Stationary Processes and Linear Systems

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PART I: The general framework

1. Introduction

Time Series Analysis: Systematic approaches to extract information from time series, i.e., from observations ordered *in time* (no permutation invariance).

time series y_t, $t = 1, \ldots, T$; $y_t \in \mathbb{R}^n$

discrete, equidistant observations

- Data driven modeling
- Signal and feature extraction

Observations may be "noisy".

Questions: Trends,

Hidden periodicities,

Dependence on time (dynamics)

Models: Stationary processes

Linear systems

2. The History of Time Series Analysis

2.1. The Early History (1772 - 1920)

- Late 18^{th} century astronomy:
  - More accurate data from observation of the orbits of the planets
  - Kepler's laws are based on the two body problem

$\longrightarrow$ Are there deviations from the elliptic shape of the orbits (beside measurement noise)?
hidden periodicities or trends

Question of secular changes (Laplace 1787; Jupiter, Saturn) Harmonic analysis:

- J. L. Lagrange (1736 - 1813), Oeuvres, Vol 6, 1772
- L. Euler (1707 - 1783)
- J. B. J. Fourier (1768 - 1831), Théorie analytique de la chaleur

- Method of least squares for fitting a line into a scatter plot: A.M. Legendre and C.F. Gauss: Early 19^{th} century.
- Periodogramm: G.C. Stokes (1879), A. Schuster (1894) Detection of hidden periodicities:

$$I_T(\lambda) = \frac{1}{T} \left| \sum_{t=1}^{T} x_t e^{-i\lambda t} \right|^2$$

  T . . . sample size

  Sunspot numbers, periodicity of earthquakes

- Empirical analysis of business cycles

  W.S. Jevons: Periodic fluctuations in economic time series ($\sim$ 1870, 1880)

  H. Moore "Economic Cycles: Their Law and Cause" 1914

  W. Beveridge 1922: Wheat price index

2.2. The formation of modern time series analysis (1920-1970)

- Business cycles: Not exactly periodic:
  Stochastic models: AR and MA process e.g.

  $$y_t = a y_{t-1} + \epsilon_t, \qquad y_t = \epsilon_t + b\epsilon_{t-1}$$

  (ϵ_t) white noise
  G.U. Yule (1921, 1927) E. Slutzky (1927)
  R. Frisch: Propagation and Impulse Problems in Dynamic Econometrics (1933)

- Theory of stationary processes:
  Concept: A. Ya Khinchin (1934)
  Spectral representation: A. N. Kolmogorov (1939, 1941)
  Wold representation: H. Wold (1938)
  Factorization of spectra; linear least squares forecasting and filtering A.N. Kolmogorov (1939, 1941)

Ergodic theory: G.D. Birkhoff (1931), A. Ya Khinchin (1932)

- Cowles Commission, Identifiability and ML estimation of (multivariate) ARX models.
  H.B. Mann and A. Wald (1943),
  T. Haavelmo (1944),
  T.C. Koopmann, H. Rubin and R.B. Leipnik (1950);
  Klein I model
- Spectral Estimation:
  Daniel (1946),
  R.B. Blackman and J. Tukey (1958),
  U. Grenander and M. Rosenblatt (1958),
  E. J. Hannan (1960)
- Asymptotic Theory for (mainly SISO) AR and ARMA estimation: Durbin, E.J. Hannan (1970), T.W. Anderson (1971)

2.3. The recent past (1970-1990)

- Box, G.E.P. and G.M. Jenkins (1970)
  Explicit instructions for SISO system identification: Differencing, Order determination, ML-estimation, validation
- Kalman: Structure theory for state space systems: Realization and parametrization. MIMO case
- Order estimation by information criteria such as AIC or BIC: Akakike, Hannan, Rissanen, Schwartz
- Asymptotic properties of ML-type estimation: E.J. Hannan (1973), W. Dunsmuir and E.J. Hannan (1976), P. Caines and L. Ljung (1979)
- Textbooks (late 80ies): Ljung, Caines, Hannan and Deistler, Söderström and Stoica

3. Areas of application

- Signal processing
- Control
- Econometrics: Macroeconometrics, finance, microeconometrics, marketing, logistics
- Medicine and biology

PART II: Stationary Processes

4. Stationary processes in time domain

For us a stochastic process is a model for random phenomena evolving in time

$(\Omega, \mathcal{A}, \mathbb{P})$ probability space

$y_t : \Omega \longrightarrow \mathbb{C}^n$ random variable

$(y_t | t \in T)$, random process, $T \subset \mathbb{R}$

in particular $T = \mathbb{Z}$

`Def.:` A stochastic process (y_t) is called (weakly) *stationary* if:

(i) $\mathbb{E} y_t^* y_t < \infty \qquad t \in \mathbb{Z}$

(ii) $\mathbb{E} y_t = m = const \qquad t \in \mathbb{Z}$

(iii) $\gamma(s) = \mathbb{E} y_{t+s} y_t^*$ does not depend on t

Covariance function

$\gamma : \mathbb{Z} \to \mathbb{C}^{n \times n} : \gamma(t) = \mathbb{E} y_t y_0^*$

describes all linear dependence relations between the one dimensional random variables $y_t^{(i)}$, $y_s^{(j)}$

γ is a covariance function if and only if γ is nonnegative definite

$y_t^{(i)} \in L_2$ (over $(\Omega, \mathcal{A}, \mathbb{P})$)

Note L_2 with inner product

$< x, y >= \mathbb{E} x \bar{y}$

is a Hilbert space

`Def.:` The *time domain* $H \subset L_2$ of a stationary process y_t is the Hilbert space spanned by $\{y_t^{(i)} | t \in Z,\, i = 1, \ldots, n\}$

5. Stationary processes in frequency domain

As a consequence of the “translation invariance” of the covariances we have:

`Theorem:` For every stationary process (y_t) there is a unique unitary operator $U : H \longrightarrow H$ such that $y_t^{(i)} = U^t y_0^{(i)}, i = 1, \ldots, n, t \in \mathbb{Z}$ holds.

From this wie obtain

`Theorem:` (Spectral representation of stationary processes). For every stationary process (y_t) there exists a process $(z(\lambda)|\lambda \in [-\pi, \pi])$ (called process with orthogonal increments) satisfying

(i) $z(-\pi) = 0$, $z(\pi) = x_0$

(ii) $\lim_{\epsilon \downarrow 0} z(\lambda + \epsilon) = z(\lambda)$

(iii) $\mathbb{E} z^*(\lambda) z(\lambda) < \infty$

(iv) $\mathbb{E}\{(z(\lambda_4) - z(\lambda_3))(z(\lambda_2) - z(\lambda_1))^*\} = 0$
for $\lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4$

such that

$$y_t = \int e^{i\lambda t} dz(\lambda)$$

holds.

Thus, every stationary process is obtained as a limit of harmonic processes

$$y_t = \sum_{j=1}^{h} e^{i\lambda_j t} z(\lambda_j)$$

Second moments in frequency domain:

Spectral distribution function

$F : [-\pi, \pi] \rightarrow \mathbb{C}^{n\times n} : \; F(\lambda) = \mathbb{E}z(\lambda)z(\lambda)^*$

Spectral representation of covariance function

$$\gamma(t) = \int e^{i\lambda t} dF(\lambda)$$

$$\gamma \longleftrightarrow F$$

Spectral density (w. r. t. L-measure)

$$F(\lambda) = \int_{-\pi}^{\lambda} f(\omega) d\omega$$

exists e.g. if $\sum \|\gamma(t)\|^2 < \infty$

$$\begin{aligned} \gamma(t) &= \int e^{i\lambda t} f(\lambda) d\lambda \\ f(\lambda) &= (2\pi)^{-1} \sum \gamma(t) e^{-i\lambda t} \end{aligned}$$

f (if it exists) contains the same information as γ but, often displayed in a more convenient form.

Peaks of f indicate dominating frequency bands.

6. Linear transformations of stationary processes

Let (x_t) be stationary; a linear transformation of (x_t) is given by $y_t = \sum_{j=-\infty}^{\infty} k_j x_{t-j}$; $k_j \in \mathbb{R}^{n \times m}$;
$\sum \|k_j\| < \infty$
then $(x_t', y_t')'$ is jointly stationary.
$(k_j | t \in \mathbb{Z})$ weighting function

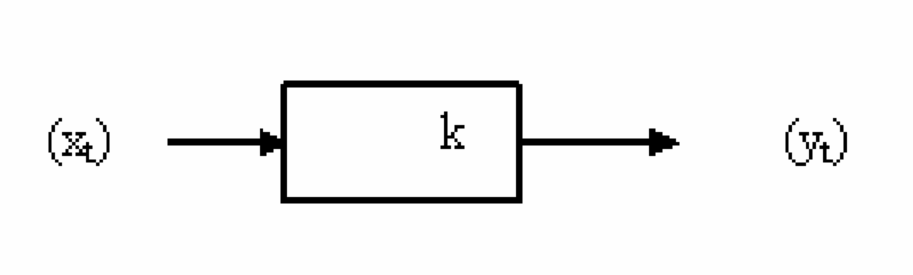
$$y_t = \int e^{i\lambda t} dz_y(\lambda) = \sum k_j \int e^{i\lambda(t-j)} dz_x(\lambda) =$$

$$\int e^{i\lambda t} \underbrace{\Big(\sum_{j=-\infty}^{\infty} k_j e^{-i\lambda j} \Big)}_{\text{transfer function } k(e^{-i\lambda})} dz_x(\lambda)$$

$$k \longleftrightarrow (k_j)$$

The transfer function describes the linear transformation in frequency domain.

Linear system

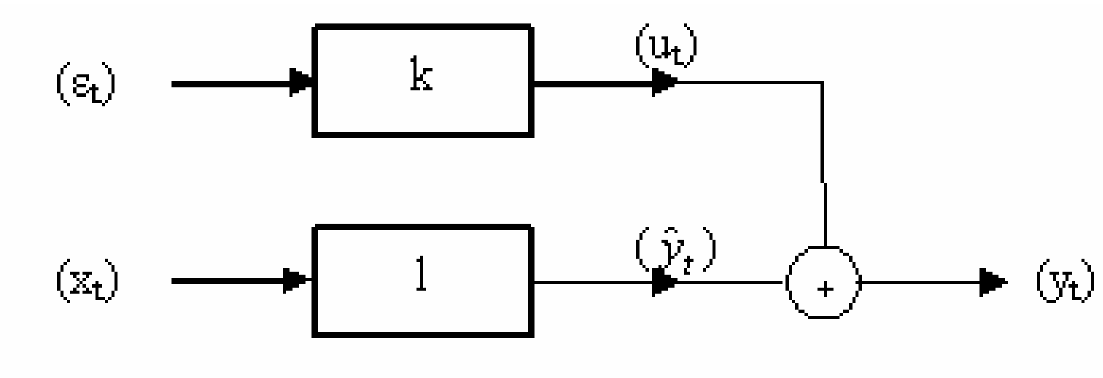


stable, time invariant

(x_t) input

(y_t) output

Linear system with noise



(ϵ_t) white noise, i.e., $\mathbb{E}\epsilon_t = 0,\ \mathbb{E}\epsilon_s\epsilon_t' = \delta_{st} \cdot \Sigma$

$f_\epsilon = (2\pi)^{-1} \cdot \Sigma$

"$dz_y(\lambda) = k(e^{-i\lambda})dz_x(\lambda)$"

Transformation of second moments in frequency domain

$$\begin{aligned} f_y &= k f_x k^* \\ f_{yx} &= k f_x \end{aligned}$$

7. The Wold decomposition

Let (x_t) be stationary
$H_x(t) = sp\{x_s^{(i)} | i = 1, \ldots, n, s \leq t\} \subset H_x$ is called the past of (x_t)

`Def.:` A stochastic process (x_t) is called (linear) *regular* if $\bigcap_t H_x(t) = \{0\}$
and (linear) *singular* if $\bigcap_t H_x(t) = H_x$

`Theorem:` (Wold decomposition)

(i) Every stationary process (x_t) can be uniquely decomposed as

$$x_t = y_t + z_t$$

where (y_t) is regular, (z_t) is singular, $\mathbb{E} y_t z_s^* = 0$, $y_t^{(i)} \in H_x(t)$, $z_t^{(i)} \in H_x(t)$

(ii) Every regular process (y_t) can be represented as

$$y_t = \sum_{j=0}^{\infty} k_j \epsilon_{t-j}, \qquad \sum \|k_j\|^2 < \infty \quad (1)$$

where (ϵ_t) is white noise satisfying $\epsilon_t^{(i)} \in H_y(t)$, $i = 1, \ldots, n$

Thus, “practically every” stationary process is obtained as an output of a linear system whose input is the “simplest” random process, namely white noise.

$(\epsilon_s,\ s \leq t)$ constitutes an "orthonormal" basis for $H_y(t)$, (1) is an abstract Fourier series.

Linear least squares forecasting:

of $y_{t+\tau}$ based on y_s, $s \leq t$

project $y^{(i)}_{t+\tau}$ on $H_y(t)$:

$$y_{t+\tau} = \underbrace{\sum_{j=h}^{\infty} k_j \epsilon_{t+\tau-j}}_{\text{forecast } \hat{y}_{t,\tau}} + \underbrace{\sum_{j=0}^{h-1} k_j \epsilon_{t+\tau-j}}_{\text{forecasting error}}$$

Spectral factorization:

The spectral density of (y_t) is given by

$$f_y = (2\pi)^{-1} \cdot \underbrace{(\sum_j k_j e^{-i\lambda j})}_{k(e^{-i\lambda})} \cdot \Sigma \cdot k(e^{-i\lambda})^* \qquad (2)$$

`Question:` Obtain k (and Σ) from f_y (in 2).

8. Estimation I

Estimation of mean

$\bar{y}_T = \frac{1}{T} \sum_{t=1}^{T} y_t$

Estimation of covariances (ass. $\mathbb{E} y_t = 0$)

$\hat{\gamma}(t) = \frac{1}{T} \sum_s y_{t+s} y_s^*$

Estimation of spectra

Periodogram

$I_T(\lambda) = \sum \hat{\gamma}(t) e^{-i\lambda t}$

is not consistent; smoothed periodogram

9. Rational spectra, ARMA and state-space systems

ARMA system

$$\underbrace{\sum_{j=0}^{p} a_j y_{t-j}}_{a(z)y_t} = \sum_{j=0}^{p} b_j \epsilon_{t-j}$$

$$a(z)y_t = b(z)\epsilon_t$$

z: backward shift as well as complex variable

Stability condition

$\det a(z) \neq 0 \qquad |z| \leq 1$

Miniphase condition

$\det b(z) \neq 0 \qquad |z| < 1$

Normalization

$a_0 = b_0$

Steady state solution

$$y_t = \underbrace{a^{-1}(z)b(z)}_{\text{Transfer function } k(z)} \epsilon_t$$

This transfer function corresponds to the Wold decomposition.

State space system

$$
\begin{aligned}
x_{t+1} &= Ax_t + B\epsilon_t \\
y_t &= Cx_t + \epsilon_t
\end{aligned}
$$

x_t: state (n-dimensional)

Stability condition
$|\lambda_{max}(A)| < 1$

Miniphase condition
$|\lambda_{max}(A - BC)| \leq 1$

Steady state solution
$y_t = (C(Iz^{-1} - A)^{-1}B + I)\epsilon_t$

again corresponds to the Wold decomposition

Theorem:

(i) Every rational and a.e. nonsingular spectral density matrix may be uniquely factorized (as in (2)), where $k(z)$ is rational (in $z \in \mathbb{C}$), analytic within a circle containing the closed unit disk, $\det k(z) \neq 0$, $|z| < 1$ and $\Sigma > 0$.

(ii) For every rational transfer function k satisfying the above mentioned properties there is a stable and miniphase ARMA system with $a_0 = b_0$ and conversely every such ARMA system has a transfer function with the properties mentioned in (i).

(iii) A completely analogous statement holds for state space systems

PART III: Identification of linear systems

10. Problem statement

Data driven modeling: Find a good model from (noisy) data

One has to specify:

- The model class, i.e. the class of all a priori feasible candidate systems to be fitted to the data. Here the model class is the set of all stable and miniphase ARMA or state space systems (for given s)
- The class of feasible data
- An identification procedure, which is a set of rules - in the fully automatized case a function - attaching to every feasible data string $y_t,\ t = 1, \ldots, T$ a system from the model class.

The theory of identification is mainly concerned with the development and evaluation of identification algorithms.

Steps in actual identification

- Data generation and preprocessing of data (e.g. removing outliers)
- Description of the model class using the prior knowledge available
- Identifying the model
- Model validation

In identification in general the following parameters have to be determined from data

- Integer-valued parameters such as the state dimension of a minimal state space system; this defines a subclass, namely the class of all systems of order n
- Real-valued parameters, such as the entries in (A,B,C)

Semi-nonparametric estimation problem

3 modules of the problem:

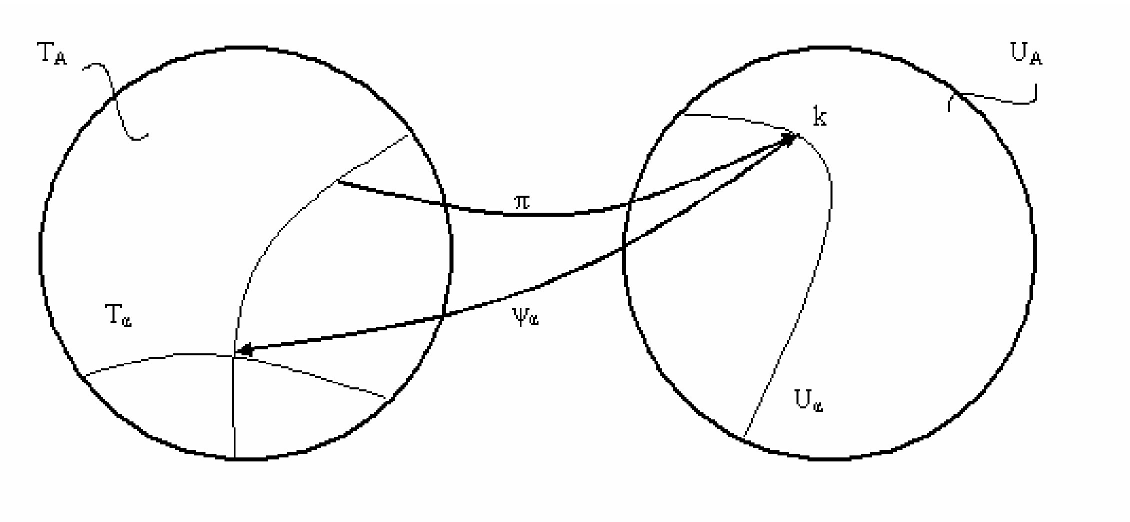
- Structure theory: Idealized identification, we commence from the population second moments of the observations or from the (“true”) transfer functions rather than from data
- Estimation of real-valued parameters for given integer-valued parameters
- Model selection: Estimation of integer-valued parameters

11. Structure Theory

Note: (2) defines a one-to-one relation between f and k, Σ under our assumptions.

We restrict ourselves to state space systems:
Let U_A denote the set of all rational $s \times s$ transfer functions $k(z)$ satisfying our assumptions and let T_A denote the set of all state space systems (A, B, C) (for fixed s but variable n) satisfying our assumptions; finally let the mapping $\pi : T_A \rightarrow U_A$ be defined by

$$\pi(A, B, C) = C(Iz^{-1} - A)^{-1}B + I$$



- π is not injective (no identifiability)
  $\pi^{-1}(k)$ the class of observationally equivalent systems
- There exists no continuous selection of representatives from the equivalence classes

"illposedness"

Identifiability and continuity of the parametrization are desirable: U_A and T_A are broken into bits U_α and T_α respectively s.t. π/T_α is injective and surjective and its inverse, the parametrization

$$\psi_\alpha : U_\alpha \to T_\alpha \subset \mathbb{R}^{d_\alpha}$$

is continuous

Free parameters $\mathbb{R}^{d_\alpha} \ni \tau_\alpha \leftrightarrow (A, B, C)$

for given T_α.

12. Estimation of real-valued parameters

(Gaussian) Likelihood function ($-2T^{-1} \times \log$)

$$L_T(\tau_\alpha, \Sigma) = T^{-1} \log \det \Gamma_T(\tau_\alpha, \Sigma) + T^{-1} y'(T) \Gamma_T^{-1}(\tau_\alpha, \Sigma) y(T)$$

where

$y(T) = (y_1', \ldots, y_T')'$ (stacked sample)

$$\underbrace{\Gamma_T(\tau_\alpha, \Sigma)}_{T \cdot s \times T \cdot s} = \left(\int e^{-i\lambda(r-t)} f_y(\lambda; \tau_\alpha, \Sigma) d\lambda\right)_{r,t=1,\ldots,T}$$

ML estimators:

$$(\hat{\tau}_{\alpha,T}, \hat{\Sigma}_T) = \arg\min_{\tau_\alpha \in T_\alpha, \Sigma \in \underline{\Sigma}} L_T(\tau_\alpha, \Sigma)$$

Coordinate free MLE: $\hat{k}_T$

Asymptotic properties:

- Consistency:

$$\begin{aligned} \hat{k}_T &\rightarrow k_0 \\ \hat{\Sigma}_T &\rightarrow \Sigma_0 \end{aligned}$$

- Asymptotic normality

$$\sqrt{T}(\hat{\tau}_{\alpha,T} - \tau_{\alpha,0}) \rightarrow N(0, V)$$

13. Model Selection

Example: Estimation of n, analogous for α

Information criteria: Tradeoff between fit and complexity

$$I_T(n) = \underbrace{\log\det\hat{\Sigma}_T(n)}_{\text{Measure of fit}} + \underbrace{(2ns)}_{\text{dimension, measures complexity}} \frac{c(T)}{T}$$

$\hat{n}_T = \arg\min I_n$

AIC criterion $c(T) = 2$

BIC criterion $c(T) = \log(T)$

BIC is consistent, AIC not

Post model selection properties