Mathematical Methods - Lecture 5

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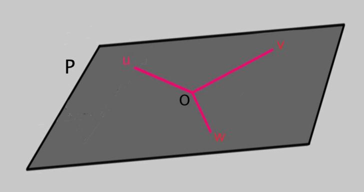
Outline

1. Linear Independence
2. Basis and Dimension
3. Eigenvalues and Eigenvectors

Linear dependence

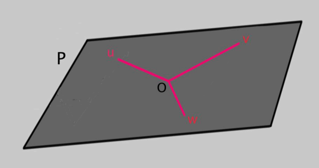


- A plane P includes the origin in $\mathbb{R}^3$ and a collection $\{u, v, w\}$ of non-zero vectors in P
- Any two vectors determine a plane
- Then there exist constants d^1, d^2 (not both zero) such that:

$$w = d^1 u + d^2 v$$

- Since w can be expressed in terms of u and v $\Rightarrow$ we say it is not independent

Linear independence



- More generally, the relationship

$$c^1 u + c^2 v + c^3 w = 0 \quad c^i \in \mathbb{R}^3, \mathrm{some}\ c^i \neq 0$$

expresses the fact that u, v, w are not all independent

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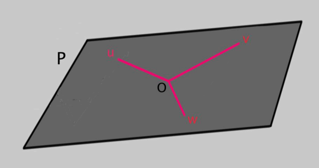

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expresses the fact that u, v, w are not all independent

- **Definition:** The vectors $v_1, v_2, \ldots, v_n$ are *linearly dependent* if there are constants $c^1, \ldots, c^n$ not all zero such that

$$c^1 v_1 + c^2 v_2 + \ldots + c^n v_n = 0$$

- Otherwise, the vectors $v_1, v_2, \ldots, v_n$ are *linearly independent*

Linear independence - Example

Are the following vectors in $\mathbb{R}^3$ linearly independent?

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

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- This system has solutions if and only if $M = (v_1 v_2 v_3)$ is singular
- $\det M =$

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- det M =0
- Nontrivial solutions exist $\Rightarrow$ vectors are linearly dependent

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- $\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right)$

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- $c^3 = \mu, c^2 = -\mu, c^1 = -2\mu$

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- $\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$
- $c^3 = \mu, c^2 = -\mu, c^1 = -2\mu$
- Any choice of μ will satisfy the linear equation
- $\mu = 1 \Rightarrow -2v_1 - v_2 + v_3 = 0$

Linear dependence theorem

- **Theorem:** A set of non-zero vectors $\{v_1, \ldots, v_n\}$ is linearly dependent if and only if:
  one of the vectors v_k is expressible as a linear combination of the preceding vectors

Linear independence - Example

Consider the vector space $P_2(t)$ of polynomials of degree less than or equal to 2. Set:

$$v_1 = 1 + t$$

$$v_2 = 1 + t^2$$

$$v_3 = t + t^2$$

$$v_4 = 2 + t + t^2$$

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- $v_4 = v_1 + v_2 \Rightarrow$ the set $\{v_1, \ldots, v_4\}$ is linearly dependent

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$$v_2 = 1 + t^2$$
$$v_3 = t + t^2$$
$$v_4 = 2 + t + t^2$$

- $v_4 = v_1 + v_2 \Rightarrow$ the set $\{v_1, \dots, v_4\}$ is linearly dependent
- The vectors $\{1 + t, 1 + t^2, t + t^2\}$ are linearly independent, and span the vector space S
- These vectors are a minimal spanning set, called a **basis** for S

Basis and dimension

- Let V be a vector space. A set S is a **basis** for V if:
  - S is linearly independent
  - $V = \mathrm{span} S$
- If S is a basis of V and S has only finitely many elements, then V is *finite-dimensional*
- The number of vectors in S is the *dimension* of V
  - The dimension of a vector space does not depend on the basis

Basis and dimension

- **Theorem:** Let $S = \{v_1, \ldots, v_n\}$ be a basis for a vector space V. Then every vector $w \in V$ can be written uniquely as a *linear combination of vectors in the basis S*:

$$w = c^1 v_1 + \cdots + c^n v_n$$

- **Lemma:** If $S = \{v_1, \ldots, v_n\}$ is a basis for a vector space V and $T = \{w_1, \ldots, w_m\}$ is a linearly independent set of vectors in V, then $m \leq n$
- For a finite-dimensional vector space V, any two bases have the same number of vectors

Standard or canonical basis for $\mathbb{R}^n$

- $\mathbb{R}^n = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$
- This basis is called the **standard or canonical basis for $\mathbb{R}^n$**
- The vector with a one in the ith position and zeros everywhere else $= e_i$

Standard or canonical basis for $\mathbb{R}^n$

- **Theorem:** Let $S = \{v_1, \ldots, v_m\}$ be a collection of vectors in $\mathbb{R}^n$. Let M be the matrix whose columns are the vectors in S. Then S is a basis for V if and only if m is the dimension of V and

$$\det M \neq 0$$

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Examples:

- $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$
- $T = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

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Examples:

- $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \Rightarrow \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \Rightarrow S$ is a basis for $\mathbb{R}^2$

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Standard or canonical basis for $\mathbb{R}^n$

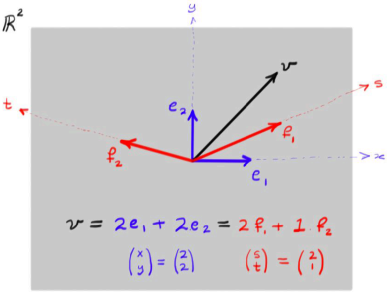
- **Theorem:** Let $S = \{v_1, \ldots, v_m\}$ be a collection of vectors in $\mathbb{R}^n$. Let M be the matrix whose columns are the vectors in S. Then S is a basis for V if and only if m is the dimension of V and

$$\det M \neq 0$$

Examples:

- $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \Rightarrow \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \Rightarrow S$ is a basis for $\mathbb{R}^2$
- $T = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \Rightarrow \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2 \Rightarrow T$ is a basis for $\mathbb{R}^2$

Components of a vector



- Vectors - elements which really exist - can be described in many ways
- In the basis $\{e_1, e_2\}$ it is the ordered pair $(x, y) = (2, 2)$
- In the basis $\{f_1, f_2\}$ it is the ordered pair $(s, t) = (2, 1)$
- Things like coordinate axes and "components of a vector" (x, y) are just math tools

Components of a vector

- Let V be a vector space with basis $S = \{e_1, \ldots, e_n\}$
- There exist constants v^i such that any vector $v \in V$ can be written as:
$$v = v^1 e_1 + v^2 e_2 + \cdots + v^n e_n$$
- The coefficients $v^1, \ldots, v^n$ are called the **components** of v in the basis $\{e_1, \ldots, e_n\}$
- It is often convenient to arrange the components v^i in a column vector:
$$v = \begin{pmatrix} e_1 & e_2 & \cdots & e_n \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix}$$

Matrix of the linear transformation - Example

- Consider a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$:

$$L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}, \quad L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

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- $L\begin{pmatrix} x \\ y \end{pmatrix} = L(x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

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- $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the matrix of L in the basis $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$

Invariant directions - Example

- Consider the linear transformation L:

$$L\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}-4\\-10\end{pmatrix},\quad L\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}3\\7\end{pmatrix}$$

- The matrix of L is $\begin{pmatrix}-4 & 3\\-10 & 7\end{pmatrix}$
- L applied to $\begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$ changes both the direction and the magnitude of the given vectors

Invariant directions - Example

- The matrix of L is $\begin{pmatrix} -4 & 3 \\ -10 & 7 \end{pmatrix}$
- $L\begin{pmatrix} 3 \\ 5 \end{pmatrix} =$

Invariant directions - Example

- The matrix of L is $\begin{pmatrix} -4 & 3 \\ -10 & 7 \end{pmatrix}$
- $L\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \cdot 3 + 3 \cdot 5 \\ -10 \cdot 3 + 7 \cdot 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

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- L fixes the direction of the vector $v_1 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

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- L fixes the direction of the vector $v_1 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$
- $L\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \cdot 1 + 3 \cdot 2 \\ -10 \cdot 1 + 7 \cdot 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$
- L fixes the direction of the vector $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ but stretches v_2 by a factor of 2

Eigenvectors and eigenvalues

- Given a linear transformation L it is sometimes possible to find a vector $v \neq 0$ and constant $\lambda \neq 0$ such that:

$$L(v) = \lambda v$$

- We call the direction of v an **invariant direction**
- Any vector pointing in the same direction satisfies: $L(cv) = \lambda cv$
- The vector v is called an **eigenvector** of L
- λ is called an **eigenvalue**

Eigenvectors and eigenvalues

- For $L = \begin{pmatrix} -4 & 3 \\ -10 & 7 \end{pmatrix}$:

- Eigenvectors are: $v_1 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; eigenvalues are 1 and 2

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- Suppose $w = rv_1 + sv_2$ for some constants r and s. Then:

$$L(w) = L(rv_1 + sv_2) = rL(v_1) + sL(v_2) = rv_1 + 2sv_2$$

- $L \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix}$

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- $L \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix}$

- Here, L is a very simple *diagonal* matrix

- This process is called **diagonalization**

Eigenvectors and eigenvalues

- How do we find eigenvectors and their eigenvalues?

Finding eigenvectors and eigenvalues - Example

Let $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : L(x, y) = (2x + 2y, 16x + 6y)$

Finding eigenvectors and eigenvalues - Example

Let $L : \mathbb{R}^2 \to \mathbb{R}^2 : L(x, y) = (2x + 2y, 16x + 6y)$

- The matrix of L : $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2 & 2 \\ 16 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

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- We want to find an invariant direction $v = \begin{pmatrix} x \\ y \end{pmatrix}$ such that

$$\begin{pmatrix} 2 & 2 \\ 16 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

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$$\Leftrightarrow \begin{pmatrix} 2 - \lambda & 2 \\ 16 & 6 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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$$\Leftrightarrow \begin{pmatrix} 2-\lambda & 2 \\ 16 & 6-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- This is a homogeneous system $\Leftrightarrow$ it only has solutions when $\begin{pmatrix} 2-\lambda & 2 \\ 16 & 6-\lambda \end{pmatrix}$ is singular

Finding eigenvectors and eigenvalues - Example

Let $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : L(x, y) = (2x + 2y, 16x + 6y)$

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Finding eigenvectors and eigenvalues - Example

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  $\Leftrightarrow (2-\lambda)(6-\lambda) - 32 = 0$

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  $\Leftrightarrow \lambda^2 - 8\lambda - 20 = 0$

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- For any square $n \times n$ matrix M, the polynomial in λ given by

$$P_M(\lambda) = \det(\lambda I - M) = (-1)^n \det(M - \lambda I)$$

  is called the **characteristic polynomial** of M.
  Its roots are eigenvalues of M

Finding eigenvectors and eigenvalues - Example

Let $L : \mathbb{R}^2 \to \mathbb{R}^2 : L(x, y) = (2x + 2y, 16x + 6y)$

- $\det \begin{pmatrix} 2-\lambda & 2 \\ 16 & 6-\lambda \end{pmatrix} = 0$

  $\Leftrightarrow (2-\lambda)(6-\lambda) - 32 = 0$

  $\Leftrightarrow \lambda^2 - 8\lambda - 20 = 0 \Leftrightarrow (\lambda - 10)(\lambda + 2) = 0$

- For any square $n \times n$ matrix M, the polynomial in λ given by

$$P_M(\lambda) = \det(\lambda I - M) = (-1)^n \det(M - \lambda I)$$

  is called the **characteristic polynomial** of M.
  Its roots are eigenvalues of M

- $\lambda_1 = 10$, $\lambda_2 = -2$

Finding eigenvectors and eigenvalues - Example

Let $L : \mathbb{R}^2 \to \mathbb{R}^2 : L(x,y) = (2x + 2y, 16x + 6y)$

- To find eigenvectors, solve $\begin{pmatrix} 2-\lambda & 2 \\ 16 & 6-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for each λ

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  - Both equations say that $y = 4x$
  - Setting $x = 1, v_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$
- $\lambda = -2 : \begin{pmatrix} 4 & 2 \\ 16 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
  - $y = -2x \quad \Rightarrow \quad v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Finding eigenvectors and eigenvalues

To find eigenvalues and eigenvectors:

1. Find the characteristic polynomial of the matrix M for L, given by $\det(\lambda I - M) = (-1)^n \det(M - \lambda I)$
2. Find the roots of the characteristic polynomial = eigenvalues of L
    - Could be real or complex or zero, and they need not all be different
3. For each λ_i, solve the linear system $(M - \lambda_i I)v = 0$ to obtain an eigenvector v associated to λ_i