Mathematical Methods - Lecture 10

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Outline

1. Complex Numbers

2. Complex Variables

The need for complex numbers

- Complex numbers occur in many branches of mathematics
- They arise most directly out of solving polynomial equations
- **Example:** Consider the quadratic equation:

$$z^2 - 4z + 5 = 0.$$

  - The solutions are:

$$z_{1,2} = \frac{4 \pm \sqrt{(-4)^2 - 4(1 \times 5)}}{2} = 2 \pm \frac{\sqrt{-4}}{2}$$

  - Both solutions contain the square root of a negative number
  - However, the **fundamental theorem of algebra** states that a quadratic equation will always have two solutions ($z_{1,2}$ in our case)

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  - Both solutions contain the square root of a negative number
  - However, the **fundamental theorem of algebra** states that a quadratic equation will always have two solutions ($z_{1,2}$ in our case)
  - The full solution is called a **complex number**. It consists of a **real** term and an **imaginary** term:

$$z_{1,2} = 2 \pm i$$

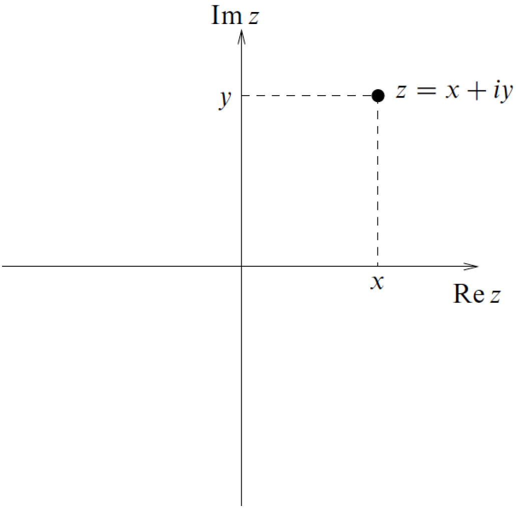
What is a complex number

- **Complex number** is a number that can be expressed in the form $z = x + iy$, where:
  - x and y are real numbers
  - i is the imaginary unit that satisfies $i^2 = -1$
- In this expression, $x = \mathrm{Re}\, z$ is the real part and $y = \mathrm{Im}\, z$ is the imaginary part
- For compactness a complex number is sometimes written in the form:

$$z = (x, y),$$

  where (x, y) may be thought of as coordinates in an xy-plot
- Such a plot is called an **Argand diagram** and is a common representation of complex numbers

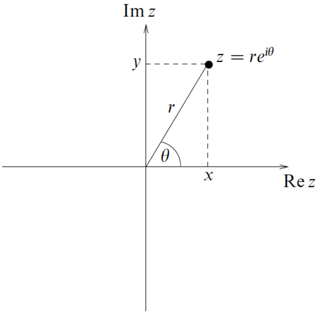
Argand diagram



Fundamental theorem of algebra

- **Fundamental theorem of algebra**: For a general polynomial $f(z)$, the equation $f(z) = 0$ will have exactly n solutions

Polar representation of a complex number



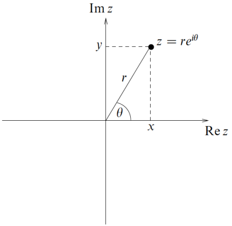
- Complex number may be represented in the **polar form**:

$$z = re^{i\theta},$$

where:

- the modulus $r = |z| = \sqrt{x^2 + y^2}$
- the argument $\theta = \arg z = \tan^{-1}(y/x)$

Polar representation of a complex number



- Complex number may be represented in the **polar form**:

$$z = re^{i\theta} \equiv re^{i(\theta+2n\pi)}$$

- The Euler's equation:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\Downarrow$$

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$

Manipulation of complex numbers

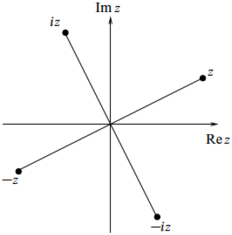
- **Addition** of two complex numbers gives another complex number:

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

- **Multiplication**:

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

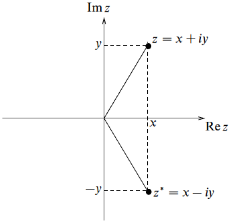
$$|z_1 z_2| = |z_1||z_2|$$



Complex conjugate and division

- For $z = x + iy$, the **complex conjugate** $z^* = x - iy$
- Complex conjugate of z is the number having the same magnitude as z that when multiplied by z leaves a real result:

$$zz^* = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$$



- **Division**:

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

Manipulation of complex numbers: Exercise

Let $z_1 = 3 - 2i$ and $z_2 = -1 + 4i$. Compute:

- $z_1 - z_2$
- $z_1 z_2$
- z_2^*
- z_1/z_2
- $|z_1|$
- $\arg z_2$

Further properties

- Multiplication and division in polar form:

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

- **De Moivre's theorem**:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

since $(e^{i\theta})^n = e^{in\theta}$

  - useful to prove trigonometric identities, solve polynomial equations with complex roots, etc.

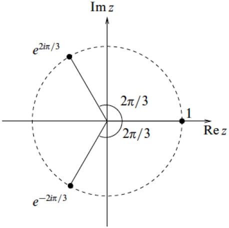
Solving polynomial equations

Task: Find the solutions to the equation $z^n = 1$

- From the fundamental theorem of algebra ⇒ the equation has n solutions
- We rewrite the equation as $z^n = e^{2ik\pi}$, where k is any integer
- Taking the nth root of each side, we find:

$$z = e^{2ik\pi/n},$$

where $k = 0, 1, 2, \ldots, n-1$



The solutions of $z^3 = 1$

Solving polynomial equations

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- If we take the cube root, we get: $z = 2^{1/3} e^{2ik\pi/3}$, $k = 0, 1, 2$
- The solutions are:

$$z_1 = 2^{1/3}, \quad z_2 = 2^{1/3} e^{2\pi i/3} = 2^{1/3}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right),$$

$$z_3 = 2^{1/3} e^{-2\pi i/3} = 2^{1/3}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right), \quad z_4 = 2i, \quad z_5 = -2i, \quad z_6 = 1$$

Solving polynomial equations

- Useful result: the roots of a polynomial with real coefficients occur in conjugate pairs

Applications to differentiation and integration

- We can use the exponential form of a complex number + de Moivre's theorem to simplify the differentiation of trigonometric functions

- **Example:** Find the derivative with respect to x of $e^{3x}\cos 4x$
  - Solution: $e^{3x}(3\cos 4x - 4\sin 4x)$

Functions of a complex variable = complex function

- The quantity $f(z)$ is said to be a **function of the complex variable** z if to every value of z in a certain domain S there corresponds one or more values of $f(z)$:

$$f(z) = u(x, y) + iv(x, y),$$

  where u and v are real and imaginary parts of $f(z)$, respectively

- We will only consider single-valued functions: for each value of z there corresponds just one value of $f(z)$

Functions of a complex variable: Example

- For the function $f(z) = 3z^2 + 7z$, we have:

$$f(x+iy) = 3(x+iy)^2 + 7(x+iy) = (3x^2 - 3y^2 + 7x) + i(6xy + 7y)$$

  - $u = 3x^2 - 3y^2 + 7x$, $v = 6xy + 7y$
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  - The domain is: $\mathbf{C} \setminus \{0\}$
  - The range is: $|z| = 1$

Differentiable functions

- A function $f(z)$ that is single-valued in some domain R is **differentiable** at the point z in R if the derivative

$$f'(z) = \lim_{\Delta z \to 0} \left[\frac{f(z + \Delta z) - f(z)}{\Delta z} \right]$$

exists and is unique, in that its value does not depend upon the direction in the Argand diagram from which Δz tends to zero

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  - $\Rightarrow f(z)$ is differentiable for all (finite) z

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where we suppose $\Delta z \to 0$ along a line through z of slope m, so that $\Delta y = m\Delta x$

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- The limit is dependent on m and hence on the direction from which $\Delta z \to 0$
- Since this conclusion is independent of the value of z, and hence true for all z, $f(z)$ is nowhere differentiable

Analytic function

- A function that is single-valued and differentiable at all points of a domain R is said to be **analytic** (or **regular**) in R
- A function may be analytic in a domain except at a finite number of points (or an infinite number if the domain is infinite)
  - In this case, it is said to be analytic except at these points, which are called the **singularities** of $f(z)$
- From the previous examples, we saw that for a function to be analytic, there must be some particular connection between its real and imaginary parts u and v

Analytic function

- **Cauchy-Riemann relations** = necessary conditions for $f(z)$ to be analytic:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

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  - Example: $f(z) = 2y + ix$

$$\frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = 1 \neq -2 = -\frac{\partial u}{\partial y}$$

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- **Sufficient** conditions for $f(z)$ to be analytic:
  1. Four partial derivatives exist,
  2. are continuous,
  3. satisfy the Cauchy-Riemann relations

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- For the first Cauchy-Neumann relation:

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- $f(z)$ is analytic only in the second and fourth quadrants

Further applications of complex variables

- Inverse Laplace transform
- Approximations to integrals
- ...
- For further reading: R. P. Agarwal, K. Perera, S. Pinelas, “An introduction to complex analysis,” *Springer*, 2010.