Approximation and interpolation in H^2: Toeplitz operators, recovery problems and error bounds

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1 Introduction

Let $\mathbb{T}$ denote the unit circle and $\mathbb{D}$ the unit disc. We write $\mathbb{T} = I \cup J$, the union of two disjoint arcs. Without loss of generality, we can take $I = (e^{-ia}, e^{ia})$ and $J = (e^{ia}, e^{i(2\pi - a)})$, where $0 < a < \pi$. We designate by H^2 the Hardy space with exponent 2 of the unit disc. For $E \subset \mathbb{T}$, we write $H^2_{|_E}$ to mean the space of traces on E of H^2 functions. For $E \subset \mathbb{T}$ or $E \subset \mathbb{R}$, we denote by $L^p(E)$ the familiar Lebesgue space and by $\|.\|_{L^p(E)}$ the corresponding norm. The symbol $(\ ,\ )_{L^2(\mathbb{T})}$ indicates the scalar product in $L^2(\mathbb{T})$. Let $\bar{H}^2_0$ be the orthogonal complement of H^2 in $L^2(\mathbb{T})$, $P_{H^2}\ : L^2(\mathbb{T}) \to H^2$ the orthogonal projection, and

$$\phi : H^2 \to H^2, \qquad g \mapsto P_{H^2}(\chi_J g) \tag{1}$$

the Toeplitz operator with symbol χ_J, the characteristic function of J.
Given $f \in L^2(I)$, $\Psi \in L^2(J)$, and $M > 0$, let us consider the following problem.

Find $g = g_\Psi \in H^2$ to minimize $\|f - g\|_{L^2(I)}$ under the constraint $\|\Psi - g\|_{L^2(J)} \leq M$.

This problem was originally taken up in [5], in a dual form with $f = 0$, and the above version in [3], after a preliminary study of the case $\Psi = 0$ in [1].

Apart from their theoretical importance, such problems have several physical motivations. For example, they occur in problems of deconvolution and linear systems identification from partial frequency-response measurements [1, 7], as well as in the study of inverse Dirichlet–Neumann problems such as those occurring in fault detection [4]. They are also closely connected with the work of Patil [8] on recovering Hardy class functions from their restrictions to a subset of the circle, as we shall discuss later.

In [1, 3, 6], the solution to the above constrained optimization problem is given implicitly in terms of a parameter $\lambda \in (-1, \infty)$. Specifically, let

$$\tilde{f} = \begin{cases} f & \text{on } I \\ 0 & \text{on } J \end{cases}, \qquad \tilde{\Psi} = \begin{cases} 0 & \text{on } I \\ \Psi & \text{on } J \end{cases}.$$

If $f \notin H^2_{|_I}$, then the optimal g_Ψ is given by

$$g_\Psi = g_\Psi(\lambda) = (1 + \lambda\, \phi)^{-1}\, P_{H^2}\left(\tilde{f} + (1+\lambda)\tilde{\Psi}\right) \tag{2}$$

for the unique $\lambda > -1$ such that $\|\Psi - g_\Psi\|_{L^2(J)} = M$.
Let

$$e_\Psi(\lambda) = \|f - g_\Psi\|^2_{L^2(I)}\,, \quad e(\lambda) = e_0(\lambda)\,,$$

and

$$M_\Psi(\lambda) = \|\Psi - g_\Psi\|_{L^2(J)}\,, \quad M(\lambda) = M_0(\lambda)\,,$$

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whenever g_Ψ is defined as above for some $\lambda \in (-1, \infty)$. Then, as $\lambda \to -1$, it is shown in [1] that $M(\lambda) \to \infty$, $(\lambda+1)M(\lambda)^2 \to \infty$ and $e(\lambda) \to 0$.

When $f \in H^2_{|_I}$, that is when the function f to be approximated is already the trace on I of an H^2 function, which we shall still call f, it turns out [3] that the family $g_\lambda = g_0(\lambda)$ of solutions of the above bounded extremal problem for $\Psi = 0$ coincides with some Carleman type interpolants to f [8]. In this case, $\|f - g_\lambda\|_{L^2(\mathbb{T})}$ goes to zero while $\lambda \to -1$, so that in particular $M(\lambda) \to \|f\|^2_{L^2(J)}$ and $e(\lambda) \to 0$.

However, estimates for the convergence rates of the quantities $M(\lambda)$ and $e(\lambda)$ are not available in general. The difficulty here can be illustrated as follows. Observe, in particular, that

$$e(\lambda) = \|P_{H^2}\chi_I(f-g_0)\|^2_{L^2(\mathbb{T})} + \|P_{\bar{H}^2_0}\chi_I(f-g_0)\|^2_{L^2(J)} + \|P_{\bar{H}^2_0}\chi_I(f-g_0)\|^2_{L^2(I)},$$

and, although it is known [2] that the two first terms of the right hand side behave as $o(1/M^3)$, this does not provide any upper bound for the last term without additional assumptions. This is to be connected with the fact that it is possible for a general H^2 (or $\bar{H}^2_0$) function to converge to 0 on $\mathbb{T}$ with some precise rate on J while going arbitrarily slowly on I, in general. Hence, the particular form of $e(\lambda)$ and of g_0 has to be taken into account.

When f belongs to some Sobolev class on I, upper estimates of the corresponding convergence rates were obtained in [2]. However, these rates are rather pessimistic, as we will show. Indeed, our aim here is to compute much better upper bounds for these rates by using a constructive diagonalization procedure for Toeplitz operators [9] as applied to the following formulae obtained in [1]:

$$M(\lambda)^2 = (\phi(1+\lambda\phi)^{-2}P_{H^2}\tilde{f}, P_{H^2}\tilde{f})_{L^2(\mathbb{T})}, \tag{3}$$

and

$$\frac{d\,e}{d\,\lambda} = -(\lambda+1)\,\frac{d\,M^2}{d\,\lambda}, \tag{4}$$

where the differentiability is understood in the strong sense: using the spectral theorem and differentiability under the integral sign, it is easily seen that $e(\lambda)$ is a C^∞ function of λ.

It will turn out that, as will be explained in Remark 3.3, the estimates we obtain are sharp in the sense that no stronger estimates can be valid in general.

2 Concrete spectral theory

The cornerstone of this work is that formula (3) can be re-expressed using the spectral measure for ϕ, which is a self-adjoint operator with spectrum contained in $[0,1]$. More precisely, following the concrete spectral theory and the diagonalization procedure for self–adjoint Toeplitz operators of multiplicity 1 [9, ch.3], we see that ϕ is unitarily equivalent to multiplication M_x by the independent variable x on $L^2([0,1], d\rho)$, where $d\rho(x) = \frac{\sin a}{\pi}dx$ (in the following, we put $C = \frac{\sin a}{\pi}$). In fact, there exists a unitary transformation $V : H^2 \to L^2([0,1], d\rho)$ such that

$$V\,\phi\,V^{-1} = M_x\,,$$

which acts on Cauchy kernels $k_\alpha(z) = 1/(1-\bar{\alpha}z)$ as

$$(Vk_\alpha)(x) = [\overline{\psi_x(\alpha)}(1-\bar{\alpha}e^{ia})^{1/2}(1-\bar{\alpha}e^{-ia})^{1/2}]^{-1}, \tag{5}$$

where, for $0 < x < 1$, we let ψ_x be the unique outer function such that

$$|\psi_x|^2 = \begin{cases} x & \text{on } I, \\ 1-x & \text{on } J, \end{cases}$$

and $\psi_x(0) > 0$, and where the principal branch of the square root is used in (5).
It follows easily by a continuity argument that, for any $h \in C([0,1])$, one has

$$(h(\phi)P_{H^2}\tilde{f}, P_{H^2}\tilde{f})_{L^2(\mathbb{T})} = C\int_0^1 h(t)|v(t)|^2\,dt.$$

where $v = V\,P_{H^2}\tilde{f}$. To obtain a representation of v, we now establish the following result that generalizes formula (5) from [9, ch.3] to a wide class of H^2 functions:

Theorem 2.1 *For every $h \in L^2(\mathbb{T})$ such that*

$$(1-e^{-i\theta}e^{ia})^{-1/2}(1-e^{-i\theta}e^{-ia})^{-1/2}\, h(e^{i\theta}) \in L^1(\mathbb{T})\,, \tag{6}$$

and $\forall x \in [0,1]$, we have:

$$(V\,(P_{H^2}\,h))\,(x) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{h(e^{i\theta})\,d\theta}{\overline{\overline{\psi_x(e^{i\theta})}\,(1-e^{-i\theta}e^{ia})^{1/2}(1-e^{-i\theta}e^{-ia})^{1/2}}}.$$

(Here, and elsewhere, for $\Gamma \subset \mathbb{T}$, the notation $\int_\Gamma$ indicates that we integrate over those θ with $e^{i\theta} \in \Gamma$.)

3 Approximation

It follows from Theorem 2.1 that, if $\tilde{f}$ satisfies (6), with $h = \tilde{f}$, then

$$v(x) = \frac{1}{2\pi}\int_I \frac{f(e^{i\theta})\psi_x(e^{i\theta})\,d\theta}{x(1-e^{-i\theta}e^{ia})^{1/2}(1-e^{-i\theta}e^{-ia})^{1/2}}. \tag{7}$$

Therefore

$$M(\lambda)^2 = C\int_0^1 \frac{t}{(1+\lambda t)^2}|v(t)|^2\,dt\,, \tag{8}$$

and, using (4),

$$\frac{d\,e}{d\,\lambda} = 2C(\lambda+1)\int_0^1 \frac{t^2}{(1+\lambda t)^3}|v(t)|^2\,dt. \tag{9}$$

Let χ be the outer function such that $\chi(0) > 0$ and

$$|\chi(e^{i\theta})| = \begin{cases} 1 & \text{on } I, \\ 1/2 & \text{on } J, \end{cases}$$

and let $\omega(e^{i\theta})$ be the argument of $\chi(e^{i\theta})$. Also let

$$\gamma(x) = \frac{\log x - \log(1-x)}{2\log 2}. \tag{10}$$

Then we have

$$v(x) = \frac{1}{2\pi}\int_I \frac{F(e^{i\theta})\exp[i\gamma(x)\omega(e^{i\theta})]\,d\theta}{\sqrt{x}} \tag{11}$$

where F is defined on I by

$$F(e^{i\theta}) = \frac{f(e^{i\theta})}{(1-e^{-i\theta}e^{ia})^{1/2}(1-e^{-i\theta}e^{-ia})^{1/2}}. \tag{12}$$

Clearly, this implies that $v(x) = O(1/\sqrt{x})$, if $F \in L^1$, which is ensured by (6).

By means of the Riemann–Lebesgue Lemma, we obtain the following more precise estimate as $x \to 1$.

Lemma 3.1 *If $f' \in L^1(I)$, then*

$$|v(x)| = o\left(\frac{1}{\log(1-x)}\right) \qquad \text{as } x \to 1. \tag{13}$$

This leads to the following result.

Theorem 3.2 *If $f \in L^2(I)$ satisfies $f' \in L^1(I)$, then as $\lambda \searrow -1$,*

$$M_\Psi(\lambda)^2 = o((\lambda+1)^{-1}\log^{-2}(\lambda+1))\,,$$

while

$$e_\Psi(\lambda) = o(\log^{-1}(1+\lambda)).$$

Hence, we can find an approximation of L^2-error $e_\Psi(\lambda)$ on I, at the expense of an L^2-error of size $M_\Psi(\lambda)^2$ on J that are linked by:

$$e_\Psi(\lambda)\,\log M_\Psi(\lambda) = o(1)\,.$$

Remark 3.3 The Riemann-Lebesgue lemma is known to be sharp, in that *every* continuous even function on $\mathbb{R}$, convex on $(0,\infty)$ and decreasing monotonically to zero, is the Fourier transform of a function in $L^1(\mathbb{R})$ (see, for example, [10, Thm. 124]). Thus the estimate in (13) cannot be improved further. This observation can be used to establish the sharpness of the estimates for M and e.

4 Interpolation

When $f \in H^2_{|_I}$, the solution $g_\lambda = g_0(\lambda)$ given by (2) coincides with Patil's sequence of interpolants [3, 7, 8] that was built for recovery purposes rather than those of approximation. In this case, if we still denote by f the H^2 function on the whole of $\mathbb{T}$ (unique whenever I is of positive Lebesgue measure), then $\|f - g_\lambda\|_{L^2(\mathbb{T})}$ goes to zero while $\lambda \to -1$. We now establish some estimates of this decay rate together with pointwise convergence results as corollaries.
In fact, the key point here is that $P_{H^2}\,\tilde{f} = (1-\phi)\, f$ so that, on using the identity

$$1 - \phi = (1 + \lambda\,\phi) - (\lambda+1)\,\phi\,, \tag{14}$$

formula (2) becomes

$$g_0 = f - (\lambda+1)\,(1+\lambda\,\phi)^{-1}\,\phi\, f\,. \tag{15}$$

Proposition 4.1 *If $f \in H^2_{|_I}$ then, as $\lambda \searrow -1$,*

$$e(\lambda) = O(1+\lambda).$$

Proposition 4.2 *If $f \in H^2_{|_I}$ verifies $f' \in L^1(\mathbb{T})$, then as $\lambda \searrow -1$,*

$$e_J(\lambda) = \|f - g_\lambda\|^2_{L^2(J)} = o(\log^{-1}(\lambda+1)) \quad ,$$

while

$$e(\lambda) = o((\lambda+1)\log^{-1}(\lambda+1))\,,$$

Hence,

$$e_J(\lambda)\,\log e(\lambda) = o(1)\,.$$

The following result is now established as a consequence of Proposition 4.1.

Theorem 4.3 *If $f \in H^2_{|_I}$, then there exists a subset $E \subset I$ of zero Lebesgue measure which verifies:*

$$\forall e^{i\theta} \in I \setminus E\,,\ \forall \varepsilon > 0\,,\ \exists \lambda_\varepsilon \in (-1,0) \text{ such that}$$

$$-1 < \lambda < \lambda_\varepsilon \ \Rightarrow |f(e^{i\theta}) - g_\lambda(e^{i\theta})| < \varepsilon\,.$$

The following equivalent expression $\hat{g}_\alpha$ of g_λ in terms of Carleman-type integral formula, that links our family of approximants to the sequences of interpolants given in [8], is also of interest. Indeed, we get from [3] that:

$$g_\lambda(z) = \hat{g}_\alpha(z) = \frac{1}{2\,i\,\pi}\int_{\mathbb{T}} \left(\frac{\varphi(\xi)}{\varphi(z)}\right)^\alpha (\chi_I f)(\xi)\,\frac{d\xi}{\xi - z}\,,\ \forall z \in \mathbb{D}\,,$$

or equivalently that:

$$g_\lambda = \hat{g}_\alpha = \frac{1}{\varphi^\alpha}\,P_{H^2}\,\left(\varphi^\alpha\,(\chi_I f)\right)\,,$$

where φ is the outer H^∞ "quenching" function of modulus equal to $\varrho > 1$ on I and to 1 on J defined by

$$\varphi(z) = \exp\left\{\frac{\log\varrho}{2\,\pi}\int_I \frac{e^{it}+z}{e^{it}-z}\,dt\right\}\,,\ z \in \mathbb{D}\,,$$

and where

$$\alpha = -\frac{\log(\lambda+1)}{2\,\log\varrho}\,,\ \text{or } \lambda = -1 + \varrho^{-2\alpha}\,. \tag{16}$$

Taking $\varrho = e^{1/2}$, Proposition 4.1 can then be rephrased as

Corollary 4.4 *If $f \in H^2_{|_I}$ then, as $\alpha \to \infty$,*

$$\|f - \hat{g}_\alpha\|^2_{L^2(I)} = O(\exp(-\alpha))\,.$$

Pointwise convergence also holds almost everywhere on J under some sufficient conditions:

Theorem 4.5 *If $f \in H^2_{|_I}$ verifies $f'' \in L^1(\mathbb{T})$, then there exists a subset $\mathcal{E} \subset J$ of zero Lebesgue measure which verifies:*

$$\forall e^{i\theta} \in J \setminus \mathcal{E}\,,\ \forall \varepsilon > 0\,,\ \exists \lambda_\varepsilon \in (-1, 0) \text{ such that}$$
$$-1 < \lambda < \lambda_\varepsilon \ \Rightarrow |f(e^{i\theta}) - g_\lambda(e^{i\theta})| < \varepsilon\,.$$

Finally, one can deduce the following pointwise convergence result of the approximants on I.

Corollary 4.6 *If $f \in L^2(I)$ verifies $f'' \in L^1(I)$, then the conclusion of Theorem 4.3 remains true.*

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