Mathematical Methods - Lecture 8

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Outline

Higher order ODEs

- Linear equations are of paramount importance in the description of physical processes
- When put into mathematical form, many natural processes appear as higher-order linear ODEs
  - ☺ most often as second-order equations

Linear ODEs

- A **linear ODE of general order** n has the form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

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- If $f(x) = 0$, the equation is called *homogeneous*
  - Otherwise, it is *inhomogeneous*

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- If $f(x) = 0$, the equation is called *homogeneous*
  - Otherwise, it is *inhomogeneous*

- The general solution will contain n arbitrary constants
  - May be determined if n boundary conditions are provided

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- To solve any equation of the form (1), we must:
  1. Find the general solution of the *complementary equation*, i.e. equation formed by setting $f(x) = 0$:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

  - Find n linearly independent functions $y_1(x), \ldots, y_n(x)$ that satisfy this equation
  - The general solution is given by their linear superposition:

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

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  - Find n linearly independent functions $y_1(x), \ldots, y_n(x)$ that satisfy (2)
  - The general solution is given by their linear superposition:

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

    where c_m are arbitrary constants $\Rightarrow$ determined from n boundary conditions

  - $y_c(x)$ is called the **complementary function** of (1)

Linear ODEs

- For n functions to be linearly independent over an interval, there must NOT exist any set of constants $c_1, \ldots, c_n$, such that:

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0$$

over the interval in question, except for the trivial case $c_1 = \cdots = c_n = 0$

Linear ODEs

- The n functions $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly independent over an interval if

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & \vdots \\ \vdots & & \ddots & \vdots \\ y_1^{(n-1)} & \ldots & \ldots & y_n^{(n-1)} \end{vmatrix} \neq 0$$

  over that interval

- $W(y_1, y_2, \ldots, y_n)$ is called the **Wronskian** of the set of functions

Linear ODEs

- A **linear ODE of general order** n has the form:

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- If the original equation (1) has $f(x) = 0$ (i.e. it is homogeneous)

$$\Downarrow$$

- The complementary function $y_c(x)$ is already the general solution

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- If the original equation (1) has $f(x) = 0$ (i.e. it is homogeneous)

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- The complementary function $y_c(x)$ is already the general solution
- If $f(x) \neq 0$, the general solution is given by:

$$y(x) = y_c(x) + y_p(x),$$

where $y_p(x)$ is the **particular solution**

  - can be any function that satisfies (1) directly, provided it is linearly independent of $y_c(x)$

Second-order linear differential equation

- The general second-order linear differential equation is given by:

$$\ddot{x} + p(t)\dot{x} + q(t)x = g(t) \tag{3}$$

where $\dot{x} = dx/dt$ and $\ddot{x} = d^2x/dt^2$

- A unique solution of (3) requires initial values $x(t_0) = x_0$ and $\dot{x}(t_0) = u_0$

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- A unique solution of (3) requires initial values $x(t_0) = x_0$ and $\dot{x}(t_0) = u_0$
- The equation *with constant coefficients* assumes that $p(t)$ and $q(t)$ are constants
- The equation is said to be **homogeneous** if $g(t) = 0$

The Euler method

- We first derive an algorithm for numerical solution
- Consider the general second-order ODE:

$$\ddot{x} = f(t, x, \dot{x})$$

- We can write this ODE as the first-order system by introducing $u = \dot{x}$:

$$\dot{x} = u \tag{4}$$
$$\dot{u} = f(t, x, u) \tag{5}$$

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- (4) gives the slope of the tangent line to the curve $x = x(t)$
- (5) gives the slope of the tangent line to the curve $u = u(t)$

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$$\dot{u} = f(t, x, u) \tag{5}$$

- Beginning at the initial values $(x, u) = (x_0, u_0)$ at time $t = t_0$, we move along the tangent lines to determine $x_1 = x(t_0 + \Delta t)$ and $u_1 = u(t_0 + \Delta t)$:

$$x_1 = x_0 + \Delta t u_0$$

$$u_1 = u_0 + \Delta t f(t_0, x_0, u_0)$$

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$$\dot{x} = u, \quad \dot{u} = f(t, x, u)$$

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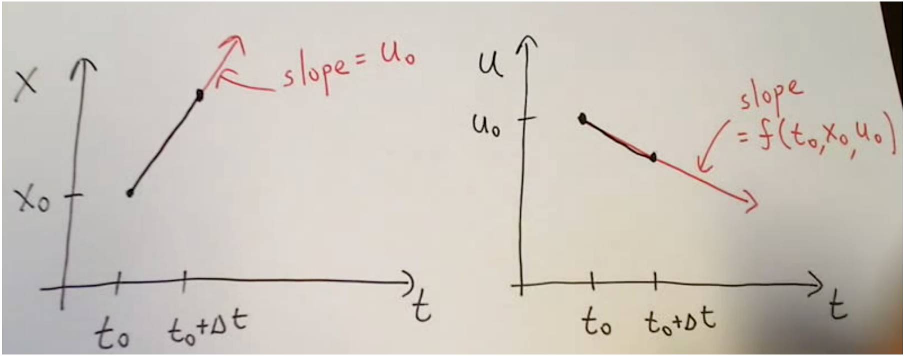
$$x_1 = x_0 + \Delta t u_0$$
$$u_1 = u_0 + \Delta t f(t_0, x_0, u_0)$$

- The values $t_1 = t_0 + \Delta t, x_1$ and u_1 are then used as new initial values to get the solution for t_2
- ...
- As $\Delta t \to 0$, the numerical solution converges to the unique solution

The Euler method

$$x_1 = x_0 + \Delta t u_0$$

$$u_1 = u_0 + \Delta t f(t_0, x_0, u_0)$$



The principle of superposition

- Consider the general second-order linear homogeneous ODE:

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0 \tag{6}$$

  and suppose that $x = X_1(t)$ and $x = X_2(t)$ are its solutions

- A linear combination of $X_1(t)$ and $X_2(t)$:

$$X(t) = c_1 X_1(t) + c_2 X_2(t), \quad c_1, c_2 = \mathrm{const}$$

- The **principle of superposition** states that $x = X(t)$ is also a solution of (6), i.e.:

Any linear combination of solutions to (6) is also a solution

The Wronskian

- Suppose we determined two solutions $x = X_1(t)$ and $x = X_2(t)$
- We then attempt to write the general solution as $X(t) = c_1 X_1(t) + c_2 X_2(t)$
- This general solution should be able to satisfy two initial conditions:

$$x(t_0) = x_0, \quad \dot{x}(t_0) = u_0$$

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- Applying these initial conditions, we get:

$$c_1 X_1(t_0) + c_2 X_2(t_0) = x_0$$

$$c_1 \dot{X}_1(t_0) + c_2 \dot{X}_2(t_0) = u_0$$

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- Solution of this system for unknowns c_1 and c_2 results in

$$c_1 = \frac{x_0 \dot{X}_2(t_0) - u_0 X_2(t_0)}{W}, \quad c_2 = \frac{u_0 X_1(t_0) - x_0 \dot{X}_1(t_0)}{W}$$

- W is called the **Wronskian** and is given by

$$W = X_1(t_0)\dot{X}_2(t_0) - \dot{X}_1(t_0) X_2(t_0)$$

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- The Wronskian must satisfy $W \neq 0$ for a solution to exist

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- The Wronskian must satisfy $W \neq 0$ for a solution to exist

- When $W \neq 0 \Rightarrow$ the two solutions are linearly independent!

2nd-order linear homogeneous ODE with const coefficients

- Let's solve a linear homogeneous, constant coefficient ODE:

$$a\ddot{x} + b\dot{x} + cx = 0, \quad a, b, c = \mathrm{const}$$

- *Applications:* position of a freely-oscillating frictional mass on a spring, damped pendulum, ...

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- *Possible solution:*
  1. Find 2 linearly independent solutions
  2. Construct a linear combination of solutions
  3. The two constants are used to satisfy two initial conditions

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- Our **ansatz**, or educated guess:

$$x = e^{rt} \quad \Rightarrow \dot{x} = re^{rt}, \ddot{x} = r^2 e^{rt}$$

where r is a constant to be determined

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- This yields:

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0 \quad \Rightarrow \quad ar^2 + br + c = 0,$$

which is a quadratic equation for the unknown constant r

2nd-order linear homogeneous ODE with const coefficients

- We solve $a\ddot{x} + b\dot{x} + cx = 0$
- $ar^2 + br + c = 0$ is called the **characteristic equation** of our ODE

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- The two solutions of this characteristic equation are:

$$r\pm = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)$$

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Three cases are possible:

1. $b^2 - 4ac > 0 \;\; \Rightarrow$ two roots are distinct and real
2. $b^2 - 4ac < 0 \;\; \Rightarrow$ two roots are distinct and complex conjugate of each other
3. $b^2 - 4ac = 0 \;\; \Rightarrow$ there is one real root

2-ord linear homog ODE w/const coef: real distinct roots

- We solve $a\ddot{x} + b\dot{x} + cx = 0$
- When $r_+ \neq r_-$ are real roots, the *general solution*:

$$x(t) = c_1 e^{r_+ t} + c_2 e^{r_- t}$$

- The unknown constants c_1 and c_2 can be determined by the initial conditions:

$$x(t_0) = x_0, \quad \dot{x}(t_0) = u_0$$

2-ord linear homog ODE w/const coef: real distinct roots

Example: Solve $\ddot{x} + 5\dot{x} + 6x = 0$ with $x(0) = 2$, $\dot{x}(0) = 3$

- We take as our ansatz $x = e^{rt}$ and obtain the characteristic equation:

$$r^2 + 5r + 6 = 0$$

- It factors to $(r+3)(r+2) = 0$
- The general solution to the ODE is:

$$x(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

- The solution for $\dot{x}$:

$$\dot{x}(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$$

2-ord linear homog ODE w/const coef: real distinct roots

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$$c_1 + c_2 = 2$$

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- $c_1 = 9$

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- $c_1 = 9$
- $c_2 = 2 - c_1 = -7$

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$$c_1 + c_2 = 2$$

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- $c_1 = 9$
- $c_2 = 2 - c_1 = -7$
- The unique solution satisfying both ODE and the initial conditions:

$$x(t) = 9e^{-2t} - 7e^{-3t}$$

2-ord linear homog ODE w/const coef: complex conj roots

- We solve $a\ddot{x} + b\dot{x} + cx = 0$
- If $b^2 - 4ac < 0$, the roots occur as complex conjugate pairs

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$$\lambda = -\frac{b}{2a}, \quad \mu = \frac{1}{2a}\sqrt{4ac - b^2},$$

the two roots are $\lambda + i\mu$ and $\lambda - i\mu$

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the two roots are $\lambda + i\mu$ and $\lambda - i\mu$
- Two complex exponential solutions to our ODE:
$$Z_1(t) = e^{\lambda t}e^{i\mu t}, \quad Z_2(t) = e^{\lambda t}e^{-i\mu t}$$

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- Two complex exponential solutions to our ODE:
$$Z_1(t) = e^{\lambda t} e^{i\mu t}, \quad Z_2(t) = e^{\lambda t} e^{-i\mu t}$$
- Any linear combination of Z_1 and Z_2 is also a solution to our ODE

2-ord linear homog ODE w/const coef: complex conj roots

- Two complex exponential solutions to our ODE:

$$Z_1(t) = e^{\lambda t} e^{i\mu t}, \quad Z_2(t) = e^{\lambda t} e^{-i\mu t}$$

- We can form two different **linear combinations** that are **real**:

$$X_1(x) = \frac{Z_1 + Z_2}{2} = e^{\lambda t} \left(\frac{e^{i\mu t} + e^{-i\mu t}}{2} \right) = e^{\lambda t} \cos \mu t,$$

$$X_2(x) = \frac{Z_1 - Z_2}{2i} = e^{\lambda t} \left(\frac{e^{i\mu t} - e^{-i\mu t}}{2i} \right) = e^{\lambda t} \sin \mu t$$

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$$X_2(x) = \frac{Z_1 - Z_2}{2i} = e^{\lambda t}\left(\frac{e^{i\mu t} - e^{-i\mu t}}{2i}\right) = e^{\lambda t}\sin \mu t$$

- Apply again the principle of superposition to get the **general solution**:

$$x(t) = e^{\lambda t}(A\cos \mu t + B\sin \mu t)$$

2-ord linear homog ODE w/const coef: complex conj roots

Example: Solve $\ddot{x} + x = 0$ with $x(0) = x_0$ and $\dot{x}(0) = u_0$

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- The characteristic equation is:

$$r^2 + 1 = 0$$

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- The derivative is:

$$\dot{x}(t) = -A\sin t + B\cos t$$

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- Applying the initial conditions:

$$x(0) = A = x_0, \quad \dot{x}(0) = B = u_0$$

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$$x(0) = A = x_0, \quad \dot{x}(0) = B = u_0$$

- The final solution is:

$$\boxed{x(t) = x_0\cos t + u_0\sin t}$$

2-ord linear homog ODE w/const coef: repeated roots

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- **Result to remember:** For the case of repeated roots, the second solution is t times the first solution

2-ord linear homog ODE w/const coef: repeated roots

Example: Solve $\ddot{x} + 2\dot{x} + x = 0$ with $x(0) = 1$ and $\dot{x}(0) = 0$

- The characteristic equation:

$$r^2 + 2r + 1 = (r+1)^2 = 0$$

- A repeated root $r = -1$
- The general solution:

$$x(t) = c_1 e^{-t} + c_2 t e^{-t}$$

- The derivative: $\dot{x}(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t}$
- Applying the initial conditions:

$$c_1 = 1$$

$$-c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = 1$$

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- The solution is:

$$\boxed{x(t) = (1 + t)e^{-t}}$$

Second-order linear inhomogeneous ODE

- The **second-order linear inhomogeneous** ODE:

$$\ddot{x} + p(t)\dot{x} + q(t)x = g(t) \tag{7}$$

with initial conditions $x(t_0) = x_0$ and $\dot{x}(0) = u_0$

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- Four-step solution:

1. Find the general solution of the homogeneous equation

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0$$

   Let us denote this solution by:

$$x_c(t) = c_1 X_1(t) + c_2 X_2(t), \quad c_1, c_2 = \text{const}$$

2. Find any particular solution x_p of (7)
3. Write the general solution of (7) as: $x(t) = x_c(t) + x_p(t)$
4. Apply the initial conditions to determine the constants

Second-order linear inhomogeneous ODE

Example: Solve $\ddot{x} - 3\dot{x} - 4x = 3e^{2t}$ with $x(0) = 1$ and $\dot{x}(0) = 0$

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- $c_1 = 1/2, \ c_2 = 1$
- The solution that satisfies both the ODE and initial conditions:

$$x(t) = \frac{1}{2}e^{4t} - \frac{1}{2}e^{2t} + e^{-t}$$

Using Laplace transform to solve ODEs

- The properties of the Laplace transform make it a great tool to solve ODEs:

1. Linearity:

$$a\ddot{x} + b\dot{x} + cx = g(t)$$

$$a\mathcal{L}\{\ddot{x}\} + b\mathcal{L}\{\dot{x}\} + c\mathcal{L}\{x\} = \mathcal{L}\{g\}$$

2. The first derivative:

$$\mathcal{L}\{\dot{x}\} = s\bar{f}(s) - x(0)$$

3. The second derivative:

$$\mathcal{L}\{\ddot{x}\} = s^2\bar{f}(s) - sx(0) - \dot{x}(0)$$

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- Taking the Laplace transform of both sides of the ODE:

$$[s^2\bar{f}(s) - sx(0) - \dot{x}(0)] - [s\bar{f}(s) - x(0)] - [2\bar{f}(s)] = 0$$

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$$\bar{f}(s) = \frac{(s-1)x(0) + \dot{x}(0)}{s^2 - s - 2}$$

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$$\bar{f}(s) = \frac{s-1}{(s-2)(s+1)}$$

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- We now need to compute the inverse Laplace transform $x(t) = \mathcal{L}^{-1}\{\bar{f}(s)\}$

Using Laplace transform to solve ODEs

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- The Laplace transformed solution:

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- Direct inversion is not possible ⇒ we can use a partial fraction expansion:

$$\frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1}$$

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  - Using the cover-up method, we multiply both sides by $s-2$ and put $s=2$:

$$a = \frac{s-1}{s+1}\Big]_{s=2} = \frac{1}{3}, \quad \text{similarly:} \quad b = \frac{s-1}{s-2}\Big]_{s=-1} = \frac{2}{3}$$

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- The solution to the ODE:

$$x(t) = \mathcal{L}^{-1}\{\bar{f}(s)\} = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$