Dirichlet/Neumann problems and Hardy classes for the planar conductivity equation

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Abstract. We study Hardy spaces H^p_ν of the conjugate Beltrami equation $\overline{\partial} f = \nu \overline{\partial f}$ over Dini-smooth finitely connected domains, for real contractive $\nu \in W^{1,r}$ with $r > 2$, in the range $r/(r-1) < p < \infty$. We develop a theory of conjugate functions and apply it to solve Dirichlet and Neumann problems for the conductivity equation $\nabla.(\sigma \nabla u) = 0$ where $\sigma = (1-\nu)/(1+\nu)$. In particular situations, we also consider some density properties of traces of solutions together with boundary approximation issues.

Key words. Hardy spaces, Boundary value problems, Second-order elliptic equations, Conjugate functions, Integral equations with kernels of Cauchy type.

1 Introduction

The foundations of pseudoanalytic function theory, that generalizes some key features of classical holomorphic function theory, go back to [14, 57] and were historically applied to boundary value problems for partial differential equations. This has recently been a topic of renewed interest [4, 26, 41, 45, 59], and reference [11] was apparently first to investigate the connections between generalized Hardy spaces on simply connected domains and Dirichlet problems for the planar conductivity equation with L^p boundary data. In [27], part of this material was carried over to annuli under more general regularity assumptions on the coefficients, and used there to approach certain mixed Dirichlet-Neumann problems. The present paper expands and generalizes the results of [11] to finitely connected domains under still weaker, conjectured optimal regularity assumptions. We also take up Neumann problems and discuss some density issues for traces of solutions. We shall consider a simple class of pseudoanalytic functions (also called generalized analytic functions), namely those satisfying the conjugate Beltrami equation:

$$\overline{\partial} f = \nu \, \overline{\partial f} \text{ a.e. in } \Omega \,, \tag{CB}$$

where $\Omega \subset \mathbb{R}^2 \simeq \mathbb{C}$ is a Dini-smooth domain. The dilation coefficient ν is *real valued* and lies in a Sobolev class $W^{1,r}(\Omega)$, $2 < r \leq \infty$, while satisfying a uniform bound of the type

$$\|\nu\|_{L^\infty(\Omega)} \leq \kappa \text{ for some } \kappa \in (0,1) \,. \tag{κ}$$

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If one writes $f = u+iv$ with real u, v, then (CB) becomes a pair of equations generalizing the Cauchy-Riemann system:

$$\begin{cases} \partial_x v = -\sigma\partial_y u, \\ \partial_y v = \sigma\partial_x u, \end{cases} \tag{1}$$

with

$$\sigma = \frac{1-\nu}{1+\nu}. \tag{2}$$

Now, a compatibility condition for (1) is the planar conductivity equation:

$$\operatorname{div}(\sigma\nabla u) = 0 \text{ in } \Omega\,, \tag{3}$$

and this fact is the main motivation to study (CB). An interesting connection, this time to a Schrödinger equation, was also pointed out in [55]. Observe that (κ) is equivalent *via* (2) to the ellipticity condition

$$0 < c \leq \sigma \leq C < \infty \quad \text{a.e. in } \Omega \tag{4}$$

for some constants c, C. Note also, if u meets (3), that the σ-conjugate function v satisfies

$$\operatorname{div}\left(\frac{1}{\sigma}\nabla v\right) = 0 \text{ in } \Omega\,. \tag{5}$$

We shall study Hardy classes of solutions to (CB) (see definition in Section 3), analyze their boundary behaviour and give a complete description of σ conjugate functions in this context. We shall apply the results to the Dirichlet problem of equation (3), and it will turn out that data in $L^p(\partial\Omega)$ for $p > r/(r-1)$ are exactly boundary values of solutions satisfying a Hardy condition. Trading smoothness of the boundary for smoothness of the coefficients, we also give an application to the Dirichlet problem with Lipschitz coefficients on piecewise $C^{1,\lambda}$ domains without outward pointing cusp, when the boundary data are integrable with respect to harmonic measure; inward pointing cusps are allowed, so that the domain may not be Lipschitz. In addition, we solve the Neumann problem with data $\sigma\partial u/\partial n \in W^{-1,p}(\partial\Omega)$.

From the point of view of regularity theory, and though we deal with two dimensions and scalar conductivity only, it is noteworthy that our assumptions are not covered by the Carleson condition set up in [24, 44]. As we rely rather extensively on complex methods, higher dimensional analogs of our results, if true at all, require new ideas of proof.

The authors' motivation for such a study originates in certain free boundary problems of Bernoulli type for equation (3) that arise naturally when trying to locate the boundary of a plasma at equilibrium in a tokamak [15]. These are genuine 2-D problem, due to rotational symmetry. Their approach *via* extremal problems, developed in [30, 31, 32], raises some density issues for traces of solutions to (CB) on subsets of $\partial\Omega$ which are interesting in their own right and deserves further studies [8].

The paper is organized as follows. After some preliminaries on conformal mapping and Sobolev spaces in Section 2, we introduce in Section 3 Hardy classes $H^p_\nu(\Omega)$ of equation (CB), along with their companion Hardy classes $G^p_\alpha(\Omega)$ of equation (13) which are of great technical importance for our approach. Dwelling on classical works [14, 57] to make connection with holomorphic Hardy spaces, we then derive the main properties of $H^p_\nu(\Omega)$. Section 4 is devoted to a decomposition theorem which identifies Hardy classes over finitely connected domains with sums of Hardy classes over simply connected domains, much like in the holomorphic case. In Section 5, we deal with analytical and topological conditions for the existence of σ-conjugate functions, and we apply our results to the Dirichlet and Neumann problems for equation (3). Finally, we discuss in Section 6 some density properties of traces of $H^p_\nu(\Omega)$-functions on $\partial\Omega$ which are relevant to inverse boundary value problems. Concluding remarks are given in Section 7. We append in Appendix some of the more technical results and proofs.

2 Notations and preliminaries

We put $\overline{\mathbb{C}} := \mathbb{C}\cup\{\infty\}$ for the extended complex plane, which identifies to the unit sphere $\mathbb{S}^2$ under stereographic projection. We let $\mathbb{D}_r$ and $\mathbb{T}_r$ designate the open disk and the circle centered at 0 of radius r; when $r = 1$ we omit the subscript. More generally, $\mathbb{T}_{a,r}$ (resp. $\mathbb{D}_{a,r}$) indicates the circle (resp. open disk) of center a and radius r.

For $\varrho \in (0, 1)$, we set $\mathbb{A}_\varrho := \mathbb{D} \setminus \overline{\mathbb{D}}_\varrho$ to be the annulus lying between $\mathbb{T}$ and $\mathbb{T}_\varrho$. For more general annuli we use the notation $\mathbb{A}_{\varrho_1,\varrho_2} := \{z;\ \varrho_1 < |z| < \varrho_2\}$. If Ω is a doubly connected domain such that no component of $\overline{\mathbb{C}} \setminus \Omega$ reduces to a single point, it is well known there is a unique ϱ making Ω conformally equivalent to $\mathbb{A}_\varrho$ [52, Thm VIII.6.1]. More generally, any finitely connected domain whose complement is infinite is conformally equivalent to $\mathbb{D}$ with finitely many closed disks removed (some of which may degenerate to points)[1]. Such a domain will be termed a normalized circular domain. Moreover, the above conformal map is unique up to a Möbius transformation.

Recall that a function h is called *Dini-continuous* if $\int_0^\varepsilon (\omega_h(t)/t)dt < +\infty$ for some, hence any $\varepsilon > 0$, where ω_h is the modulus of continuity of h. A function is *Dini-smooth* if it has Dini-continuous derivative. A domain $\Omega \subset \overline{\mathbb{C}}$ is said to be Dini-smooth if its boundary $\partial\Omega$ lies in $\mathbb{C}$ and consists of finitely many Jordan curves with nonsingular Dini-smooth parametrization. Note that a Dini-smooth domain is finitely connected by definition, and it contains ∞ if it is unbounded.

Any conformal map between Dini-smooth domains extends to a homeomorphism of their closures, and the derivative also extends continuously to the closure of the initial domain in such a way that it is never zero, *cf.* Lemma 6 in Appendix A.

We orient the boundary of a Dini-smooth domain Ω in a canonical way, i.e. Ω lies on the left side when moving along $\partial\Omega$, and the unit normal $\vec{n}$ points outward.

We denote interchangeably (the differential of) planar Lebesgue measure by

$$dm(\xi) = dt_1dt_2 = (i/2)\ d\xi \wedge d\overline{\xi}, \qquad \xi = t_1 + it_2.$$

Given a domain $\Omega \subset \mathbb{C}$, we put $\mathcal{D}(\Omega)$ for the space of C^∞-smooth complex valued functions with compact support in Ω, equiped with the usual topology[2]. Its dual $\mathcal{D}'(\Omega)$ is the space of distributions on Ω. For $1 \le p \le \infty$ and $k \in \mathbb{N}$, we let $L^p(\Omega)$ and $W^{k,p}(\Omega)$ be the familiar Lebesgue and Sobolev spaces with respect to dm; we sometimes write $L^p_{\mathbb{R}}(\Omega)$ or

[1] Indeed, any finitely connected domain is conformally equivalent to a domain whose boundary consists of circles or points [36, Sec. V.6, Thm 2]; if the complement is infinite there is at least one circle whose interior can be mapped onto $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ by a Möbius tranform.

[2] *i.e.* the inductive topology of its subspaces $\mathcal{D}_K$ comprised of functions whose support lies in a compact set K, each $\mathcal{D}_K$ being topologized by uniform convergence of a function and all its partial derivatives [53, Sec. I.2]).

$W^{k,p}_{\mathbb{R}}(\Omega)$ to emphasize restriction to real-valued functions. Recall that $W^{1,p}(\Omega)$ consists of functions in $L^p(\Omega)$ whose distributional derivatives lie in $L^p(\Omega)$ up to order k. Actually we only need $k = 1, 2$, the norms on $W^{1,p}(\Omega)$, $W^{2,p}(\Omega)$ being defined as

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\partial f\|_{L^p(\Omega)} + \|\overline{\partial} f\|_{L^p(\Omega)},$$

$$\|f\|_{W^{2,p}(\Omega)} = \|\partial f\|_{W^{1,p}(\Omega)} + \|\overline{\partial} f\|_{W^{1,p}(\Omega)} + \|f\|_{L^p(\Omega)},$$

where ∂ and $\overline{\partial}$ stand for the usual (distributional) complex derivatives, i.e.

$$\partial f := \partial_z f = \frac{1}{2}(\partial_x - i\partial_y) f \quad \text{and} \quad \overline{\partial} f := \partial_{\overline{z}} f = \frac{1}{2}(\partial_x + i\partial_y) f \qquad z = x + iy.$$

Note the obvious identity: $\overline{\partial f} = \overline{\partial}\, \overline{f}$. The closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$ is denoted by $W^{1,p}_0(\Omega)$. Recall the notation $W^{-1,p}(\Omega) = (W^{1,q}_0(\Omega))^*$, $1/p + 1/q = 1$. For basic properties of Sobolev spaces that we use, see *e.g.* [16, 58].

When $\Omega \subset \mathbb{C}$ is a bounded Dini-smooth domain, $L^p(\partial\Omega)$ is understood with respect to normalized arclength and $W^{1,p}(\partial\Omega)$ is naturally defined using local coordinates, since Lipschitz-continuous changes of variable preserve Sobolev classes. Each $f \in W^{1,p}(\Omega)$ with $1 < p \leq \infty$ has a trace on $\partial\Omega$, denoted by $\mathrm{tr}_{\partial\Omega}\, f$, lying in the so-called fractional Sobolev space $W^{1-1/p,p}(\partial\Omega)$. The latter is a real interpolation space between $L^p(\partial\Omega)$ and $W^{1,p}(\Omega)$ of exponent $1 - 1/p$, an intrinsic definition of which can be found in [1, Thm 7.48]. The trace operator defines a continuous surjection from $W^{1,p}(\Omega)$ onto $W^{1-1/p,p}(\partial\Omega)$. By the Sobolev embedding theorem, each $f \in W^{1,p}(\Omega)$ with $p > 2$ is Hölder-smooth of exponent $1-2/p$ on Ω, hence f extends continuously to $\overline{\Omega}$ in this case. The space $W^{1,\infty}(\Omega)$ identifies with Lipschitz-continuous functions on Ω.

We also introduce the spaces $L^p_{loc}(\Omega)$ and $W^{1,p}_{loc}(\Omega)$ of distributions[3] whose restriction to any relatively compact open subset Ω_0 of Ω lies in $L^p(\Omega_0)$ or $W^{1,p}(\Omega_0)$. All classes of functions we will consider are embedded in $L^p_{loc}(\Omega)$ for some $p \in (1, +\infty)$, and solutions to differential equations are understood in the distributional sense. For instance, to define distributions like $\nu\overline{\partial f}$ where $\nu \in W^{1,r}_{\mathbb{R}}(\Omega)$ and $f \in L^p_{loc}(\Omega)$ with $1/p + 1/r \leq 1$, we use Leibniz's rule:

$$< \nu\overline{\partial f}, \phi >= - \int_\Omega (\nu \overline{f}\, \overline{\partial}\phi + \overline{\partial}\nu \overline{f} \phi)\, dm, \qquad \forall\ \phi \in \mathcal{D}(\Omega).$$

where $<,>$ denotes the duality product between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$.

If in addition $r > 2$ and σ meets (4) while $u \in W^{1,p}_{\mathbb{R}}(\Omega)$ solves (3), the normal derivative $\partial_n u$ is the unique member of the dual space $W^{-1/p,p}_{\mathbb{R}}(\partial\Omega) = (W^{1-1/q,q}_{\mathbb{R}}(\partial\Omega))^*$, $1/p+1/q = 1$, such that

$$< \sigma\partial_n u, \phi >_{\partial\Omega}= \int_\Omega \sigma \nabla u \cdot \nabla g\ dm, \qquad g \in W^{1,q}(\Omega), \quad \mathrm{tr}_{\partial\Omega}\ g = \phi. \tag{6}$$

In fact, (6) defines $\sigma\partial_n u \in W^{-1/p,p}(\partial\Omega)$ and, under the stated assumptions, multiplication by σ is an isomorphism of the latter because it is an isomorphism of $W^{1-1/q,q}(\partial\Omega)$ (*e.g.* by interpolation). Clearly then, $\|\partial_n u\|_{W^{-1/p,p}(\partial\Omega)} \leq C(\Omega, \sigma, p)\|u\|_{W^{1,p}(\Omega)}$.

[3]These are topologized by the family of semi-norms $\|f_{\Omega_n}\|_{L^p(\Omega_n)}$ and $\|f_{\Omega_n}\|_{W^{1,p}(\Omega_n)}$ respectively, with $\{\Omega_n\}$ a nested family of relatively compact open subset exhausting Ω.

Sobolev spaces are naturally defined on the Riemann surface $\overline{\mathbb{C}} \sim \mathbb{S}^2$ [40], integration being understood with respect to spherical measure. We shall not be concerned with intrinsic notions: to us it suffices to say that if $\Omega \subset \overline{\mathbb{C}}$ is a (possibly unbounded) Dini-smooth domain distinct from $\overline{\mathbb{C}}$, then it can be mapped onto a bounded Dini-smooth domain Ω' by some conformal map φ and then $f \in W^{1,p}(\Omega)$ (resp. $f \in L^p(\Omega)$) if and only if $f \circ \varphi^{-1} \in W^{1,p}(\Omega')$ (resp. $L^p(\Omega')$), with equivalence of norms (the equivalence depends on φ). This is consistent with previous definitions, since by Lemma 6 (in Appendix A) conformal maps between bounded Dini-smooth domains are Lipschitz continuous. A function in $W^{1,p}(\overline{\mathbb{C}})$ is one whose restriction to any proper Dini-smooth subdomain Ω belongs to $W^{1,p}(\Omega)$.

On a bounded domain Ω, relation (CB) may be regarded as a differential equation for 1-forms in local coordinates on the Riemann surface $\mathbb{S}^2$, namely $\overline{\partial} f d\bar{z} = \nu \overline{\partial f} d\bar{z}$. Subsequently, if Ω, Ω', and φ are as before, we say that f solves (CB) on Ω if and only if $f \circ \varphi^{-1}$ satisfies a similar equation on Ω' only with ν replaced by $\nu \circ \varphi^{-1}$; this agrees with the complex chain rule when Ω is bounded [3, Sec. 1.C], and allows us to make sense of (CB) when Ω is unbounded.

If f is a function defined on Ω, the symbol $f_{|\Omega_1}$ indicates the restriction of f to $\Omega_1 \subset \Omega$. Whenever f is defined on Ω_1 and h on $\Omega_2 = \Omega \setminus \Omega_1$, the notation $f \vee h$ is used for the concatenated function defined on Ω which is equal to f on Ω_1 and to h on Ω_2.

We let ∂_t and ∂_n denote respectively the tangential and normal derivatives of a function at a smooth point on a rectifiable curve. As became customary, the same symbol (*e.g.* "C") is used many times to mean different constants.

3 Generalized Hardy classes

Hardy classes of equation (CB) over a bounded Dini-smooth simply connected domain were introduced in [11]. Their study is twined with that of Hardy classes of equation (13) further below, whose connection to (CB) was originally stressed in [14]. Hardy classes of (CB) over bounded annular domains with analytic boundary have subsequently been defined in [27]. This section carries out their generalization to arbitrary Dini-smooth domains in $\overline{\mathbb{C}}$.

Although [11] restricts to the case where $\nu \in W^{1,\infty}_{\mathbb{R}}(\Omega)$, it was observed in [27] that many results still hold when $\nu \in W^{1,r}_{\mathbb{R}}(\Omega)$ for some $r > 2$, provided that $p > r/(r-2)$. We improve on this throughout by assuming $r > 2$ and $p > r/(r-1)$, which we conjecture is the optimal range of exponents for the validity of whatever follows. When $r = \infty$, and only in this case, we cover the whole range of exponents $1 < p < \infty$.

To recap, our working assumptions will be that $\Omega \subset \overline{\mathbb{C}}$ is Dini-smooth (in particular finitely connected) and that

$$\nu \text{ meets } (\kappa), \qquad \nu \in W^{1,r}_{\mathbb{R}}(\Omega) \text{ for some } r \in (2, +\infty], \qquad r/(r-1) < p < +\infty. \tag{7}$$

Note the assumptions on ν are equivalent to require that σ given by (2) lies in $W^{1,r}_{\mathbb{R}}(\Omega)$ and satisfies (4).

3.1 $H^p_\nu(\Omega)$

In analogy to classical holomorphic Hardy spaces, the Hardy space $H^p_\nu(\mathbb{D})$ was defined in [11] to consist of those functions f in $L^p(\mathbb{D})$ satisfying (CB) in the sense of distributions and such that

$$\|f\|_{H^p_\nu(\mathbb{D})} := \operatorname*{ess\,sup}_{0<r<1} \|f\|_{L^p(\mathbb{T}_r)} = \operatorname*{ess\,sup}_{0<r<1} \left(\int_0^{2\pi} \left|f(re^{i\theta})\right|^p \frac{d\theta}{2\pi}\right)^{1/p} < +\infty\,. \tag{8}$$

Likewise, in [27], the Hardy space $H^p_\nu(\mathbb{A}_\varrho)$ was set to be comprised of functions in $L^p(\mathbb{A}_\varrho)$ solving (CB) and such that

$$\|f\|_{H^p_\nu(\mathbb{A}_\varrho)} := \operatorname*{ess\,sup}_{\varrho<r<1} \|f\|_{L^p(\mathbb{T}_r)} = \operatorname*{ess\,sup}_{\varrho<r<1} \left(\int_0^{2\pi} \left|f(re^{i\theta})\right|^p \frac{d\theta}{2\pi}\right)^{1/p} < +\infty. \tag{9}$$

Now, for Ω a Dini-smooth proper subdomain of $\overline{\mathbb{C}}$ and ν, p as in (7), we define $H^p_\nu(\Omega)$ to be comprised of those $f \in L^p_{loc}(\Omega)$ solving (CB) in the sense of distributions for which there is a sequence of domains Δ_n with $\overline{\Delta}_n \subset \Omega$, whose boundary $\partial\Delta_n$ is a finite union of rectifiable Jordan curves of uniformly bounded length, such that each compact subset of Ω is eventually contained in Δ_n, and having the property that

$$\sup_{n\in\mathbb{N}} \|f\|_{L^p(\partial\Delta_n)} < \infty. \tag{10}$$

When $\nu \equiv 0$, condition (10) defines the so-called *S*mirnov class of index p of holomorphic functions in Ω, which coincides with the Hardy class on Dini-smooth domains[4]. This class we consistently denote by $H^p(\Omega)$ (no subscript).
It is true, although not immediately clear, that $H^p_\nu(\Omega)$ is a vector space and that there is a *fixed* sequence $\{\Delta_n\}$ for which (10) yields a complete norm. It is not obvious either that (10) is equivalent to (8) or (9) for the disk or the annulus. All this is known to hold for holomorphic functions [56], [25, Sec. 10.5], but the proof when $\nu \neq 0$ will await Section 3.2. Note that H^p_ν is only a *r*eal Banach space if $\nu \neq 0$.

The definition of $H^p_\nu(\Omega)$ just given is conformally invariant: if φ conformally maps a Dini-smooth domain Ω' onto a Dini-smooth domain Ω, then $\nu \in W^{1,r}_{\mathbb{R}}(\Omega)$ and $f \in H^p_\nu(\Omega)$ if and only if $\nu \circ \varphi \in W^{1,r}_{\mathbb{R}}(\Omega')$ and $f \circ \varphi \in H^p_{\nu\circ\varphi}(\Omega')$. Indeed the $\varphi^{-1}(\Delta_n)$ form an admissible sequence of compact sets in Ω' since their boundary is eventually contained in a compact neighborhood of $\partial\Omega'$ where $|\varphi'|$ is bounded below by a strictly positive constant in view of Lemma 6. In [11, 27], conformal invariance was used to *define* $H^p_\nu(\Omega)$ on simply or doubly connected bounded Dini-smooth domains[5].
In connection with unbounded domains, the following *r*eflexion principle is useful: for $f \in L^p_{loc}(\mathbb{D})$, set

$$\check{f}(z) = \overline{f\left(\frac{1}{\overline{z}}\right)}, \quad z \in \mathbb{C} \setminus \overline{\mathbb{D}}\,. \tag{11}$$

Then

$$f \in H^p_\nu(\mathbb{D}) \Longleftrightarrow \check{f} \in H^p_{\check{\nu}}(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})\,. \tag{12}$$

Indeed, if we put $g(z) = \overline{f(\overline{z})}$ and $\mu(z) = \nu(\overline{z})$ for $z \in \mathbb{D}$, we get by definition upon using the conformal map $z \mapsto 1/z$ that $\check{f} \in H^p_{\check{\nu}}(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ if and only if $\mu \in W^{1,r}_{\mathbb{R}}(\mathbb{D})$ and

[4]The Hardy class is defined by the condition that $|f|^p$ has a harmonic majorant; the two classes coincide as soon as harmonic measure and arclength are comparable up to a multiplicative constant on $\partial\Omega$ [25, Ch. 10], [56], which is the case for Dini-smooth domains thanks to Lemma 6, Appendix A.

[5]These works do not mention the case of unbounded domains, but it requires no change as we just stressed. The paper [27] restricts to analytic boundaries, which is also unnecessary thanks to Lemma 6.

$g \in H^p_\mu(\mathbb{D})$. Clearly this is the case if and only if $\nu \in W^{1,r}_{\mathbb{R}}(\mathbb{D})$ and $f \in H^p_\nu(\mathbb{D})$, as follows from (CB) by conjugation, which proves (12).

Remark 1 *We did not define Hardy spaces of $\overline{\mathbb{C}}$ (a Dini-smooth domain with empty boundary), but this case is of little interest since no non-constant distributional solution to* (CB) *exists in $L^p_{loc}(\overline{\mathbb{C}})$. In fact, by Propositions 1, 2 further below, a function f with these properties must lie in $W^{1,k}(\overline{\mathbb{C}})$ for some $k > 2$. In particular it is bounded, so by the extended Liouville theorem [4, Cor. 3.4], $f = Ce^g$, where C is constant and g is continuous on $\overline{\mathbb{C}}$. Applying this to $f - f(0)$ we conclude the latter is identically zero, as desired.*

3.2 $G^p_\alpha(\Omega)$

When $\Omega = \mathbb{D}$ or $\Omega = \mathbb{A}_\varrho$ and $\alpha \in L^r(\Omega)$, the Hardy space $G^p_\alpha(\Omega)$ was defined in [11, 27] to consist of those $w \in L^p_{loc}(\Omega)$ such that

$$\bar{\partial} w = \alpha \overline{w} \quad \text{on } \Omega \tag{13}$$

in the distributional sense, and meeting condition (8) or (9) (with w instead of f).
On a bounded Dini-smooth domain $\Omega \subset \overline{\mathbb{C}}$, given $\alpha \in L^r(\Omega)$, we define $G^p_\alpha(\Omega)$ to consist of those $w \in L^p_{loc}(\Omega)$ meeting (10) for some admissible sequence $\{\Delta_n\} \subset \Omega$, and such that (13) holds. If we set further

$$A(z) = \frac{1}{2i\pi} \int_\Omega \frac{\alpha(\xi)}{\xi - z} \, d\xi \wedge d\bar{\xi}, \qquad z \in \Omega, \tag{14}$$

then $A \in W^{1,r}(\Omega)$ by the Sobolev embedding theorem together with standard properties of the Cauchy and Beurling transforms [42, [Ch. 1, (1.7)-(1.9)]; moreover $\alpha = \overline{\partial} A$. Rewriting $G^p_\alpha(\Omega)$ as $G^p_{\overline{\partial} A}(\Omega)$ is suggestive of a conformally invariant definition valid over arbitrary Dini-smooth domains in $\overline{\mathbb{C}}$, namely if φ conformally maps a Dini-smooth domain Ω onto a bounded Dini-smooth domain Ω' and if $A \in W^{1,r}(\Omega)$, then $w \in G^p_{\overline{\partial} A}(\Omega)$ if and only if $w \circ \varphi^{-1} \in G^p_\alpha(\Omega')$ with $\alpha = \overline{\partial}(A \circ \varphi^{-1})$.
We will prove momentarily that G^p_α is a real Banach space but first we stress the motivation behind its definition. If we let $A = \log \sigma^{1/2}$ so that $\alpha = \bar{\partial} \log \sigma^{1/2}$, an explicit connection between $H^p_\nu(\Omega)$ and $G^p_\alpha(\Omega)$ stems from a transformation introduced in [14]:

Proposition 1 *Assume that $\Omega \subset \overline{\mathbb{C}}$ is a proper Dini-smooth domain and that ν, p, r satisfy* (7)*. Let σ be as in* (2) *and define $\alpha \in L^r(\Omega)$ by*

$$\alpha = -\frac{\bar{\partial}\nu}{1 - \nu^2} = \bar{\partial} \log \left[\frac{1-\nu}{1+\nu}\right]^{1/2} = \bar{\partial} \log \sigma^{1/2}. \tag{15}$$

Then:

$$f = u + i\, v \in H^p_\nu(\Omega) \Longleftrightarrow w = \frac{f - \nu\overline{f}}{\sqrt{1-\nu^2}} = \sigma^{1/2}\, u + i\, \sigma^{-1/2}\, v \in G^p_\alpha(\Omega)\,. \tag{16}$$

For Δ_n as in (10)*, there are constants $C_1, C_2 > 0$ independent of f and w such that*

$$\sup_{n \in \mathbb{N}} \|f\|_{L^p(\partial\Delta_n)} \leq C_1 \sup_{n \in \mathbb{N}} \|w\|_{L^p(\partial\Delta_n)} \leq C_2 \sup_{n \in \mathbb{N}} \|f\|_{L^p(\partial\Delta_n)}. \tag{17}$$

The proof of Proposition 1 is a straightforward computation using Leibnitz's rule and the fact that $(1-\nu^2)^{-1/2} \in W^{1,r}(\Omega)$ since $r > 2$ and ν satisfies (κ).
This proposition entails that it is essentially equivalent to work with H^p_ν or G^p_α. However, equation (13) is technically simpler to handle because the derivative of the first order (*i.e.* $\overline{\partial} w$) is expressed in terms of the derivative of zero-th order (*i.e.* w).
A primary example of such a simplification is the factorization principle asserted in Proposition 2 below, which lies at the root of the connections between solutions to (13) and holomorphic functions. For slightly smoother classes of solutions, this principle goes back to [14] and was later extended to accomodate more general planar elliptic equations including those defining pseudo-analytic functions [59, Thm 2.3.1], see also [45], [26, Thm 2.1]; it was adapted to Hardy classes in [11, 27]. We provide in the Appendix B a proof which differs from [11] in that normalization must be argued differently in the multiply connected case (compare [27, Prop. 2.2.1]).

Proposition 2 *Assume that $\Omega \subset \overline{\mathbb{C}}$ is a proper Dini-smooth domain and that p, r satisfy* (7)*. Let $A \in W^{1,r}(\Omega)$, $\alpha = \overline{\partial} A$, and suppose $w \in L^p_{loc}(\Omega)$ is a distributional solution to* (13)*. Then w admits a factorization of the form*

$$w(z) = \exp(s(z))F(z), \qquad z \in \Omega, \tag{18}$$

where F is holomorphic and $s \in W^{1,r}(\Omega)$ satisfies

$$\|s\|_{L^\infty(\Omega)} \le C_0 \|s\|_{W^{1,r}(\Omega)} \le C_0' \|\alpha\|_{L^r(\Omega)} \le C_0'' \|A\|_{W^{1,r}(\Omega)}\,, \tag{19}$$

C_0, C_0', C_0'' being strictly positive constants depending only on r and Ω.
In particular w belongs to $W^{1,r}_{loc}(\Omega)$, and $w \in G^p_\alpha(\Omega)$ if and only if $F \in H^p(\Omega)$.
If Ω is n-connected and $\partial\Omega = \cup_{j=0}^n \Gamma_j$ where the Γ_j are disjoint Jordan curves, we may choose s so that $\operatorname{Im} s_{|\Gamma_j} = c_j$ where the c_j are constants such that $\sum_{j=0}^n c_j = 0$, one of which can be chosen arbitrarily.

Remark 2 *It follows from Proposition 2 and Sobolev's embedding theorem that $s \in C^{0,\gamma}(\overline{\Omega})$, uniformly with respect to w, and $w \in C^{0,\gamma}_{loc}(\Omega)$ with $\gamma = 1 - 2/r$.*

Proposition 2 quickly gives interior estimates for solutions to (CB):

Corollary 1 *Assume that $\Omega \subset \overline{\mathbb{C}}$ is a proper Dini-smooth domain and that ν, p, r satisfy* (7)*. Let $f \in L^p_{loc}(\Omega)$ be a distributional solution to* (CB)*. Then $f \in W^{2,r}_{loc}(\Omega)$.*

Proof. Defining α as in (15), it is straightforward to check that w given by (16) lies in $L^p_{loc}(\Omega)$ and satisfies (13). By Proposition 2, $w \in W^{1,r}_{loc}(\Omega)$, hence the same is true of f. Using this fact it is easily verified that the distributional derivative of $\nu\overline{\partial f}$ can be computed according to Leibnitz's rule. Consequently, setting $G := \partial f$ and applying ∂ to (CB), we obtain since ∂ and $\overline{\partial}$ commute, that $\overline{\partial} G = \nu \partial \overline{G} + (\partial\nu)\overline{G}$. As ν is real, conjugating this relation provides us with another expression for $\partial\overline{G}$, and solving for $\overline{\partial} G$ after substitution yields

$$\overline{\partial} G = \frac{\nu\overline{\partial}\nu}{1-\nu^2} G + \frac{\partial\nu}{1-\nu^2}\overline{G}$$

from which we deduce that $H = (1-\nu^2)^{1/2}\, G$ satisfies

$$\overline{\partial} H = \frac{\partial\nu}{1-\nu^2}\overline{H} \tag{20}$$

in the sense of distributions. As $H \in L^r_{loc}(\Omega)$, we deduce from Proposition 2 (applied with $\alpha = \partial\nu/(1-\nu^2)$) that $H \in W^{1,r}_{loc}(\Omega)$, implying that the same is true of $G = \partial f$ since $W^{1,r}_{loc}(\Omega)$ is an algebra for $r > 2$. From (CB) we see that also $\overline{\partial} f \in W^{1,r}_{loc}(\Omega)$, thereby achieving the proof. ■

We are now in position to prove that $H^p_\nu(\Omega)$ and $G^p_\alpha(\Omega)$ are indeed Banach spaces. For this, let φ conformally map Ω onto a normalized circular domain Ω', and $\delta_{\Omega'}$ be 1/2 if $\Omega' = \mathbb{D}$ and half the minimal distance between two components of $\partial\Omega'$ otherwise. Set

$$K_\varepsilon = \{z \in \Omega';\ \mathrm{dist}(z, \overline{\mathbb{C}} \setminus \Omega') \geq \varepsilon\, \delta_{\Omega'}\}, \qquad \varepsilon \leq 1,$$

where $\mathrm{dist}(z, E)$ indicates the distance from z to the set E. Then, K_ε is a compact subset of Ω' bounded by circles concentric with the components of $\partial\Omega'$. We put

$$\widetilde{\Delta}_\varepsilon = \varphi^{-1}(K_\varepsilon) \subset \Omega, \tag{21}$$

and define for $g \in H^p_\nu(\Omega)$ or $G^p_\alpha(\Omega)$:

$$\|g\|_p = \sup_{n \in \mathbb{N}} \|g\|_{L^p(\partial\widetilde{\Delta}_{1/n})}. \tag{22}$$

Although φ is not uniquely defined, different φ will give rise to equivalent $\|.\|_p$.
We use a lemma on holomorphic functions which is well known on the disk and the annulus but that we could not ferret out in the literature in the multiply connected case:

Lemma 1 *Let Ω be a Dini-smooth domain and f holomorphic in Ω. Then $f \in H^p(\Omega)$ if and only if $\|f\|_p < \infty$.*

The proof of Lemma 1 is given in Section C.

Theorem 1 *Assume that Ω is a proper Dini-smooth domain in $\overline{\mathbb{C}}$ and that ν, p, r satisfy* (7)*. Let moreover $A \in W^{1,r}(\Omega)$ and $\alpha = \overline{\partial} A$.*

- *(i) Endowed with* (22)*, $H^p_\nu(\Omega)$ and $G^p_\alpha(\Omega)$ are real Banach spaces, that coincide with those defined by* (8)*and* (9) *when $\Omega = \mathbb{D}$ or $\Omega = \mathbb{A}_\varrho$.*
- *(ii) It holds that $f \in H^p_\nu(\Omega)$ (resp. $w \in G^p_\alpha(\Omega)$) if, and only if f satisfies* (CB) *(resp. w satisfies* (13)*) and $|f|^p$ (resp. $|w|^p$) has a harmonic majorant in Ω.*

Proof. Since ν meets (κ) and $f = (w + \nu\overline{w})/\sqrt{1-\nu^2}$ by (16), it is enough to prove the result for $G^p_\alpha(\Omega)$.
By Proposition 2 and Lemma 1, it holds that $w \in G^p_\alpha$ if and only if $\|w\|_p < \infty$, in particular G^p_α is a real vector space on which $\|.\|_p$ defines a norm. To see that it is complete, it is enough to check that a Cauchy sequence w_k has a converging subsequence. Write $w_k = e^{s_k} F_k$ according to (18). By Remark 2, the sequence s_k is equicontinuous, on $\overline{\Omega}$, therefore some subsequence s_{l_k} converges uniformly there. By (19) again, $\|F_{l_k}\|_{L^p(\partial\widetilde{\Delta}_n)}$ is uniformly bounded, hence a normal family argument provides us with a subsequence F_{m_k} converging locally uniformly in Ω. Then w_{m_k} converges locally uniformly to some $w \in L^p_{loc}(\Omega)$, and it is clear from the definition of distributional derivatives that w solves (13). Moreover, passing to the limit under the integral sign for fixed n in the right hand

side of (22) shows that $w \in G^p_\alpha$ and that $\|w - w_{m_k}\|_p \to 0$ as k goes to infinity. This proves (i).
If g is holomorphic on Ω and φ conformally maps the latter on a domain Ω' with analytic boundaries, it is known that (10) holds for g for some admissible sequence Δ_n if, and only if $|g \circ \varphi^{-1}|^p (\varphi^{-1})'$ has a harmonic majorant on Ω' [25, Sec. 10.5]. In view of Lemma 6, we conclude that (10) holds if and only if $|g|^p$ has a harmonic majorant on Ω (*i.e.* the so-called Smirnov and Hardy classes coincide on Dini-smooth domains). Applying this to F in (18), point (ii) now follows from Proposition 2 since s is bounded. ∎

3.3 Basic properties of G^p_α and H^p_ν classes

Below we enumerate some properties that $G^p_\alpha(\Omega)$ and $H^p_\nu(\Omega)$ inherit from $H^p(\Omega)$ *via* Proposition 2. These generalize results stressed in [11, 27] for the simply or doubly connected case (except the last two which are not mentioned in [27]).
Recall f defined on Ω has non tangential ("n.t.") limit ℓ at $\xi \in \partial\Omega$ if and only if, for every $0 < \beta < \pi/2$, $f(z)$ tends to ℓ as $\Omega \cap S_{\xi\beta} \ni z \to \xi$, where $S_{\xi\beta}$ is the cone with vertex ξ and aperture 2β whose axis is normal to $\partial\Omega$ at ξ.
Also, the non-tangential maximal function of f at $\xi \in \mathbb{T}$ is

$$\mathcal{M}_f(\xi) := \sup_{z \in \Omega \cap S_{\xi\beta}} |f(z)|, \tag{23}$$

where we dropped the dependence of $\mathcal{M}_f$ on β.
Further (*cf.* (21)), we define a map $P_{\partial\Omega,\varepsilon} : \partial\widetilde{\Delta}_\varepsilon \to \partial\Omega$ as follows (projection on the boundary). When Ω is normalized circular, $P_{\partial\Omega,\varepsilon}(\xi)$ is the radial projection of ξ on the boundary circle nearest to ξ. When Ω is a general Dini-smooth domain in $\overline{\mathbb{C}}$, we pick a conformal map ψ onto a normalized circular domain Ω' and we set $P_{\partial\Omega,\varepsilon} = \psi^{-1} \circ P_{\partial\Omega',\varepsilon} \circ \psi$. Clearly, $P_{\partial\Omega,\varepsilon}$ is a homeomorphism. Different ψ produce different $P_{\partial\Omega,\varepsilon}$, but the results below hold for any of them.
Assumptions being as in Theorem 1, the following properties holds.

Property 1 *Any f in $H^p_\nu(\Omega)$ (resp. $w \in G^p_\alpha(\Omega)$) has a non-tangential limit almost everywhere on $\partial\Omega$, thereby defining a trace function $tr_{\partial\Omega} f \in L^p(\partial\Omega)$[6]. It holds that*

$$\lim_{\varepsilon \to 0} \|tr_{\partial\Omega} f - f \circ P^{-1}_{\partial\Omega,\varepsilon}\|_{L^p(\partial\Omega)} = 0 \qquad \Big(\text{resp. } \lim_{\varepsilon \to 0} \|tr_{\partial\Omega} w - w \circ P^{-1}_{\partial\Omega,\varepsilon}\|_{L^p(\partial\Omega)} = 0\Big). \tag{24}$$

Property 2 *The quantity $\|tr_{\partial\Omega} \,.\, \|_{L^p(\partial\Omega)}$ defines an equivalent norm on $H^p_\nu(\Omega)$ (resp. $G^p_\alpha(\Omega)$). As to the maximal function, it holds when $f \in H^p_\nu(\Omega)$ (resp. $w \in G^p_\alpha(\Omega)$) that*

$$\|\mathcal{M}_f\|_{L^p(\partial\Omega)} \le C \|tr_{\partial\Omega} f\|_{L^p(\partial\Omega)} \qquad \Big(\textit{resp. } \|\mathcal{M}_w\|_{L^p(\partial\Omega)} \le C \|tr_{\partial\Omega} w\|_{L^p(\partial\Omega)}\Big) \tag{25}$$

where C depends only on Ω, σ (resp. α), p and the aperture β used in the definition of the maximal function.

[6]There is no discrepancy in the notation since the nontangential limit coindes with the Sobolev trace whenever it exists [11, Prop. 4.3.3]

Proof. In view of Proposition 1 and the Hölder-continuity of ν, Properties 1 and 2 for f follow from their counterpart for w. The latter are consequences of (18), Remark 2, and the corresponding property in $H^p(\Omega)$, *cf.* Lemma 7.

■

Property 3 *The space* $tr_{\partial\Omega} H^p_\nu(\Omega)$ *(resp.* $tr_{\partial\Omega} G^p_\alpha(\Omega)$*) is closed in* $L^p(\partial\Omega)$*. If* $f \in H^p_\nu(\Omega)$ *(resp.* $w \in G^p_\alpha(\Omega)$*) is not identically zero, then* $tr_{\partial\Omega} f$ *(resp.* $tr_{\partial\Omega} f$*) cannot vanish on a subset of* $\partial\Omega$ *with positive measure.*

Proof. As before it is enough to prove it for G^p_α. That $\text{tr}_{\partial\Omega} G^p_\alpha(\Omega)$ is closed in $L^p(\partial\Omega)$ follows from Property 2 and Theorem 1 point (i). That $w \neq 0$ cannot vanish on a subset of $\partial\Omega$ with positive measure is immediate from (18) and the corresponding result in $H^p(\Omega)$ [25, Thm 2.2]. ■

Property 4 *If* f *(resp.*w*) is a nonzero member of* $H^p_\nu(\Omega)$ *(resp.* $G^p_\alpha(\Omega)$*), then* $\log|tr_{\partial\Omega} f|$ *(resp.* $\log|tr_{\partial\Omega} w|$*) lies in* $L^1_{\mathbb{R}}(\partial\Omega)$*. Moreover the zeros of* f *(resp.* w*) are isolated, and if we enumerate them as* ξ_j*,* $j \in \mathbb{N}$*, it holds for any* $z_0 \in \Omega$*,* $z_0 \neq \xi_j$ *for all* j*, that*

$$\sum_{j=1}^{\infty} g_\Omega(\xi_j, z_0) < \infty, \tag{26}$$

where $g_\Omega(., z_0)$ *is the Green function of* Ω *with pole at* z_0[7]*.*

When $\Omega = \mathbb{D}$, (26) is equivalent to the classical Blaschke condition $\sum_j (1 - |\xi_j|) < \infty$.

Proof. If f and w are related by (16), their log-modulus are comparable and they share the same zeros. Therefore it is enough to prove the result for w. That $\log|\text{tr}_{\partial\Omega} w| \in L^1_{\mathbb{R}}(\partial\Omega)$ unless $w \equiv 0$ follows from (18) and the corresponding result for holomorphic functions, *cf.* Lemma 7. In another connection, (18) entails that the zeros of w are those of the holomorphic function F, hence they are isolated. Moreover, since $F \in H^p(\Omega)$, it follows from the decomposition theorem [25, Sec. 10.5] and a classical result on the disk [34, Thm 5.4] that the subharmonic function $\log|F|$ has a harmonic majorant [25, Sec. 2.6, Ex. 10]. Relation (26) now follows from [50, Thm 4.5.5]. ■

Property 5 *Each* $f \in H^p_\nu(\Omega)$ *satisfies the maximum principle, i.e.* $|f|$ *cannot assume a relative maximum in* Ω *unless it is constant. More generally, a non-constant function in* $H^p_\nu(\Omega)$ *is open and the preimage of any value is discrete.*

Proof. If we let $\nu_f(z) := \nu(z)\overline{\partial f}/\partial f(z)$ if $\partial f(z) \neq 0$ and $\nu_f(z) = 0$ otherwise, then f is a pointwise a.e. solution in Ω of the *classical* Beltrami equation

$$\overline{\partial} f = \nu_f \, \partial f, \qquad |\nu_f| < \kappa < 1. \tag{27}$$

Moreover since $r > 2$, it holds that $W^{1,r}_{loc}(\Omega)$ is an algebra so that w given by (18) hence also f given by (16) lies in $W^{1,r}_{loc}(\Omega)$. It follows by Stoilov factorization [42, Thm 11.1.2] that $f = G(h(z))$, where h is a quasi-conformal topological map $\Omega \to \mathbb{C}$ satisfying (27) and G a holomorphic function on $h(\Omega)$. The conclusion now follows at once from the corresponding properties of holomorphic functions. ■

[7] $g_\Omega(., z_0)$ is the unique harmonic function in $\Omega \setminus z_0$ such that $g_\Omega(z, z_0) + \log|z - z_0|$ is bounded in a neighborhood of z_0 and $g_\Omega(z, z_0) \to 0$ when $z \to \xi \in \partial\Omega$, see [50, Sec. 4.4.].

Property 6 *To any $p_1 \in [p, 2p)$ there is a constant C depending only on Ω, ν (resp. α), and p_1 such that, for each $f \in H^p_\nu(\Omega)$ (resp. $w \in G^p_\alpha(\Omega)$),*

$$\|f\|_{L^{p_1}(\Omega)} \le C\|f\|_p\,, \qquad \Big(\text{resp. } \|w\|_{L^{p_1}(\Omega)} \le C\|w\|_p\Big). \tag{28}$$

Moreover, to each open set $\mathcal{O}$ with $\overline{\mathcal{O}} \subset \Omega$, there is a constant c depending only on Ω, $\mathcal{O}$, ν (resp. α), r, and p such that,

$$\|f\|_{W^{2,r}(\mathcal{O})} \le c\,\|f\|_p\,, \qquad \Big(\text{resp. } \|w\|_{W^{1,r}(\mathcal{O})} \le c'\,\|w\|_p\Big). \tag{29}$$

Proof. We may assume Ω is bounded. By Proposition 1 and Remark 2, inequality (28) for f follows from that on w. The latter is a consequence of (18) and the corresponding property in $H^p(\Omega)$, *cf.* Lemma 8.
Inequality (29) for $\|w\|_{W^{1,r}(\mathcal{O})}$ follows at once from Property 2 and the Cauchy formula as applied to F in (18). Observe from (16) that a similar inequality holds for $\|f\|_{W^{1,r}(\mathcal{O})}$. In addition, if we pick $\varepsilon > 0$ so small that $\overline{\mathcal{O}}$ lies interior to $\widetilde{\Delta}_\varepsilon$ (*cf.* (21)) and if, in the previous argument, we apply the Cauchy formula to F on each curve $\partial\widetilde{\Delta}_t$ for $t \in [\varepsilon, \varepsilon/2]$ and then integrate with respect to t, we obtain an inequality of the form

$$\|w\|_{W^{1,r}(\mathcal{O})} \le C\|w\|_{L^r(\widetilde{\Delta}_{\varepsilon/2}\setminus\widetilde{\Delta}_\varepsilon)}. \tag{30}$$

In view of (20), we may apply this to $(1-\nu^2)^{1/2}\partial f$ with Ω replaced by the interior of $\widetilde{\Delta}_{\varepsilon/3}$ and, since $(1-\nu^2)^{-1/2} \in W^{1,r}(\widetilde{\Delta}_{\varepsilon/2})$ which is an algebra, we deduce an inequality of the form $\|\partial f\|_{W^{1,r}(\mathcal{O})} \le C'\|f\|_{W^{1,r}(\widetilde{\Delta}_{\varepsilon/2})}$. As f satisfies (CB) a similar inequality holds for $\overline{\partial} f$, and since $\|f\|_{W^{1,r}(\widetilde{\Delta}_{\varepsilon/2})} \le C''\|f\|_p$ as pointed out already, we get that part of (29) dealing with $\|f\|_{W^{2,r}(\mathcal{O})}$. ∎

Remark 3 *If we pick $\sigma \in W^{1,2}(\mathbb{D})$ satisfying* (4) *but having no nontangential limit a.e. on $\mathbb{T}$ [21], then $\sigma^{1/2} + i\sigma^{-1/2}$ is a solution to* (13) *meeting the Hardy condition* (8) *for all $p < \infty$ (since $W^{1/2,2}(\mathbb{T}_r) \subset VMO(\mathbb{T}_r)$ [17]), but having no nontangential limit a.e. on $\mathbb{T}$. Hence the assumption that $r > 2$ is necessary for Property 1 to hold.*

We conclude this section with a parameterization of $G^p_\alpha(\Omega)$ by $H^p(\Omega)$-functions which proceeds differently from Proposition 2, and is fundamental to our approach of the Dirichlet problem. It was essentially obtained on the disk in [11] when $\sigma \in W^{1,\infty}(\Omega)$, and then carried over to the annulus in [27] under the assumption that $r > 2$ and $p > r/(r-2)$. It features the operator T_α, defined for $h \in L^p(\Omega)$ by the formula

$$T_\alpha h(z) = \frac{1}{2\pi i}\iint_\Omega \frac{\alpha(\xi)\overline{h}(\xi)}{\xi - z} d\xi \wedge d\overline{\xi}\,, \quad z \in \Omega.. \tag{31}$$

Note that $T_\alpha h(z)$ is well-defined for a.e. z when p, r satisfy (7), since $\alpha\overline{h} \in L^\gamma(\Omega)$ with $1/\gamma = 1/p + 1/r < 1$. Also, T_α is linear when $L^p(\Omega)$ is viewed as a real Banach space.

Proposition 3 *Assume that $\Omega \subset \overline{\mathbb{C}}$ is a bounded Dini-smooth domain and that $\alpha \in L^r(\Omega)$ while p, r satisfy* (7)*. Then T_α is compact from $L^p(\Omega)$ into itself, and $I - T_\alpha$ is invertible.*

It holds that $G^p_\alpha(\Omega) = (I - T_\alpha)^{-1}H^p(\Omega)$, and if $w \in G^p_\alpha(\Omega)$ then the unique $g \in H^p$ such that $w = (I - T_\alpha)^{-1}g$ is the Cauchy integral of $tr_{\partial\Omega}w$:

$$g(z) = \mathcal{C}(tr_{\partial\Omega}w) = \frac{1}{2i\pi}\int_{\partial\Omega}\frac{tr_{\partial\Omega}w(\xi)}{\xi - z}\,d\xi, \qquad z \in \Omega. \tag{32}$$

Moreover, we have that $\|w\|_p \leq C\|g\|_p$ where the constant C depends only of Ω, α, and p.

The proof of Proposition 3 is given in Appendix F and follows the lines of the proof of [11, Thm 4.4.1.1], although technicalities arise to handle the weaker assumption (7).

4 A decomposition theorem

Let $\Omega \subset \overline{\mathbb{C}}$ be Dini-smooth and write $\Omega = \overline{\mathbb{C}} \setminus \cup_{j=0}^n K_j$ where the K_j are disjoint compact sets in $\overline{\mathbb{C}}$. We establish in this section a result which, loosely speaking, asserts that every function in $H^p_\nu(\Omega)$ is a sum of members of $H^p_{\nu_j}(\overline{\mathbb{C}} \setminus K_j)$ for some appropriate extensions ν_j of ν. This result stands analogous to the decomposition theorem in $H^p(\Omega)$ [25, Sec. 10.5] but, unlike Properties 1-4 in Section 3.3, it is not an immediate consequence of the latter *via* Proposition 2.
We denote by $H^{p,0}_\nu(\Omega)$ the subspace of $H^p_\nu(\Omega)$ made of functions f such that

$$\int_{\partial\Omega} \mathrm{Im}(\mathrm{tr}_{\partial\Omega}f(s))\,|ds| = 0. \tag{33}$$

Moreover, we let $H^{p,00}_\nu(\Omega)$ be the subspace of $H^{p,0}_\nu(\Omega)$ consisting of those f for which

$$\int_{\partial\Omega} \mathrm{tr}_{\partial\Omega}f(s)\;|ds| = 0. \tag{34}$$

We record for later use a topological version of the decomposition theorem for holomorphic Hardy spaces. For normalized cicular domains it is established *e.g.* in [20, Lem. 2.1][8], and it carries over immediately to Dini-smooth domains by conformal invariance.

Lemma 2 *If $\Omega = \overline{\mathbb{C}} \setminus \cup_{j=0}^n K_j$ is a Dini-smooth domain as above, then additively*

$$H^p(\Omega) = H^p(\overline{\mathbb{C}} \setminus K_0) \oplus H^{p,00}(\overline{\mathbb{C}} \setminus K_1) \oplus \cdots \oplus H^{p,00}(\overline{\mathbb{C}} \setminus K_n),$$

where the direct sum is topological.

Recalling from (11) the notation $\check{\nu}$, we also have the following lemma which is established in [32] for $\nu \in W^{1,\infty}_{\mathbb{R}}(\mathbb{D})$.

Lemma 3 *Assume that Ω is a proper Dini-smooth domain in $\overline{\mathbb{C}}$ and that p, r, ν satisfy (7). Then the following topological decomposition holds:*

$$L^p(\mathbb{T}) = tr_{\mathbb{T}}H^p_\nu(\mathbb{D}) \oplus tr_{\mathbb{T}}H^{p,00}_{\check{\nu}}(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}).$$

[8] In this reference, Hardy spaces are defined through harmonic majorants, but we know this is equivalent to the definition based on (10) for Dini-smooth domains.

Proof. The proof of [32, Cor. 3] applies without change to $\nu \in W^{1,r}_{\mathbb{R}}(\Omega)$, granted Lemma 4 in Section 5 to come. ∎

The main result in this section is the following generalization of Lemma 2.

Theorem 2 *Let $\Omega = \overline{\mathbb{C}} \setminus \cup_{j=0}^{n} K_j$ be a Dini-smooth domain, the K_j being disjoint compact sets in $\overline{\mathbb{C}}$, and p, r, ν meet (7). Then to each j, there is $\nu_j \in W^{1,r}_{\mathbb{R}}(\overline{\mathbb{C}} \setminus K_j)$ with $\nu_{j|\Omega} = \nu$ and $\nu_{j|K_l} = \nu_{k|K_l}$ when $l \neq j, k$, satisfying (κ) and such that*

$$H^p_\nu(\Omega) = H^p_{\nu_0}(\overline{\mathbb{C}} \setminus K_0) \oplus H^{p,00}_{\nu_1}(\overline{\mathbb{C}} \setminus K_1) \oplus \cdots \oplus H^{p,00}_{\nu_n}(\overline{\mathbb{C}} \setminus K_n), \tag{35}$$

where the direct sum is topological.

Proof. It is clear that the right side of (35) is included in the left. To prove the converse, we may assume by conformal invariance (see Lemma 6) that Ω is normalized circular. First, we consider the special case where $\Omega = \mathbb{A}_\varrho$ ($n = 1$) for some $0 < \varrho < 1$. In this case, we put ν_i and ν_e for the seeked extensions of ν to $\mathbb{D}$ and $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}_\varrho}$ (the subscripts respectively stand for "interior" and "exterior"). It is standard that there exists $\widetilde{\nu} \in W^{1,r}_{\mathbb{R}}(\mathbb{D})$ such that $\widetilde{\nu}_{|\mathbb{A}_\varrho} = \nu$ [54, Sec. VI.3, Thm 5]. Letting $\varepsilon > 0$ be so small that $\|\nu\|_{L^\infty(\mathbb{A}_\varrho)} < \kappa - \varepsilon$, we set $\nu_i = \min(\widetilde{\nu}, \kappa - \varepsilon)$, so that ν_i lies in $W^{1,r}(\mathbb{D})$, extends ν, and meets (κ).
Let $f \in H^p_\nu(\mathbb{A}_\varrho)$ so that $\mathrm{tr}_{\mathbb{T}} f \in L^p(\mathbb{T})$. Lemma 3 is to the effect that

$$\mathrm{tr}_{\mathbb{T}} f = \mathrm{tr}_{\mathbb{T}} f_i + \mathrm{tr}_{\mathbb{T}} f_e\,, \tag{36}$$

for some $f_i \in H^p_{\nu_i}(\mathbb{D})$ and $f_e \in H^{p,00}_{\check{\nu}_i}(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$. Put $\mathcal{F}_i = f - f_{i|\mathbb{A}_\varrho} \in H^p_\nu(\mathbb{A}_\varrho)$, and let $\mathcal{F}_e$ and ν_e be defined on $\overline{\mathbb{C}} \setminus \varrho\overline{\mathbb{D}}$ by:

$$\mathcal{F}_e = \mathcal{F}_i \vee f_e = \begin{cases} \mathcal{F}_i \text{ on } \mathbb{A}_\varrho\,, \\ f_e \text{ on } \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}\,, \end{cases} \qquad \nu_e = \nu \vee \check{\nu}_i = \begin{cases} \nu \text{ on } \mathbb{A}_\varrho\,, \\ \check{\nu}_i \text{ on } \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}\,. \end{cases}$$

Since $\nu = \check{\nu}_i$ on $\mathbb{T}$, it holds that $\nu_e \in W^{1,r}_{\mathbb{R}}(\overline{\mathbb{C}} \setminus \varrho\overline{\mathbb{D}})$ (by absolutely continuity on lines in polar coordinates) and obviously it satisfies a condition similar to (κ).
We claim that $\mathcal{F}_e \in H^p_{\nu_e}(\overline{\mathbb{C}} \setminus \varrho\overline{\mathbb{D}})$. By construction it satisfies (10) on $\overline{\mathbb{C}} \setminus \varrho\overline{\mathbb{D}}$ and it is a solution to (CB) (with ν_e instead of ν) on $\mathbb{A}_\varrho \cup (\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$. Thus, in order to establish that (CB) holds on the whole of $\overline{\mathbb{C}} \setminus \varrho\overline{\mathbb{D}}$, it is enough to prove that $\overline{\partial}\mathcal{F}_e = \nu_e\, \overline{\partial \mathcal{F}_e}$, in the sense of distributions, on some annulus $\mathbb{A}_{r,R}$ with $\varrho < r < 1 < R$. That is, we must show that for all $\phi \in \mathcal{D}(\mathbb{A}_{r,R})$,

$$\begin{aligned} I_{r,R}(\phi) &= \iint_{\mathbb{A}_{r,R}} (-\mathcal{F}_e\,\overline{\partial}\phi + \overline{\mathcal{F}_e}\,\overline{\partial}(\nu_e\,\phi))\; dm(z) \\ &= \frac{i}{2} \iint_{\mathbb{A}_{r,R}} (-\mathcal{F}_e\,\overline{\partial}\phi + \overline{\mathcal{F}_e}\,\overline{\partial}(\nu_e\,\phi))\; dz \wedge d\overline{z} = 0. \end{aligned} \tag{37}$$

By Property 6 in Section 3.3, we get that $f_i \in L^{p_1}(\mathbb{D})$, $f_e \in L^{p_1}(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$, and $f \in L^{p_1}(\mathbb{A}_\varrho)$ hence $\mathcal{F}_i \in L^{p_1}(\mathbb{A}_\varrho)$ and $\mathcal{F}_e \in L^{p_1}(\overline{\mathbb{C}} \setminus \varrho\overline{\mathbb{D}})$, for some $p_1 > 2$. Thus, in view of (7) and Hölder's inequality, the integrand in (37) lies in $L^a(\mathbb{A}_{r,R})$ for some $a > 1$ which justifies the limiting relation:

$$\begin{aligned}
-2iI_{r,R}(\phi) &= \iint_{\mathbb{A}_r} (-\mathcal{F}_i\,\overline{\partial}\phi + \overline{\mathcal{F}_i}\,\overline{\partial}(\nu\,\phi))\;dz\wedge d\overline{z} + \iint_{\mathbb{A}_{1,R}} (-f_e\,\overline{\partial}\phi + \overline{f_e}\,\overline{\partial}(\check{\nu}_i\,\phi))\;dz\wedge d\overline{z}\\
&= \lim_{\epsilon\to 0}\underbrace{\iint_{\mathbb{A}_{r,1-\epsilon}} (-\mathcal{F}_i\,\overline{\partial}\phi + \overline{\mathcal{F}_i}\,\overline{\partial}(\nu\,\phi))\;dz\wedge d\overline{z}}_{I_{r,\epsilon}(\phi)}\\
&\qquad + \lim_{\epsilon\to 0}\underbrace{\iint_{\mathbb{A}_{1+\epsilon,R}} (-f_e\,\overline{\partial}\phi + \overline{f_e}\,\overline{\partial}(\check{\nu}_i\,\phi))\;dz\wedge d\overline{z}}_{I_{R,\epsilon}(\phi)}.
\end{aligned}$$

As $\check{\nu}_i \in W^{1,r}(\overline{\mathbb{C}}\setminus\overline{\mathbb{D}})$ and $f_e \in W^{1,r}_{loc}(\overline{\mathbb{C}}\setminus\overline{\mathbb{D}})$ by Proposition 2, we can apply Stoke's theorem:

$$I_{R,\epsilon}(\phi) = <\overline{\partial} f_e - \check{\nu}_i\overline{\partial f_e}, \phi>_{\mathbb{A}_{1+\epsilon,R}} + \int_{\partial\mathbb{A}_{1+\epsilon,R}} (f_e - \check{\nu}_i\overline{f_e})\phi\;dz.$$

Since $L^a(\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}) \ni \overline{\partial} f_e - \check{\nu}_i\overline{\partial f_e} = 0$ a.e., we are left with

$$I_{R,\epsilon}(\phi) \;=\; \int_{\partial\mathbb{A}_{1+\epsilon,R}} (f_e - \check{\nu}_i\overline{f_e})\phi\;dz = -\int_{\mathbb{T}_{1+\epsilon}} (f_e - \check{\nu}_i\overline{f_e})\phi\;dz$$

since ϕ vanishes on $\mathbb{T}_R$. Passing to the limit using Property 1 in Section 3.3 and the continuity of $\check{\nu}_i$ in $\overline{\mathbb{C}}\setminus\mathbb{D}$ yields

$$\lim_{\epsilon\to 0} I_{R,\epsilon}(\phi) = -\int_{\mathbb{T}} (f_e - \check{\nu}_i\overline{f_e})\phi\;dz.$$

Likewise one can show that

$$\lim_{\epsilon\to 0} I_{r,\epsilon}(\phi) = \int_{\mathbb{T}} (\mathcal{F}_i - \nu\overline{\mathcal{F}_i})\phi\;dz,$$

and since $\mathrm{tr}_{\mathbb{T}}\mathcal{F}_i = \mathrm{tr}_{\mathbb{T}} f_e$ by (36) while $\nu_{|\mathbb{T}} = \nu_{i|\mathbb{T}} = \check{\nu}_{i|\mathbb{T}}$, we finally conclude that

$$-2iI_{r,R}(\phi) = \lim_{\epsilon\to 0} I_{\Omega_{r,\epsilon}}(\phi) + \lim_{\epsilon\to 0} I_{\Omega_{R,\epsilon}}(\phi) = 0,$$

which is (37). This proves the claim that $\mathcal{F}_e \in H^p_{\nu_e}(\overline{\mathbb{C}}\setminus\varrho\overline{\mathbb{D}})$. Now,

$$f = \left(f_{i|\mathbb{A}_\varrho} + \int_{\mathbb{T}_\varrho} \mathrm{tr}_{\mathbb{T}_\varrho}\mathcal{F}_e\right) + \left(\mathcal{F}_{e|\mathbb{A}_\varrho} - \int_{\mathbb{T}_\varrho} \mathrm{tr}_{\mathbb{T}_\varrho}\mathcal{F}_e\right) \tag{38}$$

is the decomposition we look for on $\mathbb{A}_\varrho$.
The (not yet topological) existence of (35) on $\Omega = \mathbb{A}_\varrho$ implies its existence on any Dini-smooth doubly connected domain by conformal invariance.
Subsequently, we get it over any normalized circular domain by induction on n: if $\Omega = \mathbb{D}\setminus\cup_{j=1}^n\overline{\mathbb{D}}_{a_j,r_j}$ and ϱ is close enough to 1 that $\mathbb{D}_\varrho \supset \cup_{j=1}^n\overline{\mathbb{D}}_{a_j,r_j}$, we decompose $f_{|\mathbb{A}_\varrho} = f_{0|\mathbb{A}_\varrho} + f_{e|\mathbb{A}_\varrho}$ with $f_0 \in H^p_{\nu_0}(\mathbb{D})$ and $f_e \in H^p_{\nu_e}(\overline{\mathbb{C}}\setminus\varrho\overline{\mathbb{D}})$ for suitable extensions ν_0, ν_e of ν to $\mathbb{D}$ and $\overline{\mathbb{C}}\setminus\varrho\overline{\mathbb{D}}$ respectively. As it coincides with f_e on $\mathbb{A}_\varrho$, the function $f - f_0$ lies in $H^p_{\nu_0\vee\nu_e}(\Omega')$ where $\Omega' = \overline{\mathbb{C}}\setminus\cup_{j=1}^n\overline{\mathbb{D}}_{a_j,r_j}$ is $n-1$-connected, hence we can carry out the induction step.

Observe that in the latter the ν_j will coincide with ν_e on $\overline{\mathbb{C}} \setminus \varrho\overline{\mathbb{D}}$ for $1 \leq j \leq n$, thereby proving the existence of decomposition (35) in general.
To see that the sum is direct, write

$$f = \sum_{j=0}^{n} f_j \qquad \text{with} \;\; f_0 \in H^p_{\nu_0}(\mathbb{D}) \;\; \text{and} \;\; f_j \in H^{p,00}_{\nu_j}(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_{a_j,r_j}), \;\; 1 \leq j \leq n.$$

Suppose that $f_0 = -\sum_{j=1}^{n} f_j$ on Ω. Then $h = f_0 \vee (-\sum_{j=1}^{n} f_j)_{|\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}}$ lies in $W^{1,p}(\overline{\mathbb{C}})$ and satisfies $\overline{\partial} h = \widetilde{\nu}\,\overline{\partial h}$ with $\widetilde{\nu} = \nu_0 \vee \nu_1 \vee \cdots \vee \nu_n \in W^{1,r}(\overline{\mathbb{C}})$ (remember two ν_j coincide wherever they are both defined). By the arguments in Remark 1, we deduce that h, thus also f_0 is a constant, say C_0. If we put $\widetilde{f_1} = f_1 + C_0$, then $\widetilde{f_1} = -\sum_{j=2}^{n} f_j$ and arguing the same way we find that $\widetilde{f_1}$, thus also f_1, is in turn a constant. Proceeding inductively each f_j is a constant C_j, and $C_j = 0$ for $j \geq 1$ by (34). Therefore $f_j = 0$ for $j \geq 1$ and then $f_0 = 0$ as well.
Finally, having shown that the natural map

$$H^p_{\nu_0}(\overline{\mathbb{C}} \setminus K_0) \oplus H^{p,00}_{\nu}(\overline{\mathbb{C}} \setminus K_1) \oplus \cdots \oplus H^{p,00}_{\nu_1}(\overline{\mathbb{C}} \setminus K_n) \longrightarrow H^p_{\nu_n}(\Omega)$$

is injective and surjective, we observe from Property 2 in Section 3.3 that it is continuous, hence a homeomorphism by the open mapping theorem.

■

5 The Dirichlet problem

For Ω a Dini-smooth domain, we let $\mathcal{U}^p(\Omega)$ consist of those functions U satisfying (3) for which $\|U\|_p < \infty$ (*cf.* (22)). In this section, we investigate the solvability of the Dirichlet problem for the class $\mathcal{U}^p(\Omega)$ with boundary data in $L^p(\partial\Omega)$. In other words, the solution is understood to meet condition (10) for some admissible sequence Δ_n, and to converge non-tangentially on $\partial\Omega$ to some prescribed member of $L^p(\partial\Omega)$.
The existence of non-tangential estimates of the form (25) for functions in $\mathcal{U}^p(\Omega)$ will make these requirements equivalent, in the present context, to standard notions of solvability [29]. Note that $\sigma \in W^{1,r}(\Omega)$ with p, r as in (7) is an assumption which is not covered by the Carleson condition set up in [24, 44][9].
We study the Dirichlet problem for (3) in relation to the issue of finding a function in $H^p_\nu(\Omega)$ with prescribed real part on $\partial\Omega$, ν being as in (2). Slightly abusing terminology, we call this issue the Dirichlet problem in $H^p_\nu(\Omega)$. Clearly, a solution to the Dirichlet problem for (3) is obtained from a solution to the Dirichlet problem in $H^p_\nu(\Omega)$ by taking the real part. However, we shall see that the Dirichlet problem in $H^p_\nu(\Omega)$ is not always solvable on multiply connected domains, whereas the Dirichlet problem for (3) is solvable. On simply connected domains the two problems are equivalent as follows from the next lemma. When $\sigma \in W^{1,\infty}_{\mathbb{R}}(\Omega)$, this is essentially proved in [11] except for estimate (39).

[9]The condition is that the *sup* on the Carleson domain $B(z, d(z,\partial\Omega)/2)$ of $d(.,\partial\Omega)|\nabla\sigma(.)|^2$, when viewed as a function of z, should be the density of a Carleson (or vanishing Carleson) measure. Now, for χ_k the caracteristif function of $1 - 1/k - e^{-k^2} < |z| < 1 - 1/k + e^{-k^2}$, $k \geq 2$, the radial function $\psi(r,\theta) = \sum_{k=2}^{\infty} e^k \chi_k$ lies in $L^r(\mathbb{D})$ for each $r \in (2,\infty)$. If we put $\sigma(r,\theta) = 1 + \int_0^r \psi(\rho)\,d\rho$, the corresponding density is not even integrable on Carleson domains.

Lemma 4 *Let Ω be a Dini-smooth simply connected domain and $\sigma \in W^{1,r}(\Omega)$ satisfy (4) with p, r as in (7). For every $u \in L^p_{\mathbb{R}}(\partial\Omega)$, the following assertions hold.*
(i) There exists a unique solution U to (3) in Ω such that $\|U\|_p < \infty$ (cf. (22)) and the non-tangential limit of U on $\partial\Omega$ is u. The function U satisfies non-tangential estimates of the form (25). Moreover, to every open set O with compact closure $\overline{O} \subset \Omega$, there is a constant $c = c(\Omega, O, \sigma, r, p) > 0$ such that:

$$\|U\|_{W^{2,r}(O)} \leq c \, \|u\|_{L^p(\partial\Omega)} \,. \tag{39}$$

(ii) For ν defined by (2), there exists a unique $f \in H^{p,0}_\nu(\Omega)$ such that $Re\ tr_{\partial\Omega} f = u$. Moreover $\|tr_{\partial\Omega} f\|_{L^p(\partial\Omega)} \leq c\|u\|_{L^p(\partial\Omega)}$ for some constant $c = c(\Omega, p, \sigma) > 0$.

We prove Lemma 4 in Appendix G.

5.1 Conjugate functions

A σ-harmonic conjugate to $U \in \mathcal{U}^p(\Omega)$ is a real function V such that $U + iV \in H^p_\nu(\Omega)$. If it exists, a σ-harmonic conjugate is unique up to an additive constant and is a solution to (5), see (1).
We also say that $u \in L^p_{\mathbb{R}}(\partial\Omega)$ has σ-harmonic conjugate $v \in L^p_{\mathbb{R}}(\partial\Omega)$ if $u + iv = \mathrm{tr}_{\partial\Omega} f$ for some $f \in H^p_\nu(\Omega)$. If $U \in \mathcal{U}^p(\Omega)$ has σ-harmonic conjugate V, then clearly $\mathrm{tr}_{\partial\Omega} U$ has σ-harmonic conjugate $\mathrm{tr}_{\partial\Omega} V$. Theorem 3 further below asserts that each $u \in L^p_{\mathbb{R}}(\partial\Omega)$ is uniquely the trace of some $U \in \mathcal{U}^p(\Omega)$, so the two notions of conjugacy (in the domain and on the boundary) will soon be proven equivalent.
Lemma 4 entails that if Ω is simply connected, then each $U \in \mathcal{U}^p(\Omega)$ (resp. $u \in L^p(\partial\Omega)$) has a σ-harmonic conjugate. If Ω is multiply connected it is not so as we will now see.

Lemma 5 *Let Ω be a Dini-smooth domain and $\sigma \in W^{1,r}(\Omega)$ satisfy (4) with p, r as in (7). Each $U \in \mathcal{U}^p(\Omega)$ lies in $W^{2,r}_{loc}(\Omega)$, and if $\Gamma \subset \Omega$ is a rectifiable Jordan curve then $\int_\Gamma \sigma \partial_n U$ depends only on the homotopy class of Γ. Moreover, the function U has a σ-harmonic conjugate if and only if this integral is zero for all Γ.*

Proof. By Lemma 6 we may assume that $\Omega = \mathbb{D} \setminus \cup_{j=1}^n \overline{\mathbb{D}}_{a_j,r_j}$ is normalized circular. Since $\mathcal{U}^p(\Omega) \subset L^p(\Omega)$, the proofs of Lemma 4 and Corollary 1 imply that U has a σ-harmonic conjugate in $W^{2,r}_{loc}(\Omega_1)$ on every Dini-smooth simply connected relatively compact subdomain $\Omega_1 \subset \Omega$. In particular $U \in W^{2,r}_{loc}(\Omega)$ so that ∇U is continuous, hence $\int_\Gamma \sigma \partial_n U$ is well-defined. For small ε, the circles $\mathbb{T}_{a_j,r_j+\varepsilon}$ for $1 \leq j \leq n$ form a homotopy basis in Ω. Hence in order to prove that $\int_\Gamma \sigma \partial_n U$ depends only on the homotopy class of Γ, it is enough to assume Γ is homotopic to $\mathbb{T}_{a_j,r_j+\varepsilon}$ and to show that $\int_\Gamma \sigma \partial_n U = \int_{\mathbb{T}_{a_j,r_j+\varepsilon}} \sigma \partial_n U$. If Γ is smooth and disjoint from $\mathbb{T}_{a_j,r_j+\varepsilon}$, or intersects it transversally, the result follows immediately from the Green formula. The genericity of transversal intersections [39] now implies the result for every smooth Γ by continuity. If Γ is merely rectifiable with parametrization $\gamma \in W^{1,1}(\mathbb{T})$,we approximate the latter in the Sobolev sense by a smooth function on $\mathbb{T}$, which gives us the result.
Now, if U has σ-harmonic conjugate V, then: by (1) $\int_\Gamma \sigma \partial_n U = \int_\Gamma dV = 0$ for every Γ. Conversely, if $\int_\Gamma \sigma \partial_n U = 0$ for every Γ, then by continuation along any path we can define globally in Ω a real-valued function V such that $f = U + iV \in W^{2,r}_{\mathbb{R},loc}(\Omega)$ satisfies (CB). We have to prove that $f \in H^p_\nu(\Omega)$. By Proposition 2, it is equivalent to show that if w

given by (16) gets factored as in (18), then $F \in H^p(\Omega)$. The question localizes around each component of $\partial\Omega$, *i.e.* it is enough to establish for all $1 \leq j \leq n$ that $\|F\|_{L^p(\mathbb{T}_{a_j,r_j+\eta})}$ is bounded independently of η when the latter is small enough, and that $\|F\|_{L^p(\mathbb{T}_{1-\eta})}$ is likewise bounded independently of η. Consider this last case, the others being similar. Since $U \in \mathcal{U}^p(\Omega)$ we know that $\|\mathrm{Re}\, f\|_{L^p(\mathbb{T}_{1-\eta})}$ is bounded independently of η, so by (16) the same is true of $\|\mathrm{Re}\, w\|_{L^p(\mathbb{T}_{1-\eta})}$. As $\|\mathrm{Im}\, F\|_{L^p(\mathbb{T}_{1-\eta})} \leq C_1 + C_2 \|\mathrm{Re}\, F\|_{L^p(\mathbb{T}_{1-\eta})}$ by Lemma 9, we can argue as in the proof of Lemma 4 (see (G.1) and after) to the effect that $\|F\|_{L^p(\mathbb{T}_{1-\eta})}$ is bounded independently of η, as desired. ∎

Next, we single out a particular subspace of $L^p(\partial\Omega)$ no element of which has a conjugate except the zero function, and whose "periods" on a homotopy basis can be assigned arbitrarily. Namely, if $\partial\Omega = \cup_{j=0}^n \Gamma_j$ where the Γ_j are disjoint Jordan curves, we set

$$\mathcal{S}_\Omega = \left\{(u_0, u_1, \cdots, u_n) \in \Pi_{j=0}^n L^p_{\mathbb{R}}(\Gamma_j);\ u_j \equiv C_j \in \mathbb{R} \ \text{ with } \Sigma_{j=0}^n C_j = 0\right\} \subset L^p_{\mathbb{R}}(\partial\Omega).$$

Proposition 4 *Let Ω be Dini-smooth and $\sigma \in W^{1,r}(\Omega)$ satisfy (4) with p, r as in (7). If $u \in \mathcal{S}_\Omega$ has a σ-harmonic conjugate, then $u \equiv 0$. Each $u \in \mathcal{S}_\Omega$ is uniquely the trace on $\partial\Omega$ of some $U \in W^{2,r}_{\mathbb{R}}(\Omega)$ satisfying (3), and $\|U\|_{W^{2,r}(\Omega)} \leq C\|u\|_{L^p(\partial\Omega)}$ for some constant C independent of u. If $\lambda_1, \cdots, \lambda_n$ are real numbers and $\gamma_1, \cdots, \gamma_n$ is a homotopy basis for Ω, then there exists a unique $u \in \mathcal{S}_\Omega$ such that $\int_{\gamma_j} \partial_n U = \lambda_j$.*

Proof. By Lemma 6 we may assume that $\Omega = \mathbb{D} \setminus \cup_{j=1}^n \overline{\mathbb{D}}_{a_j,r_j}$ is normalized circular. Set by convention $a_0 = 0$, $r_0 = 1$, so that $\mathbb{T} = \mathbb{T}_{a_0,r_0}$. Assume that $u \in \mathcal{S}_\Omega$ has a σ-harmonic conjugate $v \in L^p_{\mathbb{R}}(\partial\Omega)$, *i.e.* $u + iv = \mathrm{tr}_{\partial\Omega} f$ for some $f \in H^p_\nu(\Omega)$. Denote by $\check{f}_j, \check{\nu}_j$ the reflections of f, ν across $\mathbb{T}_{a_j,r_j}$, $0 \leq j \leq n$, that is

$$\check{f}_j(z - a_j) = 2C_j - \overline{f\left(\frac{r_j^2}{\overline{z-a_j}}\right)},\ \ \check{\nu}_j(z - a_j) = \nu\left(\frac{r_j^2}{\overline{z-a_j}}\right). \tag{40}$$

By (12), it holds that $\check{f}_j \in H^p_{\check{\nu}_j}(\check{\Omega}_j)$ where, for $1 \leq j \leq n$, we put $\check{\Omega}_j \subset \mathbb{D}_{a_j,r_j}$ (resp. $\check{\Omega}_0 \subset \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$) for the reflection of Ω across $\mathbb{T}_{a_j,r_j}$ (resp. across $\mathbb{T}$). Put $\widetilde{f} = f \vee \check{f}_0 \vee \check{f}_1 \cdots \vee \check{f}_n$, $\widetilde{\nu} = \nu \vee \check{\nu}_0 \vee \check{\nu}_1 \cdots \vee \check{\nu}_n$, and $\check{\Omega} = \overline{\Omega} \cup \check{\Omega}_0 \cdots \cup \check{\Omega}_n \dagger)$. Since $\mathrm{tr}_{\mathbb{T}_{a_j,r_j}} \check{f}_j = \mathrm{tr}_{\mathbb{T}_{a_j,r_j}} f$ and $\mathrm{tr}_{\mathbb{T}_{a_j,r_j}} \check{\nu}_j = \mathrm{tr}_{\mathbb{T}_{a_j,r_j}} \nu$, the argument leading to (37) in the proof of Theorem 2 shows that $\widetilde{f} \in H^p_{\widetilde{\nu}}(\check{\Omega})$. Thus $f = U + iV \in W^{2,r}(\Omega)$ by Property 6, hence $\sigma^{-1}\nabla V \in W^{1,r}_{\mathbb{R}}(\Omega)$ since the latter is an algebra for $r > 2$. Moreover $\partial_n V \in W^{1-1/r,r}(\partial\Omega) \subset L^r(\partial\Omega)$, which grants us enough smoothness, in view of (7), to apply the divergence formula:

$$0 = \iint_\Omega V\, \nabla \cdot (\sigma^{-1}\nabla V)\ dm = \int_{\partial\Omega} \sigma^{-1}\, v\, \partial_n V\ |ds| - \iint_\Omega \sigma^{-1}|\nabla V|^2\ dm\,, \tag{41}$$

where the first equality comes from (5). Since u takes constant values a.e. on each $\mathbb{T}_{a_j,r_j}$, $0 \leq j \leq n$, we have that $\partial_t u = 0$ on $\partial\Omega$ whence $\partial_n V = 0$ by (1). Taking this into account in (41), we obtain

$$\iint_\Omega \sigma^{-1}|\nabla V|^2\ dm = 0$$

implying by (4) that $\nabla V = 0$ a.e. in Ω, hence $\nabla U = 0$ by (1). Thus U is constant, in particular all C_j are equal, and since they add up to zero we obtain $u \equiv 0$, as announced.

In another connection, since $\sigma \in W^{1,r}(\Omega) \subset VMO(\Omega)$, it follows from elliptic regularity theory [7, 28] that any $u \in W^{1-1/l,l}(\partial\Omega)$ for some $l \in (1,\infty)$ is uniquely the trace of some $U \in W^{1,l}(\Omega)$ meeting (3) with $\|U\|_{W^{1,l}(\Omega)} \leq C(l,\Omega,\sigma)\|u\|_{W^{1-1/l,l}(\partial\Omega)}$. In particular, since $u \in \mathcal{S}_\Omega$ is constant on each component of $\partial\Omega$, we have that

$$\|U\|_{W^{1,p}(\Omega)} \leq C(p,\Omega,\sigma)\|u\|_{W^{1-1/p,p}(\partial\Omega)} = C(p,\Omega,\sigma)\|u\|_{L^p(\partial\Omega)}. \tag{42}$$

Moreover, if we let C_j (resp. C_0) be the constant value that u assumes on $\mathbb{T}_{a_j,r_j}$ (resp. $\mathbb{T}$), the reflected function $\check{U}_j$ defined *via* (40) (with U in place of f and σ in place of ν) allows us to define $\check{U} = U \vee \check{U}_0 \cdots \vee \check{U}_n \in W^{1,p}(\check{\Omega})$ meeting (3) in $\check{\Omega}$ with conductivity $\check{\nu} = \sigma \vee \check{\sigma}_0 \vee \cdots \vee \check{\sigma}_n$. Let Ω_1 be a bounded Dini-smooth open set, $\overline{\Omega} \subset \Omega_1 \subset \overline{\Omega_1} \subset \check{\Omega}$. By compactness, we can cover $\overline{\Omega}$ with finitely many disks $D_k = \mathbb{D}_{\xi_k,\rho_k}$ such that $\overline{\mathbb{D}}_{\xi_k,2\rho_k} \subset \Omega_1$. Using (39) and the trace theorem, we obtain

$$\|U\|_{W^{2,r}(D_k)} \leq c_k\|\mathrm{tr}_{\mathbb{T}_{\xi_k,2\rho_k}} U\|_{L^p(\mathbb{T}_{\xi_k,2\rho_k})} \leq c_k\|\mathrm{tr}_{\mathbb{T}_{\xi_k,2\rho_k}} U\|_{W^{1-1/p,p}(\mathbb{T}_{\xi_k,2\rho_k})} \leq c_k'\|U\|_{W^{1,p}(\mathbb{D}_{\xi_k,2\rho_k})}$$

so that $\|U\|_{W^{2,r}(\Omega)} \leq C\|U\|_{W^{1,p}(\Omega_1)}$. Moreover, from the very form of (40), it is easy to check that $\|U\|_{W^{1,p}(\Omega_1)} \leq C\|U\|_{W^{1,p}(\Omega)}$. Therefore by (42) $\|U\|_{W^{2,r}(\Omega)} \leq C\|u\|_{L^p(\Omega)}$, as desired.

For $0 \leq j \leq n$, let υ_j be equal to 1 on $\mathbb{T}_{a_j,r_j}$ and to 0 on $\mathbb{T}_{a_k,r_k}$, $k \neq j$. Each $u \in \mathcal{S}_\Omega$ decomposes uniquely as $u = \sum_j C_j\upsilon_j$ with $\sum_j C_j = 0$. In addition, if $\Upsilon_j \in W^{2,r}(\Omega)$ is the solution to (3) such that $\mathrm{tr}_{\partial\Omega}\Upsilon_j = \upsilon_j$, then $U = \sum_j C_j\Upsilon_j$ is the solution to (3) with trace u on $\partial\Omega$. Let γ_k be a homotopy basis for Ω, $1 \leq k \leq n$, and $\lambda_1, \cdots, \lambda_n$ be real constants. The relations $\int_{\gamma_k} \partial_n U = \lambda_k$ are equivalent to the linear system of equations:

$$\sum_{j=0}^{n} C_j \int_{\gamma_k} \partial\partial_n \Upsilon_j = \lambda_k, \quad 1 \leq k \leq n, \qquad \sum_{j=0}^{n} C_j = 0. \tag{43}$$

When all the λ_k are zero, it follows from Lemma 5 and the first part of the proof that $C_j = 0$ for all j is the only solution. Therefore, by elementary linear algebra, (43) has a unique solution $C_0, C_1, \ldots, C_n$ for each n-tuple $\lambda_1, \cdots, \lambda_n$. ■

Remark 4 *If $u \in \mathcal{S}_\Omega$ and $U \in W^{2,r}(\Omega)$ is the solution to* (3) *such that $\mathrm{tr}_{\partial\Omega}U = u$, then U certainly satisfies non-tangential estimates of the form* (25). *Indeed, by the Sobolev embedding theorem and Proposition 4, we get that*

$$\|U\|_{L^\infty(\Omega)} \leq C\|U\|_{W^{1,r}(\Omega)} \leq C\|U\|_{W^{2,r}(\Omega)} \leq C'\|u\|_{L^p(\partial\Omega)}. \tag{44}$$

5.2 Dirichlet problem for the conductivity equation

The result below generalizes point (i) of Lemma 4 to multiply connected domains.

Theorem 3 *Let Ω be Dini-smooth and $\sigma \in W^{1,r}(\Omega)$ satisfy* (4) *with p, r as in (7). To each $u \in L^p_{\mathbb{R}}(\partial\Omega)$, there is a unique $U \in \mathcal{U}^p(\Omega)$ whose non-tangential limit on $\partial\Omega$ is u. The function U satisfies non tangential estimates of the form* (25). *Moreover, to every open set $\mathcal{O}$ with compact closure $\overline{\mathcal{O}} \subset \Omega$, there is a constant $c = c(\mathcal{O},\Omega,\sigma,r,p) > 0$ such that*

$$\|U\|_{W^{2,r}(\mathcal{O})} \leq c\,\|u\|_{L^p(\partial\Omega)}\,. \tag{45}$$

Proof. By Lemma 6 we may assume that $\Omega = \mathbb{D} \setminus \cup_{j=1}^{n} \overline{\mathbb{D}}_{a_j,r_j}$ is normalized circular, and we set by convention $a_0 = 0$, $r_0 = 1$. Let ν be as in (2), so that ν meets (κ). For $0 \leq j \leq n$, we let $\nu_j \in W^{1,r}_{\mathbb{R}}(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_{a_j,r_j})$ extend ν and satisfy (κ), as in Theorem 2. Subsequently, we let $\sigma_j \in W^{1,r}_{\mathbb{R}}(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_{a_j,r_j})$ be associated to ν_j through (2), so that it meets (4).
Put $u = (u_0, u_1, \cdots, u_n) \in \Pi_{j=0}^{n} \mathbb{T}_{a_j,r_j} = L^p_{\mathbb{R}}(\partial\Omega)$. By Lemma 4 point (i), there exists a unique $U_0 \in \mathcal{U}^p(\mathbb{D})$ (with conductivity σ_0) such that $\mathrm{tr}_{\mathbb{T}} U_0 = u_0$. Similarly, for $1 \leq j \leq n$, we let $U_j \in \mathcal{U}^p(\mathbb{C} \setminus \overline{\mathbb{D}}_{a_j,r_j})$ (with conductivity σ_j) be such that $\mathrm{tr}_{\mathbb{T}_{a_j,r_j}} U_j = u_j$. Define operators $K_{i,j} : L^p_{\mathbb{R}}(\mathbb{T}_{a_i,r_i}) \to L^p_{\mathbb{R}}(\mathbb{T}_{a_j,r_j}), 0 \leq i \neq j \leq n$, by

$$K_{i,j}(u_i) = U_{i|\mathbb{T}_{a_j,r_j}} \ , \quad i \neq j \ .$$

Observe that $K_{i,j}$ is compact, for (39) and the trace theorem imply that it maps continuously $L^p_{\mathbb{R}}(\mathbb{T}_{a_i,r_i})$ into $W^{2-1/r,r}_{\mathbb{R}}(\mathbb{T}_{a_j,r_j}) \subset W^{1,r}_{\mathbb{R}}(\mathbb{T}_{a_j,r_j})$ which is compactly included in $L^l_{\mathbb{R}}(\mathbb{T}_{a_j,r_j})$ for all $l \in [1, \infty]$ by the Rellich-Kondratchov theorem.
Consider now the operator $\mathfrak{U}$ from $L^p(\partial\Omega)$ into itself given by

$$\mathfrak{U}(u_0, \cdots, u_n) = \sum_{j=0}^{n} U_{j|\partial\Omega} = \Big(\sum_{j=0}^{n} U_{j|\mathbb{T}} \, , \, \sum_{j=0}^{n} U_{j|\mathbb{T}_{a_1,r_1}} \, , \cdots , \, \sum_{j=0}^{n} U_{j|\mathbb{T}_{a_n,r_n}} \Big). \tag{46}$$

If we let $f_0 \in H^{p,0}_{\nu_0}(\mathbb{D})$ and $f_j \in H^{p,0}_{\nu_j}(\overline{\mathbb{C} \setminus \overline{\mathbb{D}_{a_j,r_j}}}), 1 \leq j \leq n$ be the functions granted by Lemma 4 point (ii), with boundary condition $\mathrm{Re}\ f_{j|\mathbb{T}_{a_j,r_j}} = u_j$, we observe that $\mathfrak{U}(u) = \sum_{j=0}^{n} \mathrm{Re}\ f_{j|\partial\Omega}$ is the real part of $\mathrm{tr}_{\partial\Omega} f$, where $f = \sum_{j=0}^{n} f_{j|\Omega}$ lies in $H^{p,0}_{\nu}(\Omega)$ since $\nu_{j|\Omega} = \nu$ and we may use the characterization by harmonic majorants in Theorem 1. In particular

$$\mathrm{Ran}\, \mathfrak{U} \subset \mathrm{Re}\, \mathrm{tr}_{\partial\Omega} H^p_{\nu}(\Omega) \subset \mathrm{tr}_{\partial\Omega} \mathcal{U}^p(\Omega) \, . \tag{47}$$

Moreover, we have the following matrix relation

$$\mathfrak{U} = (I + K) \tag{48}$$

where $K : L^p(\partial\Omega) \to L^p(\partial\Omega)$ is given by

$$\big(K(u_0, u_1, \cdots, u_n)\big)^T = \begin{pmatrix} 0 & K_{1,0} & K_{2,0} & \cdots & K_{n,0} \\ K_{0,1} & 0 & K_{2,1} & \cdots & K_{n,1} \\ K_{0,2} & K_{1,2} & 0 & \cdots & K_{n,2} \\ \vdots & & \ddots & \ddots & \vdots \\ K_{0,n} & \cdots & \cdots & K_{n-1,n} & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

Because the $K_{i,j}$ are compact, so is K from $L^p_{\mathbb{R}}(\partial\Omega)$ into itself.
Next, we prove that $\mathrm{Ker}\, \mathfrak{U} = \mathcal{S}_{\Omega}$. Indeed let $u = (u_0, \cdots, u_n) \in L^p_{\mathbb{R}}(\partial\Omega)$ be such that $\mathfrak{U}u = 0$. Using the notations introduced above, this means that

$$\mathrm{Re}\ \mathrm{tr}_{\partial\Omega} f = \mathrm{Re}\ \mathrm{tr}_{\partial\Omega} \sum_{j=0}^{n} f_{j|\Omega} = 0 \, . \tag{49}$$

Define further $\tilde{f} = f - i \int_{\mathbb{T}} \mathrm{Im}\ \mathrm{tr}_{\mathbb{T}} f$ which lies in $H^p_{\nu}(\Omega)$ and set

$$w = \frac{\tilde{f} - \nu \overline{\tilde{f}}}{\sqrt{1 - \nu^2}} \in G^p_{\alpha}(\Omega), \qquad w = e^s F \, ,$$

for the functions w, s, F associated to $\tilde f$ through (16) and (18), with $\mathrm{Im}\ s_{|\mathbb{T}} = 0$ and, say, $\mathrm{Im}\ s_{|\mathbb{T}_{a_j,r_j}} = \theta_j$ for $1 \le j \le n$, see Proposition 2. Clearly (46) and (49) entail that $\mathrm{Re}\ \mathrm{tr}_{\mathbb{T}_{a_j,r_j}} w = 0$ for $0 \le j \le n$. Thus, from the boundary conditions for s, we see that F has constant argument $-\theta_j + \pi/2$ modulo π on each $\mathbb{T}_{a_j,r_j}$. By Morera's theorem [34, II, Ex. 12], this allows one to reflect F across each $\mathbb{T}_{a_j,r_j}$ according to the rule

$$\check F_j(z-a_j) = -e^{-2i\theta_j}\overline{F\left(\frac{r_j^2}{\overline{z-a_j}}\right)}, \tag{50}$$

so that F is in fact analytic on a neighborhood of $\overline{\Omega}$. In particular, $F(\mathbb{T}_{a_j,r_j})$ is a segment S_j on the line through the origin defined by $\{\arg z = -\theta_j + \pi/2 \bmod \pi\}$, and if $z_0 \in \mathbb{C}$ belongs to none of the S_j there is a single-valued branch of $\log(F - z_0)$ on each $\mathbb{T}_{a_j,r_j}$, $0 \le j \le n$. Thus, by the argument principle, F cannot assume the value z_0 in Ω, but if F is not constant $F(\Omega)$ is open, therefore it contains some $z_0 \notin \cup_j S_j$. Hence F is constant, all θ_j are equal to 0, and $F = ic$, $c \in \mathbb{R}$, is a pure imaginary constant on $\mathbb{T}$. Because $\mathrm{Im}\ \mathrm{tr}_{\mathbb{T}}\tilde f$ has vanishing mean by construction, and since

$$\mathrm{Im}\ \mathrm{tr}_{\mathbb{T}}\tilde f = \mathrm{Im}\ \mathrm{tr}_{\mathbb{T}}\left(w\sqrt{\frac{1-\nu}{1+\nu}}\right) = c\, e^{\mathrm{Re}\ s_{|\mathbb{T}}}\sqrt{\frac{1-\nu}{1+\nu}}$$

we must have $c = 0$ whence $F = w = \tilde f \equiv 0$. It follows that

$$\sum_{j=0}^{n} f_{j|\Omega} = f = iC\ , \quad C \in \mathbb{R}\ . \tag{51}$$

Now, since $f_j \in H^{p,0}_{\nu_j}(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_{a_j,r_j})$ we find upon writing

$$f = \underbrace{\left(f_{0|\Omega} + \sum_{j=1}^{n}\int_{\mathbb{T}_{a_j,r_j}} \mathrm{Re}\ f_{j|\Omega}\right)}_{h_0} + \sum_{j=1}^{n}\underbrace{\left(f_{j|\Omega} - \int_{\mathbb{T}_{a_j,r_j}} \mathrm{Re}\ f_{j|\Omega}\right)}_{h_j},$$

that $f = h_0 + \sum_{j=1}^n h_j$ is the direct sum decomposition of f furnished by Theorem 2. However, in view of (51), $f = iC + \sum_{j=1}^n 0$ is also such a decomposition, therefore by uniqueness $h_0 = iC$ and $h_j = 0$, $1 \le j \le n$. Hence $U_{j|\Omega} = \mathrm{Re}\ f_{j|\Omega} = c_j \in \mathbb{R}$ for $j = 0, 1, \cdots, n$ and $c_0 = -\sum_{j=1}^n c_j$ by (49), i.e. $u \in \mathcal{S}_\Omega$.
The reverse inclusion $\mathcal{S}_\Omega \subset \mathrm{Ker}\,\mathfrak{U}$ is immediate from (46) for U_j is a constant when u_j is.
Finally, let us show that the Riesz number of the operator $\mathfrak{U}$ is equal to 1:

$$\mathrm{Ker}\,\mathfrak{U} = \mathrm{Ker}\,\mathfrak{U}^2\,, \tag{52}$$

or equivalently $\mathrm{Ker}\,\mathfrak{U}^2 \subset \mathrm{Ker}\,\mathfrak{U}$. Indeed, let $x \in \mathrm{Ker}\,\mathfrak{U}^2$. Then

$$u = \mathfrak{U}\,x \in \mathrm{Ker}\,\mathfrak{U} \cap \mathrm{Ran}\,\mathfrak{U} = \mathcal{S}_\Omega \cap \mathrm{Ran}\,\mathfrak{U} \subset \mathcal{S}_\Omega \cap \mathrm{Re}\,\mathrm{tr}_{\partial\Omega} H^p_\nu(\Omega)\,,$$

in view of (47). By Proposition 4 it holds that $\mathcal{S}_\Omega \cap \mathrm{Re}\,\mathrm{tr}_{\partial\Omega} H^p_\nu(\Omega) = \{0\}$, therefore $u = 0$ hence $x \in \mathrm{Ker}\,\mathfrak{U}$, thereby establishing (52)

In view of what precedes, a theorem of F. Riesz[19, Thm 1.16] implies the decomposition

$$L^p_{\mathbb{R}}(\partial\Omega) = \operatorname{Ker}\mathfrak{U} \oplus \operatorname{Ran}\mathfrak{U} = \mathcal{S}_\Omega \oplus \operatorname{Ran}\mathfrak{U}. \tag{53}$$

The existence of U now follows from (53), (47), (25), Theorem 2, (39), Proposition 4 and Remark 4.
To prove uniqueness, suppose that $U \in \mathcal{U}^p(\Omega)$ satisfies $\mathrm{tr}_{\partial\Omega}U = 0$ and let γ_j be a homotopy basis for Ω, $1 \leq j \leq n$. By Proposition 4, there is $\upsilon \in \mathcal{S}_\Omega$ and $\Upsilon \in W^{2,r}(\Omega)$ satisfying (3) such that $\mathrm{tr}_{\partial\Omega}\Upsilon = \upsilon$ and $\int_{\gamma_j} \partial_n(U - \Upsilon) = 0$ for all j. From Lemma 5 it follows that $W = U - \Upsilon$ has a σ-harmonic conjugate V, *i.e.* $W + iV = f \in H^p_\nu(\Omega)$. Since $\mathrm{Re}(\mathrm{tr}_{\partial\Omega}f)$ is equal to some constant C_j on $\mathbb{T}_{a_j,r_j}$, the reflection formula (40) and the argument thereafter shows that $f \in W^{2,r}(\Omega)$. Then $U \in W^{1,r}(\Omega)$ *a fortiori*, therefore $U \equiv 0$ by uniqueness of $W^{1,r}(\Omega)$-solutions to the Dirichlet problem [7, 28]. ∎

Note that (53) and Proposition 4 immediately imply:

Corollary 2 *Ran* $(\mathfrak{U}) = Re\, tr_{\partial\Omega} H^p_\nu(\Omega)$.

Remark 5 *From* (53), *Corollary 2, Theorem 1 point* (*ii*), *and Proposition 4, we see that the condition "*$U \in \mathcal{U}^p(\Omega)$*" may be replaced by "* U *solves* (3) *and* $|U|^p$ *has a harmonic majorant".*

It is standard in regularity theory that smoothness of the boundary may be traded for smoothness of the coefficients. Here is an application of Theorem 3 to the Dirichlet problem for equation (3) on non smooth domains. Given a weight $W \geq 0$ on $\partial\Omega$, we denote by $L^p(\partial\Omega, W)$ the weighted space of functions h for which $|h|^pW \in L^1(\partial\Omega)$.

Corollary 3 *Let* $D \subset \overline{\mathbb{C}}$ *be a finitely connected domain whose boundary is a piecewise* $C^{1,\lambda}$ *polygon,* $0 < \lambda \leq 1$, *with* N *vertices* $W_1, \cdots, W_N$. *Let* $\lambda_j\pi$ *be the jump of the oriented tangent at* W_j, $-1 \leq \lambda_j \leq 1$, *and assume that* $\mu = \max\{\lambda_j\} < 1$ *(i.e. there is no outward-pointing cusp). Define*

$$W(z) = \Pi_{j=1}^N |z - W_j|^{\lambda_j}.$$

If $\sigma \in W^{1,\infty}_{\mathbb{R}}(D)$ *meets* (4) *and* $u \in L^p_{\mathbb{R}}(\partial D, W)$ *for some* $p > 2/(2 - \max\{0, \mu\})$, *there is a unique solution* U *to* (3) *in* D *such that* $|U|^p$ *has a harmonic majorant and* U *has non tangential limit* u *a.e. on* ∂D.

Proof: Let φ map a circular domain Ω onto D. By Remark 5, the statement is equivalent to the existence of a unique solution to (3) in $\mathcal{U}^p(\Omega)$ with σ replaced by $\sigma \circ \varphi$ and u by $u \circ \varphi$. By [12, Prop. 4.2], $(W \circ \varphi)|\varphi'|$ extends continuously to $\overline{\Omega}$ and is never zero there[10]. Hence $\sigma \circ \varphi \in W^{1,r}(\Omega)$ for $r \in (2, 2/\max\{0, \mu\})$, and the assumptions imply that $p > r/(r-1)$ for r close enough to $2/\max\{0, \mu\}$. Finally, the condition on p implies that $u \circ \varphi \in L^p(\partial\Omega)$, so that we can apply Theorem 3. ∎

Although $\partial_n U$ needs not be a distribution on $\partial\Omega$ when $U \in \mathcal{U}^p(\Omega)$ (unless $p = \infty$), Theorem 3 allows us to define $\sigma\partial_n U$. Recall from (21) the definition of $\widetilde{\Delta}_\epsilon$.

[10]This expresses that $W(z)|dz|$ is comparable to harmonic measure on $\partial\Omega$.

Corollary 4 *Let Ω be a Dini-smooth domain and p, r, σ satisfy* (2) *and* (7). *To each $U \in \mathcal{U}^p(\Omega)$, there is a unique distribution $\sigma\partial_n U \in W_{\mathbb{R}}^{-1,p}(\partial\Omega)$ such that*

$$\int_{\partial\Omega} \sigma\partial_n U \, \varphi := \lim_{\epsilon\to 0} \iint_{\widetilde{\Delta}_\epsilon} \sigma\nabla U \cdot \nabla\varphi, \qquad \varphi \in \mathcal{D}(\mathbb{R}^2). \tag{54}$$

Further, there is a constant $C = C(\Omega, \sigma, r, p)$ such that

$$\|\sigma\partial_n U\|_{W^{-1,p}(\partial\Omega)} \le C \|tr_{\partial\Omega} U\|_{L^p(\partial\Omega)} . \tag{55}$$

Proof: we may assume that Ω is bounded. If $\mathrm{tr}_{\partial\Omega}\, U \in \mathcal{S}_\Omega$, then $U \in W^{2,r}(\Omega)$ by Proposition 4 and $\partial_n U$ may be defined a.e. on $\partial\Omega$ as the scalar product of $\mathrm{tr}_{\partial\Omega}\nabla U$ with the unit normal to $\partial\Omega$. Thus, (54) follows from the divergence formula (the limit on the right is equal to $\iint_\Omega \sigma\nabla U.\nabla\varphi$ by dominated convergence) while (55) drops out from (44) and the Sobolev embedding theorem (because $\|\nabla U\|_{L^\infty(\Omega)} \le C\|U\|_{W^{2,r}(\Omega)}$). By (53) and Corollary 2, it remains to handle the case where $U = \mathrm{Re} f$ with $f = U + iV \in H^{p,0}_\nu(\Omega)$.
In this case we know from (1) that $\sigma\partial_n U = \partial_t V$ on $\partial\widetilde{\Delta}_\epsilon$. Hence, using the Green formula and integrating by parts, we obtain for $\varphi \in \mathcal{D}(\mathbb{R}^2)$:

$$\iint_{\widetilde{\Delta}_\epsilon} \sigma\nabla U \cdot \nabla\varphi = \int_{\partial\widetilde{\Delta}_\epsilon} \sigma\partial_n U \, \varphi = -\int_{\partial\widetilde{\Delta}_\epsilon} \mathrm{tr}_{\partial\widetilde{\Delta}_\epsilon} V \partial_t\varphi. \tag{56}$$

Consequently, by Property 1 in section 3.3 and Lemma 6, we obtain

$$\lim_{\epsilon\to 0} \iint_{\widetilde{\Delta}_\epsilon} \sigma\nabla U \cdot \nabla\varphi = -\int_{\partial\Omega} \mathrm{tr}_{\partial\Omega} V \, \partial_t\varphi \, . \tag{57}$$

As $\|\mathrm{tr}_{\partial\Omega} V\|_{L^p(\partial\Omega)} \le c\|\mathrm{tr}_{\partial\Omega} U\|_{L^p(\partial\Omega)}$ by Theorem 4, the right hand side of (57) indeed defines a distribution $\sigma\partial_n U \in W_{\mathbb{R}}^{-1,p}(\partial\Omega)$ satisfying our requirements. ■

5.3 The Dirichlet problem in $H^p_\nu(\Omega)$

We are now in position to solve the "Dirichlet problem" for equation (CB). Given Ω a Dini-smooth domain, we put $\partial\Omega = \cup_{j=0}^n \Gamma_j$ where the Γ_j are disjoint Dini-smooth Jordan curves. Introduce for $U \in \mathcal{U}^p(\Omega)$ the compatibility condition

$$(H_{\Omega,\sigma}): \quad \textstyle\int_{\Gamma_j} \sigma\partial_n U = 0 \, , \; 0 \le j \le n$$

where the normal derivative is understood in the sense of Corollary 4. Note that the relation $\int_{\partial\Omega} \sigma\partial_n U = 0$, which follows from (54) when $\varphi = 1$ on Ω, is to the effect that $(H_{\Omega,\sigma})$ holds as soon as $\int_{\Gamma_j} \sigma\partial_n U = 0$ for every j *but one*.

Theorem 4 *Let Ω be a Dini-smooth domain, $\partial\Omega = \cup_{j=0}^n \Gamma_j$, and σ, p, r, ν meet (2) and (7). If $u \in L^p_{\mathbb{R}}(\partial\Omega)$ satisfies $(H_{\Omega,\sigma})$, and only in this case, there exists a unique $f \in H^{p,0}_\nu(\Omega)$ such that Re $tr_{\partial\Omega} f = u$ a.e. on $\partial\Omega$. Moreover, there is a constant $c = c_{p,\alpha,\nu} > 0$ such that $\|tr_{\partial\Omega} f\|_{L^p(\partial\Omega)} \le c\|u\|_{L^p(\partial\Omega)}$.*

Proof. Let $u \in L^p(\partial\Omega)$ and $U \in \mathcal{U}^p(\Omega)$ be such that $u = \mathrm{tr}_{\partial\Omega} U$, see Theorem 3. From (21) and the definition of $P_{\partial\Omega,\epsilon}$ (after equation (23)), we see that the connected components of $\partial\widetilde{\Delta}_\varepsilon$ form a system of smooth Jordan curves $\gamma_{j,\varepsilon}$, $0 \le j \le n$, with $\gamma_{j,\varepsilon}$ homotopic to

$\gamma_{j,\varepsilon'}$ for any two $\varepsilon, \varepsilon'$ small enough. Hence $\int_{\gamma_{j,\varepsilon}} \partial_n U$ is independent of ε by Lemma 5, and letting φ in equation (54) be 1 on a neighborhood of Γ_j and 0 on a neighborhood of Γ_k for $k \neq j$, we deduce from Corollary 4 (see (56)) that $\int_{\gamma_{j,\varepsilon}} \sigma\partial_n U = \int_{\Gamma_j} \sigma\partial_n U$. Since the $\gamma_{j,\varepsilon}$ are a homotopy basis of Ω for $1 \leq j \leq n$, we conclude from Lemma 5 that U has a σ-harmonic conjugate V if and only if $(H_{\Omega,\sigma})$ holds. Adding a constant to V if necessary, we can ensure that $f = U + iV \in H^{p,0}_\nu(\Omega)$. Uniqueness of f follows from uniqueness of $\mathcal{U}$ and the fact that any two σ-harmonic conjugates differ by a constant. ∎

Remark 6 *Let $E(p,\sigma) \subset L^p_{\mathbb{R}}(\partial\Omega)$ denote the closed subspace of functions with zero mean meeting $(H_{\Omega,\sigma})$. Taking into account that $f \in H^p_\nu(\Omega)$ if, and only if $(if) \in H^p_{-\nu}(\Omega)$, it follows from Theorem 4 that the σ-conjugating map $\mathcal{H}(u) = \mathrm{Im}\, tr_{\partial\Omega} f$ is an isomorphism from $E(p,\sigma)$ onto $E(p,1/\sigma)$ satisfying $\mathcal{H}^2 = -Id$.*

5.4 Neumann problem for the conductivity equation

Theorem 4 allows us to solve a weighted Neumann problem for (3), where data consist of the normal derivative of u multiplied by the conductivity on $\partial\Omega$:

Theorem 5 *Let Ω be a Dini-smooth domain and p, r, $\sigma \in W^{1,r}(\Omega)$ satisfy (4) and (7). To each $\phi \in W^{-1,p}_{\mathbb{R}}(\partial\Omega)$ such that $\int_{\partial\Omega} \phi = 0$, there is $U \in \mathcal{U}^p(\Omega)$, unique up to an additive constant, such that $\sigma\partial_n U = \phi$.*

Proof: by Proposition 4 and the proof of Theorem 4, there is $\Upsilon \in \mathcal{U}^p(\Omega)$ with $\mathrm{tr}_{\partial\Omega} U \in \mathcal{S}_\Omega$ such that $\psi = \phi - \sigma\partial_n\Upsilon$ satisfies $\int_{\Gamma_j} \psi = 0$ for $0 \leq j \leq n$. Because $\psi_{|\Gamma_j}$ lies in $W^{-1,p}(\Gamma_j)$ and annihilates the constants, ψ is of the form $\partial_t v$ for some $v \in L^p(\Omega)$. By Proposition 4 and Theorems 3, 4 applied with $1/\sigma$ and $-\nu$ rather than σ and ν, we can add to v an element of $\mathcal{S}_\Omega$ (this does not change $\partial_t v$) so that $v = \mathrm{Im}\,\mathrm{tr}_{\partial\Omega} f$ for some $f = W + iV \in H^{p,0}_\nu$. From (56), (54), and Property 1, it follows that

$$\int_{\partial\Omega} \sigma\partial_n W\, \varphi = \int_{\partial\Omega} \partial_t v\, \varphi = \int_{\partial\Omega} \psi\, \varphi\,, \qquad \varphi \in \mathcal{D}(\mathbb{R}^2),$$

hence $\sigma\partial_n W = \psi$. Then $U = W + \Upsilon$ satisfies our requirements. In another connection, if $U \in \mathcal{U}^p(\Omega)$ is such that $\sigma\partial_n U = 0$ then $U = \mathrm{Re} f$ with $f = U + iV \in H^{p,0}_\nu(\Omega)$ by Theorem 4. Thus, we deduce from (57) and Property 1 that $\int_{\partial\Omega} \mathrm{tr}_{\partial\Omega} V\, \partial_t\varphi = 0$ for $\varphi \in \mathcal{D}(\mathbb{R}^2)$. Since in addition $\mathrm{tr}_{\partial\Omega} V$ has zero mean by contruction, it is the zero function so that $V \equiv 0$ by the uniqueness part of Theorem 3 (applied with $1/\sigma$ rather than σ). Finally, U must be constant by (1). ∎

6 Density of traces, approximation issues

On a rectifiable curve, we let $|E|$ indicate arclength of a measurable set E.
We consider the following
Conjecture *Let Ω be a Dini-smooth domain and p, r, ν satisfy (7). If $E \subset \partial\Omega$ satisfies $|E| < |\partial\Omega|$, then $\left(tr_{\partial\Omega} H^p_\nu(\Omega)\right)_{|E}$ is dense in $L^p(E)$.*

When $\nu = 0$[11] and E is closed, this is an easy consequence of Runge's theorem. Still when $\nu = 0$, but this time E is arbitrary, it was proven to hold in [9] when $\Omega = \mathbb{D}$, hence it is true for all simply connected Ω by conformal invariance of Hardy spaces and Lemma 6. These results are of key importance to the approach of bounded extremal problems developed in [9, 20, 22, 32, 43].
When $\nu \in W^{1,\infty}(\Omega)$ and Ω is simply connected, the answer is again positive as established on the disk in [11]. Below, we prove that if Ω is multiply connected but E is contained in a single connected component of $\partial\Omega$, then the statement is indeed correct:

Theorem 6 *Let Ω be a multiply connected Dini-smooth domain, $\partial\Omega = \cup_{j=0}^{n}\Gamma_j$, $n > 0$, and ν, p, r, ν meet (7). Let $E \subset \Gamma_{j_0}$ for some $j_0 \in \{0, \cdots, n\}$. If $|E| < |\Gamma_{j_0}|$, then restrictions to E of traces of $H^p_\nu(\Omega)$-functions are dense in $L^p(E)$.*

To prove Theorem 6, we establish in Appendix H the following result which may be of independent interest. Note that the arguments of proof are in fact to the effect that the result holds if there is at least one connected component of the boundary that does not intersect E.

Proposition 5 *Let $\Omega = \mathbb{D}$ and assume that ν, p, r satisfy* (7). *Let moreover $\alpha \in L^r(\mathbb{D})$ and $\varrho \in (0, 1)$.*
Then $\left(H^p_\nu(\mathbb{D})\right)_{|\mathbb{D}_\varrho}$ (resp. $\left(G^p_\alpha(\mathbb{D})\right)_{|\mathbb{D}_\varrho}$) is dense in $H^p_{\nu_{|\mathbb{D}_\varrho}}(\mathbb{D}_\varrho)$ (resp. $G^p_{\alpha_{|\mathbb{D}_\varrho}}(\mathbb{D}_\varrho)$).

Proof of Theorem 6. Assume without loss of generality that $\Omega = \mathbb{D} \setminus \cup_{j=1}^{n}\overline{\mathbb{D}}_{\xi_j, r_j}$ and that $j_0 = 1$. Extending if necessary α by zero to each $\mathbb{D}_{\xi_j, r_j}$ with $2 \leq j \leq n$, we are back to the case $n = 1$ and then to $\Omega = \mathbb{A}_\varrho$. Clearly, it is enough to consider $E = \mathbb{T}_\varrho$.
Let $\nu_e \in W^{1,r}(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_\varrho)$ extend ν and satisfy (κ) (see proof of Theorem 2), and set $\nu_i \in W^{1,r}(\overline{\mathbb{D}})$ to be $\nu \vee \check{\nu}_e$ where $\check{\nu}_e(z) = \nu_e(\varrho^2/\overline{z})$ for $z \in \mathbb{D}_\varrho$. By Lemma 3, any $\psi \in L^p(\mathbb{T}_\varrho)$ can be written as $\mathrm{tr}_{\mathbb{T}_\varrho}\psi_1 + \mathrm{tr}_{\mathbb{T}_\varrho}\psi_2$ where $\psi_1 \in H^p_{\check{\nu}_e}(\mathbb{D}_\varrho)$ and $\psi_2 \in H^{p,00}_{\nu_e}(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_\varrho)$. By Proposition 5, to each $\varepsilon > 0$ there is $\psi_3 \in H^p_{\nu_i}(\mathbb{D})$ such that $\|\mathrm{tr}_{\mathbb{T}_\varrho}\psi_1 - \mathrm{tr}_{\mathbb{T}_\varrho}\psi_3\|_{L^p(\mathbb{T}_\varrho)} < \varepsilon$. Since $(\psi_2 + \psi_3)_{|\mathbb{A}_\varrho} \in H^p_\nu(\mathbb{A}_\varrho)$, the result follows. ■

We shall illustrate the use of Theorem 6 in certain bounded extremal problems (BEP) in $H^p_\nu(\mathbb{A}_\varrho)$ which play an important role in the works [30, 32, 31, 33] where inverse boundary problems for equation (3) are considered.
We let now $\Omega = \mathbb{A}_\varrho$ for some $\varrho \in (0, 1)$ and we assume that p, r, ν meet (7). Fix $I \subset \mathbb{T}_\varrho$ with $|I| > 0$, and define $J = \partial\mathbb{A}_\varrho \setminus I$. To $M > 0$ and $\phi \in L^p_{\mathbb{R}}(J)$, we associate the following subset of $L^p(I)$.

$$\mathcal{B}^{\mathbb{A}_\varrho}_p = \left\{ g_{|I} :\; g \in \mathrm{tr}_{\partial\mathbb{A}} H^p_\nu(\mathbb{A}_\varrho);\; \|\mathrm{Re}\, g - \phi\|_{L^p(J)} \leq M \right\} \subset L^p(I)\,.$$

Note that a function $g \in H^p_\nu(\mathbb{A}_\varrho)$ is completely determined by $g_{|I}$ in view of Property 3, Section 3.3.
The theorem below extends to annular geometry with weaker smoothness assumptions a result obtained on the disk with Lipschitz-continuous ν in [32, Thm 3] (see also [9, 20, 22, 43] for the case $\nu = 0$).

[11]We deal then with holomorphic Hardy spaces in which case we may take $r = \infty$ and $p \in (1, \infty)$.

Theorem 7 *Let notations and assumptions be as above. Then, to every $F_d \in L^p(I)$, there exists a unique function $g_* \in \mathcal{B}_p^{\mathbb{A}_\varrho}$ such that*

$$\|F_d - g_*\|_{L^p(I)} = \min_{g \in \mathcal{B}_p^{\mathbb{A}_\varrho}} \|F_d - g\|_{L^p(I)} . \tag{BEP}$$

Moreover, if $F_d \notin \left(\mathcal{B}_p^{\mathbb{A}_\varrho}\right)_{|I}$, then $\|Re\, g_* - \phi\|_{L^p(J)} = M$.

Proof. Since $\mathcal{B}_p^{\mathbb{A}_\varrho}$ is a convex subset of the uniformly convex Banach space $L^p(I)$, it is enough, in order to prove existence and uniqueness of g_*, to check that $\mathcal{B}_p^{\mathbb{A}_\varrho}$ is closed in $L^p(I)$ [13, Prop. 5]. Let $\varphi_k \in H^p_\nu(\mathbb{A}_\varrho)$ be such that $\varphi_{k|_I} \in \mathcal{B}_p^{\mathbb{A}_\varrho}$ converges to some function φ_I in $L^p(I)$ as $k \to \infty$. Then $\mathrm{tr}_{\partial\mathbb{A}_\varrho}\varphi_k$ is bounded in $L^p(\partial\mathbb{A}_\varrho)$ by definition of $\mathcal{B}_p^{\mathbb{A}_\varrho}$. Hence, extracting a subsequence if necessary, we may assume that $(\mathrm{tr}_{\partial\mathbb{A}_\varrho}\varphi_k)$ converges weakly in $L^p(\partial\mathbb{A}_\varrho)$ to $(\mathrm{tr}_{\partial\mathbb{A}_\varrho}\psi)$ for some $\psi \in H^p_\nu(\partial\mathbb{A}_\varrho)$ by Properties 2-3 in Section 3.3 and Mazur's theorem, [16]. Because $(\mathrm{tr}_J\varphi_k)$ *a fortiori* converges weakly to $\mathrm{tr}_J\psi)$ in $L^p(J)$, we deduce from the weak-* compactness of balls that $\psi_I \in \mathcal{B}_p^{\mathbb{A}_\varrho}$. Moreover, we must have $\psi_{|_I} = \varphi_I$ by the strong convergence of $(\varphi_k)_{|I}$ in $L^p(I)$, hence $\mathcal{B}_p^{\mathbb{A}_\varrho}$ is indeed closed.

To prove that $\|\mathrm{Re}\ g_* - \phi\|_{L^p(J)} = M$ when $F_d \notin \mathcal{B}_p^{\mathbb{A}_\varrho}$ assume for a contradiction that $\|\mathrm{Re}\ g_* - \phi\|_{L^p(J)} < M$. By Theorem 6, there is a function $h \in \mathrm{tr}_{\partial\mathbb{A}_\varrho} H^p_\nu(\mathbb{A})$ such that

$$\|F_d - g_* - h\|_{L^p(I)} < \|F_d - g_*\|_{L^p(I)} ,$$

and by the triangle inequality we have

$$\|F_d - g_* - \lambda h\|_{L^p(I)} < \|F_d - g_*\|_{L^p(I)}$$

for all $0 < \lambda < 1$. Now for $\lambda > 0$ sufficiently small we have $\|\mathrm{Re}\ (g_* + \lambda h) - \phi\|_{L^p(J)} \leq M$, contradicting the optimality of g_*. ■

7 Conclusion

We developped in this paper a theory of Hardy spaces and conjugate functions on Dini smooth domains for the conjugate Beltrami equation that runs parallel to the holomorphic case. We conjecture the assumptions $\nu \in W^{1,r}$, $r > 2$ and $p > r/(r-1)$ are best possible for the above mentioned results to hold. We applied our results to Dirichlet and Neumann problems for the conductivity equation with L^p and $W^{-1,p}$ data. Whether those continue to hold in higher dimension [6] and for matrix-valued conductivity coefficients is an interesting open question.

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Appendix

A Conformal maps of Dini-smooth annular domains

As is well-known [49, Thm 3.5], a conformal map between Dini-smooth simply connected domains extends to a homeomorphism of their closures, and the derivative extends continuously to the closure of the initial domain in such a way that it is never zero. This remains true in the multiply connected case, but the authors could not locate the result in the literature which is why we provide a proof.

Lemma 6 *Let φ conformally map a Dini-smooth domain Ω onto a Dini-smooth domain Ω'. Then φ extends to a homeomorphism from $\overline{\Omega}$ onto $\overline{\Omega'}$ whose derivative also extends continuously to $\overline{\Omega}$ and is never zero there.*

Proof. That φ extends to a homeomorphism from $\overline{\Omega}$ onto $\overline{\Omega'}$ can be proved as in the simply connected case (see *e.g.* [51, Thm 14.18]), granted that each boundary point of Ω is accessible, by Dini-smoothness of $\partial\Omega$, and that every bounded analytic function on Ω has nontangential limits at almost every boundary point [25, Thms 10.3, 10.12]. We are thus left to show that ϕ' extends in a continuous nonvanishing manner to $\overline{\Omega}$.
For this, observe that it is enough to consider the doubly connected case. For if J is one of the Jordan curves composing $\partial\Omega$ and J' is another Dini-smooth Jordan curve contained in Ω, disjoint from J, such that the annular region $A(J,J')$ between J and J' lies entirely in Ω, then φ conformally maps $A(J,J')$ onto some annular region in Ω' whose boundary consists of two Dini-smooth Jordan curves, one of which is a connected component of $\partial\Omega'$ (by what precedes). If φ' continuously extends to J in a nonvanishing manner, we will be done since J was an arbitrary connected component of $\partial\Omega$.
Now, let Ω be doubly connected and lie between two Dini-continuous Jordan curves Γ_1, Γ_2, the latter being interior to the former. Let ψ_1 map the interior Ω_1 of Γ_1 onto the unit disk $\mathbb{D}$. Because Γ_1 is Dini-smooth, ψ_1 extends to a homeomorphism from $\overline{\Omega_1}$ onto $\overline{\mathbb{D}}$ and the derivative ψ_1' extends continuously to $\overline{\Omega_1}$ and is never zero there. Clearly $\psi_1(\Gamma_2)$ is a Dini-smooth Jordan curve. Let Ω_2 indicate the interior of Γ_2 and ψ_2 conformally map $\overline{\mathbb{C}}\setminus\psi_1(\overline{\Omega_2})$ onto $\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$. Then $\psi_3 := \psi_2\circ\psi_1$ maps Ω onto an annular region Ω_3 bounded by analytic Jordan curves (namely a cicle and an analytic image of a circle), ψ_3 extends to a homeomorphism of the closures, and by the chain rule ψ_3' extends continuously to $\overline{\Omega}$ where it is never zero. Let $\varrho\in(0,1)$ be such that ψ_4 conformally maps Ω_3 onto $\mathbb{A}_\varrho$. Then ψ_4 extends to a homeomorphisn from $\overline{\Omega_3}$ onto $\overline{\mathbb{A}_\varrho}$, and since $\partial\Omega_3$ consists of analytic curves it follows from the reflexion principle that ψ_4 extends analytically and locally injectively to a neighborhood of $\overline{\Omega_3}$. Altogether, we constructed a conformal map from Ω onto $\mathbb{A}_\varrho$, namely $\psi_4\circ\psi_3$, that extends continuously from $\overline{\Omega}$ onto $\overline{\mathbb{A}_\varrho}$, and whose derivative extends continuously to $\overline{\Omega}$ where it is never zero. Because self-conformal maps of $\mathbb{A}_\varrho$ must be Möbius transforms, similar properties hold for any conformal map from Ω onto $\mathbb{A}_\varrho$. The same is true of Ω' which is conformally equivalent to Ω and therefore to the same $\mathbb{A}_\varrho$. Factoring φ into a conformal map from Ω onto $\mathbb{A}_\varrho$ followed by a conformal map from $\mathbb{A}_\varrho$ onto Ω' (*e.g.* $(\psi_4\circ\psi_3)\circ(\psi_4\circ\psi_3)^{-1}\varphi)$, we get the desired result. ■

B Proof of Proposition 2

We may assume that Ω is bounded. Set by convention $\overline{w(\xi)}/w(\xi) = 0$ if $w(\xi) = 0$ and define

$$\lambda(z) = \frac{1}{2i\pi} \iint_\Omega \frac{\overline{w(\xi)}}{w(\xi)} \frac{\alpha(\xi)}{\xi - z} \, d\xi \wedge d\bar{\xi}, \qquad z \in \Omega.$$

As in (14), we find that $\lambda \in W^{1,r}(\Omega)$ with $\overline{\partial}\lambda = \alpha\overline{w}/w$ and $\|\partial\lambda\|_{L^r(\Omega)} \leq C_1 \|\alpha\overline{w}/w\|_{L^r(\Omega)} \leq C_1\|\alpha\|_{L^r(\Omega)}$ for some constant $C_1 = C_1(r)$, see [42, Ch. 1, (1.7)-(1.9)]. Thus, if we set $s(z) = \lambda(z) - \iint_\Omega \lambda \, dm/m(\Omega)$, we obtain by Poincaré's inequality that

$$\|s\|_{W^{1,r}(\Omega)} \leq C_2(\|\partial s\|_{L^r(\Omega)} + \|\overline{\partial} s\|_{L^r(\Omega)}) = C_2(\|\partial\lambda\|_{L^r(\Omega)} + \|\overline{\partial}\lambda\|_{L^r(\Omega)}) \leq 2C_2C_1\|\alpha\|_{L^r(\Omega)} \tag{B.1}$$

for some constant $C_2 = C_2(r, \Omega)$. Now (19) follows from the Sobolev embedding theorem, since $r > 2$.

Next, we show that $F = e^{-s}w \in L^p_{loc}(\Omega)$ is in fact holomorphic. By Weyl's lemma, it is enough to check that $\bar{\partial}F = 0$ as a distribution. Let $\psi \in \mathcal{D}(\Omega)$ and ψ_n a sequence in $\mathcal{D}(\mathbb{R}^2)_{|\Omega}$ converging to s in $W^{1,r}(\Omega)$. As $r > 2$, ψ_n converges uniformly to s on Ω by Sobolev's embedding theorem, hence by dominated convergence and since $\alpha\overline{w} \in L^1_{loc}(\Omega)$

$$\langle \bar{\partial}F, \psi\rangle = -\langle e^{-s}w, \bar{\partial}\psi\rangle = -\lim_n \langle w, e^{-\psi_n}\bar{\partial}\psi\rangle = -\lim_n \langle w, \bar{\partial}(e^{-\psi_n}\psi) + \psi e^{-\psi_n}\bar{\partial}\psi_n\rangle$$

$$= \lim_n \langle \alpha\overline{w}, e^{-\psi_n}\psi\rangle - \lim_n \langle w, \psi e^{-\psi_n}\bar{\partial}\psi_n\rangle = \langle e^{-s}(\alpha\overline{w} - w\bar{\partial}s), \psi\rangle = 0$$

since $w\bar{\partial}s = \alpha\overline{w}$, where we used in the fourth equality that $e^{-\psi_n}\psi \in \mathcal{D}(\Omega)$. This proves (18). Because $s \in W^{1,r}(\Omega)$ is bounded we have that $e^s \in W^{1,r}(\Omega)$, and as F is locally smooth we get that $w \in W^{1,r}_{loc}(\Omega)$, as announced.

Clearly F satisfies (10) if and only if w does by (19), *i.e.* $F \in H^p(\Omega)$ if and only if $w \in G^p_\alpha(\Omega)$.

As for the normalization, let $u \in W^{1,r}_{\mathbb{R}}(\Omega)$ be harmonic in Ω with $u_{|\partial\Omega} = \operatorname{Im} s_{|\partial\Omega} \in W^{1-1/r,r}_{\mathbb{R}}(\partial\Omega)$. Such a function uniquely exists with $\|u\|_{W^{1,r}(\Omega)} \leq C_3\|\operatorname{Im} s\|_{W^{1-1/r,r}(\partial\Omega)}$, where $C_3 = C_3(r, \Omega)$ [18]. Thus, by (B.1) and continuity of the trace, it holds that

$$\|u\|_{W^{1,r}(\Omega)} \leq C_4\|\alpha\|_{L^r(\Omega)}, \quad \text{with } C_4 = C_4(r, \Omega).$$

Set $a_j = \int_{\Gamma_j} \partial_n u$, $0 \leq j \leq n$, where $\partial_n u \in W^{-1/r,r}_{\mathbb{R}}(\partial\Omega)$. Note that $\sum_j a_j = 0$ by (6) (applied with $\sigma = g \equiv 1$). We can find a function ω, harmonic on Ω and C^1-smooth on $\overline{\Omega}$, which is constant on each Γ_j and such that $\int_{\Gamma_j} \partial_n\omega = a_j$, see [2, Sec. 6.5.1] and Lemma 6. By construction

$$\|\omega\|_{W^{1,r}(\Omega)} \leq C_5\|\partial_n u\|_{W^{-1/r,r}(\partial\Omega)} \leq C_5\|u\|_{W^{1,r}(\Omega)} \leq C_6\|\alpha\|_{L^r(\Omega)} \quad \text{with } C_6 = C_6(r, \Omega).$$

The harmonic function $v = u - \omega$ lies in $W^{1,r}(\Omega)$ and its conjugate differential $d^*v = -\partial v/\partial y dx + \partial v/\partial x dx$ is exact[12]. Thus, there is a harmonic conjugate $\tilde{v}$ in Ω, unique up to an additive constant, such that $G = v + i\tilde{v}$ is holomorphic; if we normalize $\tilde{v}$ so that

[12] Indeed, since $d^*v = \partial_n v|dz|$ along any curve, its integral over a cycle $\gamma \subset \Omega$ is zero by Green's formula (6) (applied with $\sigma = g \equiv 1$ on the domain bounded by γ and all the Γ_j located inside γ) because $\int_{\Gamma_j} \partial_n v = 0$ for all j by construction, see [2, Sec. 4.6].

$\int_\Omega \tilde{v} dm = 0$, it is immediate from the Cauchy-Riemann equation and Poincaré's inequality that $\|\tilde{v}\|_{W^{1,r}(\Omega)} \leq C_7 \|v\|_{W^{1,r}(\Omega)}$ with $C_7 = C_7(r, \Omega)$. Altogether, $\|G\|_{W^{1,r}(\Omega)} \leq C_8 \|\alpha\|_{L^r(\Omega)}$, and since $r > 2$ we see that G is bounded by the Sobolev embedding theorem. Finally, setting $\tilde{s} = s - iG$ and $\tilde{F} = e^{iG} F$, we find that $w = e^{\tilde{s}} \tilde{F}$ is a factorization of the form (18) in which $\operatorname{Im} \tilde{s}$ is constant on each Γ_j. Clearly, we may impose the value of this constant on any given Γ_j upon renormalizing $\tilde{F}$. ■.

C Proof of Lemma 1

We must show that if f is holomorphic in Ω and (10) holds for some sequence of admissible compact sets Δ_n, then it holds for $\widetilde{\Delta}_n$ defined in (21) as well. When Ω is simply connected, this a well-known consequence of Carathéodory's kernel convergence theorem, see [25, Thm 10.1].

Assume next that Ω is m-connected. By Lemma 6 and the change of variable formula, it is enough to prove the result when Ω is a normalized circular domain (so that $\Omega = \Omega'$ and φ is the identity map in definition (21)). Let $\mathbb{T}_{a_j,r_j}$, $1 \leq j \leq m$ denote the connected components of $\partial\Omega$ lying inside $\mathbb{D}$. By the decomposition theorem [25, Sec. 10.5], we can write $f = f_1 + \cdots + f_{m+1}$ with $f_{m+1} \in H^p(\mathbb{D})$ and $f_j \in H^p(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}_{a_j,r_j}})$ for $1 \leq j \leq m$. The result just quoted in the simply connected case inplies that

$$\sup_{n \in \mathbb{N}} \|f_j\|_{L^p(\mathbb{T}_{a_j,r_j+\delta_\Omega/n})} < \infty,$$

and since the $\mathbb{T}_{a_k,r_k+\delta_\Omega/n}$ are compactly embedded in $\mathbb{C} \setminus \overline{\mathbb{D}_{a_j,r_j}})$ when $k \neq j$, the inequality

$$\sup_{n \in \mathbb{N}} \|f_j\|_{L^p(\mathbb{T}_{a_k,r_k+\delta_\Omega/n})} < \infty,$$

follows from Hölder's inequality and the Cauchy representation formula for f_j from its values on $\mathbb{T}_{a_j,r_j+\delta_\Omega/n_0}$ for some fixed n_0. Likewise $\|f_{m+1}\|_{L^p(\mathbb{T}_{1-\delta_\Omega/n})}$, $\|f_{m+1}\|_{L^p(\mathbb{T}_{a_j,r_j+\delta_\Omega/n})}$ are uniformly bounded, so that $\|f\|_p < \infty$ as desired. ■

D Traces of holomorphic functions

Lemma 7 *Let Ω be a Dini-smooth domain, and $g \in H^p(\Omega)$. Then g has a non-tangential limit a.e. on $\partial\Omega$ defining a trace function $tr_{\partial\Omega} g \in L^p(\partial\Omega)$. In fact*

$$\lim_{\varepsilon \to 0} \|tr_{\partial\Omega} g - g \circ P^{-1}_{\partial\Omega,\varepsilon}\|_{L^p(\partial\Omega)} = 0, \tag{D.1}$$

and if g is not identically zero then $\log|tr_{\partial\Omega} g| \in L^1(\partial\Omega)$.
The quantity $\|tr_{\partial\Omega} g\|_{L^p(\partial\Omega)}$ defines a norm on $H^p(\Omega)$ which is equivalent to $\|g\|_p$. Moreover

$$\|\mathcal{M}_g\|_{L^p(\partial\Omega)} \leq C \|tr_{\partial\Omega} f\|_{L^p(\partial\Omega)}, \tag{D.2}$$

where C depends on Ω, p and the aperture β used in the definition of the maximal function.

Proof. It is well-known that functions in $H^p(\Omega)$ (recall from Theorem 1 that it coincides both with the Hardy and Smirnov class) have non-tangential limit in $L^p(\partial\Omega)$ of which g

is the Cauchy integral [25, Thm 10.4, Sec. 10.5]. By Lemma 6, we may assume that Ω is a normalized circular domain. When $\Omega = \mathbb{D}$ all properties stated are standard, see [25, Thms 1.6, 2.2, 2.6], [34, Thm 3.1] and the remarks thereafter. By reflection, they also hold for Hardy spaces of the complement of a disk. Next, assume that $\Omega = \mathbb{D} \setminus \cup_{j=1}^n \overline{\mathbb{D}}_{a_j,r_j}$, with $a_j \in \mathbb{D}$ and $0 < r_j < 1 - |a_j|$. The decomposition theorem [25, Sec. 10.5] tells us that $g = \sum_{j=0}^n g_j$ with $g_0 \in H^p(\mathbb{D})$ and $g_j \in H^p(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}_{a_j,r_j}})$, $1 \le j \le n$. From the known result in the simply connected case and the smoothness of holomorphic functions on their domain of analyticity, we thus obtain

$$\lim_{\varepsilon\to 0} \left\| \left(\mathrm{tr}_{\mathbb{T}} g_0 + \sum_{j=1}^n g_{j|\mathbb{T}} \right) - \left(\sum_{j=0}^n g_j \right) \circ P_{\mathbb{T},\varepsilon}^{-1} \right\|_{L^p(\mathbb{T})} = 0,$$

and by a similar argument we also get for $1 \le j \le n$ that

$$\lim_{\varepsilon\to 0} \left\| \left(\mathrm{tr}_{\mathbb{T}_{a_j,r_j}} g_j + \left(\sum_{k\neq j}^n g_k \right)_{|\mathbb{T}_{a_j,r_j}} \right) - \left(\sum_{j=0}^n g_j \right) \circ P_{\mathbb{T}_{a_j,r_j},\varepsilon}^{-1} \right\|_{L^p(\mathbb{T}_{a_j,r_j})} = 0,$$

from which (D.1) follows.

Next, observe from Hardy's convexity theorem (see [25, Thms 1.5, 1.6] and the remark thereafter) that $\log \|g\|_{L^p(\mathbb{T}_r)}$ is a convex function of $\log r$ for $r \in (1-\delta_\Omega, 1)$, hence by (D.1)

$$\sup_{1-\delta_\Omega \le r < 1} \|g\|_{L^p(\mathbb{T}_r)} \le \max\{\|g\|_{L^p(\mathbb{T}_{1-\delta_\Omega})}, \|\mathrm{tr}_{\mathbb{T}} g\|_{L^p(\mathbb{T})}\}. \tag{D.3}$$

Likewise, for $j = 1, \cdots, n$,

$$\sup_{r_j+\delta_\Omega \ge r > r_j} \|g\|_{L^p(\mathbb{T}_{a_j,r_j})} \le \max\{\|g\|_{L^p(\mathbb{T}_{a_j,r_j+\delta_\Omega})}, \|\mathrm{tr}_{\mathbb{T}_{a_j,r_j}} g\|_{L^p(\mathbb{T}_{a_j,r_j})}\}. \tag{D.4}$$

From (D.3)-(D.4) and Hölder's inequality applied to the representation of g as the Cauchy integral of $\mathrm{tr}_{\partial\Omega} g$, we deduce that $\|g\|_p \le C \|\mathrm{tr}_{\partial\Omega} g\|_{L^p(\partial\Omega)}$ where the constant C depends only on p and Ω. In the other direction, the inequality $\|\mathrm{tr}_{\partial\Omega} g\|_{L^p(\partial\Omega)} \le \|g\|_p$ follows from the Fatou lemma applied to (D.1), hence $\|\mathrm{tr}_{\partial\Omega} g\|_{L^p(\partial\Omega)}$ is equivalent to $\|g\|_p$.

In the same vein, (D.2) is easily obtained from the known simply connected case, the decomposition theorem and Hölder's inequality applied to the representation of g as the Cauchy integral of $\mathrm{tr}_{\partial\Omega} g$.

To prove that $\log|\mathrm{tr}_{\partial\Omega} g| \in L^1(\partial\Omega)$ if g is not identically zero, we observe that Ω can be decomposed as a finite union of Dini-smooth simply connected domains Ω_l such that $\partial\Omega \subset \cup_l \Omega_l$. By Theorem 1 $g_{|\Omega_l} \in H^p(\Omega_l)$ since $|g|^p$ has a harmonic majorant on Ω, *a fortiori* on Ω_l. The result now follows from the one in the simply connected case. ∎

Lemma 8 *Let Ω be a Dini-smooth domain, and $g \in H^p(\Omega)$. To each $p_1 \in [p, 2p)$, there is a constant c depending only on Ω and p_1 such that $\|g\|_{L^{p_1}(\Omega)} \le c\|g\|_p$.*

Proof. When $\Omega = \mathbb{D}$, this is established in [11, appendix, proof of lem. 5.2.1]. So, by conformal mapping, we get it for simply connected Ω[13]. In the multiply connected case, the result follows from its simply connected version and Lemma 2. ∎

[13]Remember summability is understood with respect to area measure on the sphere.

Lemma 9 *Let f be holomorphic in the annulus $\mathbb{A}_\varrho$, $0<\varrho<1$. To each $p\in(1,\infty)$, there are constants C_1, C_2, depending on f, ϱ and p, such that*

$$\|Im\, f\|_{L^p(\mathbb{T}_r)} \leq C_1\|Re\,\|_{L^p(\mathbb{T}_r)} + C_2, \quad \varrho<r<1.$$

Proof. Set $r_1=(1-\varrho)/3$, and pick $(1-\varrho)/2<r_2<1$. For $(1-\varrho)/2\leq|z|<r_2$, we get by the Cauchy formula

$$f(z)=F_2(z)-F_1(z), \qquad F_2(z)=\frac{1}{2i\pi}\int_{\mathbb{T}_{r_2}}\frac{f(\xi)}{\xi-z}\,d\xi, \quad F_1(z)=\frac{1}{2i\pi}\int_{\mathbb{T}_{r_1}}\frac{f(\xi)}{\xi-z}\,d\xi.$$

Thus, setting $r=|z|$, it holds that

$$\|\mathrm{Im}\, f\|_{L^p(\mathbb{T}_r)} \leq C(\rho,f,p)+\|\mathrm{Im}F_2\|_{L^p(\mathbb{T}_r)},$$

and since F_2 is holomorphic in $\mathbb{D}_{r_2}$ we obtain from the M. Riesz theorem

$$\|\mathrm{Im}\, f\|_{L^p(\mathbb{T}_r)} \leq C(\rho,f,p)+C(p)\|\mathrm{Re}F_2\|_{L^p(\mathbb{T}_r)}$$

$$\leq C(\rho,f,p)+C(p)\big(\|\mathrm{Re}f\|_{L^p(\mathbb{T}_r)}+\|\mathrm{Re}F_1\|_{L^p(\mathbb{T}_r)}\big)\leq C'(\rho,f,p)+C(p)\|\mathrm{Re}f\|_{L^p(\mathbb{T}_r)}.$$

A similar estimate holds for $\varrho<|z|\leq(1-\varrho)/2$ upon swaping the role of F_1 and F_2. ■

E A lemma on Sobolev functions

Lemma 10 *Let Ω be a bounded Dini-smooth domain and assume that p, r satisfy* (7). *Let $p_1\in[p,2p)$ be such that $2/p_1-1/p<1-2/r$ and set $1/\beta=1/p_1+1/r$. Then*

(i) $W^{1-1/\beta,\beta}(\partial\Omega)$ is compactly included in $L^p(\partial\Omega)$;

(ii) $\widetilde{\Delta}_{1/n}$ being as in (21), *there is a constant C depending only of Ω, p, and β such that for each $h\in W^{1,\beta}(\Omega)$*

$$\sup_{n\in\mathbb{N}}\|tr_{\partial\widetilde{\Delta}_{1/n}}h\|_{L^p(\partial\widetilde{\Delta}_{1/n})}<C\|h\|_{W^{1,\beta}(\Omega)}. \tag{E.1}$$

Proof: let φ conformally map Ω onto a normalized circular domain Ω'. By Lemma 6 it is clear that $\|h\|_{W^{1,\beta}(\Omega)}$ and $\|h\circ\varphi^{-1}\|_{W^{1,\beta}(\Omega')}$ are comparable. Likewise (see (21)), for any $l\in(1,\infty)$ and any smooth Φ, $\|\mathrm{tr}_{\partial\widetilde{\Delta}_{1/n}}\Phi\|_{L^l(\partial\widetilde{\Delta}_{1/n})}$ and $\|\mathrm{tr}_{\partial K_{1/n}}\Phi\circ\varphi^{-1}\|_{L^l(\partial K_{1/n})}$ on the one hand, $\|\mathrm{tr}_{\partial\widetilde{\Delta}_{1/n}}\Phi\|_{W^{1,l}(\partial\widetilde{\Delta}_{1/n})}$ and $\|\mathrm{tr}_{\partial K_{1/n}}\Phi\circ\varphi^{-1}\|_{W^{1,l}(\partial K_{1/n})}$ on the other hand are comparable. Hence, by interpolation, $\|\mathrm{tr}_{\partial\widetilde{\Delta}_{1/n}}\Phi\|_{W^{1-1/l,l}(\partial\widetilde{\Delta}_{1/n})}$ and $\|\mathrm{tr}_{\partial K_{1/n}}\Phi\circ\varphi^{-1}\|_{W^{1-1/l,l}(\partial K_{1/n})}$ are also comparable. Altogether, we may assume for the proof that Ω is normalized circular. Moreover, in view of the extension theorem for Sobolev functions [54, Sec. VI.3, Thm 5], we may proceed componentwise on the boundary so it is enough to consider the case where $\Omega=\mathbb{D}$.
From [23, Thm 4.54], we know if $\beta\geq2$ that the inclusion $W^{1-1/\beta,\beta}(\mathbb{T})\subset L^l(\mathbb{T})$ is compact for all $l\in(1,\infty)$, while if $\beta<2$ the inclusion $W^{1-1/\beta,\beta}(\mathbb{T})\subset L^{\beta/(2-\beta)}(\mathbb{T})$ is compact. One can check that $\beta/(2-\beta)>p$ when $2/p_1-1/p<1-2/r$, thereby proving (i).
From (i) and the trace theorem, there is a constant $c=c(p,\beta)$ such that

$$\|\mathrm{tr}_{\mathbb{T}}h\|_{L^p(\mathbb{T})}<c\|h\|_{W^{1,\beta}(\mathbb{D})}, \qquad h\in W^{1,\beta}(\mathbb{D}).$$

Picking $r \in (0,1)$ and applying the above inequality to $h_r(z) = h(rz)$, we obtain (remember arclength is normalized)

$$\|\mathrm{tr}_{\mathbb{T}_r} h\|_{L^p(\mathbb{T}_r)} < c\|h_r\|_{W^{1,\beta}(\mathbb{D})} \leq \frac{c}{r^{2/\beta}}\|h\|_{W^{1,\beta}(\mathbb{D})}.$$

Since $\partial K_{1/n} = \mathbb{T}_{1-1/2n}$ in the present case, assertion (ii) follows. ■

F Proof of Proposition 3

By Hölder's inequality and standard properties of the Cauchy and Beurling transforms [42, Ch. 1, (1.7)-(1.9)], T_α maps $L^p(\Omega)$ into $W^{1,rp/(r+p)}(\Omega)$. By the Rellich-Kondratchov theorem, either $rp/(r+p) \geq 2$ in which case $W^{1,rp/(r+p)}(\Omega)$ is compactly embedded in $L^\lambda(\Omega)$, $1 \leq \lambda < \infty$, or else $rp/(r+p) < 2$ and then $W^{1,rp/(r+p)}(\Omega)$ is compactly embedded in every $L^\lambda(\Omega)$ with $1 \leq \lambda < 2rp/(2(r+p) - rp)$. Since $2rp/(2(r+p) - rp) > p$ when $r > 2$, this proves that T_α is compact from $L^p(\Omega)$ into itself.
Next, we show that $I - T_\alpha$ is injective. Indeed, if $h = T_\alpha h$, we get from what precedes that $h \in L^\lambda(\Omega)$ for $1 \geq 1/\lambda > \max(0, 1/p + 1/r - 1/2)$. Thus, by Hölder's inequality, $\alpha\overline{h} \in L^t(\Omega)$ for every t such that $1 \geq 1/t > \max(1/r, 1/p + 2/r - 1/2)$, and in turn $h \in W^{1,t}(\Omega)$ for all such t. Therefore, by the Sobolev embedding theorem, $h \in L^\lambda(\Omega)$ for each λ such that $1 \geq 1/\lambda > \max(0, 1/p + 2(1/r - 1/2))$. Iterating, we find that $h \in L^\lambda(\Omega)$ whenever $1 \geq 1/\lambda > \max(0, 1/p + k(1/r - 1/2))$ for some $k \geq 1$, and that $h \in W^{1,t}(\Omega)$ for $1 \geq 1/t > \max(1/r, 1/p + 1/r + k(1/r - 1/2))$. Since $r > 2$, we deduce that $h \in W^{1,t}(\Omega)$ as soon as $1 \leq t < r$, in particular we may pick $t > 2$. The rest of the argument proceeds as in [11, App. A]: we put

$$H(z) = \frac{1}{2\pi i}\iint_\Omega \frac{\alpha(\xi)\overline{h}(\xi)}{\xi - z} d\xi \wedge d\overline{\xi}\,, \quad z \in \mathbb{C}\,,$$

noting that $h = H_{|\Omega}$ and $H \in W^{1,t}_{loc}(\mathbb{C})$ by what precedes. Clearly $\overline{\partial} H = (\alpha \vee 0)H$ on $\mathbb{C}$, and since $t > 2$ while H vanishes at infinity we can apply the extended Liouville theorem [4, Prop. 3.3] to the effect that $H \equiv 0$ hence $h \equiv 0$, as desired.
It now follows from a theorem of F. Riesz [19, Thm 1.16] that $I - T_\alpha$ is an isomorphism of $L^p(\Omega)$.
In another connection, let $w \in G^p_\alpha(\Omega)$ and set $g = w - T_\alpha w$. Then $\overline{\partial} g = 0$ because $\overline{\partial} T_\alpha w = \alpha\overline{w}$, hence g is holomorphic in Ω. Moreover, we know from Property 2 that $w \in L^{p_1}(\Omega)$ for $p \leq p_1 < 2p$, hence for such p_1 it holds that $T_\alpha w \in W^{1,\beta}(\Omega)$ with $1/\beta = 1/p_1 + 1/r$. Choosing p_1 such that $2/p_1 - 1/p < 1 - 2/r$, Lemma 10 point (ii) implies that $T_\alpha w$ satisfies (10), hence so does g, that is, $g \in H^p$.
Conversely, for $g \in H^p$, let us put $w = (I - T_\alpha)^{-1} g$. Since $g \in L^{p_1}(\Omega)$ for $p \leq p_1 < 2p$ and (7) continues to hold with p replaced by p_1, we get from the previous part of the proof that w lies in $L^{p_1}(\Omega)$ and consequently that $T_\alpha w$ satisfies (10). Hence w in turn meets (10), and since $\overline{\partial}(I - T_\alpha)w = 0$ it is a solution to (13), hence a member of $G^p_\alpha(\Omega)$.
Equation (32) simply means that $\mathcal{C}(\mathrm{tr}_{\partial\Omega} T_\alpha w)(z) = 0$ for $z \in \Omega$. To see this, let

$$F(z) = \frac{1}{2\pi i}\iint_\Omega \frac{\alpha(\xi)\overline{w}(\xi)}{\xi - z} d\xi \wedge d\overline{\xi}\,, \quad z \in \mathbb{C}.\,.$$

By what precedes $F(z) \in W^{1,\beta}_{loc}(\mathbb{C})$, $F_{|\Omega} = T_\alpha w$, and clearly F is holomorphic in $\overline{\mathbb{C}} \setminus \overline{\Omega}$ with $F(\infty) = 0$. Lemma 10 point (ii), applied with $h = F$ on $(\overline{\mathbb{C}} \setminus \overline{\Omega}) \cap \mathbb{D}_R$ where R is a large positive number, shows that $F \in H^p(\overline{\mathbb{C}} \setminus \overline{\Omega})$, and we have that $\mathrm{tr}_{\partial\Omega} F = \mathrm{tr}_{\partial\Omega} T_\alpha w$. Consequently, by Cauchy's theorem, it holds for any system of rectifiable Jordan curves Γ homotopic to $\partial\Omega$ in $\overline{\mathbb{C}} \setminus \Omega)$ that

$$\mathcal{C}(\mathrm{tr}_{\partial\Omega} T_\alpha w)(z) = \frac{1}{2\pi i} \int_\Gamma \frac{F(\xi)}{\xi - z} d\xi\,, \quad z \in \Omega.\,.$$

Deforming the inner components of $\partial\Omega$ to a point and the outer component to ∞, we see that the above integral is zero, as desired.
Finally, since $\int_{\partial\Omega} T_\alpha w = 0$ by what we just said, we get from the Poincaré inequality, Hölder's inequality, the continuity of $(I - T_\alpha)^{-1}$ and Lemma 8 that

$$\|T_\alpha w\|_{W^{1,\beta}(\Omega)} \le c_1 \|\alpha \overline{w}\|_{L^\beta(\Omega)} \le c_2 \|\alpha\|_{L^r(\Omega)} \|w\|_{L^{p_1}(\Omega)} \le c_3 \|g\|_{L^{p_1}(\Omega)} \le c_4 \|g\|_p.$$

Thus, by the trace theorem and Lemma 10 point (i), we obtain $\|\mathrm{tr}_{\partial\Omega} T_\alpha w\|_{L^p(\partial\Omega)} \le c_5 \|g\|_p$, hence $\|\mathrm{tr}_{\partial\Omega} w\|_{L^p(\partial\Omega)} \le c_6 \|g\|_p$ since $w = g + T_\alpha w$. Property 2 now yields $\|w\|_p \le C \|g\|_p$, as announced. ■

G Proof of Lemma 4

By Lemma 6, we may assume that $\Omega = \mathbb{D}$. Let $\mathbf{P}_+(h) = \mathrm{tr}_\mathbb{T} \mathcal{C}(h)$ denote the Riesz projection which discards Fourier coefficients of non-positive index. It is continuous from $L^p(\mathbb{T})$ onto $\mathrm{tr}_\mathbb{T}\, H^p(\mathbb{D}) \subset L^p$ [34, Sec. III.1]. Moreover, to each $u \in L^p(\mathbb{T})$ there uniquely exists $\widetilde{u} \in L^p(\mathbb{T})$ such that $u + i\widetilde{u} \in \mathbf{P}_+(H^p)$ and $\int_\mathbb{T} \widetilde{u} = 0$.
For $u \in L^p(\mathbb{T})$ and $c \in \mathbb{R}$, let $w_{u,c} \in G^p_\alpha(\mathbb{D})$ satisfy $(u + i(\widetilde{u} + c)) = P_+ w_{u,c}$. Such a function uniquely exists by Proposition 3 and depends continuously on u and c. Define

$$A(u, c) := \left(\mathrm{Re} \left(\mathrm{tr}_\mathbb{T}\, w_{u,c} \right)\ ,\ \mathrm{Im} \int_\mathbb{T} \mathrm{tr}_\mathbb{T}\, \sigma^{1/2} w_{u,c} \right) \in L^p_\mathbb{R}(\mathbb{T}) \times \mathbb{R}.$$

Since $(I - T_\alpha) w_u = g$ where $g \in H^p$ satisfies $\mathrm{tr}_\mathbb{T}\, g = u + i(\widetilde{u} + c)$, we can decompose the operator A as $A(u, c) = (u, c) + B(u, v)$ where

$$B(u, c) := \left(\mathrm{Re} \left(\mathrm{tr}_\mathbb{T}\, T_\alpha(w_{u,c}) \right)\ ,\ \mathrm{Im} \int_\mathbb{T} \sigma^{1/2} \mathrm{tr}\, w_{u,c} - c \right).$$

From the proof of Proposition 3, we know that $(u, c) \mapsto T_\alpha w_{u,c}$ is continuous from $L^p(\mathbb{T}) \times \mathbb{R}$ into $W^{1,\beta}(\mathbb{D})$ when $1/\beta = 1/p_1 + 1/r$ for some $p_1 \in [p, 2p)$, hence B is compact from $L^p(\mathbb{T}) \times \mathbb{R}$ into itself by Lemma 10 point (i) and the trace theorem.
In another connection, if $w \in G^p_\alpha(\mathbb{D})$ is such that $\mathrm{Re}\,(\mathrm{tr}_\mathbb{T}\, w) = 0$ and $\mathrm{Im} \int_\mathbb{T} \sigma^{1/2} \mathrm{tr}_\mathbb{T}\, w = 0$, then $w = 0$. Indeed, normalizing $\mathrm{tr}_\mathbb{T}\, s$ to be real in (18) we find that $F \in H^p$ has zero real part on $\mathbb{T}$, hence it is an imaginary constant, and in fact $F = 0$ as the mean on $\mathbb{T}$ of $\sigma^{1/2} F$ must vanish. Consequently A is injective, hence a homeomorphism of $L^p(\mathbb{T}) \times \mathbb{R}$ by Riesz's theorem. This shows one can impose uniquely in $L^p(\mathbb{T})$ the real part of $w \in G^p_\alpha(\mathbb{D})$ on $\mathbb{T}$ together with the mean of $\sigma^{1/2}$ times its imaginary part there, and that

$$\|\mathrm{tr}_\mathbb{T}\, w\|_{L^p(\mathbb{T})} \le c_1 \|\mathrm{Re} \mathrm{tr}_\mathbb{T}\, w\|_{L^p(\mathbb{T})} + c_2 \left| \int_\mathbb{T} \sigma^{1/2} \mathrm{Im} \mathrm{tr}_\mathbb{T}\, w \right|.$$

From this and (16), it follows easily that one can impose uniquely the real part $u \in L^p(\mathbb{T})$ of $f \in H^p_\alpha(\mathbb{D})$ on $\mathbb{T}$ and the mean of its imaginary part there. In addition, if the latter is taken to be zero, there is an inequality of the form $\|\mathrm{tr}_{\partial\Omega} f\|_{L^p(\partial\Omega)} \leq c\|u\|_{L^p(\partial\Omega)}$ which proves assertion (ii).
Now, in view of Properties 1,2, and 6 in section 3.3, taking real parts in assertion (ii) yields assertion (i) except for the uniqueness part. To establish the latter, assume $u = 0$ and let us prove that $U = 0$. As u satisfies (3), there is a distribution V such that (1) holds. Since $\|U\|_p < \infty$, hence *a fortiori* $U \in L^p(\mathbb{D})$, we observe much as in the proof of [11, Thm 4.4.2.2] that $V \in L^p(\mathbb{D})$; the only difference is that, in order to obtain equation (63) *loc. cit.*, one must know whenever Φ is smooth with compact support that $\|\Phi\nabla\sigma\|_{L^q(\mathbb{D})} \leq c\|\nabla\Phi\|_{L^q(\mathbb{D})}$, $1/p + 1/q = 1$, which follows easily from the Hölder and the Sobolev inequalities for $r > 2$. Then $f = U + iV$ satisfies (CB), so that w given by (16) satisfies (13). Since $U = \mathrm{Re} f$ satisfies (8) by assumption, so does $\mathrm{Re}\, w$. By Proposition 2 w assumes the form (18) where $\mathrm{tr}_{\mathbb{T}}\, \mathrm{Im}\, s = 0$ and, say, $F = a + ib$ is holomorphic. Assume for a contradiction that $\|a\|_{L^p(\mathbb{T}_\rho)}$, $0 \leq \rho < 1$ is unbounded. Then it must tend to $+\infty$ as it increases with ρ [25, Thms 1.5, 1.6]. By the continuity of s (*cf.* Remark 2), to each $\varepsilon \in (0, 1)$ there is $\rho_0 \in (0, 1)$ such that $|\mathrm{Im}\, \exp(s(z))| < \varepsilon|\exp(s(z))|$ as soon as $\rho_0 < |z| \leq 1$. For such z, we deduce from (18) that

$$|\mathrm{Re}\, w(z)| \geq e^{-\|s\|_{L^\infty(\mathbb{D})}}((1 - \varepsilon^2)^{1/2}|a(z)| - \varepsilon|b(z)|). \tag{G.1}$$

By a theorem of M. Riesz $\|b - b(0)\|_{L^p(\mathbb{T}_\rho)} \leq C\|a\|_{L^p(\mathbb{T}_\rho)}$ with $C = C(p)$, uniformly with respect to $\rho \in (\rho_0, 1]$ [25, Thm 4.1]. Hence integrating (G.1), we get

$$\|\mathrm{Re}\, w\|_{L^p(\mathbb{T}_\rho)} \geq e^{-\|s\|_{L^\infty(\mathbb{D})}}\Big((1 - \varepsilon^2)^{1/2} - |b(0)|/\|a\|_{L^p(\mathbb{T}_\rho)} - \varepsilon C\Big)\|a\|_{L^p(\mathbb{T}_\rho)}$$

and taking ε small enough we find this absurd since the left hand-side is bounded while the right hand side goes to infinity when $\rho \to 1$. Hence $\|a\|_{L^p(\mathbb{T}_\rho)}$ is bounded and so is $\|b\|_{L^p(\mathbb{T}_\rho)}$, by the M. Riesz theorem again, in other words $F \in H^p(\mathbb{D})$. Therefore $w \in G^p_\alpha(\mathbb{D})$ by Proposition 2, thus $f \in H^p_\nu(\mathbb{D})$. Moreover, since $\mathrm{Re}\, w$ has nontangential limit 0 on $\mathbb{T}$, so does $\mathrm{Re}\, F$ since $\mathrm{tr}_{\mathbb{T}}\, e^s > 0$, thus F is an imaginary constant. However V was defined up to an additive constant only, and since we have just shown that $\mathrm{tr}_{\mathbb{T}} V \in L^p(\mathbb{T})$ we can pick this constant so that $\int_{\mathbb{T}} V\sqrt{(1 + \nu)/(1 - \nu)} = 0$. Then $\mathrm{tr}_{\mathbb{T}}\mathrm{Im}\, w$ has zero mean by (16), consequently $F = 0$, hence $w = f = 0$. In particular $U = 0$, as desired. ∎

H Proof of Proposition 5

By Proposition 1, the conclusion for $H^p_\nu(\mathbb{D})$ follows from the result for $G^p_\alpha(\mathbb{D})$ which we now prove.
Define a Hermitian duality pairing on $L^p(\mathbb{D}) \times L^q(\mathbb{D})$, $1/p + 1/q = 1$, by the formula

$$< h, g >_{\mathbb{D}} = \frac{1}{2\pi i} \iint_{\mathbb{D}} h(z)\overline{g(z)}\, dz \wedge d\overline{z}.$$

If $A : L^p(\mathbb{D}) \to L^p(\mathbb{D})$ is antilinear (*i.e.* real linear such that $A(\lambda h) = \bar{\lambda} A(h)$), then $A(h) = B(h) + iB(ih)$ where $B = \mathrm{Re} A$. We let $A^\sharp$ designate the antilinear operator on $L^q(\mathbb{D})$ such that $< Ah, g > = < A^\sharp g, h >$. It is easy to check that $A^\sharp(g) = B^*(g) + iB^*(ig)$,

where B^* is the adjoint of B when $L^p(\mathbb{D})$, $L^q(\mathbb{D})$ are viewed as real vector spaces endowed with the pairing Re $< .,. >$.
For $\alpha \in L^r(\mathbb{D})$ and $h \in L^p(\mathbb{D})$, define functions $T_\alpha(h)$ and $\mathcal{T}_\alpha(h)$ on $\mathbb{D}$ and $\mathbb{C}$ by

$$\frac{1}{2\pi i}\iint_{\mathbb{D}} \frac{\alpha(\xi)\overline{h(\xi)}}{\xi - z} d\xi \wedge d\bar{\xi} = \begin{cases} \mathcal{T}_\alpha h(z) & \text{for } z \in \mathbb{C}\,, \\ T_\alpha h(z) & \text{for } z \in \mathbb{D} \,:\, T_\alpha h = (\mathcal{T}_\alpha h)_{|\mathbb{D}}\,. \end{cases} \tag{H.1}$$

From Proposition 3, we get that T_α is compact from $L^p(\mathbb{D})$ into itself and clearly it is antilinear. Moreover $I - T_\alpha$ is an isomorphism of $L^p(\mathbb{D})$ and the restriction map $I - T_\alpha : G^p_\alpha(\mathbb{D}) \to H^p(\mathbb{D})$ is an isomorphism which coincides with the analytic (Cauchy) projection, see (32).
By a theorem of Schauder $(\mathrm{Re}\, T_\alpha)^*$ is compact [16, Thm. VI.4], hence also $T^\sharp_\alpha$. In addition $I - T^\sharp_\alpha$ is injective, for if $g = T^\sharp_\alpha(g)$ we get from the definition of $A^\sharp$

$$\langle (I - T_\alpha)h, g\rangle = \langle h, g\rangle - \langle g, h\rangle, \qquad h \in L^p(\mathbb{D}),$$

which is absurd if $g \neq 0$ since the right hand side is pure imaginary whereas $I - T_\alpha$ is surjective. Hence $I - T^\sharp_\alpha$ is an isomorphism of $L^q(\mathbb{D})$ by Riesz's theorem. By Fubini's theorem, we obtain for $h \in L^p(\mathbb{D})$ and $g \in L^q(\mathbb{D})$ that

$$\begin{aligned} < T^\sharp_\alpha g, h >_{\mathbb{D}} = < T_\alpha h, g >_{\mathbb{D}} &= -\frac{1}{4\pi^2}\iint_{\mathbb{D}} \left(\iint_{\mathbb{D}} \frac{\alpha(\xi)\overline{h(\xi)}}{\xi - z} d\xi \wedge d\bar{\xi} \right) \overline{g(z)}\, dz \wedge d\bar{z} \\ &= -\frac{1}{4\pi^2}\iint_{\mathbb{D}} \left(\iint_{\mathbb{D}} \frac{\overline{g(z)}}{\xi - z} dz \wedge d\bar{z} \right) \alpha(\xi)\overline{h(\xi)}\, d\xi \wedge d\bar{\xi} \\ &= -\frac{1}{4\pi^2}\iint_{\mathbb{D}} \left(-\alpha(\xi) \iint_{\mathbb{D}} \frac{\overline{g(z)}}{z - \xi} dz \wedge d\bar{z} \right) \overline{h(\xi)}\, d\xi \wedge d\bar{\xi} \end{aligned}$$

so that

$$T^\sharp_\alpha g = -\alpha T_{\chi_{\mathbb{D}}} g \tag{H.2}$$

where $\chi_{\mathbb{D}}$ is the characteristic function of $\mathbb{D}$.
Let α_ϱ be 0 on $\mathbb{D}_\varrho$ and α on $\mathbb{A}_\varrho$. If $p_1 \geq p$, then T_{α_ϱ} maps $L^{p_1}(\mathbb{D})$ into $W^{1,\beta}(\mathbb{D})$ with $1/\beta = 1/r + 1/p_1$, see proof of Proposition 3. Besides, $\left(\mathrm{Ran} T_{\alpha_\varrho}\right)_{|\mathbb{D}_\varrho}$ consists of holomorphic functions. Therefore, for p_1 as in Lemma 10, we get from Property 6 and (E.1) that

$$\left(T_{\alpha_\varrho}(I - T_\alpha)^{-1} H^p(\mathbb{D})\right)_{|\mathbb{D}_\varrho} = \left(T_{\alpha_\varrho} G^p_\alpha(\mathbb{D})\right)_{|\mathbb{D}_\varrho} \subset \left(T_{\alpha_\varrho} L^{p_1}(\mathbb{D})\right)_{|\mathbb{D}_\varrho} \subset H^p(\mathbb{D}_\varrho). \tag{H.3}$$

Moreover, the operator, $A_\varrho = I + T_{\alpha_\varrho}(I - T_\alpha)^{-1}$ maps $H^p(\mathbb{D})$ into $L^{p_1}(\mathbb{D})$. Let us introduce the operator $B_\varrho = J \circ A_\varrho$, where $J : L^{p_1}(\mathbb{D}) \to L^{p_1}(\mathbb{D}_\varrho)$ is the natural restriction. In view of (H.3), B_ϱ maps continuously $H^p(\mathbb{D})$ into $H^p(\mathbb{D}_\varrho)$.

Lemma 11 *The operator $B_\varrho : H^p(\mathbb{D}) \to H^p(\mathbb{D}_\varrho)$ has dense range.*

Proof: It is equivalent to prove that if $\Psi \in H^{q,00}(\mathbb{C} \setminus \overline{\mathbb{D}}_\varrho)$ satisfies

$$\frac{1}{2i\pi}\int_{\mathbb{T}_\varrho} \Psi(z)\, \mathrm{tr}_{\mathbb{T}_\varrho} B_\varrho g(z)\, dz = 0, \qquad \forall g \in H^p(\mathbb{D}), \tag{H.4}$$

then $\Psi = 0$; indeed, the line integral of the product over $\mathbb{T}_\varrho$ identifies $H^{q,00}(\mathbb{C}\setminus\overline{\mathbb{D}}_\varrho)$ (non isometrically) with the dual of $H^p(\mathbb{D}_\varrho)$ [25, Theorem 7.3]. For small $\varepsilon > 0$, since Ψ and g are smooth on $\overline{\mathbb{A}}_{\varrho+\varepsilon,1-\varepsilon}$ while $A_\varrho g - g \in W^{1,\beta}(\mathbb{D})$, we get from Stokes' theorem

$$\frac{1}{2i\pi}\int_{\partial\mathbb{A}_{\varrho+\varepsilon,1-\varepsilon}} \Psi(z)\,\mathrm{tr}A_\varrho g(z)\ dz = -\frac{1}{2i\pi}\iint_{\mathbb{A}_{\varrho+\varepsilon,1-\varepsilon}} \overline{\partial}\big(\Psi(z)A_\varrho g(z)\big)\ dz\wedge d\overline{z}.$$

where the trace of $A_\varrho g$ in the first integral is on $\partial\mathbb{A}_{\varrho+\varepsilon,1-\varepsilon}$. Because Ψ, g are holomorphic in $\mathbb{A}_\varrho$ and $\overline{\partial}T_{\alpha_\varrho}h = \alpha_\varrho\overline{h}$ for $h \in L^p(\mathbb{D})$ by standard properties of the Cauchy transform, we may compute the surface integral using the definition of A_ϱ to obtain

$$0 = \frac{1}{2i\pi}\int_{\partial\mathbb{A}_{\varrho+\varepsilon,1-\varepsilon}} \Psi(z)\,\mathrm{tr}A_\varrho g(z)\ dz + \frac{1}{2i\pi}\iint_{\mathbb{A}_{\varrho+\varepsilon,1-\varepsilon}} \Psi(z)\alpha(z)\overline{(I-T_\alpha)^{-1}g(z)}\ dz\wedge d\overline{z}.$$

Now, $\Psi_{|\mathbb{A}_\varrho} \in H^q(\mathbb{A}_\varrho)$ and $A_\varrho g \in H^p(\mathbb{D}) + W^{1,\beta}(\mathbb{D})$, while $(I-T_\alpha)^{-1}g \in G^p_\alpha$. Let Ψ_ϱ be 0 on $\mathbb{D}_\varrho$ and Ψ elsewhere. Pick $q_1 \in (2, 2q)$ with $2/q_1 - 1/q < 1 - 2/r$ and recall from Lemma 8 that $\Psi_{|\mathbb{A}_\varrho} \in L^{q_1}(\mathbb{A}_\varrho)$, so that $\alpha\Psi_\varrho \in L^\delta(\mathbb{D})$ where $1/\delta = 1/r + 1/q_1 < (p_1-1)/p_1$. Thus, letting $\varepsilon \to 0$, we get from Property 1, 6, Lemma 10, and Hölder's inequality that

$$\frac{1}{2i\pi}\int_{\mathbb{T}_\varrho} \Psi(z)\,\mathrm{tr}_{\mathbb{T}_\varrho}B_\varrho g(z)\,dz = \frac{1}{2i\pi}\int_{\mathbb{T}} \Psi(z)\,\mathrm{tr}_{\mathbb{T}}A_\varrho g(z)dz + <\alpha\Psi_\varrho, (I-T_\alpha)^{-1}g>_{\mathbb{D}}, \tag{H.5}$$

where we took into account that $\mathrm{tr}_{\mathbb{T}_\varrho}B_\varrho g = \mathrm{tr}_{\mathbb{T}_\varrho}A_\varrho g$ on $\mathbb{T}_\varrho$. Put $b = (I-T^\sharp_\alpha)^{-1}(\alpha\Psi_\varrho) \in L^{p_1/(p_1-1)}(\mathbb{D})$. Then, by definition of $A^\sharp_\varrho$, it holds that

$$\begin{aligned} <\alpha\Psi_\varrho, (I-T_\alpha)^{-1}g>_{\mathbb{D}} &= <(I-T^\sharp_\alpha)b, (I-T_\alpha)^{-1}g>_{\mathbb{D}} \\ &= <b, (I-T_\alpha)^{-1}g>_{\mathbb{D}} - <T_\alpha(I-T_\alpha)^{-1}g, b> \\ &= <g, (I-T^\sharp_\alpha)^{-1}(\alpha\Psi_\varrho)>_{\mathbb{D}} + 2i\mathrm{Im} <b, (I-T_\alpha)^{-1}g>_{\mathbb{D}}\ . \end{aligned} \tag{H.6}$$

Representing $g \in H^p(\mathbb{D})$ by the Cauchy integral of $\mathrm{tr}_{\mathbb{T}}g$, we further have that

$$\begin{aligned} <g, (I-T^\sharp_\alpha)^{-1}(\alpha\Psi_\varrho)>_{\mathbb{D}} &= \frac{1}{2i\pi}\iint_{\mathbb{D}}\left(\int_{\mathbb{T}}\frac{\mathrm{tr}_{\mathbb{T}}g(z)}{z-\xi}\ dz\right)\overline{(I-T^\sharp_\alpha)^{-1}(\alpha\Psi_\varrho)(\xi)}\ d\xi\wedge d\overline{\xi} \\ &= \frac{1}{2i\pi}\int_{\mathbb{T}}\mathrm{tr}_{\mathbb{T}}g(z)\left(\iint_{\mathbb{D}}\frac{\overline{(I-T^\sharp_\alpha)^{-1}(\alpha\Psi_\varrho)(\xi)}}{z-\xi}\ d\xi\wedge d\overline{\xi}\right)\ dz\ , \end{aligned}$$

and letting $\mathfrak{I} = 2i\mathrm{Im} <b, (I-T_\alpha)^{-1}g>_{\mathbb{D}}$ we get in view of (H.6)

$$<\alpha\Psi_\varrho, (I-T_\alpha)^{-1}g>_{\mathbb{D}} = -\frac{1}{2i\pi}\int_{\mathbb{T}}\mathrm{tr}_{\mathbb{T}}g(z)\mathrm{tr}_{\mathbb{T}}T_{\chi_{\mathbb{D}}}(I-T^\sharp_\alpha)^{-1}(\alpha\Psi_\varrho)(z)\ dz + \mathfrak{I}\ . \tag{H.7}$$

Next, by (H.1), the function $\mathcal{T}_{\alpha_\varrho}h \in W^{1,\beta}_{loc}(\mathbb{C})$ has the same trace on $\mathbb{T}$ as $T_{\alpha_\varrho}h$ for all $h \in L^{p_1}(\mathbb{D})$. Hence, the first integral in the right hand side of (H.5) can be rewritten as

$$\frac{1}{2i\pi}\int_{\mathbb{T}}\Psi(z)\,\mathrm{tr}_{\mathbb{T}}g(z)\ dz + \frac{1}{2i\pi}\int_{\mathbb{T}}\Psi(z)\,\mathrm{tr}_{\mathbb{T}}\mathcal{T}_{\alpha_\varrho}(I-T_\alpha)^{-1}g(z)\ dz\ . \tag{H.8}$$

Moreover, an argument similar to the one that led us to (H.3) easily yields that

$$\big(\mathcal{T}_{\alpha_\varrho}L^{p_1}(\mathbb{D})\big)_{|\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}} \subset H^{p,00}(\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}), \tag{H.9}$$

therefore $\Psi_{|\mathbb{T}}\mathrm{tr}_{\mathbb{T}}\mathcal{T}_{\alpha_\varrho}(I-T_\alpha)^{-1}g$ is the trace on $\mathbb{T}$ of a function in $H^{1,00}(\overline{\mathbb{C}}\setminus\overline{\mathbb{D}})$ and the second integral in (H.8) is zero by Cauchy's theorem (deform $\mathbb{T}$ to infinity). Thus,

$$\frac{1}{2i\pi}\int_{\mathbb{T}}\Psi(z)\,\mathrm{tr}_{\mathbb{T}}A_\varrho g(z)\;dz=\frac{1}{2i\pi}\int_{\mathbb{T}}\Psi(z)\,\mathrm{tr}_{\mathbb{T}}g(z)\;dz\;, \tag{H.10}$$

and in view of (H.5), (H.7), and (H.10), we conclude if (H.4) holds that

$$\frac{1}{2i\pi}\int_{\mathbb{T}}\mathrm{tr}_{\mathbb{T}}g(z)\;\big(\Psi(z)-\mathrm{tr}_{\mathbb{T}}T_{\chi_{\mathbb{D}}}(I-T^\sharp_\alpha)^{-1}(\alpha_\varrho\Psi)(z)\big)\;dz=-\Im\;,\quad g\in H^p(\mathbb{D}). \tag{H.11}$$

Then, observe from (H.1) that $\mathcal{T}_{\chi_{\mathbb{D}}}h\in W^{1,\delta}_{loc}(\mathbb{C})$ has the same trace on $\mathbb{T}$ as $T_{\chi_{\mathbb{D}}}h$ for all $h\in L^\delta(\mathbb{D})$. In addition, since either $\delta>2$ or $\delta/(2-\delta)>q$, we get from [23, Thm 4.54] as in the proof of Lemma 10 point (i) that

$$(\mathcal{T}_{\chi_{\mathbb{D}}}L^{q_1}(\mathbb{D}))_{|\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}}\subset H^{q,00}(\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}) \tag{H.12}$$

(compare (H.9)). Hence (H.11) can be rewritten as

$$\frac{1}{2i\pi}\int_{\mathbb{T}}\mathrm{tr}_{\mathbb{T}}g(z)\;\big(\Psi(z)-\mathrm{tr}_{\mathbb{T}}\mathcal{T}_{\chi_{\mathbb{D}}}(I-T^\sharp_\alpha)^{-1}(\alpha\Psi_\varrho)(z)\big)\;dz=-\Im\;,\quad g\in H^p(\mathbb{D}), \tag{H.13}$$

and the left hand side of (H.13) is a complex linear form $\mathcal{L}(g)$ on $H^p(\mathbb{D})$ while the right hand side is always pure imaginary. Therefore $\mathcal{L}$ is the zero form, which means that

$$\Psi-\mathcal{T}_{\chi_{\mathbb{D}}}(I-T^\sharp_\alpha)^{-1}(\alpha\Psi_\varrho)\in H^{q,00}(\overline{\mathbb{C}}\setminus\overline{\mathbb{D}})(\alpha\Psi_\varrho)$$

must be the zero function. Translating back to $\mathbb{D}$, this amounts to say that the function

$$G(z)=\Psi_\varrho(z)-T_{\chi_{\mathbb{D}}}(I-T^\sharp_\alpha)^{-1}\,(\alpha\Psi_\varrho)\,, \tag{H.14}$$

which lies in $\in\Big(H^q(\mathbb{A}_\varrho)\vee 0_{|\mathbb{D}_\varrho}\Big)+W^{1,\delta}(\mathbb{D})$, satisfies $\mathrm{tr}_{\mathbb{T}}G=0$. A short calculation using (H.2) and the identity $(A-B)^{-1}=A^{-1}+A^{-1}B(A-B)^{-1}$ shows that

$$\alpha G=\alpha\Psi_\varrho+T^\sharp_\alpha(I-T^\sharp_\alpha)^{-1}(\alpha\Psi_\varrho)=(I-T^\sharp_\alpha)^{-1}(\alpha\Psi_\varrho), \tag{H.15}$$

hence

$$G=\Psi_\varrho-T_{\chi_{\mathbb{D}}}\,(\alpha G)=\Psi_\varrho-T_{\bar\alpha}\,(G) \tag{H.16}$$

by (H.14) and (H.15). Note, since $q_1>2$ and either $\delta>2$ or $2\delta/(2-\delta)>2q$, that $G\in L^\lambda(\mathbb{D})$ for some $\lambda>r/(r-1)$ by Lemma 8 and the Sobolev embedding theorem. Set for simplicity $G_1=G_{\mathbb{A}_\varrho}$. Applying $\overline{\partial}$ to (H.16), we find that $\overline{\partial}G_1=-\overline{\alpha}\,\overline{G_1}$. Thus, by Proposition 2, it holds that $G_{|\mathbb{A}_\varrho}=e^sF$ where s is bounded and F is holomorphic in $\mathbb{A}_\varrho$. Moreover, as we noticed already that either $\delta>2$ or $\delta/(2-\delta)>q$, we get as in the proof of Lemma 10 point (i) that $\|G\|_q<\infty$ on $\mathbb{A}_\varrho$. In particular $F\in H^q(\mathbb{A}_\varrho)$, and since $\mathrm{tr}_{\mathbb{T}}F=0$ we have that $F=0$ so that $G_1=0$. Plugging this in (H.16), we get that $(I+T_{\bar\alpha})(0_{\mathbb{A}_\varrho}\vee G_{\mathbb{D}_\varrho})=0$ hence $G_{\mathbb{D}_\varrho}=0$ since $(I+T_{\bar\alpha})$ is injective on $L^\lambda(\mathbb{D})$ by Proposition 3. Altogether $G=0$, hence $\Psi_\varrho=0$, and finally $\Psi=0$, by analytic continuation. ■

Proof of Proposition 5: for $0<r\leq 1$, we let for simplicity $T_{\alpha,r}=\mathbb{T}_{\alpha_{|\mathbb{D}_r}}:L^p(\mathbb{D}_r)\to L^p(\mathbb{D}_r)$, so that $\mathbb{T}_{\alpha,1}=T_\alpha$. so that $T_{\alpha,1}=T_\alpha$. Recall also from (32) the notation $\mathcal{C}$ for Cauchy

integrals. By Proposition 3, a function $w_\varrho \in G^p_{\alpha_{\mathbb{D}_\varrho}}(\mathbb{D}_\varrho)$ lies in $\big(G^p_\alpha(\mathbb{D})\big)_{|\mathbb{D}_\varrho}$ if, and only if there is $g \in H^p(\mathbb{D})$ such that

$$(I - T_{\alpha,\varrho})^{-1}\mathcal{C}(\mathrm{tr}_{\mathbb{T}_\varrho} w_\varrho) = \big((I - T_\alpha)^{-1} g\big)_{|\mathbb{D}_\varrho}. \tag{H.17}$$

Define $\widetilde{\alpha}_\varrho = \chi_{\mathbb{D}_\varrho}\alpha$. Since $T_{\alpha,\varrho}(h_{|\mathbb{D}_\varrho}) = (T_{\widetilde{\alpha}_\varrho} h)_{|\mathbb{D}_\varrho}$ for $h \in L^p(\mathbb{D})$, equation (H.17) means that

$$\mathcal{C}(\mathrm{tr}_{\mathbb{T}_\varrho} w_\varrho) = \Big((I - T_{\widetilde{\alpha}_\varrho})(I - T_\alpha)^{-1} g\Big)_{|\mathbb{D}_\varrho}. \tag{H.18}$$

Observe that $T_\alpha = T_{\alpha_\varrho} + T_{\widetilde{\alpha}_\varrho}$ where the notation $\alpha_\varrho = \chi_{|\mathbb{A}_\varrho}\alpha$ was introduced in the proof of Lemma 11. Hence, using the identity $(A - B)^{-1} = A^{-1} + A^{-1}B(A - B)^{-1}$, we obtain

$$(I - T_\alpha)^{-1} = (I - T_{\widetilde{\alpha}_r})^{-1} + (I - T_{\widetilde{\alpha}_r})^{-1} T_{\alpha_r}(I - T_\alpha)^{-1}.$$

Substituting in (H.18), we get

$$\mathcal{C}(\mathrm{tr}_{\mathbb{T}_\varrho} w_\varrho) = B_\varrho g,$$

and we conclude from Lemma 11 that $\big(G^p_\alpha(\mathbb{D})\big)_{|\mathbb{D}_\varrho} = (I - T_{\alpha,\varrho})^{-1}\mathrm{Ran} B_\varrho$ is dense in $G^p_{\alpha_{\mathbb{D}_\varrho}}(\mathbb{D}_\varrho)$, as desired. ∎

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