Mathematical Methods - Lecture 2

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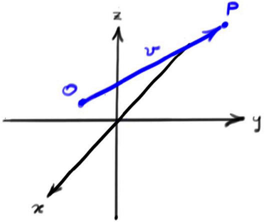
Outline

What is *n*-vector?

- Think about the space $\mathbb{R}^n$ as the space of points with n coordinates
- Specify a favorite point in $\mathbb{R}^n$ as an origin O
- Given any other point P, we can draw a **vector** v from O to P
- Just as in $\mathbb{R}^3$, a vector has a *magnitude* and a *direction*

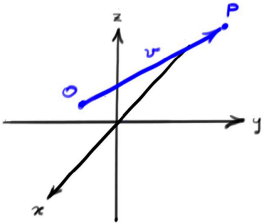


What is n-vector?

- Given any other point P, we can draw a **vector** v from O to P
- If O has coordinates $(o^1, \ldots, o^n)$ and P has coordinates $(p^1, \ldots, p^n)$, the *components* of the vector v are

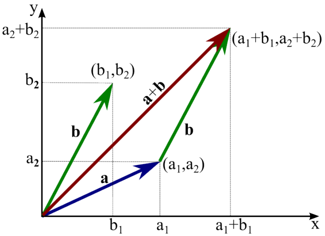
$$\begin{pmatrix} p^1 - o^1 \\ p^2 - o^2 \\ \vdots \\ p^n - o^n \end{pmatrix}$$

- This construction allows us to put the origin anywhere in $\mathbb{R}^n$



Some operations on vectors

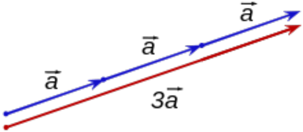
- We can *add* vectors $a = \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^n \end{pmatrix}$ and $b = \begin{pmatrix} b^1 \\ b^2 \\ \vdots \\ b^n \end{pmatrix}$
- The sum is: $a + b = \begin{pmatrix} a^1 + b^1 \\ a^2 + b^2 \\ \vdots \\ a^n + b^n \end{pmatrix}$



Some operations on vectors

- We can *multiply a vector* $a = \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^n \end{pmatrix}$ by a scalar

- The scalar multiple is: $\lambda a = \begin{pmatrix} \lambda a^1 \\ \lambda a^2 \\ \vdots \\ \lambda a^n \end{pmatrix}$



Some operations on vectors - Example

- $a = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$
- $a + b = \begin{pmatrix} \dots \\ \dots \\ \dots \\ \dots \end{pmatrix}$
- $3a - 2b = \begin{pmatrix} \dots \\ \dots \\ \dots \\ \dots \end{pmatrix}$

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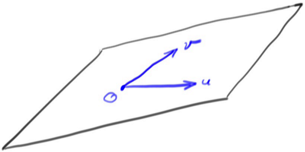
- $a = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$
- $a + b = \begin{pmatrix} 5 \\ 5 \\ 5 \\ 5 \end{pmatrix}$
- $3a - 2b = \begin{pmatrix} -5 \\ 0 \\ 5 \\ 10 \end{pmatrix}$

Zero vector

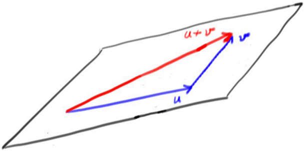
- A special vector is the **zero vector** connecting the origin to itself
- All of its components are zero
- Respecting the notions of Euclidean geometry:
  - it is the only vector with zero magnitude
  - it is the only vector which points in no particular direction
- Any single vector determines a line, except the zero vector
- Any scalar multiple of a non-zero vector lies in the line determined by that vector

Plane determined by two vectors

- Given two non-zero vectors u and v, they will usually determine a plane
  - unless both vectors are in the same line (one vector is a scalar multiple of the other)



- If u and v determine a plane, their sum lies in this plane



Parametric notation

- In 3 variables x, y and z an equation for a plane looks like

$$Ax + By + Cz = D,$$

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$$Ax + By + Cz = D,$$

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- **Example:** $x + 2y + 5z = 3$
- This is a system of linear equations, which has a set of solutions (plane)

Parametric notation

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- Augmented matrix: $\left(\begin{array}{ccc|c} 1 & 2 & 5 & 3 \end{array}\right)$

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- $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$

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- This describes a plane in parametric coordinates $\{P + su + tv | s, t \in \mathbb{R}\}$

Parametric notation

- The plane determined by two vectors u and v can be written as

$$\{P + su + tv | s, t \in \mathbb{R}\}$$

- Example:

$$\left\{ \begin{pmatrix} 3 \\ 1 \\ 4 \\ 1 \\ 5 \\ 9 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \middle| s, t \in \mathbb{R} \right\}$$

describes a plane in 6-dimensional space parallel to the xy-plane

We can generalize the notion of a plane

- **Definition:** A set of k vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$ with $k \leq n$ determines a k-dimensional *hyperplane*, unless any of the vectors v_i lives in the same hyperplane determined by the other vectors

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- If the vectors determine a k-dimensional hyperplane, any point in this hyperplane can be written as:

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- When k is not specified, one usually assumes $k = n - 1$ for a hyperplane inside $\mathbb{R}^n$

Directions and magnitudes - Definitions

- The **dot product** of two vectors $u = \begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix}$ and $v = \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$ is:

$$u \cdot v = u^1 v^1 + \cdots + u^n v^n$$

- The **length** or **norm** or **magnitude** of a vector:

$$||v|| = \sqrt{v \cdot v}$$

- The **angle** θ between two vectors is determined by:

$$u \cdot v = ||u|| ||v|| \cos \theta$$

Properties of the dot product

1. The dot product is *symmetric*, so

$$u \cdot v = v \cdot u$$

2. *Distributive* so

$$u \cdot (v + w) = u \cdot v + u \cdot w$$

3. *Bilinear*, which is to say, linear in both u and v. Thus

$$u \cdot (cv + dw) = cu \cdot v + du \cdot w$$

and

$$(cu + dw) \cdot v = cu \cdot v + dw \cdot v$$

4. *Positive definite*:

$$u \cdot u \geq 0,$$

and $u \cdot u = 0$ only when u is the 0-vector

Vector products

- The dot product determines the *Euclidean length and angle* between two vectors
- There are other useful ways to define lengths of vectors
- Other definitions of length and angle arise from *inner products* $< u, v >$
  - have all the properties listed above
  - in some contexts the positive definite requirement is relaxed
  - Example: The Lorentzian inner product on $\mathbb{R}^4$ is given by

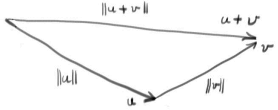
$$< u, v >= u^1 v^1 + u^2 v^2 + u^3 v^3 - u^4 v^4$$

Triangle Inequality

- **Theorem (Triangle Inequality)**: Given vectors u and v, we have:

$$||u+v|| \leq ||u|| + ||v||$$

- Proof



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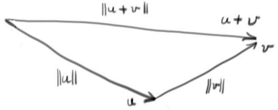
$$||u+v||^2 = (u+v)\cdot(u+v) = u\cdot u + 2u\cdot v + v\cdot v =$$

$$||u||^2 + ||v||^2 + 2||u||||v||\cos\theta = (||u|| + ||v||)^2 + 2||u||||v||(\cos\theta - 1)$$
$$\leq (||u|| + ||v||)^2$$



Example

- Let $a = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$ $\Rightarrow a + b = \begin{pmatrix} 5 \\ 5 \\ 5 \\ 5 \end{pmatrix}$
- $a \cdot a = b \cdot b = 1 + 2^2 + 3^2 + 4^2 = 30$
- $||a|| = ||b|| = \sqrt{30}$ and $(||a|| + ||b||)^2 = (2\sqrt{30})^2 = 120$
- $||a + b||^2 = 5^2 + 5^2 + 5^2 + 5^2 = 100 < 120 = (||a|| + ||b||)^2$ as predicted by the triangle inequality

Vector space

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  - (+i) Additive closure: $u + v \in V$
  - (+ii) Additive commutativity: $u + v = v + u$
  - (+iii) Additive associativity: $(u + v) + w = u + (v + w)$
  - (+iv) There is a vector $0_V \in V$ such that $u + 0_V = u$ for all $u \in V$
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    - ($\cdot$ i) Multiplicative closure: $c \cdot v \in V$
    - ($\cdot$ ii) Distributivity: $(c + d) \cdot v = c \cdot v + d \cdot v$
    - ($\cdot$ iii) Distributivity: $c \cdot (u + v) = c \cdot u + c \cdot v$
    - (+iv) Associativity: $(cd) \cdot v = c \cdot (d \cdot v)$
    - (+iv) Unity: $1 \cdot v = v$ for all $v \in V$

Vector space - Example

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  - Example: $f = \{0, 1, 8, 27, \ldots, n^3, \ldots\}$
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  - Zero: We need to propose a zero vector. $g(n) = 0$ works:
    $f(n) + g(n) = f(n)$

Vector space - Example

- Another important example of a vector space is the space of all differentiable functions:

$$V = \{f | f : \mathbb{R} \to \mathbb{R}, \frac{d}{dx} f \text{exists}\}$$

- The sum of any two differentiable functions is differentiable
  - since the derivative distributes over addition
- A scalar multiple of a function is also differentiable
  - since the derivative commutes with scalar multiplication
- The zero function is just the function that $0(x) = 0$ for every x
- ...

Vector spaces over other fields

- Above, we defined vector spaces over the real numbers
- One can define vector spaces over any field
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- A *field* is a collection of "numbers" satisfying a number of properties
- One example of a field is the complex numbers

$$\mathbb{C} = \{x + iy | i^2 = -1, x, y \in \mathbb{R}\}$$

- Example: Vector space

$$V = \left\{ \begin{pmatrix} \lambda \\ \mu \end{pmatrix} | \lambda, \mu \in \mathbb{C} \right\}$$

described states of an electron

Linear transformations

- Suppose we have two vector spaces V and W and a map L between them:

$$L : V \to W$$

- Both V and W satisfy properties of vector addition and scalar multiplication
- It would be ideal if the map L *preserved* these operations, i.e. for any $u, v \in V$ and $c \in \mathbb{R}$:

$$L(u + v) = L(u) + L(V)$$

$$L(cv) = cL(v)$$

- Combining these two requirements into one equation, we get the definition of a *linear function* or a *linear transformation*

Linear transformation

- **Definition:** A function $L : V \to W$ is linear if for all $u, v \in V$ and $r, s \in \mathbb{R}$ we have

$$L(ru + sv) = rL(u) + sL(v)$$

- On the left the addition and scalar multiplication occur in V, while on the right the operations occur in W
- This is often called the *linearity property* of a linear transformation

Linear transformation - Example

- Check linearity for $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by:

$$L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ y + z \\ 0 \end{pmatrix}$$

Linear transformation in the matrix form

- $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by:

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y+z \\ 0 \end{pmatrix}$$

- We can write this linear transformation using a matrix like so:

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y+z \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- Matrix multiplication on vectors is a linear transformation:

$$M(ru+sv) = rMu + sMv$$