POLYNOMIAL AND RATIONAL APPROXIMATION

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Polynomial Approximation & Interpolation

f analytic at $z = 0$, $s_n(z) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k$

Properties of Taylor polynomials

Interp. Property: $s_n^{(k)}(0) = f^{(k)}(0)$, $k = \overline{0, n}$.

Least Squares Property:

$$\Gamma : |z| = 1 \, , \qquad (g, h) := \frac{1}{2\pi} \int_\Gamma g(z) \overline{h(z)} |dz|$$

$1, \ z, \ z^2, \ldots$ orthogonal

$$\begin{aligned} \left(f, z^k\right) &= \frac{1}{2\pi} \int_\Gamma f(z) \overline{z}^k |dz| = \frac{1}{2\pi} \int_\Gamma \frac{f(z)}{z^k} \frac{dz}{iz} \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z^{k+1}} \, dz = \frac{f^{(k)}(0)}{k!} \, . \end{aligned}$$

Best $L^2(\Gamma)$ approx to f from $\mathcal{P}_n$:

$$\sum_{k=0}^{n} \frac{(f, z^k)}{(z^k, z^k)} \, z^k = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \, z^k = s_n(z) \, .$$

Minimal L^∞-norm Projection Property:

$$\Delta : |z| \le 1\,, \qquad \|\cdot\|_\Delta = \text{sup norm on } \Delta$$

$$\mathcal{A}(\Delta) := \{f \in C(\Delta) : f \text{ analytic in } |z| < 1\}$$

$\mathbf{P} : \mathcal{A}(\Delta) \to \mathcal{P}_n$ Projection operator

$(\mathbf{S}_n f)(z) = s_n(z)$ Taylor projection operator

Claim: $\|\mathbf{S}_n\| \le \|\mathbf{P}\|$ for all $\mathbf{P}$.

Let $\mathcal{B}_t : f(z) \to f(tz)$, $|t| = 1$

$$\left(\mathcal{B}_{\bar{t}}\mathbf{P}\mathcal{B}_t\right)(z^k) = \begin{cases} z^k & 0 \le k \le n \\ t^{k-n}\mathbf{P}\left(z^k\right) & k > n \end{cases}$$

$$(\mathbf{S}_n f)(z) = \frac{1}{2\pi i}\int_{|t|=1} \left(\mathcal{B}_{\bar{t}}\mathbf{P}\mathcal{B}_t f\right)(z)\frac{dt}{t}$$

$$\Rightarrow \|\mathbf{S}_n f\| \le \frac{1}{2\pi}\int_{|t|=1} \left\|\mathcal{B}_{\bar{t}}\mathbf{P}\mathcal{B}_t\right\| \|f\||dt| \le \|\mathbf{P}\|\|f\|\,.$$

Maximal Convergence Property

f analytic on $\Delta : |z| \le 1$. Then

$$\limsup_{n\to\infty} \|f - s_n\|_\Delta^{1/n} = \frac{1}{\rho} < 1\,,$$

where ρ is the radius of the largest open disk about $z = 0$ in which f is analytic. Moreover, $s_n \longrightarrow f$ in $|z| < \rho$.

Proof. Let $1 < r < \rho$. Then by Hermite Interp. Formula:

$$f(z) - s_n(z) = \frac{1}{2\pi i} \int\limits_{|t|=r} \frac{z^{n+1} f(t)}{t^{n+1}(t-z)}\, dt\,, \qquad |z| < r$$

$$\|f - s_n\|_\Delta \le \frac{1}{2\pi} \frac{M_r 2\pi r}{r^{n+1}(r-1)}\,, \qquad M_r := \max_{|t|=r} |f(t)|$$

$$\Rightarrow \limsup_{n\to\infty} \|f - s_n\|_\Delta^{1/n} \le \frac{1}{r} \left(\to \frac{1}{\rho} \right).$$

Equality later. $\square$

Polynomial Approximation on Compact Sets

Given: $E \subset \mathbb{C}$ compact, $\overline{\mathbb{C}} \setminus E$ connected, f analytic on E.

Problem: Construct "good" poly approximations to f on E.

Runge: $\exists$ polys $\{p_n\}$ such that $p_n \to f$ uniformly on E.

Remark: Not true if E separates the plane.

Popular Methods: Faber polys, interpolating polys, CF (AAK) methods

Assume $\overline{\mathbb{C}} \setminus E$ is simply connected

$w = \varphi(z) : \overline{\mathbb{C}} \setminus E \to \{|w| > 1\}$,

$\varphi(\infty) = \infty, \quad \varphi'(\infty) > 0$

$\varphi(z) = \frac{z}{c} + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots, \qquad c = \mathrm{cap}(E) > 0$

$$\{w^n\}_0^\infty \leftrightarrow \{\varphi^n(z)\}_0^\infty$$

$$\begin{aligned} \varphi^n(z) &= \left(\frac{z}{c} + b_0 + \frac{b_1}{z} + \cdots\right)^n \\ &= \left(\frac{z^n}{c^n} + \cdots\right) + \frac{1}{z} M_n(z) \\ &= \underbrace{F_n(z)}_{\textbf{Faber polys}} + \frac{1}{z} M_n(z)\,. \end{aligned}$$

Goal: Expand $f(z)$ analytic on E

$$f(z) = a_0 F_0(z) + a_1 F_1(z) + a_2 F_2(z) + \cdots$$

$z = \psi(w)$ inverse of φ

$\Gamma_r : |\varphi(z)| = r \ (> 1)$ level curves, $C_r : |w| = r$.

NOTE:

$$F_n(z) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{\varphi^n(t)\, dt}{t - z} = \frac{1}{2\pi i} \int_{C_r} \frac{s^n \psi'(s)\, ds}{\psi(z) - z}$$

Write

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(t)\, dt}{t - z} = \frac{1}{2\pi i} \int_{C_r} \frac{f(\psi(s))\psi'(s)\, ds}{\psi(z) - z}$$

$$f(\psi(s)) = \sum_{-\infty}^{\infty} a_n s^n \qquad \text{for } 1 < |s| < R$$

$$f(z) = \sum_{-\infty}^{\infty} a_n \frac{1}{2\pi i} \int_{C_r} \frac{s^n \psi'(s)\, ds}{\psi(z) - z} = \sum_{0}^{\infty} a_n F_n(z) \, .$$

Maximal Convergence: f analytic on E.

$$\limsup_{n\to\infty} \|f - \sum_0^n a_k F_k\|_E^{1/n} = \frac{1}{\rho} < 1\,,$$

where ρ is the largest index such that f is analytic inside Γ_ρ. Moreover, Faber series converges to f inside Γ_ρ.

Ex: $E = [-1,1]$, $\varphi(z) = z + \sqrt{z^2-1}$,

Γ_r : Ellipse foci ± 1
semi – major axis length
$(r + r^{-1})/2$

For $n \geq 1$, $F_n(x) = \cos n\theta$, $x = \cos\theta$

Faber series $\Leftrightarrow$ Chebyshev expansion

INTERPOLATION

Determine points of E

$$
\begin{array}{ccccccc}
\beta_0^{(0)} & & & & & & \\
\beta_0^{(1)}, & \beta_1^{(1)} & & & & & \\
\cdot & \cdot & \cdot & & & & \\
\cdot & \cdot & \cdot & \cdot & & & \\
\beta_0^{(n)}, & \beta_1^{(n)}, & \cdot & \cdot & \cdot & , \beta_n^{(n)} & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

so that interpolating polys $p_0, p_1, \ldots, p_n, \ldots$ converge **maximally** for every f analytic on E.

Recall Hermite Formula

$$f(z) - p_n(z) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{w_n(z) f(t)}{w_n(t)(t-z)} \, dt$$

$$w_n(z) := \prod_{k=0}^{n} \left(z - \beta_k^{(n)} \right)$$

(Walsh) Get maximal convergence for every f analytic on E **iff** the w_n's have asymptotically minimal norm:

$$\lim_{n\to\infty} \|w_n\|_E^{1/n} = \text{cheb}(E) = \text{cap}(E). \tag{1}$$

EX: $E : |z| \leq 1$, $w_n(z) = z^{n+1}$, $w_n(z) = z^{n+1} - 1$

EX: E bounded by smooth Jordan arc or curve Γ. Take images of equally spaced points (roots of unity)

EX: $E = [-1, 1]$

zeros of Chebyshev, *not* equally spaced

EX: Fekete Points

Let $V_n(z_0, z_1, \ldots, z_n) := \prod_{i<j} (z_i - z_j)$.

Choose $\beta_k^{(n)} = z_k \in E$ for which

$$\max \{|V_n(z_0, z_1, \ldots, z_n)| : z_0, z_1, \ldots, z_n \in E\}$$

is attained.

Remark $\mathcal{F}_n : C(E) \to \mathcal{P}_n$ denotes **poly interpolation operator in** $n+1$ **Fekete points.** Then

$$\|\mathcal{F}_n\| \leq n + 1 \,.$$

$$(\mathcal{F}_n f)(z)$$

$$= \sum_{k=0}^{n} f\left(\beta_k^{(n)}\right) \frac{V_n(z_0, \ldots, z_{k-1}, z, z_{k+1}, \ldots, z_n)}{V_n(z_0, z_1, \ldots, z_n)}$$

$$\|\mathcal{F}_n f\|_E \leq \sum_{k=0}^{n} |f\left(\beta_k^{(n)}\right)| \leq (n+1)\|f\|_E$$

From logarithmic potential theory, we know that for the Fekete points $\beta_k^{(n)}$,

$$\lim_{n\to\infty} \left\| \prod_{k=0}^{n} \left(z - \beta_k^{(n)} \right) \right\|_E^{1/n} = \mathrm{cap}(E)$$

and

$$\frac{1}{n+1} \sum_{k=0}^{n} \delta\left(\beta_k^{(n)}\right) \xrightarrow{\;*\;} \mu_E\,,$$

where $\delta(x)$ is unit point mass supported at x and μ_E is the equilibrium measure for E.

So **interpolation in Fekete points gives maximal convergence.**

Also true when E is not a continuum, as long as $\overline{\mathbb{C}} \setminus E$ is connected and regular.

Level curves:

$$\Gamma_r : g(z;\infty) = \log r\,, \qquad r > 1$$

where $g(z;\infty)$ is Green Function with pole at ∞ for $\overline{\mathbb{C}} \setminus E$.

$$g(z;\infty) = \log \frac{1}{\mathrm{cap}(E)} - U^{\mu_E}(z)\,,$$

where

$$U^{\mu_E}(z) := \int_E \log \frac{1}{|z-t|}\, d\mu_E(t)\,.$$

Convergence Rate

$\mathcal{A}(E) := \{f \in C(E) : f$ analytic in interior of $E\}$

Extension of Weierstrass Thm:
(Mergelyan, 1951) If E is a compact set that does not separate the plane and $f \in \mathcal{A}(E)$, then for each $\epsilon > 0$, $\exists$ poly p such that

$$\|f - p\|_E < \epsilon\,.$$

Remark If $E = [a, b]$, then $\mathcal{A}(E) = C(E)$, so Weierstrass $\subset$ Mergelyan.

Geometric Rates of Convergence

Let

$$\begin{aligned} E_n(f) &:= \inf\{\|f - p\|_E : p \in \mathcal{P}_n\} \\ &= \|f - p_n^*\|_E\,, \quad p_n^* \in \mathcal{P}_n\,. \end{aligned}$$

THM Let E be a compact set with connected and regular complement, and $f \in A(E)$. Then f is analytic on some open set $G \supset E$ iff

$$\limsup_{n\to\infty} E_n(f)^{1/n} < 1\,.$$

Proof. ($\Rightarrow$ Fekete points) , ($\Leftarrow$ B-W Lemma)

Bernstein-Walsh Lemma. If $P \in \mathcal{P}_n$, and $|P(z)| \leq M$ for $z \in E$, then $|P(z)| \leq M r^n$, for z on $\Gamma_r : |\varphi(z)| = r$ $(r > 1)$.

Proof. $P(z)/\varphi^n(z)$ analytic outside E, even at ∞.

$$\begin{aligned} \left|\frac{P(z)}{\varphi^n(z)}\right| &\leq M \quad \text{as } z \to \partial E\,,\ z \in \overline{\mathbb{C}} \setminus E \\ &\leq M \text{ in } \overline{\mathbb{C}} \setminus E\,, \quad \text{by Max. Principle}. \end{aligned}$$

□

To complete proof of theorem, assume

(2) $$\limsup_{n\to\infty} E_n(f)^{1/n} < 1\,,$$

and we shall show that f has an analytic extension.

From (2),

$$\|f - p_n^*\|_E < \frac{1}{\rho^n}\,,\ \ n \geq n_0\,,\ \text{ for some } \rho > 1.$$

$$\|f - p_{n+1}^*\|_E < \frac{1}{\rho^{n+1}} \Rightarrow \|p_{n+1}^* - p_n^*\|_E < \frac{2}{\rho^n}\,.$$

By B-W Lemma,

$$\|p_{n+1}^* - p_n^*\|_E < \frac{2r^{n+1}}{\rho^n}\,, \qquad z \text{ on } \Gamma_r$$

$$\Rightarrow p_0 + \sum_0^\infty (p_{k+1}^* - p_k^*)$$

converges uniformly inside Γ_r $(r < \rho)$ to an analytic function. □

COR f analytic on E

$$\Rightarrow \limsup_{n\to\infty} E_n(f)^{1/n} = \frac{1}{\rho},$$

where ρ is the largest index such that f is analytic inside Γ_ρ.

How to Construct Polys of Near Best Uniform Approximation

$p_n^* \in \mathcal{P}_n$ best uniform approx. to $f \in \mathcal{A}(E)$.

$\text{card}(E) \geq n+1$ implies p_n^* unique.

Kolmogoroff Characterization:
Let $\mathcal{M} := \{z \in E : |f(z) - p_n^*(z)| = \|f - p_n^*\|_E\}$.
Then, for all $q \in \mathcal{P}_n$

$$\min_{z \in \mathcal{M}} \mathfrak{Re}\{\overline{(f(z) - p_n^*(z))}q(z)\} \leq 0\,.$$

Construction: E bounded by a Jordan curve Γ.

$$f \in \mathcal{A}(E)\,, \qquad \|f - p\|_E = \|f - p\|_\Gamma\,.$$

Perfect Circularity: If $f \in \mathcal{A}(E)$, $p \in \mathcal{P}_n$, $(f-p)(\Gamma)$ is perfect circle about 0 with winding $\# \geq n+1$, then $p = p_n^*$.

Proof. If not, $\exists\, q \in \mathcal{P}_n$ such that

$$\|f-q\|_E < \|f-p\|_E.$$

But then, for $z \in \Gamma$,

$$|(f-p)(z)-(q-p)(z)| = |(f-q)(z)|$$
$$< \|f-p\|_E = |(f-p)(z)|$$

via $\Rightarrow$ Rouché	$q-p$ and $f-p$ have same number of zeros inside Γ $\therefore$ $q-p$ has $\geq n+1$ zeros, so $q \equiv p$, a contradiction.

□

Ex: $E : |z| \leq 1$, $f(z) = z^{n+1}$, $p_n^*(z) \equiv 0$.

Ex: If error = Blaschke product

$$= \prod_{k=0}^{n} \frac{(z-\alpha_k)}{(1-\overline{\alpha}_k z)}, \quad |\alpha_k| < 1\,.$$

Near circularity

winding number of $(f-p)(\Gamma) \geq n+1$,

$$\max_{\Gamma} |(f-p)(z)| \quad - \quad \min_{\Gamma} |(f-p)(z)| \quad \text{small}$$

$$\Rightarrow \quad p \text{ near } p_n^* .$$

For a large class of functions f, near circularity occurs as $n \to \infty$.

Algorithm (Trefethen) based on

Carathéodory-Fejér Thm: Given

$$p(z) = \sum_{k=0}^{\nu} c_k z^k ,$$

$\exists!$ power series extension

$$p(z) + \sum_{k=\nu+1}^{\infty} c_k^* z^k =: B(z)$$

analytic in $\Delta : |z| \leq 1$ that minimizes $\|B\|_\Delta$ among all such extensions.

$B(z)$ is a finite Blaschke product

$$B(z) = \lambda \frac{\bar{b}_\nu + \bar{b}_{\nu-1} z + \cdots + \bar{b}_0 z^\nu}{b_0 + b_1 z + \cdots + b_\nu z^\nu},$$

$\lambda =$ modulus of largest eigenvalue, b_k's components of eigenvector, of Hankel matrix formed from c_k's (real).

Finding $f - p_n^*$ for $f(z) = \sum_0^\infty a_k z^k$ on Δ $\Leftrightarrow$

$$\min \left\{ \left\| \sum_0^n c_k z^k + \sum_{n+1}^\infty a_k z^k \right\|_{|z|=1} : \ (c_0, \ldots, c_n) \right\}$$

By truncating $f(z) = \sum_0^\infty a_k z^k$ and inverting $z \to 1/z$ and solving CF problem we get nearly circular error curve. That is, we solve CF for

$$\left\| a_m + a_{m-1} z + \cdots + a_{n+1} z^{m-n-1} + \cdots \right\|_\Delta, \ m >> n.$$