LOGARITHMIC POTENTIALS WITH EXTERNAL FIELDS

E. B. Saff
Center for Constructive Approximation
Department of Mathematics
Vanderbilt University

Outline

I) Background

II) Classical Theory of Log Potentials

III) External Fields (varying weights)

IV) Applications

  (a) Generalized Weierstrass Problem

  (b) Optimal Nodes for Bivariate Interpolation

  (c) Minimal Energy Points on the Sphere

I) Background

G. G. Lorentz, 1976 $\qquad P(x) = \sum_{k=s}^{n} a_k x^k, \quad s > 0$

DEF If $s/n \geq \theta$ $(0 < \theta < 1)$, then $P(x)$ is said to be **incomplete of type** θ $(P \in I_\theta)$.

$$x + 3x^2 \in I_{1/2}, \qquad x^3 + 5x^4 - x^6 \in I_{1/2}$$

I_θ is closed under multiplication, but not addition.

THM (Lorentz, '76). If $\{P_n\} \subset I_\theta$, $\deg P_n \to \infty$, and

$$\|P_n\|_{[0,1]} = \max_{x\in[0,1]} |P_n(x)| \leq M \qquad \forall n,$$

then

$$P_n(x) \to 0 \quad \text{for} \quad x \in \left[0, \theta^2\right).$$

Problem 1: Is this result sharp?

G. Freud, 1976

Let $W_\alpha(x) = e^{-|x|^\alpha}$, $\alpha > 0$, on $(-\infty, \infty)$

$\{p_n\}$ orthonormal polynomials w.r.t. W_α,

$$\int_{-\infty}^{\infty} p_m(x) p_n(x) e^{-|x|^\alpha} dx = \delta_{mn}.$$

$\alpha = 2$ Hermite polynomials

3-term recurrence for the p_n

$$\boxed{x p_n(x) = a_{n+1} p_{n+1}(x) + a_n p_{n-1}(x).}$$

Freud Conjecture: $\lim_{n\to\infty} n^{-1/\alpha} a_n$ exists

Problem 2: Resolve this conjecture.

Common Thread

Weighted Polynomials $w(x)^n P_n(x)$, $\deg P_n \le n$.

Lorentz Problem:

$$\|w(x)^n P_n(x)\|_{[0,1]} \le M, \quad w(x) = x^{\theta/(1-\theta)}, \ \deg P_n \le n.$$

Freud Problem:

$$\int_{-\infty}^{\infty} p_n^2(x) e^{-|x|^\alpha} dx = 1$$

$$x \to n^{1/\alpha} x, \quad p_n(x) \to P_n(x) = n^{1/2\alpha} p_n\left(n^{1/\alpha} x\right),$$

$$\|w^n P_n\|_{L_2(\mathbb{R})} = 1, \quad w(x) = e^{-|x|^\alpha/2}, \ \deg P_n \le n.$$

Problem 3:

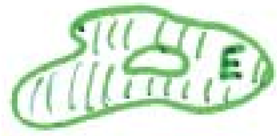
Generalized Weierstrass Approx. Problem
For $E \subset \mathbb{R}$ closed, $w : E \to [0, \infty)$, characterize those functions $f \in C(E)$ that are uniform limits on E of some $\{w^n P_n\}$, $\deg P_n \le n$.

J. E. Littlewood

"I constantly meet people who are doubtful, generally without due reason, about their **potential capacity**."

II) Classical Logarithmic Potential Theory

Let $E \subset \mathbb{C}$ be compact.



cap(E),	$\tau(E)$,	cheb(E)
↑	↑	↑
electrostatics problem	geometric problem	approximation problem

Logarithmic Capacity: cap(E).
Electrostatics Problem: Place a unit charge on E so that equilibrium is reached in the sense of minimum energy.

$$\mathcal{M}(E) := \{\text{unit measures supported on } E\}$$

$$I(\mu) := \int\int \log \frac{1}{|z-t|} d\mu(z) d\mu(t) \qquad \textbf{Energy}$$

$$V_E := \inf_{\mu \in \mathcal{M}(E)} I(\mu)$$

If $I(\mu) = \infty \quad \forall \mu \in \mathcal{M}(E)$, then cap($E$) $:= 0$.

If $I(\mu) < \infty$ for some $\mu \in \mathcal{M}(E)$, then V_E is finite

$$\boxed{\text{cap}(E) := e^{-V_E}.}$$

Frostman: If $\text{cap}(E) > 0$, $\exists$ a unique $\mu_E \in \mathcal{M}(E)$ such that $I(\mu_E) = V_E$.

μ_E is called **equilibrium measure** for E.

Proof follows from 3 basic properties:

(i) Compactness of $\mathcal{M}(E)$ in weak* topology;

(ii) Strict convexity of $I(\mu)$ on $\mathcal{M}(E)$;

(iii) Lower semi-continuity of $I(\mu)$.

Potential

$$U^{\mu}(z) = \int \log \frac{1}{|z-t|} \, d\mu(t) \quad \text{superharmonic}$$

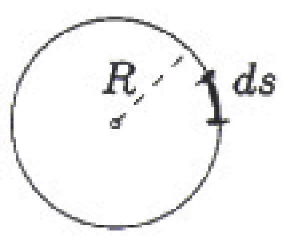
Properties

$S(\mu_E) = \text{supp}(\mu_E) \subseteq \partial_\infty E$;

$\partial_\infty E \setminus S(\mu_E)$ is polar ($\text{cap} = 0$);

$U^{\mu_E}(z) = V_E$ q.e. on E.

Ex.



$E: |z| \le R$

$\text{cap}(E) = R$

$$d\mu_E = \frac{1}{2\pi R} ds$$

$E: [a, b]$

$\text{cap}(E) = (b-a)/4$

$$d\mu_E = \frac{1}{\pi} \frac{dx}{\sqrt{(x-a)(b-x)}}$$

Transfinite diameter: $\tau(E)$.

Geometric problem: Place n points on E so that they are "as far apart" from each other as possible.

2 points: $\max_{z_1, z_2 \in E} |z_1 - z_2| = \operatorname{diam} E$

3 points: $\max_{z_1, z_2, z_3 \in E} (|z_1 - z_2|\,|z_1 - z_3|\,|z_2 - z_3|)^{1/3}$

$\vdots \qquad \vdots$

n points: $\max_{z_1, \dots, z_n \in E} \left(\prod_{1 \le i < j \le n} |z_i - z_j| \right)^{2/n(n-1)} =: \delta_n$

$\mathcal{F}_n = \left\{ z_1^{(n)}, \dots, z_n^{(n)} \right\}$ where max is attained is n-**point Fekete set**

$\mathcal{F}_n \subset \partial_\infty E$

FACT: $\delta_n \downarrow$

$$\tau(E) := \lim_{n \to \infty} \delta_n$$ **transfinite diameter**

Ex.

$E : \lvert z \rvert \le 1$	$E : [-1, 1]$
$\mathcal{F}_n = \left\{ \sqrt[n]{1} \right\}$	$\mathcal{F}_n = \left\{ (1 - x^2)\, P_{n-2}^{(1,1)}(x) = 0 \right\}$
$\tau(E) = 1$	$\tau(E) = 1/2$

Chebyshev Constant: cheb(E)

Polynomial Extremal Problem: Determine the minimum sup norm for monic polys on E.

$$t_n(E) := \min_{p_{n-1}\in\mathcal{P}_{n-1}} \|z^n - p_{n-1}(z)\|_E \,.$$

$T_n(z) = z^n + \cdots$ such that $\|T_n\|_E = t_n(E)$ is called *Chebyshev polynomial*.

$$t_{m+n}(E) = \left\|T_{n+m}\right\| \le \|T_m\|_E \, \|T_n\|_E = t_m(E)t_n(E).$$

So

$$\boxed{\lim_{n\to\infty} t_n(E)^{1/n} = \inf_{n\ge 1} t_n(E)^{1/n} =: \text{cheb}(E).}$$

Ex.

$E : \lvert z\rvert \le R$	$E : [-1,1]$
$T_n(z) = z^n$	$T_n(x) = \frac{1}{2^{n-1}}\cos(n \arccos x)$
$t_n(E) = R^n$	$t_n(E) = 1/2^{n-1}$
$\text{cheb}(E) = R$	$\text{cheb}(E) = \frac{1}{2}$

Fundamental Theorem (Fekete, Szegő, Frostman)

$$\boxed{\mathrm{cap}(E) = \tau(E) = \mathrm{cheb}(E).}$$

Moreover, Fekete point sets $\mathcal{F}_n$ have asymptotic distribution μ_E and Fekete polynomials are asymptotically optimal for the Chebyshev problem.

Essential Observation

$$P(z) = \prod_{k=1}^{n} (z - z_k)$$

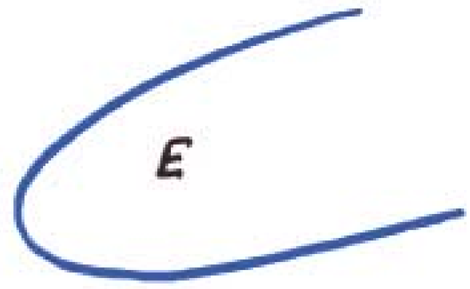
$$\frac{1}{n} \log \frac{1}{|P(z)|} = \int \log \frac{1}{|z-t|} \, d\mu(t) = U^{\mu}(z),$$

where

$$\mu := \frac{1}{n} \sum_{k=1}^{n} \delta_{z_k}, \quad \delta_{z_k} \text{ unit mass at } z_k.$$

III) External Fields

$E \subset \mathbb{C}$ closed



$w : E \to [0, \infty)$ upper semi-continuous,
> 0 on subset of positive cap,
$|z|w(z) \to 0$ as $z \to \infty$ if
E is unbded.

New distance function

$$\boxed{|z - t| \to |z - t| w(z) w(t)}$$

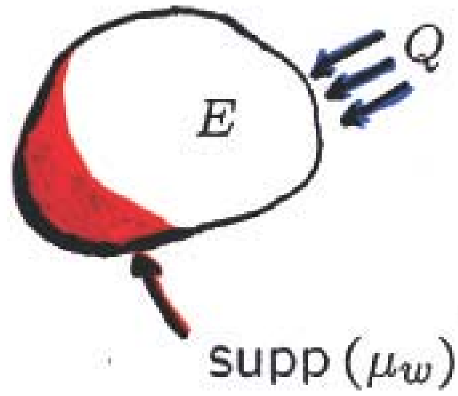
gives rise to

$$\text{cap}(w, E), \qquad \tau(w, E), \qquad \text{cheb}(w, E)$$

Weighted Capacity: cap(w, E)

Electrostatics in the presence of the external field

$$Q := \log\left(\frac{1}{w}\right)$$



$\mu \in \mathcal{M}(E) :=$ {unit measures supported on E}

$$I_w(\mu) := \iint \log \frac{1}{|z-t|w(z)w(t)} d\mu(z)d\mu(t)$$
$$= \iint \log \frac{1}{|z-t|} d\mu(z)d\mu(t) + 2\int Q(z)d\mu(z)$$

$$V_w := \inf_{\mu \in \mathcal{M}(E)} I_w(\mu)$$

$$\text{cap}(w, E) := e^{-V_w}$$

$\exists$ a unique $\mu_w \in \mathcal{M}(E)$ called **weighted equilibrium measure** such that

$$I_w(\mu_w) = V_w \, .$$

Remark: $S(\mu_w) = \text{supp}(\mu_w)$ need not lie entirely on $\partial_\infty E$

Weighted Transfinite Diameter: $\tau(w, E)$

$$\delta_n(w) := \max_{z_1,\ldots,z_n \in E} \left\{ \prod_{1 \le i < j \le n} |z_i - z_j| w(z_i) w(z_j) \right\}^{2/n(n-1)}$$

$\delta_n \downarrow$, $\boxed{\tau(w, E) := \lim_{n\to\infty} \delta_n(w)}$

$\mathcal{F}_n(w) = \left\{ z_1^{(n)}, \ldots, z_n^{(n)} \right\}$ points of E

at which max is attained are called

weighted Fekete points.

$\mathcal{F}_n(w) \subset S(\mu_w)^* := \{z \in E : U^{\mu_w}(z) + Q(z) = F_w\}$.

Weighted Chebyshev Constant: $\mathrm{cheb}(w, E)$

$$t_n(w) := \min_{p \in \mathcal{P}_{n-1}} \| w(z)^n [z^n - p(z)] \|_E ,$$

$$\boxed{\mathrm{cheb}(w, E) := \lim_{n\to\infty} t_n(w)^{1/n}.}$$

FUNDAMENTAL THM (Mhaskar & S.)

$$\text{cap}(w, E) = \tau(w, E) = e^{-\int Q d\mu_w} \text{cheb}(w, E),$$

$$Q = \log(1/w).$$

Moreover, weighted Fekete points $\mathcal{F}_n(w)$ have asymptotic distribution μ_w as $n \to \infty$, and weighted Fekete polynomials are asymptotically optimal in the weighted Chebyshev problem.

The **BIG** question: **How to find** μ_w?

$\mu = \mu_w$ is characterized by the conditions that for some constant F_w,

$$\begin{cases} U^\mu(z) + Q(z) = F_w & z \in S(\mu) \\ U^\mu(z) + Q(z) \geq F_w & z \in E \end{cases}$$

(here we assume E regular, Q continuous)

Essential Problem: Determine $S(\mu_w)$.

Anonymous

" To achieve goals in life,
an important first step is
to find the needed **support**."

Properties of the support $S(\mu_w)$.

a) Sup norm of weighted polys
"lives" on $S(\mu_w)$, i.e., if E is regular

$$\|w(z)^n P_n(z)\|_E = \|w(z)^n P_n(z)\|_{S(\mu_w)}, \quad \forall n, \ \forall P_n \in \mathcal{P}_n$$

b) The outer boundary of $S(\mu_w)$ maximizes
F-functional

$$F(K) := \log[\text{cap}(K)] - \int_K Q\, d\mu_K$$

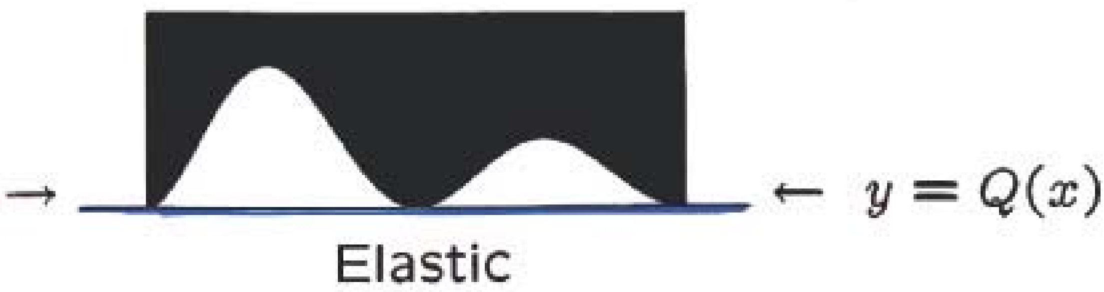
over compact sets $K \subset E$;

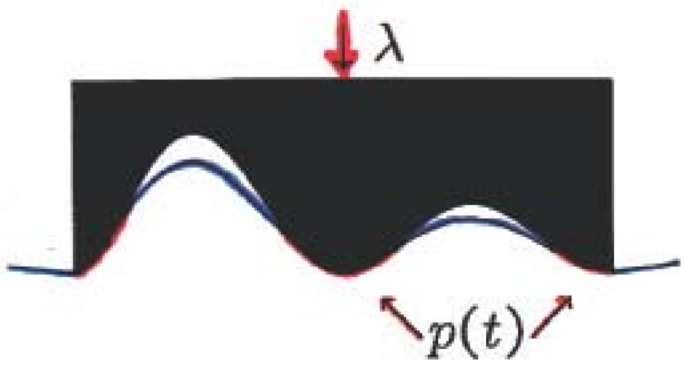
$F(K) \leq F(S(\mu_w))$, Equality $\Rightarrow$ K surrounds $S(\mu_w)$

c) $S(\mu_w)$ is the set of **peaking points**
for weighted polys $w(z)^n P_n(z)$ on E.

d) If $E \subset \mathbb{R}$, "iterated balayage methods"
generate compact sets $K_n \supset S(\mu_w)$
with $\cap K_n = S(\mu_w)^*$.

Contact Problem of Elasticity





$S = S(Q, \lambda) =$ {points where stamp touches elastic}

$p(t)$ pressure on contact region

$$\int_S p(t)dt = \lambda$$

$$\int \log|x-t|p(t)dt \begin{cases} = Q(x) - D, & x \in S \\ < Q(x) - D, & x \notin S \end{cases}$$

$\Rightarrow$ For external field $Q(x)/\lambda$, $w(x) = e^{-Q/\lambda}$,

$d\mu_w = p(t)dt/\lambda$, $S = S(\mu_w)$.

Ex. **Incomplete polynomials**

$$w(x) = x^{\theta/(1-\theta)} \text{ on } [0,1],\ 0 < \theta < 1,$$

$$S(\mu_w) = \left[\theta^2, 1\right].$$

Ex. **Freud Weights**

$$w(x) = e^{-|x|^\alpha},\ \alpha > 0, \quad \text{on } \mathbb{R}$$

$$S(\mu_w) = [-a_\alpha, a_\alpha],$$

where

$$a_\alpha := \left[\frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{\alpha+1}{2}\right)}\right]^{1/\alpha}.$$

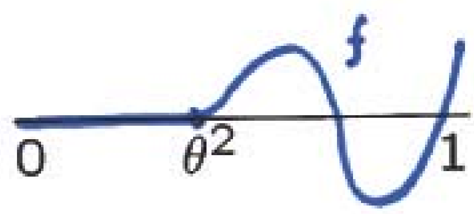
Generalized Weierstrass Approximation.

Let $E \subset \mathbb{R}$ be closed.

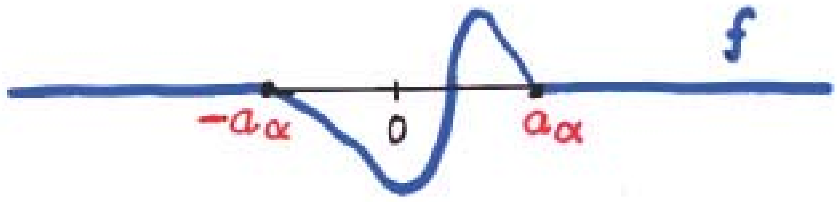
> *Conjecture.* $f \in C(E)$ is the uniform limit on E of some sequence $\{w^n p_n\}_0^\infty$, $\deg p_n \le n$, iff $f(x) = 0$ in $E \setminus S(\mu_w)$.

False in general!
But many important cases when true:

Lorentz Case: (S. & Varga) $f \in C[0,1]$ is the uniform limit on $[0,1]$ of incomplete polynomials of type θ iff $f = 0$ on $\left[0, \theta^2\right] = [0,1] \setminus S(\mu_w)$, $w(x) = x^{\theta/(1-\theta)}$.



COR Lorentz result is sharp!



Freud Case (Lubinsky & S.) If $w(x) = e^{-|x|^\alpha}$, $\alpha > 1$, on $\mathbb{R}$, then $f \in C(\mathbb{R})$ is the uniform limit on $\mathbb{R}$ of a sequence

$$e^{-n|x|^\alpha} p_n(x), \quad \deg p_n \le n, \quad n \to \infty$$

iff $f = 0$ in $\mathbb{R} \setminus [-a_\alpha, a_\alpha] = \mathbb{R} \setminus S(\mu_w)$.

COR (Mhaskar, Lubinsky, S.) Freud's conjecture is true.

THM (Totik) If $E \subset \mathbb{R}$ is an interval and $Q = \log(1/w)$ is convex on E, then conjecture is true.

See also Benko.

Elliptic Fekete Points on $\mathbb{S}^2$

$\mathbb{S}^2 := \left\{ \mathbf{x} \in \mathbb{R}^3 \, : \, |\mathbf{x}| = 1 \right\}$, $\quad |\cdot| =$ Euclidean dist.

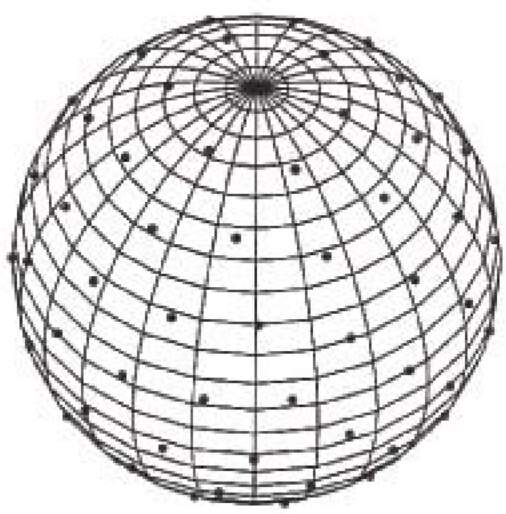
For each $N \geq 2$, let $\left\{ \mathbf{x}^*_{1,N}, \mathbf{x}^*_{2,N}, \ldots, \mathbf{x}^*_{N,N} \right\}$ **maximize**

$$\prod_{1 \leq i < j \leq N} \left| \mathbf{x}_i - \mathbf{x}_j \right|$$

over all $\{\mathbf{x}_i\}_1^N \subset \mathbb{S}^2$.

$\left\{ \mathbf{x}^*_{i,N} \right\}$ are called *elliptic Fekete points.*

Shub & Smale (Prob. #7 for next century)



$N = 100$

Expect $\left\{\mathbf{x}_{i,N}^*\right\}_1^N$ to be *well-separated*, i.e.,

$\exists C$ such that

$$d_N := \min_{i \neq j} \left|\mathbf{x}_{i,N}^* - \mathbf{x}_{j,N}^*\right| \geq \frac{C}{\sqrt{N}}, \quad \forall N \geq 2.$$

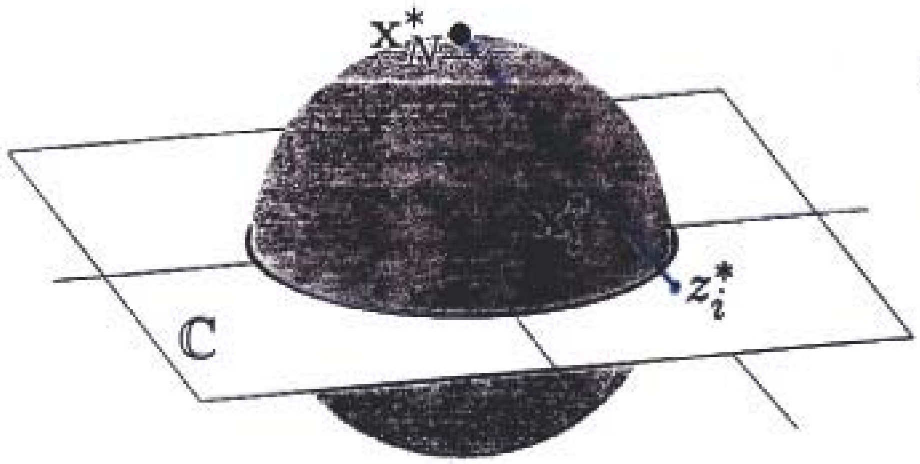
From **results** on best-packing on $\mathbb{S}^2$
(due to Habicht & van der Waerden)

$$C \leq \sqrt{8\pi/\sqrt{3}} \approx 3.81\,.$$

THM (Dragnev) Elliptic Fekete points are well-separated with $C = 2$; in fact,

$$d_N \geq \frac{2}{\sqrt{N-1}}, \quad N \geq 2.$$

Proof. Assume $\mathbf{x}_N^* = \mathbf{x}_{N,N}^*$ is the North Pole



z_i^* = stereographic projection, $i = 1, 2, \ldots, N-1$

Then

$$\prod_{1 \le i < j \le N} \left| \mathbf{x}_i^* - \mathbf{x}_j^* \right|$$

$$= \prod_{1 \le i < j \le N-1} \frac{2 \left| z_i^* - z_j^* \right|}{\sqrt{1 + \left| z_i^* \right|^2} \sqrt{1 + \left| z_j^* \right|^2}} \prod_{1 \le i \le N-1} \frac{2}{\sqrt{1 + \left| z_i^* \right|^2}}$$

$$= 2^{N(N-1)/2} \prod_{1 \le i < j \le N-1} \left| z_i^* - z_j^* \right| w_N\left(z_i^*\right) w_N\left(z_j^*\right),$$

where

$$w_N(z) := \left(\frac{1}{\sqrt{1 + |z|^2}} \right)^{(N-1)/(N-2)}.$$

Thus $\left\{ z_i^* \right\}$ are weighted Fekete points for $w_N(z)$.

Since $w_N(z)$ is radially symmetric it is easy to determine (from the F-functional) that

$$S(\mu_{w_N}) = \left\{ z \in \mathbb{C} : |z| \le \sqrt{N-2} \right\},$$

and, moreover, that all the weighted Fekete points $\left\{z_i^*\right\}_1^{N-1}$ lie on $S(\mu_{w_N})$. Thus

$$|\mathbf{x}_N^* - \mathbf{x}_i^*| = \frac{2}{\sqrt{1+\left|z_i^*\right|^2}} \ge \frac{2}{\sqrt{1+N-2}} = \frac{2}{\sqrt{N-1}}.$$

But $\mathbf{x}_N^*$ was arbitrarily chosen. ■

Other Applications

(1) eigenvalues of random matrices
(2) fast decreasing polynomials
(3) rational approximation and interpolation
(4) extremal problems with constraints

Reference

E. B. Saff, V. Totik

Logarithmic Potentials with External Fields
Springer-Verlag (1997)
ISBN: 3–540–57078–0