $((\iota\to 0)\to 0)$ defined by:

$$\forall_{\iota}(f)=\begin{cases}\text{tt (if } fx=\text{tt for all } x \text{ in } D_{\iota})\\ \\ \text{ff (otherwise).}\end{cases}$$

Now $\forall_{\iota}$ is permutation-invariant; however if $|D_{\iota}|\geq 3$ it is not λ-definable from $=_{\iota}$. To see this let $R_{\iota}=\{\langle 0,0\rangle,\langle 1,1\rangle\}$ where 0,1 are distinct elements of D_{ι}. $R(=_{\iota},=_{\iota})$ is true, but if $f,g\in D_{(\iota\to 0)}$ are such that f(x) is always tt but g(x) is tt iff x is 0 or 1 then R/

$R(f,g)$ but not $R(\forall_\iota(f),\forall_\iota(g))$. Incidentally, if $|D|<3$, $\forall_\iota$ <u>is</u> λ-definable from $=_\iota$.

We can only characterise definability using logical relations, for types of level ≤ 2.

<u>Theorem 1</u> Suppose τ has the form $(\tau_1,\ldots,\tau_n,\iota)$ where each τ_i has the form $(\iota,\ldots,\iota,\iota)$. Then if $|D_\iota|\geq \aleph_0$ and $f \in D_\tau$ satisfies every 2-logical relation, it is λ-definable.

<u>Proof</u> We will just give two cases since this should give the idea without overmuch detail.

Suppose $\tau=(\iota,\iota,\iota)$. Let $x,y,0,1$ be elements of D, with 0 and 1 distinct and take $R_\iota=\{\langle x,1\rangle,\langle y,0\rangle\}$. Then $R(fxy,f10)$. So for every $x,y \in D_\iota$ either $fxy=x$ and $f10=1$ or else $fxy=y$ and $f10=0$. Therefore either $f10=1$ or $f10=0$ and so, since $1\neq 0$ either $f=[\![\lambda\alpha^\iota_0\,\lambda\alpha^\iota_1\,\alpha^\iota_0]\!]$ or $f=[\![\lambda\alpha^\iota_0\,\lambda\alpha^\iota_1\,\alpha^\iota_1]\!]$.

The other case we consider is $\tau=((\iota,\iota),\iota,\iota)$. Identify the integers, with a subset of D_ι and let the restriction of s to the integers be the successor function. Given g in $D_{(\iota\to\iota)}$ and $x \in D_\iota$ let $R_\iota=\{\langle g^n(x),s^n(0)\rangle \mid n\geq 0\}$. Clearly $R(x,0)$ and $R(g,s)$. Therefore $R(f(g)(x),f(s)(0))$ and so for every $g \in D_{(\iota\to\iota)}$ and $x \in D_\iota$ there is an n such that $f(g)(x)=g^n(x)$ and $f(s)(0)=s^n(0)$. Since $s^n(0)=s^{n'}(0)$ iff $n=n'$, n must be independent of g and x and so for some n,

$$f=[\![\lambda\alpha_0^{(\iota\to\iota)}\lambda\alpha_0^\iota\,\underbrace{\alpha_0^{(\iota\to\iota)}(\ldots\ldots(\alpha_0^{(\iota\to\iota)}}_{n\text{ times}}(\alpha_0^\iota))\ldots\ldots)]\!].$$

We believe the theorem holds without the restriction on D_ι. The simplest type which has us baffled when $|D_\iota|\geq 2$ is $(((\iota\to\iota)\to\iota)\to\iota)$.

Some characterisation of definability can be obtained by strengthening the implication in the definition of R_σ to an intuitionistic one, à la Kripke [1].

To this end, suppose we have a set W (of worlds) a reflexive, transitive binary relation $\leq$ which is a subset of W^2 (alternativeness) and a relation $R_\iota \subseteq D^3\times W$ such that:

$\forall$/