# Distribution of the Roots of Random Real Polynomials. Application to Spectral Analysis.

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# Introduction

$$P_n(z) \equiv \sum_{k=0}^{n} a_k z^k$$

real random variables

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*PLAN :*

*1- Distribution of Real Roots*

*2- Complex roots of Homogeneous and Monic Polynomials*

*3- Szegö Polynomials and Spectral Analysis*

# real roots -1 : average density

number of distinct real roots of $P_n$ : $N_n \equiv \int_{\mathbb{R}} dt\, \sigma_n(t)$

counting measure $\sigma_n(t) \equiv \sum_{k=0}^{N_n} \delta(t - t_k^{(n)}) = |P_n'(t)|\, \delta(P_n(t))$

real zeros of $P_n$

change of variable in the Dirac distribution

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mathematical expectation of the density of roots
$\varrho_n(t) \equiv \langle \sigma_n(t) \rangle = \iint_{\mathbb{R}} dP dP' \;\mathscr{P}(P, P')\; |P'|\; \delta(P)$

$\forall t$ fixed, $P_n(t) = \sum_{k=0}^{n} a_k\, t^k$ and $P_n'(t) = \sum_{k=0}^{n} k\, a_k\, t^{k-1}$ can be considered as 2 coupled random variables.

# real roots -2 : gaussian case

Simple case : $a_k$ iid $\mathscr{N}(0,1)$

$P_n(t)$ and $P_n'(t)$ are gaussian variables with zero mean and joint pdf

$$\mathscr{P}(P,P') \equiv \frac{1}{2\pi\sqrt{\Delta}} \exp\Big\{ -\frac{1}{2}(P,P')\mathbf{C}^{-1}\begin{pmatrix} P \\ P' \end{pmatrix}\Big\}$$

$\Delta = \det(\mathbf{C})$

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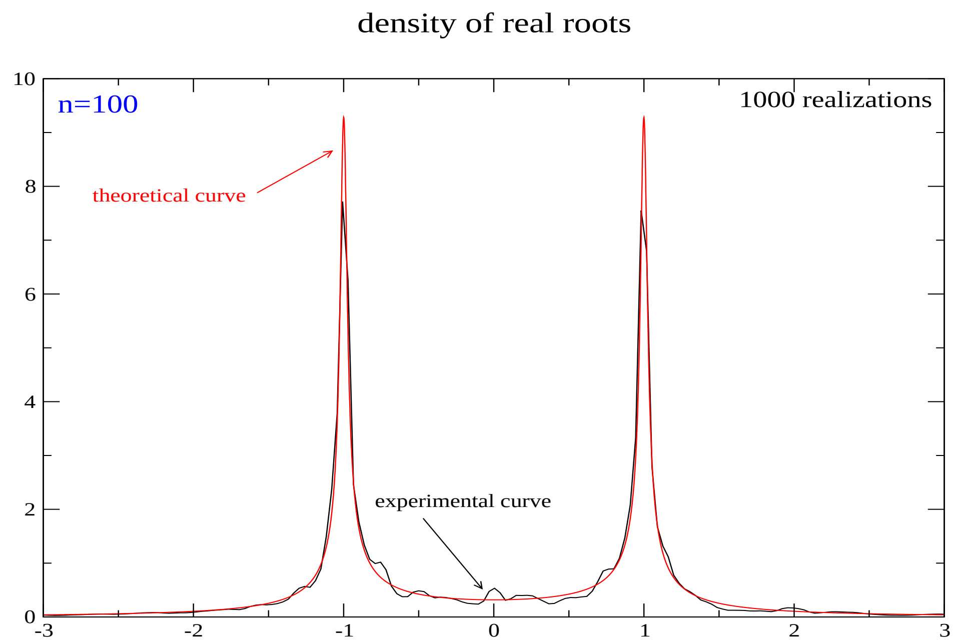
gaussian integral

$$\Rightarrow \varrho_n(t) = \frac{1}{2\pi\sqrt{\Delta}} \int_{\mathbb{R}} dP' \; |P'| \; \exp\{-\frac{1}{2\Delta}\langle P^2\rangle P'^2\} = \frac{\sqrt{\Delta}}{\pi\langle P^2\rangle}$$

# real roots -3 : Kac formula (1943)

$$\varrho_n(t) = \frac{1}{\pi}\Big\{\frac{1}{(1-t^2)^2} - (n+1)^2\frac{t^{2n}}{(1-t^{2n+2})^2}\Big\}^{1/2}$$



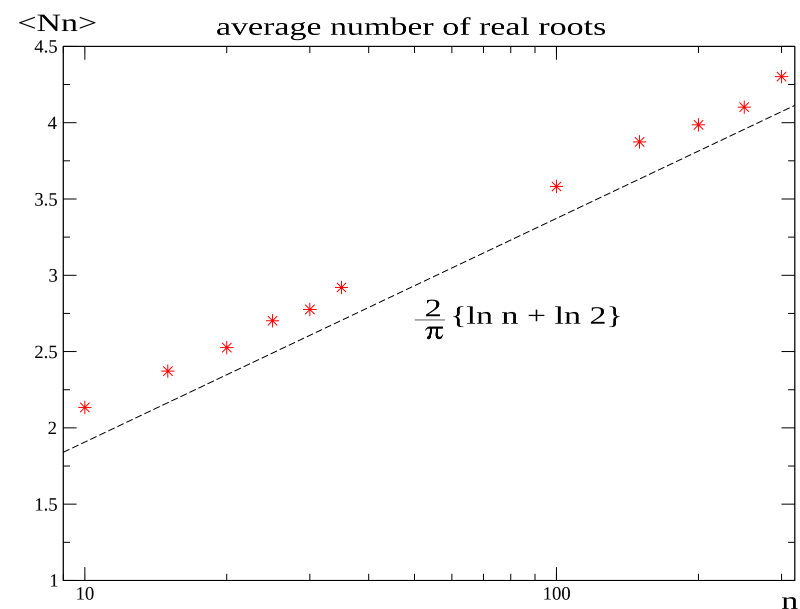
# real roots -4 : asymptotics

average number of real roots $\langle N_n \rangle = \int_{\mathbb{R}} dt\ \varrho_n(t)$

$$\frac{2}{\pi}\{\ln n + \ln(2 - \frac{1}{n})\} \ \leq \ \langle N_n \rangle \ \leq \ \frac{2}{\pi}\{\ln n + \ln 2 + 4\sqrt{3}\}$$

$\Rightarrow \langle N_n \rangle \simeq \frac{2}{\pi} \ln n,$

$\ln n \gg 1$



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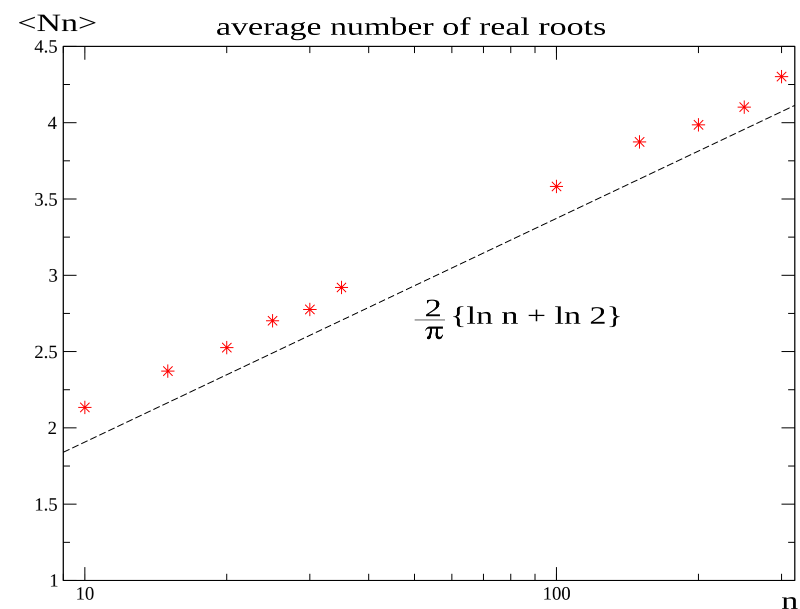



\+ *Universality of this behaviour*

$\rightarrow$ *Singularity of the real axis.*

# complex roots -1

$z \in \mathbb{C}$ fixed $\Rightarrow$ 4 r.v. : real & imaginary parts of $P$ and $P'$.
2-dimension Dirac distrib. + Cauchy-Riemann equations

$$\Rightarrow \sigma_n(z) = \sum_k \delta^{(2)}(z - z_k^{(n)}) = |P_n'(z)|^2 \, \delta^{(2)}(P_n(z))$$

average density $\varrho_n(z) = \int d^2P' \; |P'|^2 \; \underbrace{\mathscr{P}}_{\text{4-dimensional gaussian}}(0, P')$

$$\langle N_n(\Omega)\rangle = \int_{\Omega\subset\mathbb{C}} d^2z \, \varrho_n(z)$$

# generalized monic polynomials

$$P_n(z) \equiv \Phi(z) + \sum_{k=0}^{n-1} a_k f_k(z)$$

holomorphic functions

Cases of particular interest :

$\Phi = 0$, $\quad f_k = z^k \rightarrow$ Homogeneous random polynomial.

$\Phi =$ polynomial, $f_k = z^k \rightarrow$ Monic-type random polynomial.

# complex roots -2 : gaussian case

$\{a_k\}$ iid $\mathscr{N}(0,1) \Rightarrow$ closed formula (Mezincescu & al., 1997)

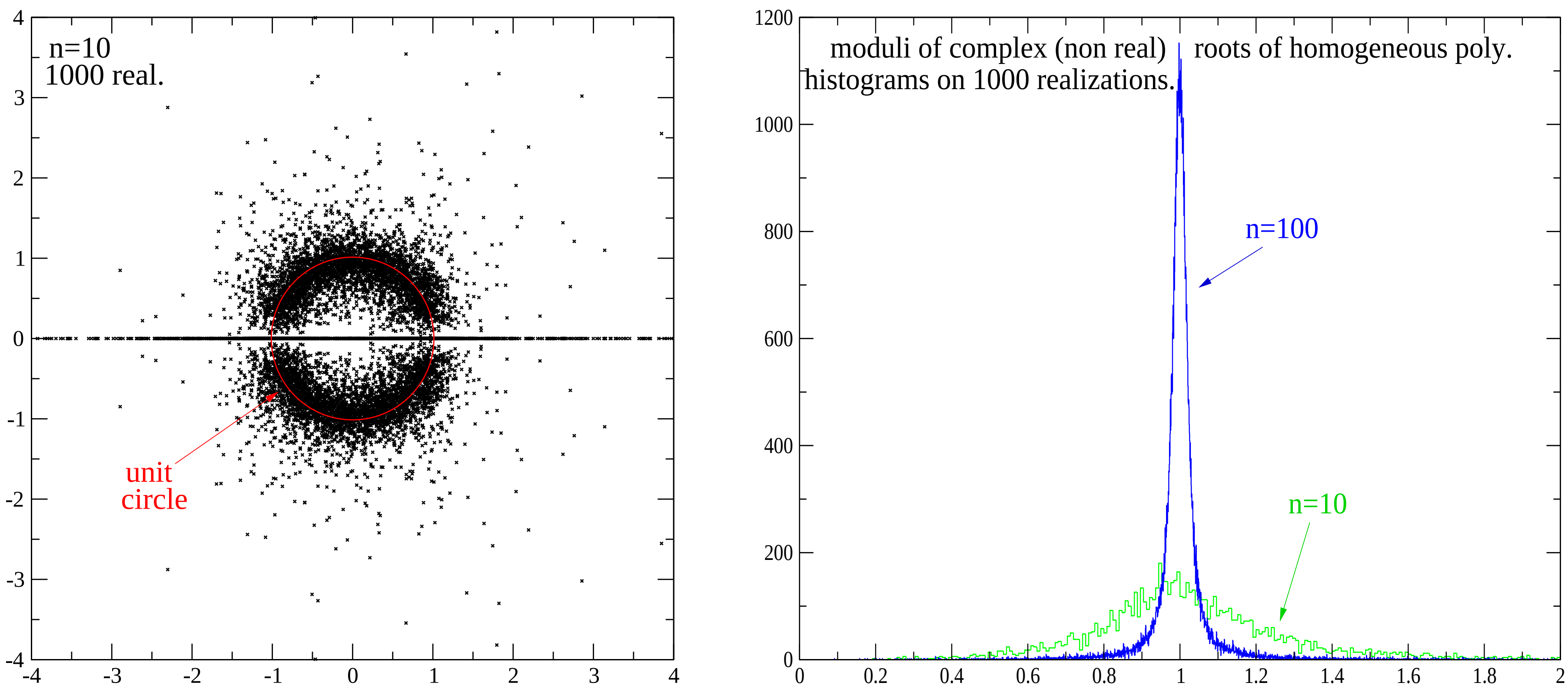
$$\varrho_n(z) = \frac{1}{2\pi}\frac{1}{\sqrt{\det(f,f)}}\exp\{-\frac{1}{2}\Phi.(f,f)^{-1}\Phi\} \times$$

$$\times\left\{Tr[(f',f') - (f',f)(f,f)^{-1}(f,f')] - \|\Phi' - (f',f)(f,f)^{-1}\Phi\|^2\right\}$$

for $\det(f,f) \neq 0$.

# complex roots -3 : homogeneous case

$$\left|\ln r\,\frac{1+r^{2n}}{1-r^{2n}}\right| \ll |\sin\theta| \Rightarrow \varrho_n(re^{i\theta}) \simeq \frac{1}{\pi}\left\{\frac{1}{(\ln r^2)^2} - \frac{n^2 r^{2n}}{(1-r^{2n})^2}\right\}$$



# complex roots -4 : monic polynomials

*Positions of the maxima of the density of complex roots ?*

$\Lambda =$ order of magnitude of $\Phi$ parameters

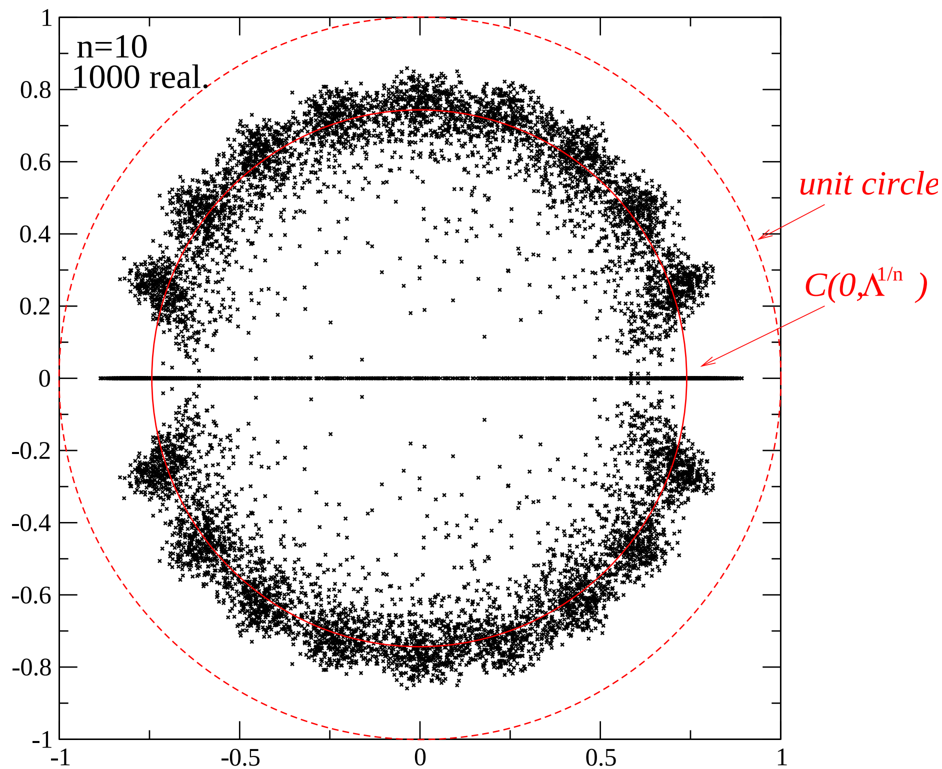
**High disorder regime :** $\Lambda = \mathcal{O}(1) \rightarrow$ cf. homogeneous case.

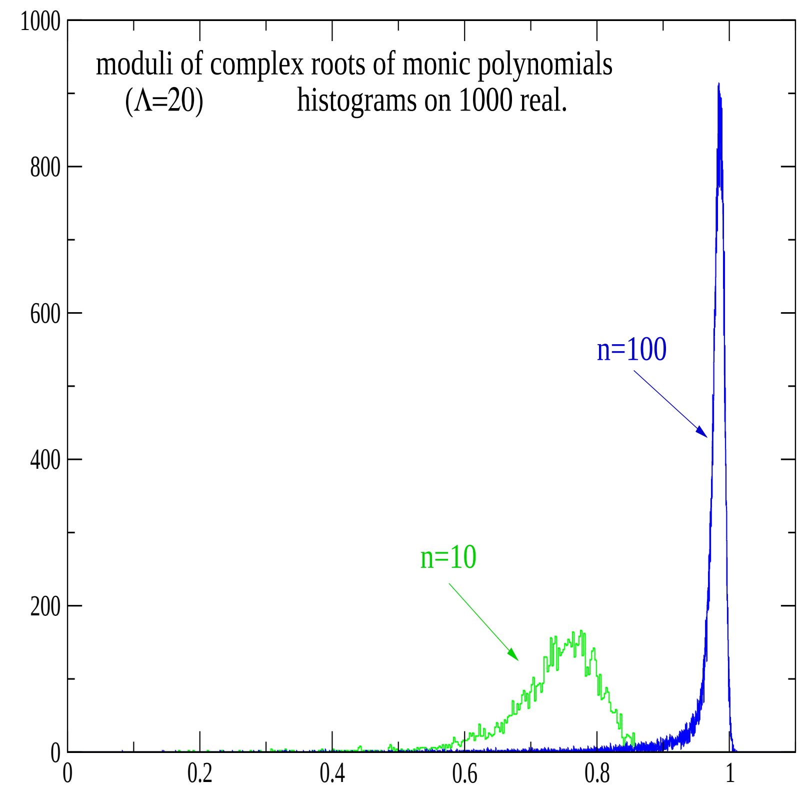
**Weak disorder regime :** $\Lambda \gg 1$.

* Near the non-zero roots of $\Phi$ :
$\varrho_n(z) \propto \exp\{-\Lambda^2 C^{ste}(z - z_0)^2\} \Rightarrow$ peaks of width $\mathcal{O}(\Lambda^{-1})$.

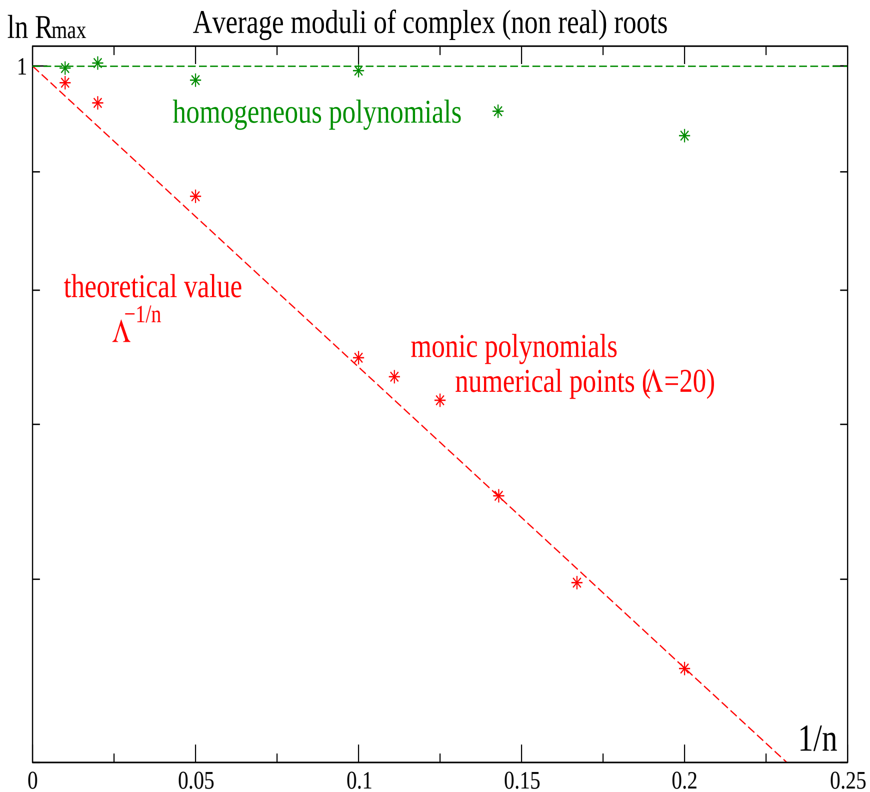
* root of order $n_0$ at the origin : peaks near the points situated on a circle of radius $\propto \Lambda^{-1/n_0}$, at the angles $\frac{2k(+1)\pi}{n_0}$, of width $\mathcal{O}(\Lambda^{-1})$.

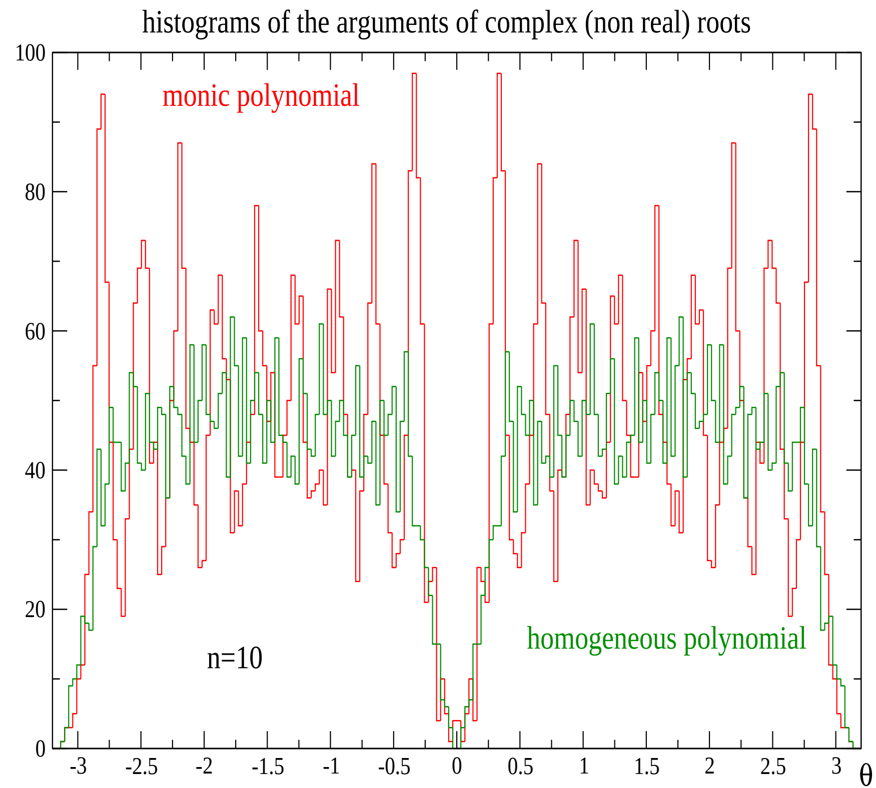
# complex roots -5: monic, low disorder



# complex roots -6



# Szegö polynomials-1 : definition

Orthogonal polynomials ; scalar product on the unit circle

$$\langle f | g \rangle \equiv \int_{|z|=1} f(z)\overline{g(z)}\, \mathscr{E}(z) \frac{dz}{iz} \qquad \langle S_m | S_n \rangle = \delta_m^n$$

Power spectrum of the signal $\{X_m\}_{0 \le m \le N-1}$

$$\mathscr{E}(z) \equiv \sum_{k=-(N-1)}^{N-1} C_k z^{-k} \;=\; \widehat{X}(z)\widehat{X}(1/z)$$

autocorrelation $C_k \equiv \sum_{m=0}^{N-1-k} X(m)X(m+k)$

$Z$-transform $\widehat{X}(z) \equiv \sum_{m=0}^{N-1} X(m)z^{-m}$

# Szegö-2 : AR system generated signal

$\{X\}$ generated by an AR system of order $A$

$$X(m) = a_1 X(m-1) + \cdots + a_A X(m-A) + \phi(m)$$

gaussian white noise

$Z$-transform

$$\widehat{L_A^\dagger}(\frac{1}{z})\ \widehat{X}(z) = \widehat{\phi}(z)$$

Reciprocal characteristic polynomial

$$L_A^\dagger(z) \equiv z^A - a_1 z^{A-1} \cdots - a_A$$

# Szegö-3 : AR(A)

Power spectrum

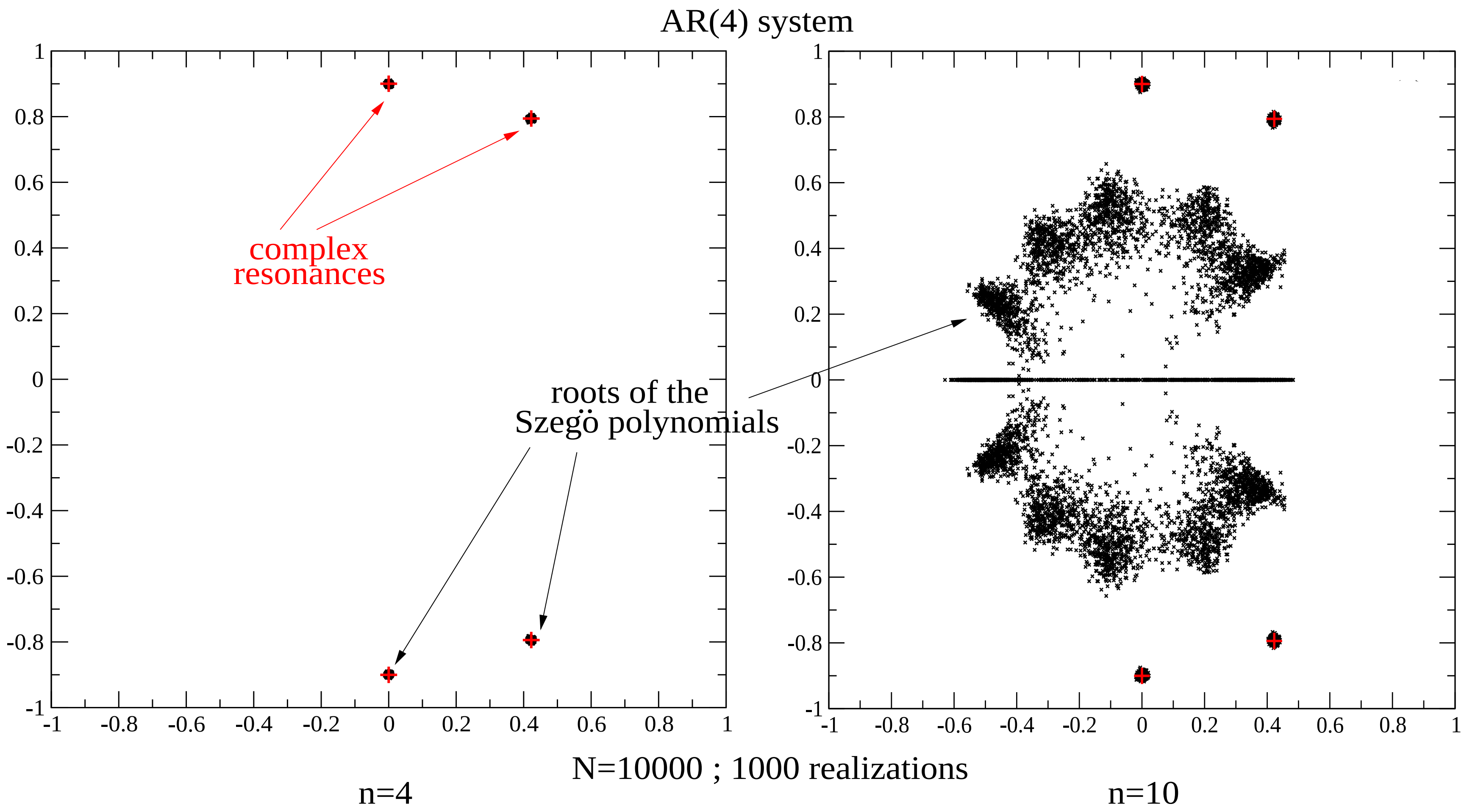
$$\mathscr{E}(z) \; = \; \frac{1}{L^{\dagger}_{A}(z) L^{\dagger}_{A}(1/z)} + \mathcal{O}^{f}(\frac{1}{\sqrt{N}}), \qquad N \gg 1$$

Associated Szegö polynomial

$$\Rightarrow S_n(z) = z^{n-A} L^{\dagger}_{A}(z) + \frac{1}{\sqrt{N}} S^{f}_{n-1}(z) \quad \forall n \geq A$$

zero-mean fluctuating part

# Szegö-4 : AR(4)


AR(4) system
complex resonances
roots of the Szegö polynomials
N=10000 ; 1000 realizations
n=4
n=10

# Szegö-5 : deterministic signal

trigonometric signal ;
no noise, no dissipation.

$$X(m) = \sin\frac{\pi}{6}m + \sin\frac{\pi}{3}m + \sin\frac{3\pi}{4}m$$

(Jones, Saff & al., 1990)