Mathematical Methods - Lecture 4

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Outline

What is determinant?

Given a square matrix, how to know if it is invertible?

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  - M is invertible if and only if $m_1^1 m_2^2 - m_2^1 m_1^2 \neq 0$
- **Determinant** $\det M = |M| = \det \begin{pmatrix} m_1^1 & m_2^1 \\ m_1^2 & m_2^2 \end{pmatrix} = m_1^1 m_2^2 - m_2^1 m_1^2$

Determinant of a 3×3 matrix

- $|M| = \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} =$
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- A **minor** of an $n \times n$ matrix m is the determinant of any square matrix obtained from M by deleting rows and columns

- The determinant can be computed using (Laplace) **expansion by minors**

Determinant of an $n \times n$ matrix

- The Leibniz formula for the determinant of an $n \times n$ matrix M:

$$\det M = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} m^{i}_{\sigma_i}$$

- The sum is computed over all *permutations* σ of the set $\{1, 2, \ldots, n\}$.
- Example: $\sigma = [2, 3, 1]$ with $\sigma_1 = 2$, $\sigma_2 = 3$ and $\sigma_3 = 1$
- $\operatorname{sgn}(\sigma) = 1$ if σ is achieved by an even number of interchanges, and $\operatorname{sgn}(\sigma) = -1$ otherwise

Determinant of an $n \times n$ matrix - Example

$$\begin{vmatrix} m_1^1 & m_2^1 & m_3^1 \\ m_1^2 & m_2^2 & m_3^2 \\ m_1^3 & m_2^3 & m_3^3 \end{vmatrix} = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} m_{\sigma_i}^i =$$

$$\operatorname{sgn}([1,2,3]) \prod_{i=1}^{n} m_{[1,2,3]_i}^i + \operatorname{sgn}([1,3,2]) \prod_{i=1}^{n} m_{[1,3,2]_i}^i + \operatorname{sgn}([2,1,3]) \prod_{i=1}^{n} m_{[2,1,3]_i}^i +$$

$$\operatorname{sgn}([2,3,1]) \prod_{i=1}^{n} m_{[2,3,1]_i}^i + \operatorname{sgn}([3,1,2]) \prod_{i=1}^{n} m_{[3,1,2]_i}^i + \operatorname{sgn}([3,2,1]) \prod_{i=1}^{n} m_{[3,2,1]_i}^i =$$

$$m_1^1 m_2^2 m_3^3 - m_1^1 m_3^2 m_2^3 - m_2^1 m_1^2 m_3^3 + m_2^1 m_3^2 m_1^3 + m_3^1 m_1^2 m_2^3 - m_3^1 m_2^2 m_1^3$$

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- **Theorem:** If M has two identical rows, then: $\det M = 0$

Elementary matrices

- **Elementary matrix** is a matrix which differs from the identity matrix by one single elementary row operation

Elementary matrices

- **Goal:** Find a matrix that, when it multiplies a matrix M, rows i and j of M are swapped

- Let R^1 through R^n denote the rows of M. Then:

$$M = \begin{pmatrix} \vdots \\ R^i \\ \vdots \\ R^j \\ \vdots \end{pmatrix}, \text{ and } M' = \begin{pmatrix} \vdots \\ R^j \\ \vdots \\ R^i \\ \vdots \end{pmatrix}$$

- $M' = E^i_j M$, where E^i_j is the identity matrix with rows i and j swapped

- E^i_j is called an **elementary matrix**

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- $\det E^i_j M = \det E^i_j \det M$

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- $\det R^i(\lambda) = \lambda$
- $\det M' = \det R^i(\lambda)M = \lambda \det M$

Elementary matrices

- The final row operation is adding λR^j to R^i.
- Let $S^i_j(\lambda)$ be I with λ in the i,j position. Then:

$$S^i_j(\lambda)M = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & \lambda & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} \vdots \\ R^i \\ \vdots \\ R^j \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ R^i + \lambda R^j \\ \vdots \\ R^j \\ \vdots \end{pmatrix}$$

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- $\det S^i_j(\lambda) = 1$
- $\det M' = \det S^i_j(\lambda)M = \det M$

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- **Theorem:** If E is any of the elementary matrices $E^i_j, R^i(\lambda), S^i_j(\lambda)$, then:

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- If M is not invertible, $\det \mathrm{RREF}(M) = 0$
- Notice that $\det \mathrm{RREF}(M) = \det(E_1 E_2 \cdots E_k M)$, where E_i is an elementary matrix
- Then: $\det \mathrm{RREF}(M) = \det(E_1) \cdots \det(E_k) \det M$
- Since $\det(E_i) \neq 0$, then $\det \mathrm{RREF}(M) = 0$ if and only if $\det M = 0$

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- Since $\det(E_i) \neq 0$, then $\det \mathrm{RREF}(M) = 0$ if and only if $\det M = 0$
- **Theorem:** For any square matrix M, $\det M \neq 0$ if and only if M is invertible

Exercise

- Compute $\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix}$

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- Compute $\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix} = 24$

Properties of the Determinant

- For any matrices M and N:

$$\det(MN) = \det M \det N$$

Properties of the Determinant

- For any square matrix M:

$$\det M^T = \det M$$

Determinant of the inverse

- $\det M^{-1} = ?$

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- **Theorem:**

$$\det M^{-1} = \frac{1}{\det M}$$

Adjoint of a matrix

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  - taking the determinant of the resulting matrix, multiplied by $(-1)^{i+j}$

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- **Theorem:** If M is invertible,

$$M^{-1} = \frac{1}{\det M}\mathrm{adj} M$$