Mathematical Methods - Lecture 1

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INVENTORS FOR THE DIGITAL WORLD

Outline

Methods course details

- Course title: Mathematical Methods
- Course lecturer: Yuliya Tarabalka (yuliya.tarabalka@inria.fr)
- Lectures:
    - 3 hours $\times$ 8 times
    - From Sept. 17 to Sep. 28
- Assessment at the end
- Recommended books:
    - Mathematical methods for science students. G Stephenson, 1973.
    - Analysis: with an introduction to proof. Steven R Lay, 2005.
    - Mathematical methods for physics and engineering. K. F. Riley, M. P. Hobson and S. J. Bence, 2006.
    - Introduction to differential equations. J. R. Chasnov, 2016.
    - Linear algebra in twenty five lectures. T. Denton and A. Waldron, 2012.
    - Tons of materials online.

Goals of the course

- Revise and learn the most important mathematical concepts
- Maths underpins most computing concepts/applications in data science & business analystics, e.g.:
    - stock market models
    - information search and retrieval
    - computer vision
    - neural computing
    - ...

What is Linear Algebra?

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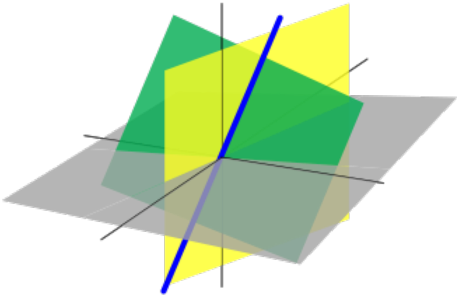
- **Linear algebra** is the branch of maths concerning *vector spaces* and *linear mappings* between such spaces [Wikipedia]



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$$\begin{pmatrix} 9 & 4 & \cdots \\ 10 & 6 & \\ \vdots & & \ddots \end{pmatrix}$$

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- We can replace this array by an abstract symbol M, called a **matrix**
- M is an example of a more general abstract structure called **linear transformation**

Example 1

- The theory of linear transformations is incredibly useful!
  - Think that we can replace "likebility ratings" with the number of times internet websites link to one another
- Three main topics in the Linear Algebra course:
  - Linear Systems
  - Vector Spaces
  - Linear Transformations

Example 2

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- To keep track of the number of apples & oranges:
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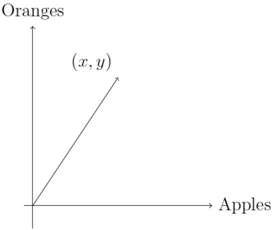
- We can represent this vector with a point in a 2-D plane with the corresponding coordinates



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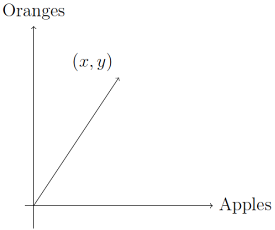
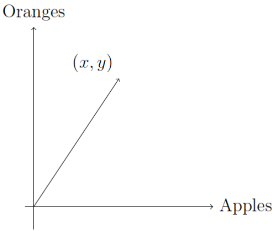


- In the plane, we can imagine each point as some combination of apples & oranges (or parts thereof for non-int coordinates)
- Then each point corresponds to some vector
- The collection of all such vectors is an example of a **vector space**

Example 2



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Vector space (also called a linear space) is a collection of objects called vectors, which may be added together and multiplied by numbers

Exercise

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$$y = 2x \tag{2}$$

- This is an example of a **linear system**

$$\begin{cases} f_1(x_1, \cdots, x_m) = a_1 \\ \vdots \\ f_n(x_1, \cdots, x_m) = a_n \end{cases}$$

Linear function

- **Definition:** Let X and Y be vectors and α and β be scalars. A function is *linear* if

$$f(\alpha X + \beta Y) = \alpha f(X) + \beta f(Y)$$

Exercise

There are 27 pieces of fruit in a barrel, and twice as many oranges as apples. How many oranges and apples in the barrel?

- We can re-write the question mathematically:

$$\begin{aligned} x + y &= 27 \\ y &= 2x \end{aligned}$$

- By manipulating the equations:

$$\begin{aligned} x + y &= 27 \\ -2x + y &= 0 \end{aligned} \quad \Rightarrow \quad 3x = 27 \quad \Rightarrow \quad \begin{aligned} x &= 9 \\ y &= 18 \end{aligned}$$

Exercise

- Let us express this set of equations

$$x + y = 27$$
$$2x - y = 0$$

with a matrix:

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 27 \\ 0 \end{pmatrix}$$

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- To get back the linear system, we apply the multiplication rule:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

Exercise

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- The matrix is an example of a **linear transformation**: it takes one vector and turns it into another in a linear way

Next task:

Solve linear systems

$\Downarrow$

We'll learn a general method called Gaussian Elimination

Notation for linear systems

- We've found that we can express the linear system

$$\begin{aligned} x + y &= 27 \\ 2x - y &= 0 \end{aligned} \qquad \text{as} \qquad \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 27 \\ 0 \end{pmatrix}$$

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- Likewise, we can write the solution as:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 18 \end{pmatrix}$$

- $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the **Identity Matrix**: $Iv = v$

Augmented matrix notation

- A useful shorthand for a linear system is an **Augmented Matrix**:

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 27 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \left(\begin{array}{cc|c} 1 & 1 & 27 \\ 2 & -1 & 0 \end{array}\right)$$

- The solution to the linear system looks like this:

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Augmented matrix notation

- Another example of an augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & 0 & 9 \\ 6 & 2 & 0 & -2 & 0 \\ -1 & 0 & 1 & 1 & 3 \end{array}\right)$$

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- General case:

$$\left(\begin{array}{cccc|c} a_1^1 & a_2^1 & \cdots & a_k^1 & b^1 \\ a_1^2 & a_2^2 & \cdots & a_k^2 & b^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1^r & a_2^r & \cdots & a_k^r & b^r \end{array}\right)$$

  - number of rows r = number of equations
  - number of columns k = number of unknowns

Idea of Gaussian elimination (equivalence relations)

- From the example:

$$\left(\begin{array}{cc|c} 1 & 1 & 27 \\ 2 & -1 & 0 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & 18 \end{array}\right)$$

- Idea:

Turn a general augmented matrix into a simple augmented matrix consisting of the identity matrix on the left and a bunch of numbers (solution) on the right

Equivalence relations

- From the example:

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- Two augmented matrices *that actually have solutions* are said to be (row) *equivalent* if they have the *same* solutions

$$\left(\begin{array}{cc|c} 1 & 1 & 27 \\ 2 & -1 & 0 \end{array}\right) \quad \sim \quad \left(\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & 18 \end{array}\right)$$

Reduced Row Echelon Form

- **Reduced Row Echelon Form** (RREF) = canonical form for augmented matrices:

$$\left(\begin{array}{ccccccc|c} 1 & * & 0 & * & 0 & \cdots & 0 & b^1 \\ 0 & & 1 & * & 0 & \cdots & 0 & b^2 \\ 0 & & 0 & & 1 & \cdots & 0 & b^3 \\ \vdots & & \vdots & & \vdots & & 0 & \vdots \\ & & & & & & 1 & b^k \\ 0 & & 0 & & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots \\ 0 & & 0 & & 0 & \cdots & 0 & 0 \end{array}\right)$$

  - *Pivot* = the first non-zero entry in each row
  - The asterisks = arbitrary content (1 or several columns long)
  - In RREF, the pivot is always 1
  - The pivot is the only non-zero entry in its column
  - The pivot of any row is always to the right of the pivot of the row above it

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- **Theorem**: Every augmented matrix is row-equivalent to a unique augmented matrix in RREF

Reduced Row Echelon Form: Example

- RREF:

$$\left(\begin{array}{cccc|c} 1 & 0 & 7 & 0 & 4 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- We can read off the solution set directly from the RREF:

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ -3 \\ 1 \\ 0 \end{pmatrix}$$

Gauss-Jourdan (Gaussian) elimination

- **Idea:** Begin with an arbitrary matrix & apply operations that respect row equivalence until we have RREF

- **3 elementary row operations:**

  1. (Row Swap) Exchange any two rows
  2. (Scalar Multiplication) Multiply any row by a non-zero constant
  3. (Row Sum) Add a multiple of one row to another row

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- **Theorem:** Gauss-Jourdan Elimination produces a unique augmented matrix in RREF

Gaussian elimination - Example

$$
\begin{array}{rcrcrcl}
 & & & & 3x_3 & = & 9 \\
x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\
\frac{1}{3}x_1 & + & 2x_2 & & & = & 3
\end{array}
$$

Gaussian elimination - Example

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$R_1 \leftrightarrow R_3$

$R_1 =$
$R_1 - 6R_2$

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Gaussian elimination - Example

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$R_1 = R_1 + 12R_3$

$-R_2$

Gaussian elimination - Example

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$R_2 = R_2 - R_1$

$-R_2$

$$\left(\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 3 & 9 \end{array}\right)$$

$R_1 = R_1 - 6R_2$

$1/3R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & -12 & -33 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$R_1 = R_1 + 12R_3$

$R_2 = R_2 - 2R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Gaussian elimination - Example

$$\begin{array}{rcrcrcl} & & & & 3x_3 & = & 9 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ \frac{1}{3}x_1 & + & 2x_2 & & & = & 3 \end{array} \sim \left(\begin{array}{ccc|c} 0 & 0 & 3 & 9 \\ 1 & 5 & -2 & 2 \\ \frac{1}{3} & 2 & 0 & 3 \end{array}\right)$$

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$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array} \right) \quad \Rightarrow$$

Solution:

$x_1 = 3$

$x_2 = 1$

$x_3 = 3$

Gaussian elimination - Example 2

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ 0 & -1 & -2 & 3 & -3 \\ 5 & 2 & -1 & 4 & 1 \end{array}\right)$$

$R_2 - R1; R_4 - 5R_1$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & -1 \\ 0 & 1 & 2 & -3 & 3 \\ 0 & -1 & -2 & 3 & -3 \\ 0 & 2 & 4 & -6 & 6 \end{array}\right)$$

$R_3 + R2; R_4 - 2R_3$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & -1 \\ 0 & 1 & 2 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Gaussian elimination - Example 2

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & -1 \\ 0 & 1 & 2 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- Variables x_3 and x_4 are undetermined
- The solution is not unique
- Let's set $x_3 = \lambda$ and $x_4 = \mu$. The solution is set is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$