Mathematical Methods - Lecture 7

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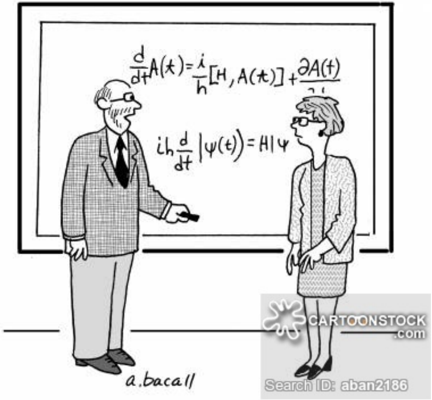
Outline

What is a differential equation?



"I can understand Heisenberg's equation and Schrodinger's equation for quantum mechanics but I cannot understand derivative trading."

What is a differential equation?

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- An **ordinary differential equation (ODE)**:
    - contains only ordinary derivatives (no partial derivatives)
    - describes the relationship between these derivatives of the *dependent variable* (ex: y) with respect to the *independent variable* (ex: x)

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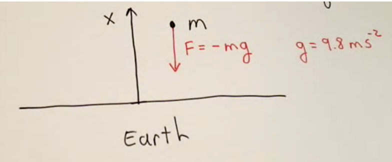
- An **ordinary differential equation (ODE)**:
  - contains only ordinary derivatives (no partial derivatives)
  - describes the relationship between these derivatives of the *dependent variable* (ex: y) with respect to the *independent variable* (ex: x)
- The solution to such an ODE is a function of x: $y(x)$

The simplest type of differential equation



- The simplest ODEs have the form:

$$\frac{d^n x}{dt^n} = G(t),$$

where $G(t)$ depends only on the independent variable t

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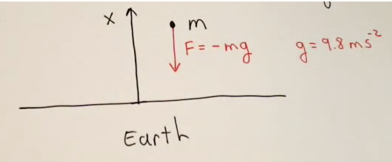

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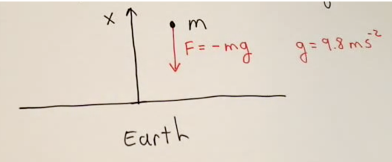
- Newton's law: $F = ma \Rightarrow$

$$m\frac{d^2 x}{dt^2} = -mg,$$

x = height of the object over the ground, m = its mass,
g = constant gravitational acceleration

The simplest type of differential equation - Example



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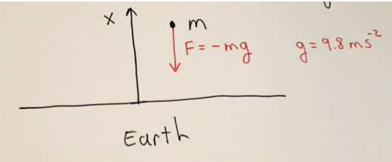

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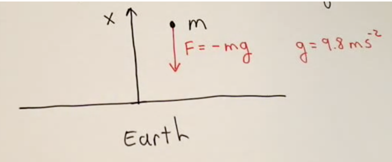
- The first integration yields:

$$\frac{dx}{dt} = A - gt,$$

with A the *constant of integration*

The simplest type of differential equation - Example



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- The first integration yields:

$$\frac{dx}{dt} = A - gt,$$

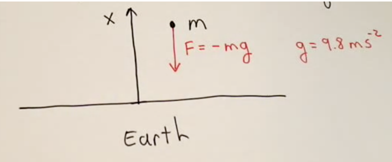
  with A the *constant of integration*

- The second integration yields the general solution:

$$x = B + At - \frac{1}{2}gt^2,$$

  with B the second constant of integration

The simplest type of differential equation - Example

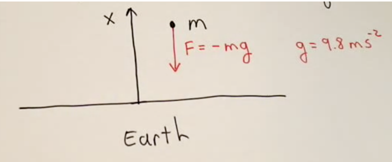


$$x = B + At - \frac{1}{2}gt^2$$

- Two constants of integration A and B can be found from *initial conditions*:

$$x(0) = x_0, ; \quad \frac{dx}{dt}(0) = v_0$$

The simplest type of differential equation - Example



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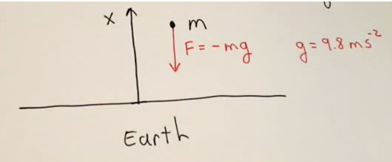
$$x(0) = x_0 = B$$

The simplest type of differential equation - Example



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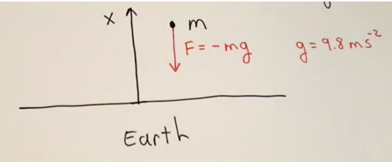
$$\frac{dx}{dt}(0) = v_0 = A$$

The simplest type of differential equation - Example



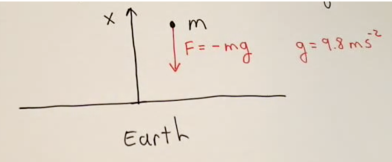
- **General** solution:

$$x = B + At - \frac{1}{2}gt^2$$

- **Particular** solution (unique solution that satisfies both the ODE and the initial conditions):

$$x(t) = x_0 + v_0 t - \frac{1}{2}gt^2$$

The simplest type of differential equation - Example



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- **Particular** solution (unique solution that satisfies both the ODE and the initial conditions):

$$x(t) = x_0 + v_0 t - \frac{1}{2}gt^2$$

- Example: We drop a ball off from the top of a 50 meter building with $v_0 = 0$. When will the ball hit the ground?

Order of differential equations

- The **order** of an ODE is the order of the highest derivative it contains

$$\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2}$$

$$\frac{d^3y}{dx^3}$$

First-order differential equations

- The general first-degree **first-order** differential equation for the function $y = y(x)$ can be written in either of two standard forms:

$$\frac{dy}{dx} = f(x, y), \quad A(x, y)dx + B(x, y)dy = 0,$$

$f(x, y), A(x, y), B(x, y)$ are in general functions of both x and y

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- **How to solve?**

The Euler method

- The general **first-order** differential equation for the function $y = y(x)$:

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

The Euler method

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$$\frac{dy}{dx} = f(x, y) \quad (1)$$

☹ It is not always possible to find an analytical solution of (1) for $y = y(x)$

The Euler method

- The general **first-order** differential equation for the function $y = y(x)$:

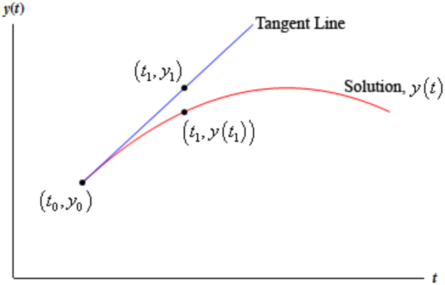
$$\frac{dy}{dx} = f(x, y) \tag{1}$$

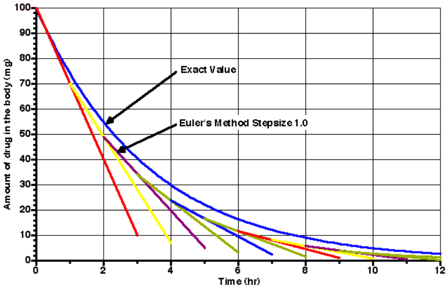
☹ It is not always possible to find an analytical solution of (1) for $y = y(x)$

☺ It is always possible to determine a unique **NUMERICAL** solution given:

- an initial vaue $y(x_0) = y_0$
- provided $f(x, y)$ is a well-behaved function

The Euler method = the simplest Runge-Kutta method





- ODE $\frac{dy}{dx} = f(x, y)$ gives the slope $f(x_0, y_0)$ of the tangent line to the solution curve $y = y(x)$ at the point (x_0, y_0):

$$\frac{dy}{dx}(0) = \lim_{\Delta x \to 0} \frac{y(x_0 + \Delta x) - y(x_0)}{\Delta x} = f(x_0, y_0)$$

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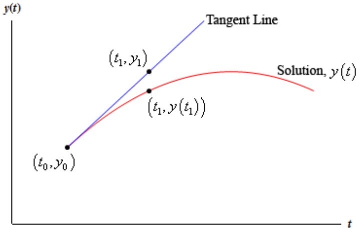
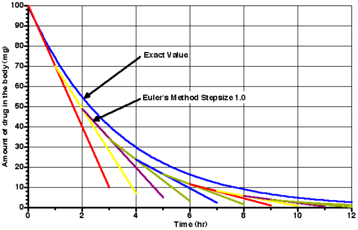
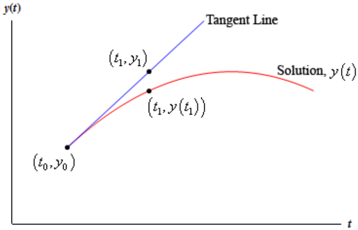
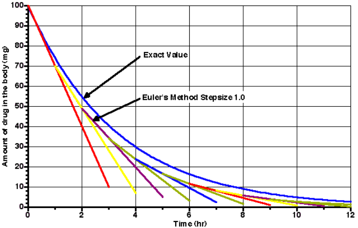




1. The first point: (x_0, y_0)

The Euler method = the simplest Runge-Kutta method

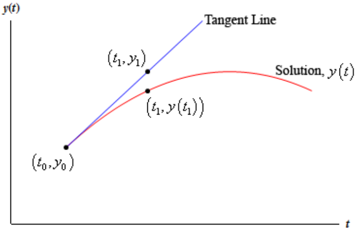


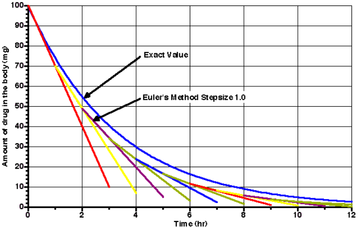


1. The first point: (x_0, y_0)
2. The next point is obtained by choosing a small Δx, and computing the next y-coordinate (along the tangent line):

$$x = x_0 + \Delta x, \quad y(x_0 + \Delta x) = y(x_0) + \Delta x f(x_0, y_0)$$

The Euler method = the simplest Runge-Kutta method





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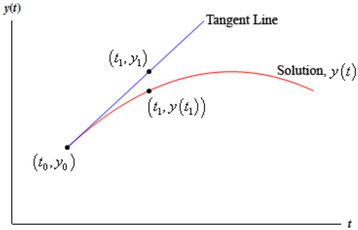
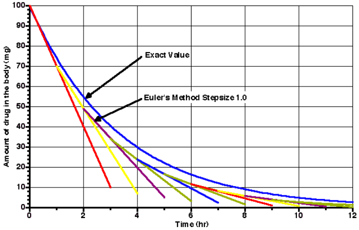
3. $(x_0 + \Delta x, y_0 + \Delta y)$ becomes the initial condition and we repeat step 2

The Euler method = the simplest Runge-Kutta method





- For small enough Δx, the numerical solution converges to the exact solution!

Analytical solution

- The general **first-order** differential equation for the function $y = y(x)$:

$$\frac{dy}{dx} = f(x, y),$$

- Some special forms of this equation can be solved analytically:
  - separable equations
  - exact equations
  - inexact equations
  - linear equations
  - ...

Separable equations

- A first-order ODE is **separable** if it can be written in the form:

$$g(y)\frac{dy}{dx} = f(x)$$

or

$$\frac{dy}{dx} = f(x)g(y)$$

where $f(x)$ is independent of y and $g(y)$ is indepedent of x

Separable equations

A first-order ODE is **separable** if it can be written in the form:

$$g(y)\frac{dy}{dx} = f(x)$$

1. Rearrange (factorise if needed) the equation so that the terms depending on x and y appear on opposite sides:

$$g(y)dy = f(x)dx$$

2. Integrate:

$$\int g(y)dy = \int f(x)dx$$

Separable equations - Example

Solve:

$$\frac{dy}{dx} = x + xy$$

Separable equations - Example

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- Since $x + xy = x(1 + y)$:

$$\int \frac{dy}{1 + y} = \int x dx$$

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$$\Downarrow$$

$$\ln(1+y) = \frac{x^2}{2} + c, \quad c = \mathrm{const}$$

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$$\Downarrow$$

$$\ln(1+y) = \frac{x^2}{2} + c, \quad c = \text{const}$$

$$\Downarrow$$

$$1 + y = \exp\left(\frac{x^2}{2} + c\right) = A \exp\left(\frac{x^2}{2}\right), \quad A = \text{const}$$

Separable equations - Example

Solve:

$$\frac{dy}{dx} = \frac{2\cos 2x}{3+2y}, \quad y(0) = -1$$

Separable equations - Example

Solve:

$$\frac{dy}{dx} = \frac{2\cos 2x}{3+2y}, \quad y(0) = -1$$

Solution:

$$y(x) = -\frac{3}{2} + \frac{1}{2}\sqrt{1+4\sin 2x}$$

Linear equations

- The first-order **linear differential equation** (linear in y and its derivative) can be written in the form:

$$\frac{dy}{dx} + p(x)y = g(x)$$

with (optional) the initial condition $y(x_0) = y_0$

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- These equations can be integrated using an integrating factor $\mu(x)$

Linear equations

$$\frac{dy}{dx} + p(x)y = g(x) \quad \Rightarrow \quad \mu(x)\left[\frac{dy}{dx} + p(x)y\right] = \mu(x)g(x)$$

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$$\mu(x)\left[\frac{dy}{dx} + p(x)y\right] = \frac{d}{dx}[\mu(x)y]$$

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- Using $\mu(x_0) = \mu_0$ and $y(x_0) = y_0$:

$$\mu(x)y - \mu_0 y_0 = \int_{x_0}^{x} \mu(x)g(x)dx$$

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$$\Downarrow$$

$$y = \frac{1}{\mu(x)}\left(\mu_0 y_0 + \int_{x_0}^{x} \mu(x)g(x)dx\right)$$

Linear equations

- Next step: Determine $\mu(x)$ from $\mu(x)\left[\frac{dy}{dx}+p(x)y\right]=\frac{d}{dx}[\mu(x)y]$

Linear equations

- Next step: Determine $\mu(x)$ from $\mu(x)\left[\frac{dy}{dx} + p(x)y\right] = \frac{d}{dx}[\mu(x)y]$
- Using product rule for differentiation:

$$\mu\frac{dy}{dx} + p\mu y = \frac{d\mu}{dx}y + \mu\frac{dy}{dx}$$

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$$\Downarrow$$

$$\frac{d\mu}{dx}=p\mu$$

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$$\Downarrow$$

$$\frac{d\mu}{dx} = p\mu$$

- This equation is separable:

$$\int_{\mu_0}^{\mu}\frac{d\mu}{\mu} = \int_{x_0}^{x}p(x)dx$$

Linear equations

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$$\ln\frac{\mu}{\mu_0} = \int_{x_0}^{x} p(x)dx \quad \Rightarrow \quad \mu(x) = \mu_0 \exp\left(\int_{x_0}^{x} p(x)dx\right)$$

Linear equations

- The first-order linear differential equation:

$$\frac{dy}{dx} + p(x)y = g(x)$$

- Its solution satisfying the initial condition $y(x_0) = y_0$ is written as:

$$y = \frac{1}{\mu(x)}\left(y_0 + \int_{x_0}^{x} \mu(x)g(x)dx\right)$$

with

$$\mu(x) = \exp\left(\int_{x_0}^{x} p(x)dx\right)$$

- **Frequent use in applied mathematics!**

Linear equations - Example

Solve: $\dfrac{dy}{dx} + 2y = e^{-x}$, with $y(0) = 3/4$

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Linear equations - Example

Solve: $\frac{dy}{dx} + 2y = e^{-x}$, with $y(0) = 3/4$

- This equation is not separable
- With $p(x) = 2$ and $g(x) = e^{-x}$, we have:

$$\mu(x) = \exp\left(\int_0^x 2dx\right)$$

Linear equations - Example

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$$= e^{2x}$$

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and

$$y = e^{-2x}\left(\frac{3}{4} + \int_0^x e^{2x}e^{-x}dx\right) = e^{-2x}\left(\frac{3}{4} + \int_0^x e^{x}dx\right)$$

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$$= e^{-2x}\left(\frac{3}{4} + (e^x - 1)\right) = e^{-2x}\left(e^x - \frac{1}{4}\right)$$

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$$= e^{-2x}\left(\frac{3}{4} + (e^x - 1)\right) = e^{-2x}\left(e^x - \frac{1}{4}\right)$$

$$e^{-x}\left(1 - \frac{1}{4}e^{-x}\right)$$

Example 2

Solve: $\dfrac{dy}{dx} - 2xy = x$, with $y(0) = 0$

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Solve: $\frac{dy}{dx} - 2xy = x$, with $y(0) = 0$

- Solution:

$$y = \frac{1}{2}\left(e^{x^2} - 1\right)$$

Bernoulli's equation

- **Bernoulli's equation** has the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \quad \text{where } n \neq 0 \text{ or } 1$$

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- **Bernoulli's equation** has the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \quad \text{where } n \neq 0 \text{ or } 1$$

- The equation can be made linear by substituting $v = y^{1-n}$ and correspondingly:

$$\frac{dy}{dx} = \left(\frac{y^n}{1-n}\right)\frac{dv}{dx}$$

- Bernoulli's equation becomes linear:

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

Business analytics application: compound interest

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  - The interest awarded after 1 month is $r\Delta tS = (0.06/12) \times \$10000 = \$50$

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- It can be solved with the initial condition $S(0) = S_0$
  - S_0 = initial capital

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Business analytics application: compound interest

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$$\frac{dS}{dt} = rS + k$$

- The solution:

$$S = S_0 e^{rt} + \frac{k}{r} e^{rt} (1 - e^{-rt})$$

  - The first term on the right-hand side comes from the initial invested capital
  - The second term comes from deposits (or withdrawals)

- Compounding results in the exponential growth of an investment

Business analytics application: compound interest

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- **Exercise:** A 25-year old plans to set aside a fixed amount every year, invests at a real return of 6%, and retires at age 65. How much must he invest every year to have \$8,000,000 at retirement?

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- Solution:

$$k = \frac{rS(t)}{e^{rt} - 1}$$

$$k = \frac{0.06 \times 8,000,000}{e^{0.06 \times 40} - 1} = \$47,889 \mathrm{year}^{-1}$$