# From $H^2$ approximation / interpolation to identification results for 2D Laplace inverse problems: determination of an exchange coefficient

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## 1 Introduction

We consider the problem of determining an unknown exchange function involved in a Robin-type boundary condition from partially overdetermined boundary measurements of the solution of some 2D Laplace equation. This is one of the classical inverse problems of parameter identification arising in non destructive control issues [10, 12, 14], from either thermical or electrical measurements. We illustrate here the fact that analytic function theory together with constructive approximation results provide original solutions to this coefficient recovery issue, as it has already been shown for a class of geometrical inverse problems (cracks determination) [6].
We first give an identifiability result ensuring that an unknown Robin coefficient involved in a 2D Laplace equation, (a heat exchange rate, in the thermical framework) is uniquely determined from the knowledge of Neumann data (flux) and of additional Dirichlet data (temperature), on suitable proper parts of the boundary.
We then approach constructively the issue of determining the unknown coefficient from the available boundary data. The point here is that the above problem amounts to that of recovering an analytic function from its trace on a *proper subset of the boundary of its analyticity domain.* In order to ensure stability and robustness properties of the recovery procedure, we are led to state this interpolation issue as an approximation one in Hardy classes.
After conformally mapping the involved domain into the unit disk $\mathbb{D}$ of the complex plane, we handle these problems using complex analysis tools and analytic approximation results [9, 5, 3].
This finally allows us to make use of the approximation software *Hyperion* to compute the desired coefficient.

## 2 Uniqueness

Let $\Omega$ be a simply connected bounded domain of $\mathbb{R}^2$ with a simple closed $\mathcal{C}^{1,\beta}$ boundary $\partial\Omega$, $\beta \in ]0,1[$. Let then $\gamma$, $\Gamma_D$ and $\Gamma_N$ be three open disjoint subsets of $\partial\Omega$:

$$\partial\Omega = \overline{\gamma} \cup \overline{\Gamma}_D \cup \overline{\Gamma}_N \,. \tag{1}$$

The inverse problem we address is the following.

(PR) Being given a prescribed flux $\phi \in L^2(\Gamma_N)$ ($\phi \not\equiv 0$ on $\Gamma_N$), together with measures of the= temperature $f$ on some subset $K \subset \Gamma_N$, find an unknown heat-exchange function $\varphi \in \mathcal{C}^0(\overline{\gamma})$ such that the solution $u$ of:

$$\left\{ \begin{array}{rcll} \Delta u & = & 0 & \text{in} \quad \Omega, \\ u & = & 0 & \text{on} \quad \Gamma_D, \\ \dfrac{\partial u}{\partial n} & = & \phi & \text{on} \quad \Gamma_N, \\ \dfrac{\partial u}{\partial n} + \varphi u & = & 0 & \text{on} \quad \gamma, \end{array} \right.$$

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satisfies:

$$u_{|_K} = f\,.$$

That the above inverse problem is well-posed as soon as the additional measurements are available on "big enough" sets $K$ is guaranteed by the following "optimal" uniqueness result which improves Holmgren's theorem for the Laplacian case in 2D [9].

**Theorem 1**
*For $\Omega$ as above, assume that $K \subset \partial\Omega$ is an infinite subset having at least an accumulation point in $\partial\Omega$. If $u \in W^{1,2}(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ verifies :*

$$\left\{ \begin{array}{rclcl} \Delta u &=& 0 & in & \Omega, \\ \dfrac{\partial u}{\partial n} &=& 0 & on & \overline{K}, \\ u &=& 0 & on & K, \end{array} \right. \tag{2}$$

*then $u \equiv 0$ in $\Omega$.*

As a consequence, we get an uniqueness result improving in 2D situations the one obtained in [12]:

**Theorem 2**
*Let $\varphi_1, \varphi_2 \in \mathcal{C}^0(\overline{\gamma})$, $\varphi_1, \varphi_2 > c$ for some constant $c$ depending on the partition* (1)*. Let $u_i$ be the unique solution of* (PR) *associated to $\varphi = \varphi_i$, $i = 1, 2$. It holds that, if $u_1\,|_K = u_2\,|_K$, then $\varphi_1 = \varphi_2 =$ on $\overline{\gamma}$.*

## 3 Constructive identification by $H^2$ approximation

We now express (PR) in the unit disk $\mathbb{D}$.

### 3.1 Conformal mapping onto $\mathbb{D}$

Under the present hypotheses on $\Omega$, it is shown in [16] that there exists a conformal mapping $\Upsilon$ from the unit disk $\mathbb{D}$ into $\Omega$ having a $C^{1,\beta}$ extension to $\overline{\mathbb{D}}$ (which we still write $\Upsilon$). Put:

$$v = u \circ \Upsilon \text{ in } \mathbb{D}, \quad \left\{ \begin{array}{rcl} T_D &=& \Upsilon^{-1}(\Gamma_D), \\ T_N &=& \Upsilon^{-1}(\Gamma_N), \\ T_R &=& \Upsilon^{-1}(\gamma), \end{array} \right. \quad \left\{ \begin{array}{rcll} \tilde{\phi} &=& |\Upsilon'|\, \phi \circ \Upsilon & \text{on } T_N, \\ \tilde{\varphi} &=& |\Upsilon'|\, \varphi \circ \Upsilon & \text{on } T_R. \end{array} \right. \tag{3}$$

In particular, the unit circle $\mathbb{T} = T_D \cup T_N \cup T_R$ and the problem (PR) can be formulated as follows [9]:

given $\tilde{\phi} \not\equiv 0 \in L^2(T_N)$, $K \subset T_N$, and $f \in \mathcal{C}^{0,\alpha}(K)$, for $0 < \alpha < 1$, find $\tilde{\varphi}$ such that the solution $v$ of:

$$\left\{ \begin{array}{rclcl} \Delta v &=& 0 & \text{in} & \mathbb{D} \\ v &=& 0 & \text{on} & T_D \\ \dfrac{\partial v}{\partial n} &=& \tilde{\phi} & \text{on} & = T_N \\ \dfrac{\partial v}{\partial n} + \tilde{\varphi}\, v &=& 0 & \text{on} & T_R. \end{array} \right. \tag{4}$$

satisfies:

$$v_{|_K} = f\,.$$

Hence, from the last equation (3), the inverse problem (PR) of determining $\varphi$ is equivalent[1] to that of solving (4) and of finding $\tilde{\varphi}$. Thus, it now takes place in $\mathbb{D}$ where we now formulate it as an interpolation / approximation issue in the Hardy space $H^2$.

In the sequel, we assume that *both the measurement part $K \subset T_N \subset \mathbb{T}$ and its complementary set $\mathbb{T} \setminus K$ have positive Lebesgue measure.*

[1]Note that, although the following procedure actually provides a computational scheme to solve for (PR) in $\mathbb{D}$, we do not approach here the issue of building the conformal map $\Upsilon$.

### 3.2 Interpolation and bounded extremal problems in $H^2$

From the knowledge of the flux $\tilde{\phi} \in L^2(T_N)$ and of the temperature $f \in \mathcal{C}^{0,\alpha}(K)$, we can in principle build the trace on $K$ of an analytic function in $\mathbb{D}$. This holds because $v$ is harmonic in $\mathbb{D}$ and Cauchy-Riemann equations ensure that, if we note $\int \tilde{\phi}$ for some primitive of $\tilde{\phi}$ on $K$:

$$\mathcal{F} = f + i \int \tilde{\phi} \text{ on } K,$$

is actually the trace on $K$ of a unique function $g$ analytic in $\mathbb{D}$: $\mathcal{F} = g_{|K}$. Moreover, smoothness preserving properties of the harmonic conjugation operator implies that this analytic function $g$ is in fact bounded in $\overline{\mathbb{D}}$ [2, 13].
Our aim is then to recover $g$ in the whole of $\mathbb{D}$, or at least on $T_R \subset \mathbb{T} \setminus K$, from the knowledge of its trace $\mathcal{F}$ on $K$. Indeed, this function would solve (4) since

$$v = \operatorname{Re} g \text{ in } \mathbb{D},$$

and then

$$\tilde{\varphi} = \frac{\dfrac{\partial(\operatorname{Im}(g))}{\partial \tau}}{\operatorname{Re}(g)} \text{ on } T_R, \tag{5}$$

where the above equality should be properly understood (non tangential limits of the right hand side). Now, $\varphi = |\Upsilon'|^{-1} \tilde{\varphi} \circ \Upsilon^{-1}$ on $T_R$ is the expected solution to (PR).
Moreover, we want the recovery procedure to be convergent in order to carry some stability and robustness properties. Our concern here is that, in practice, one may not know exactly $f$ nor $\mathcal{F}$ on $K$: for example, pointwise values of $f$ only might be available through experimental devices that necessarily induce noises and perturbations of the measurements. Also, the knowledge of $\mathcal{F}$ requires that of some "primitive of the flux", which is to be computed numerically.
A more realistic problem is thus to approximately and robustly recover $g$ in the whole of $\mathbb{D}$ from the knowledge of a perturbation $\mathcal{F}_\varepsilon$ of its trace $\mathcal{F}$ on $K \subset \mathbb{T}$, where $\varepsilon$ is a small parameter that stands for a (deterministic) measure of the perturbation.
However, classical analytic interpolation or extrapolation results from part of the boundary (Carleman integrals, for example) do not possess any stability properties on their own and are not suitable to ensure robustness with respect to perturbation, as shows the next proposition from [7]. This is the reason why we need a constrained approximation framework.
Now, despite the following approximation questions should be approached in uniform norm, it is simpler to handle them in the Hardy space $H^2$ which possesses an hilbertian structure. Also, for various reasons, robustness properties are easier to ensure there. Recall that $H^2$ can be viewed as the space of functions analytic in $\mathbb{D}$ that are square-summable on circles of radius less than 1 centered at 0. It is a consequence of this definition that traces on $\mathbb{T}$ of $H^2$ functions belong to $L^2(\mathbb{T})$, and in this sense $H^2 \subset L^2(\mathbb{T})$ inherits the inner product, see e.g. [13, 17] for definitions and properties of Hardy spaces. Note that $H^2$ strictly contains the functions that are analytic and uniformly bounded in $\mathbb{D}$. For any subset $E \subset \mathbb{T}$, we will denote by $\| \, \|_E$ the $L^2(E)$-norm and by $\chi_E$ the characteristic function of $E$.

**Proposition 1** *Let $K \subset \mathbb{T}$ such that $K$ and $\mathbb{T} \setminus K$ have positive Lebesgue measure.*
*(i) $H^2_{|K}$ is dense in $L^2(K)$.*
*(ii) Let $h \in L^2(K)$ and suppose $(g_n)$ is a sequence of $H^2$ converging to $h$ in $L^2(K)$. If $h$ is not the trace of an $H^2$ function, then $\|g_n\|_{\mathbb{T} \setminus K} \to \infty$.*

In view of this, we see that as soon as a perturbation is involved in the measurements on $K$ and prevents the data from being truely the trace of an $H^2$ function (mainly by loose of analyticity), any $H^2$ interpolating procedure from $K$ *only* will exhibit a wild behaviour outside $K$.
A remedy for this unstable behaviour is to state the above recovery problem as a (best) constrained approximation issue which is a bounded generalization on subsets of $\mathbb{T}$ of classical (dual) extremal problems in $H^2$.

(BEP) Being given $h \in L^2(K)$ and $M > 0$, find $g_0 = g_0(h, M) \in H^2$, $\|g_0\|_{\mathbb{T}\setminus K} \leq M$ such that

$$\|h - g_0\|_K = \min_{\substack{g \in H^2 \\ \|g\|_{\mathbb{T}\setminus K} \leq M}} \|h - g\|_K .$$

Under the norm constraint, this problem becomes well-posed. Existence and uniqueness of solutions to (BEP) are established in [3, 5] as well as a constructive formula to compute $g_0$. Denote by $P_{H^2} : L^2(\mathbb{T}) \to H^2$ the orthogonal (analytic) projection and by $\mathcal{T}$ the Toeplitz operator with symbol $\chi_{\mathbb{T}\setminus K}$

$$\begin{array}{rcl} \mathcal{T} : H^2 & \to & H^2 \\ g & \mapsto & P_{H^2}\, \chi_{\mathbb{T}\setminus K}\, g \end{array}$$

($\mathcal{T}$ is a semi-infinite Toeplitz matrix: $\mathcal{T}_{k,l} = \mathcal{T}_{k-l}\ ,\ k, l \geq 0$). Hence,

$$g_0 = g_0(\lambda) = (I + \lambda\, \mathcal{T})^{-1}\, P_{H^2}\, \chi_K\, h\,,$$

for the unique value of the (Lagrange type) parameter $\lambda > -1$ such that $\|g_0\|_{\mathbb{T}\setminus K} = M$, excepted if $h$ already belongs to the approximants class ($h \in H^2_{|K}$, $\|h\|_{\mathbb{T}\setminus K} \leq M$) in which case $g_0 = g_0(-1) = h$. Note that integral formulas of Carleman type are also available to represent $g_0(\lambda)$.

The behaviour with respect to $\lambda$ of the error $e(\lambda)$ and of the constraint $M(\lambda)$ defined by:

$$\begin{array}{l} e(\lambda) = \|h - g_0(\lambda)\|^2_K\,, \\ M(\lambda) = \|g_0(\lambda)\|_{\mathbb{T}\setminus K}\,, \end{array}$$

is smooth and monotonous. In particular,as $\lambda \searrow -1$,

$$\begin{array}{ll} e(\lambda) \searrow 0\,, & \\ M(\lambda) \nearrow \infty\,, & \text{if } h \notin H^2_{|K}\,, \\ M(\lambda) \nearrow \|h\|_{\mathbb{T}\setminus K}\,, & \text{if } h \in H^2_{|K}\,. \end{array}$$

Hence, whenever $h \in H^2_{|K}$, this approximation process on $K$ gives rise to interpolation on the whole $\mathbb{T}$. Applying this procedure to the perturbed boundary data $\mathcal{F}_\varepsilon$ and building the solution $g_0(\mathcal{F}_\varepsilon, \lambda)$ to (BEP) gives rise to approximate solutions to (4), with $v_{\epsilon,\lambda} = \mathrm{Re} g_0(\mathcal{F}_\varepsilon, \lambda)$ and to the associated Robin coefficient $\tilde{\varphi}_{\epsilon,\lambda}$. Robustness and continuity are also ensured, since:

$$g_0(\mathcal{F}_\varepsilon, \lambda) \to g \text{ in } H^2$$

as $\varepsilon \to 0$ ($\mathcal{F}_\varepsilon \to \mathcal{F}$ in $L^2(K)$) and $\lambda \to -1$, although some care has to be taken as $\lambda$ approaches $-1$. Sharp estimates of the convergence rates of $e$ and $M$ with respect to $\lambda$ have recently been obtained [8].

Numerical results obtained with this technique will be provided from the software *Hyperion* [4].

## References


[1] G. Alessandrini (1997): *Examples of instability in inverse boundary-value problems*, Inverse Problems **13**, 887-897.

[2] L.V. Ahlfors (1953): *Complex analysis*, McGraw-Hill, New-York.

[3] D. Alpay, L. Baratchart, J. Leblond (1992): *Some extremal problems linked with identification from partial frequency data. In 10th Conference on Analysis and optimization of systems, Sophia-Antipolis*, Lecture Notes in Control and Information Science **185**, Springer-Verlag, 563-573.

[4] L. Baratchart, J. Grimm, J. Leblond, M. Olivi, F. Seyfert, F. Wielonsky (1998): *Identification d'un filtre hyperfréquences par approximation dans le domaine complexe*, INRIA technical report, n$^o$ 219.

[5] L. Baratchart, J. Leblond (1998): *Hardy approximation to $L^p$ functions on subsets of the circle with $1 \leq p < \infty$*, Constructive Approximation **14**, 41-56.

[6] L. Baratchart, J. Leblond, F. Mandréa, E.B. Saff(1999): *How can the meromorphic approximation help to solve some 2D inverse problems for the Laplacian?*, Inverse Problems **15**, 79-90.

[7] L. Baratchart, J. Leblond, J.R. Partington (1996): *Hardy approximation to $L^\infty$ functions on subsets of the circle*, Constructive Approximation **12**, 423-436.

[8] L. Baratchart, J. Leblond, J.R. Partington: *Asymptotic estimates for interpolation and constrained approximation in $H^2$ by diagonalization of Toeplitz operators*, in preparation.

[9] S. Chaabane (1999): *Étude de quelques problèmes inverses*, PHD thesis, École Nationale d'Ingénieurs de Tunis (ENIT), Univ. Tunis II.

[10] A. Ben Abda, S. Chaabane, F. El Dabaghi, M. Jaoua (1998): *On the identification of unknown boundaries submitted to Signorini boundary conditions : identifiability and stability*, Math. Meth. in Appl. Sci. **21**, 1379-1398.

[11] S. Chaabane, M. Jaoua (1999): *Dérivation par rapport au domaine pour un problème aux limites avec condition de Signorini*, C.R. Acad. Sci. Paris, Série **1**, **328**, 395-400.

[12] S. Chaabane, M. Jaoua(1999): *Identification of Robin coefficients by the means of boundary measurements*, to appear in Inverse problems.

[13] J.B. Garnett: *Bounded analytic functions*, Academic Press, New-York.

[14] V. Isakov (1993): *Uniqueness and stability in multi-dimensional inverse= problems*, Inverse Problems **9**, 579-621.

[15] G. Inglese (1997): *An inverse problem in corrosion detection*, Inverse Problems= **13**, 977-994.

[16] C. Pommerenke (1992): *Boundary behavior of conformal maps*, Springer, Berlin.

[17] W. Rudin (1987): *Analyse réelle et complexe*, Masson, Paris.