Bounded extremal problems in Hardy spaces for the conjugate Beltrami equation in simply-connected domains

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Abstract. Techniques of constrained approximation are used to recover solutions to elliptic partial differential equations from incomplete and corrupted boundary data. The approach involves the use of generalized Hardy spaces of functions whose real and imaginary parts are related by formulae similar to the Cauchy–Riemann equations. A prime motivation for this is the modelling of plasma confinement in a tokamak reactor. Constructive and numerical aspects are also discussed.

1 Introduction

1.1 Problems, motivation

Let Ω be a bounded open domain in $\mathbb{R}^2$ with C^1 boundary and

$$\sigma \in W^{1,\infty}(\Omega) \qquad \text{such that} \qquad 0 < c \le \sigma \le C \quad \text{a.e. in} \quad \Omega\,, \tag{1}$$

for two constants $0 < c < C < +\infty$. The elliptic equation we look at is

$$\operatorname{div}(\sigma \nabla u) = 0 \text{ in } \Omega. \tag{2}$$

The divergence and gradient operators are understood in the sense of weak derivatives (with respect to real variables x, y in $\mathbb{R}^2$). We are interested in the following inverse problem of Cauchy type: assume that u and $\sigma \partial_n u$ (where $\partial_n u$ denotes the normal derivative of u on $\partial\Omega$) are given on an open subset $I \subset \partial\Omega$. From these data, we want to recover u on the complementary subset $J = \partial\Omega \setminus I$, also assumed to be of non-empty interior. This problem is ill-posed in the sense of Hadamard, for instance in Sobolev spaces $W^{1-1/p,p}(I)$ [17], $1 < p < +\infty$, but also in $L^p(I)$, where it still makes sense [9]. There, the associated (direct) Dirichlet or Neumann problems on $\partial\Omega$ have a unique (up to normalisation) solution, which depends continuously (Lebesgue, Sobolev, or Hardy norms) on the data, a

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stability property. Further, well-posedness of the inverse problem is ensured if a constraint is added on J and the extrapolation issue expressed as a best L^p approximation problem on I (in the class of bounded extremal problems (BEP), which will be explained later). When σ is constant, this problem amounts to recovering or approximating the values of a holomorphic function in a domain of analyticity from part of its boundary values. This problem was extensively studied in Hardy spaces on simply and doubly connected domains (see [3, 7, 8, 19]).

In the present paper, we are interested in Hardy classes of the *conjugate* Beltrami equation, which is

$$\overline{\partial} f = \nu \, \overline{\partial f} \qquad \text{a.e. in} \quad \Omega \,, \tag{CB}$$

where

$$\nu \in W^{1,\infty}(\Omega) \qquad \text{is } \textit{real valued,} \qquad \text{and} \quad \|\nu\|_{L^\infty(\Omega)} \leq \kappa < 1 \,, \tag{κ}$$

for some $\kappa \in (0,1)$. There, ∂ and $\overline{\partial}$ respectively denote the derivation operators with respect to complex variables $z = x + iy$ and $\overline{z}$ in $\mathbb{C} \simeq \mathbb{R}^2$ (recall that $\partial \, \overline{\partial} = \overline{\partial} \, \partial = \Delta = \operatorname{div}(\nabla)$).

The equation (CB) is related to (2), in fact to a system of second order elliptic equations in divergence form, in the following way [9, Sec. 3.1] which generalizes the well-known link between holomorphic (analytic) and harmonic functions.

Let $p \in (1, +\infty)$ and $f \in W^{1,p}(\Omega)$ be a solution of (CB). If $f = u + iv$ where u and v are real-valued, then $u, v \in W^{1,p}(\Omega)$, u satisfies equation (2) and (its σ-harmonic conjugate function) v satisfies

$$\operatorname{div}\left(\frac{1}{\sigma} \nabla v\right) = 0 \text{ in } \Omega \,, \tag{3}$$

where $\sigma = (1 - \nu)/(1 + \nu)$. It follows from ($\kappa$) that there exist $c, C > 0$ such that (1) holds.

Conversely, let $\sigma \in W^{1,\infty}(\Omega)$ satisfying (1) and $u \in W^{1,p}(\Omega)$ be a real valued solution of (2), [17]. Then, if Ω is simply connected, there exists a real-valued function $v \in W^{1,p}(\Omega)$, unique up to an additive constant, such that $f = u + iv \in W^{1,p}(\Omega)$ satisfies (CB), with $\nu = (1 - \sigma)/(1 + \sigma)$. Indeed, with this definition of ν, (CB) is equivalent to the system of generalized Cauchy–Riemann equations:

$$\begin{cases} \partial_x v = -\sigma \partial_y u, \\ \partial_y v = \sigma \partial_x u, \end{cases} \tag{4}$$

where ∂_x and ∂_y stand for the partial derivatives with respect to x and y, respectively. System (4) admits a solution because $\partial_y(-\sigma \partial_y u) = \partial_x(\sigma \partial_x u)$ by (2), and Ω is simply connected. Assumption (1) implies that ν satisfies (κ) for some $\kappa \in (0,1)$.

Even more general classes of “analytic” functions were studied in [13, 33], that are strongly related to quasi-conformal maps [2]. The above link between (2) and (CB) is the basis of the approaches of [4, 5, 24] and of the work [9] where fundamental properties of associated generalized Hardy classes of solutions (in the disc) are established, and Dirichlet-type issues considered. Here we shall complete some of these results, set up and solve best constrained approximation (“bounded extremal”) problems in these classes, constructively in the Hilbertian framework of L^2.

An important practical motivation for such issues comes from plasma confinement in tokamaks, for thermonuclear controlled fusion, when trying to recover the boundary of

the plasma, or for numerical simulations. Indeed, in such a toroidal domain, a classical symmetry assumption (with respect to the $(0, y)$ axis, see Fig. 1 in Section 4) is to the effect that the magnetic quantities involved in Maxwell equations do not depend on the corresponding "poloidal" plane section. This allows to formulate the issue in 2 dimensions rather than in 3. There, the vacuum is an annular domain Ω (with $0 \notin \Omega$) lying between the plasma and the chamber where, from the so-called Grad–Shafranov equation, the magnetic flux is a solution to the conductivity equation (2) with the conductivity $\sigma(x, y) = \sigma_*(x, y) = 1/x$ (see Fig. 1).
However, in the present work, we stick to the situation of (Dini) smooth simply connected domains; the more involved one of annular domains will be analysed in a forthcoming study. Up to conformal mappings, this allows us to handle the issue in the open unit disc (see [9, Sec. 6]), where we consider Hardy classes.

The overview is as follows.
- Notation and definitions: Section 2.
- Properties of generalized Hardy classes, bounded extremal problems (BEP): Section 3.
- Complete families of solutions : Section 4.
- Constructive aspects: Section 5.
- Concluding remarks: Section 6.

2 Notation and definitions

We first present the notation that is used in the next section and later on.

2.1 General settings

Throughout the paper, $\mathbb{D}$ refers to the open unit disc and $\mathbb{T}$ to the unit circle of the complex plane $\mathbb{C}$, both centered at 0.

Let $\mathcal{O}$ be a bounded open set in $\mathbb{R}^d$, $d = 1$ or 2 with a Lipschitz-continuous boundary. The generic point in $\mathcal{O}$ is denoted by x in dimension $d = 1$ and by $\mathbf{x} = (x, y)$ in the dimension $d = 2$. For each p, $1 \le p < +\infty$, we introduce the space

$$L^p(\mathcal{O}) = \{\varphi : \mathcal{O} \to \mathbb{C} \text{ measurable}\,;\, \|\varphi\|_{L^p(\mathcal{O})} = \left(\frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} |\varphi(\mathbf{x})|^p d\mathbf{x}\right)^{1/p} < +\infty\}\,.$$

Note that we include a normalization factor, so that the norm is taken as an L^p mean value. Now, let $\mathcal{D}(\mathcal{O})$ designate the space of complex-valued C^∞ functions with compact support in $\mathcal{O}$. Its dual $\mathcal{D}'(\mathcal{O})$ is the usual space of distributions on $\mathcal{O}$. If $\mathbf{k}$ denotes a d-tuple $(k_1, ..., k_d)$ of nonnegative integers, we use the notation $\partial^k\varphi$ for the partial derivative (in the distribution sense) of a function φ of total order $|\mathbf{k}| = k_1 + ... + k_d$ and partial order k_i with respect to the i-th variable.
So, for each nonnegative integer m and each p with $1 \le p < +\infty$, we consider the space $W^{m,p}(\mathcal{O})$ of functions such that all their partial derivatives of total order $\le m$ belong to $L^p(\mathcal{O})$, i.e.,

$$W^{m,p}(\mathcal{O}) = \{\varphi \in \mathcal{D}'(\mathcal{O})\,;\, \|\varphi\|_{W^{m,p}(\mathcal{O})} = (\sum_{|k|\le m} \|\partial^k\varphi\|^p_{L^p(\mathcal{O})})^{1/p} < +\infty\}\,.$$

Next, the Sobolev spaces of non-integer order can be defined by an intrinsic norm [1]. Indeed, any positive real number s which is not an integer can be written $[s]+\alpha$, where $[s]$ denotes its integral part while its fractional part α satisfies $0<\alpha<1$. Then, for any p with $1 \le p < +\infty$, the space $W^{s,p}(\mathcal{O})$ is the space of distributions φ in $\mathcal{D}'(\mathcal{O})$ such that

$$\|\varphi\|_{W^{s,p}(\mathcal{O})} = \left(\|\varphi\|^p_{W^{[s],p}(\mathcal{O})} + \sum_{|k|=[s]} \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|\partial^k \varphi(\mathbf{x}) - \partial^k \varphi(\mathbf{x}')|^{\mathbf{p}}}{|\mathbf{x}-\mathbf{x}'|^{\alpha \mathbf{p}+\mathbf{d}}} \, d\mathbf{x} \, d\mathbf{x}' \right)^{1/p} < +\infty \, .$$

Observe that such Sobolev spaces may also be defined in several other ways, for instance by interpolation methods [1, Ch. VII].
The subscript $\mathbb{R}$ (for instance in $L^p_{\mathbb{R}}$ or in $W^{s,p}_{\mathbb{R}}$) indicates real-valued function spaces.
We let $L : L^p(\mathbb{T}) \to \mathbb{C}$ be the (bounded) mean value operator on $\mathbb{T}$ (recall that $p>1$):

$$L\,\phi = \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{i\theta}) \, d\theta \, , \qquad \forall \phi \in L^p(\mathbb{T}) \, .$$

2.2 Generalized Hardy classes

If $1 < p < +\infty$ and $\nu \in W^{1,\infty}_{\mathbb{R}}(\mathbb{D})$ satisfies (κ), we define the "generalized" Hardy space $H^p_\nu(\mathbb{D}) = H^p_\nu$ to consist of those Lebesgue measurable functions f on $\mathbb{D}$ such that

$$\operatorname*{ess\,sup}_{0<r<1} \|f\|_{L^p(\mathbb{T}_r)} = \operatorname*{ess\,sup}_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right)^{1/p} < +\infty$$

(which therefore belong to $L^p(\mathbb{D})$) and satisfying (CB) in the sense of distributions on $\mathbb{D}$, see [9]. Equipped with the norm

$$\|f\|_{H^p_\nu(\mathbb{D})} := \operatorname*{ess\,sup}_{0<r<1} \|f\|_{L^p(\mathbb{T}_r)} \, ,$$

H^p_ν is a Banach space. When $\nu = 0$, $H^p_0 = H^p = H^p(\mathbb{D})$ is the classical Hardy space of holomorphic functions on the unit disc (see [20, 21]).
By analogy with classical Hardy spaces on $\mathbb{C} \setminus \overline{\mathbb{D}}$, let $\overline{H^p_\nu} = \overline{H^p_\nu}(\mathbb{D})$ be the space of functions f such that

$$\operatorname*{ess\,sup}_{r>1} \|f\|_{L^p(\mathbb{T}_r)} < +\infty \, ,$$

and satisfying (CB) in $\mathbb{C} \setminus \overline{\mathbb{D}}$. For each function $f \in L^p(\mathbb{D})$, set

$$\check{f}(z) = \overline{f\left(\frac{1}{\bar{z}}\right)} \, , \quad z \in \mathbb{C} \setminus \overline{\mathbb{D}} \, .$$

A direct computation shows that

$$f \in H^p_\nu \Longleftrightarrow \check{f} \in \overline{H^p_{\check{\nu}}}, \tag{5}$$

for the dilation $\check{\nu} \in W^{1,\infty}_{\mathbb{R}}(\mathbb{C} \setminus \overline{\mathbb{D}})$ (since ν is real-valued, $\check{\nu}(z) = \nu(1/\bar{z})$).
We introduce $H^{p,0}_\nu \subset H^p_\nu$ (resp. $\overline{H^{p,0}_\nu} \subset \overline{H^p_\nu}$), the (closed) subset of functions $f \in H^p_\nu$ (resp. $\overline{H^p_\nu}$) subject to the normalization condition $L(\operatorname{Im}(\operatorname{tr} f)) = 0$. Let $\overline{H^{p,00}_\nu}$ be the subspace of functions $f \in \overline{H^{p,0}_\nu}$ such that

$$\int_0^{2\pi} \operatorname{tr} f(e^{i\theta}) \, d\theta = 0 = L(\operatorname{tr} f) \, . \tag{6}$$

Finally, let $\Omega \subset \mathbb{C}$ be a simply connected bounded Dini-smooth domain (its boundary being a Jordan curve with nonsingular Dini-smooth parametrization). Let ν satisfying (κ) on Ω. We introduce the Hardy classes $H^p_\nu(\Omega)$ as the space of functions f on Ω such that $f \circ \psi \in H^p_{\nu\circ\psi}(\mathbb{D})$, for some conformal transformation ψ from $\mathbb{D}$ onto Ω. A straightforward computation [2, Ch. 1] shows that this class does not depend on the particular choice of ψ and consists of distributional solutions to (CB).

3 Properties of Hardy classes, approximation issues

Let $1 < p < \infty$ and assume that ν satisfies (κ). We first state and discuss some fundamental properties from [4, 5, 9].

3.1 Background

- Classes H^p_ν:

Basic properties of the above classes H^p_ν were established in the preliminary work [9], of which we give a brief account below. These H^p_ν spaces mainly share properties of the classical Hardy spaces H^p. From [9, Prop. 4.3.1, 4.3.2] we have:

Proposition 1
- Any function f in H^p_ν has a non-tangential limit almost everywhere on $\mathbb{T}$, which we call the trace of f and denote by $\operatorname{tr} f$*; moreover,* $\operatorname{tr} f \in L^p(\mathbb{T})$ *and its $L^p(\mathbb{T})$ norm is equivalent to the H^p_ν norm of f.*
- The space $\operatorname{tr} H^p_\nu$ *is closed in $L^p(\mathbb{T})$.*
- If $f \in H^p_\nu$ and $f \not\equiv 0$, then its zeros are isolated in $\mathbb{D}$ and $\log|\operatorname{tr} f| \in L^1(\mathbb{T})$ *(in particular,* $\operatorname{tr} f$ *does not vanish on a subset of $\mathbb{T}$ having positive Lebesgue measure).*
- Each $f \in H^p_\nu$ satisfies the maximum principle (in modulus).
- Let $f \in H^{p,0}_\nu$. If $\operatorname{Re}(\operatorname{tr} f) = 0$ *a.e. on $\mathbb{T}$, then $f \equiv 0$.*

Concerning the Dirichlet problem, we have [9, Thm 4.4.2.1]:

Theorem 1 *For all $\varphi \in L^p_{\mathbb{R}}(\mathbb{T})$, there exists a unique $f \in H^{p,0}_\nu$ such that, a.e. on $\mathbb{T}$,* $\operatorname{Re}(\operatorname{tr} f) = \varphi$. *Moreover, there exists $c_{p,\nu} > 0$ such that: $\|f\|_{H^p_\nu(\mathbb{D})} \le c_{p,\nu} \|\varphi\|_{L^p(\mathbb{T})}$.*

This result allows us to define a generalized conjugation operator $\mathcal{H}_\nu$ from $L^p_{\mathbb{R}}(\mathbb{T})$ into itself. Indeed, to each $\varphi \in L^p_{\mathbb{R}}(\mathbb{T})$, associate the unique function $f \in H^{p,0}_\nu$ such that $\operatorname{Re}\operatorname{tr} f = \varphi$, and set $\mathcal{H}_\nu \varphi = \operatorname{Im}\operatorname{tr} f \in L^p(\mathbb{T})$. Note that $\mathcal{H}_\nu$ was introduced on $W^{1/2,2}(\mathbb{T})$ in [5] as the ν-Hilbert transform. When $\nu = 0$, we observe that $\mathcal{H}_0\, \varphi$ is just the harmonic conjugate of φ normalized to have zero mean on $\mathbb{T}$.
Finally, the following density property of traces hold from [9, Thm 4.5.2.1]:

Theorem 2 *Let $I \subset \mathbb{T}$ be a measurable subset such that $\mathbb{T} \setminus I$ has positive Lebesgue measure. The restrictions to I of traces of H^p_ν-functions are dense in $L^p(I)$.*

- Conjugation operator $\mathcal{H}_\nu$:

It follows from Theorem 1 that [9, Cor. 4.4.2.1]:

Corollary 1 *The operator $\mathcal{H}_\nu$ is bounded on $L^p_{\mathbb{R}}(\mathbb{T})$.*

Further, the following properties hold [4, 5]:
(i) On $L^p_{\mathbb{R}}(\mathbb{T})$, $\mathcal{H}_{-\nu} \circ \mathcal{H}_\nu \, u = \mathcal{H}_\nu \circ \mathcal{H}_{-\nu} \, u = -u + Lu$: $(\mathcal{H}_\nu + L)^{-1} = -\mathcal{H}_{-\nu} + L$. This follows immediately from the fact that $i H^p_\nu = H^p_{-\nu}$ (f is a solution to (CB) $\Leftrightarrow$ $g = i\, f$ is a solution to $\overline{\partial} g = -\nu \, \overline{\partial g}$).
(ii) It is easily checked that, by definition: $\mathcal{H}_\nu \circ L = L \circ \mathcal{H}_\nu = 0$ on $L^p_{\mathbb{R}}(\mathbb{T})$.
(iii) So far, $\mathcal{H}_\nu(u)$ was defined only for real-valued functions. It is however natural to set as in [4, 5]: $\mathcal{H}_\nu(i\, u) = i\, \mathcal{H}_{-\nu}(u)$, thereby extending $\mathcal{H}_\nu$ to complex-valued functions, as an $\mathbb{R}$-linear bounded operator on $L^p(\mathbb{T})$.

- Projection operator P_ν:

(i) On $L^2(\mathbb{T})$ let:

$$P_\nu \, f = \frac{1}{2}(I + i\, \mathcal{H}_\nu)\, f + \frac{1}{2} L(f) \, .$$

This defines a generalized Riesz projection operator P_ν from $L^p(\mathbb{T})$ onto $\operatorname{tr} H^p_\nu$, [5, 9]. In particular P_ν is a bounded operator on $L^2(\mathbb{T})$. Equivalently, for $f = u + iv$, $u, v \in L^2_{\mathbb{R}}(\mathbb{T})$,

$$P_\nu \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u - \mathcal{H}_{-\nu} v \\ \mathcal{H}_\nu u + v \end{pmatrix} + \frac{1}{2} \begin{pmatrix} Lu \\ Lv \end{pmatrix}$$

(ii) Observe that $P_\nu \circ L = L \circ P_\nu = L$.
(iii) Let $u_1, u_2 \in L^2_{\mathbb{R}}(\mathbb{T})$.

$$P_\nu(u_1 + i \mathcal{H}_\nu u_2 + i\, c) = P_\nu \begin{pmatrix} u_1 \\ \mathcal{H}_\nu u_2 + c \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u_1 + u_2 + L(u_1 - u_2) \\ \mathcal{H}_\nu(u_1 + u_2) + 2c \end{pmatrix} .$$

Remark 1 *Observe that on the subspace $L^{2,0}(\mathbb{T}) \subset L^2(\mathbb{T})$ of functions with vanishing mean value L, the above formulas are to the effect that we have $\mathcal{H}_\nu^{-1} = -\mathcal{H}_{-\nu}$ while:*

$$P_\nu \, f = \frac{1}{2}(I + i\, \mathcal{H}_\nu) \operatorname{tr} \, f \ \text{ or } P_\nu \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u - \mathcal{H}_{-\nu} v \\ \mathcal{H}_\nu u + v \end{pmatrix} ,$$

and

$$P_\nu(u_1 + i \mathcal{H}_\nu u_2) = P_\nu \begin{pmatrix} u_1 \\ \mathcal{H}_\nu u_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u_1 + u_2 \\ \mathcal{H}_\nu(u_1 + u_2) \end{pmatrix} .$$

3.2 Further properties of Hardy classes

By definition of the Hilbert transform $\mathcal{H}_\nu$, we have the following M. Riesz-like result, which is also a generalization of [25, Lem. 3]:

Corollary 2 *Let $f = u + i\, v \in L^p(\mathbb{T})$ $(u\, , \, v \in L^p_{\mathbb{R}}(\mathbb{T}))$.*
(i) *$f \in \operatorname{tr} H^{p,0}_\nu \ \Leftrightarrow v = \mathcal{H}_\nu u$. Further, there exists an absolute constant $c = c_{p,\nu} > 0$ such that:*

$$\forall f = u + i\, v \in \operatorname{tr} H^{p,0}_\nu \, , \quad \|f\|_{L^p(\mathbb{T})} \leq c \, \|u\|_{L^p(\mathbb{T})} \, .$$

(ii) *$f \in \operatorname{tr} H^p_\nu \ \Leftrightarrow v = \mathcal{H}_\nu u + Lv$. If $I \subset \mathbb{T}$ is a connected subset of $\mathbb{T}$ such that both I and $J = \mathbb{T} \setminus I$ have positive Lebesgue measure, there exists an absolute constant $c = c_{p,\nu,I} > 0$ such that:*

$$\forall f = u + i\, v \in \operatorname{tr} H^p_\nu \, , \quad \|f\|_{L^p(\mathbb{T})} \leq c \, (\|u\|_{L^p(\mathbb{T})} + \|v\|_{L^p(I)}) \, .$$

Proof. (i) Use Theorem 1 and Corollary 1.
(ii) Let $f = u + i\,v \in \operatorname{tr} H^p_\nu$. Apply point (i) to $f - i\,Lv \in \operatorname{tr} H^{p,0}_\nu$. We then have $v = \mathcal{H}_\nu u + \alpha$, for some constant $\alpha = Lv$, and there exists $C = C_{p,\nu} > 0$ such that

$$\|v - \alpha\|_{L^p(\mathbb{T})} \leq C\ \|u\|_{L^p(\mathbb{T})} \, .$$

Then

$$\|f - i\alpha\|_{L^p(I)} \leq 2\,C\ \|u\|_{L^p(\mathbb{T})} \, ,$$

whence

$$\|\alpha\|_{L^p(I)} \leq 2\,C\ \|u\|_{L^p(\mathbb{T})} + \|f\|_{L^p(I)} \, .$$

This gives the required bound on α. □

As another consequence of Theorem 1, we get:

Corollary 3 $L^p(\mathbb{T}) = \operatorname{tr} H^p_\nu \oplus \operatorname{tr} \overline{H^{p,00}_{\check{\nu}}}$*, and the decomposition is topological.*

Proof. In view of (5), Proposition 1 also holds for $\overline{H^p_{\check{\nu}}}$ functions (while $f \in H^{p,0}_\nu \iff \check{f} \in \overline{H^{p,0}_{\check{\nu}}}$ as well). In particular, $\operatorname{tr} H^p_\nu + \operatorname{tr} \overline{H^{p,0}_{\check{\nu}}} \subset L^p(\mathbb{T})$. Note also that $\operatorname{Re}(\operatorname{tr} \check{f}) = \operatorname{Re}(\operatorname{tr} f)$, $\operatorname{Im}(\operatorname{tr} \check{f}) = -\operatorname{Im}(\operatorname{tr} f)$ a.e. on $\mathbb{T}$.
Conversely, it follows first from Theorem 1 that $L^p_\mathbb{R}(\mathbb{T}) \subset \operatorname{Re}(\operatorname{tr} H^{p,0}_\nu) + \operatorname{Re}(\operatorname{tr} \overline{H^{p,0}_{\check{\nu}}})$. Indeed, assume that $\varphi \in L^p_\mathbb{R}(\mathbb{T})$. In view of Theorem 1, there exists $f \in H^{p,0}_\nu$ such that, a.e. on $\mathbb{T}$: $\operatorname{Re}(\operatorname{tr} f) = \varphi/2$. Hence, $\check{f} \in \overline{H^{p,0}_{\check{\nu}}}$ and a.e. on $\mathbb{T}$, $\operatorname{Re}(\operatorname{tr} \check{f}) = \varphi/2$, while $\operatorname{Re}\operatorname{tr}(f + \check{f}) = \varphi$, and $\operatorname{Im}\operatorname{tr}(f + \check{f}) = 0$.
Next, consider $\Phi = \varphi + i\psi \in L^p(\mathbb{T})$, with $\varphi,\ \psi \in L^p_\mathbb{R}(\mathbb{T})$. Let f and $\check{f}$ as above, and $g \in H^{p,0}_{-\nu}$ such that $\operatorname{Re}(\operatorname{tr} g) = \psi/2$ a.e. on $\mathbb{T}$; thus $\operatorname{Re}\ \operatorname{tr}(g + \check{g}) = \psi$, and $\operatorname{Im}\operatorname{tr}(g + \check{g}) = 0$. Then, one can check that $ig \in H^p_\nu$ and $i\check{g} \in \overline{H^p_{\check{\nu}}}$, while $\operatorname{Im} i\ \operatorname{tr}(g + \check{g}) = \psi$ and $\operatorname{Re} i\ \operatorname{tr}(g + \check{g}) = 0$ a.e. on $\mathbb{T}$. Finally, a.e. on $\mathbb{T}$,

$$\Phi = \operatorname{tr}(f + \check{f}) + i\ \operatorname{tr}(g + \check{g}) = \operatorname{tr}(f + i\,g) + c_{f,g} + \operatorname{tr}(\check{f} + i\,\check{g}) - c_{f,g} \in \operatorname{tr} H^p_\nu + \operatorname{tr} \overline{H^{p,00}_{\check{\nu}}} \, ,$$

with

$$c_{f,g} = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\ \operatorname{tr} f(e^{i\theta})\, d\theta + \frac{i}{2\pi} \int_0^{2\pi} \operatorname{Re}\ \operatorname{tr} g(e^{i\theta})\, d\theta \, .$$

To check that this decomposition is direct, let $f \in H^p_\nu$ and $g \in \overline{H^{p,00}_{\check{\nu}}}$ such that $\operatorname{tr} f = \operatorname{tr} g$ a.e. on $\mathbb{T}$. Hence $f + \check{g} \in H^p_\nu$ and $\operatorname{Im}\operatorname{tr}(f + \check{g}) = 0$. Theorem 1 implies that $f + \check{g}$ is identically equal in $\mathbb{D}$ to a real-valued constant: $f + \check{g} = 2\operatorname{Re}\operatorname{tr} f = 2\operatorname{Re}\operatorname{tr} g = c$. From the normalization assumption (6), we necessarily have that $\operatorname{Re}\operatorname{tr} f = \operatorname{Re}\operatorname{tr} g = 0$ a.e. on $\mathbb{T}$, hence also $\operatorname{Im}\operatorname{tr} f = \operatorname{Im}\operatorname{tr} g = 0$ a.e. on $\mathbb{T}$. □

Remark 2 *Let $\Omega \subset \mathbb{C}$ be a simply connected bounded Dini-smooth domain and ν satisfying (κ) on Ω. The present results still hold in the Hardy class $H^p_\nu(\Omega)$. Note that $\nu \circ \psi \in W^{1,\infty}(\mathbb{D})$ with $\|\nu \circ \psi\|_{L^\infty(\mathbb{D})} \leq \kappa < 1$, and that any conformal transformation ψ from $\mathbb{D}$ onto Ω extends continuously from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$ together with its derivative, in such a way that the latter is never zero [32, Thm 3.5].*

3.3 Bounded extremal problems (BEP) in the disc

Let $I \subset \mathbb{T}$ be a connected subset of $\mathbb{T}$ such that both I and $J = \mathbb{T} \setminus I$ have positive Lebesgue measure. For $M \geq 0$, $\phi \in L^p_{\mathbb{R}}(J)$, define

$$\mathcal{C}_p = \mathcal{C}_p(\phi, M) = \left\{ g \in \mathrm{tr}\, H^p_\nu ;\ \|\mathrm{Re}\, g - \phi\|_{L^p(J)} \leq M \right\} |_I \subset L^p(I) \,.$$

Theorem 3 *Fix $M > 0$, $\phi \in L^p_{\mathbb{R}}(J)$. For every function $F \in L^p(I)$, there exists a unique function $g_* \in \mathcal{C}_p$ such that*

$$\|F - g_*\|_{L^p(I)} = \min_{g \in \mathcal{C}_p} \|F - g\|_{L^p(I)} \,. \tag{BEP}$$

Moreover, if $F \notin \mathcal{C}_p$, then $\|\mathrm{Re}\, g_* - \phi\|_{L^p(J)} = M$.

Proof. We argue as in [7, 19]. Since $\mathcal{C}_p$ is clearly convex, to prove the existence and the uniqueness of g_*, it is enough to check that $\mathcal{C}_p$ is closed in $L^p(I)$, since this Banach space is uniformly convex.
Let $\varphi_{k|_I} \in \mathcal{C}_p$, $\varphi_{k|_I} \to \varphi_I$ in $L^p(I)$ as $k \to \infty$. Put $\varphi_k = u_k + i v_k \in \mathrm{tr}\, H^p_\nu$. By assumption, (u_k) is bounded in $L^p(\mathbb{T})$ and (v_k) in $L^p(I)$. Hence, we get from Corollary 2, (ii), that (φ_k) is bounded in $L^p(\mathbb{T})$ (hence (v_k) is bounded in $L^p(J)$) and thus, up to extracting a subsequence, weakly converges to $\psi \in L^p(\mathbb{T})$; necessarily $\psi_{|_I} = \varphi_I$.
Finally, $\varphi_k \in \mathrm{tr}\, H^p_\nu$, which is closed thus weakly closed in $L^p(\mathbb{T})$ (Mazur's theorem, [22]); this implies that $\psi \in \mathrm{tr}\, H^p_\nu$. Because $\mathrm{Re}\, \varphi_k = u_k$ satisfies the constraint on J, so does $\mathrm{Re}\, \psi$, whence $\varphi_I \in \mathcal{C}_p$.
Let us now prove that, if $F \notin \mathcal{C}_p$, then $\|\mathrm{Re}\, g_* - \phi\|_{L^p(J)} = M$. Assume for a contradiction that $\|\mathrm{Re}\, g_* - \phi\|_{L^p(J)} < M$. By Theorem 2, since $\|F - g_*\|_{L^p(I)} > 0$, there is a function $h \in \mathrm{tr}\, H^p_\nu$ such that

$$\|F - g_* - h\|_{L^p(I)} < \|F - g_*\|_{L^p(I)} \,,$$

and by the triangle inequality we have

$$\|F - g_* - \lambda h\|_{L^p(I)} < \|F - g_*\|_{L^p(I)}$$

for all $0 < \lambda < 1$. Now for $\lambda > 0$ sufficiently small we have $\|\,\mathrm{Re}(g_* + \lambda h) - \phi\|_{L^p(J)} \leq M$, contradicting the optimality of g_*. □

Remark 3 *Theorem 3 still holds if the constraint on J is replaced by $\|g - \phi\|_{L^p(J)} \leq M$, for $\phi \in L^p(J)$, as in [7, 8], or more generally by a combination:*

$$\alpha \,\|\mathrm{Re}\, g - \phi_r\|_{L^p(J)} + \beta \,\|\mathrm{Im}\, g - \phi_i\|_{L^p(J)} \leq M \,,$$

for ϕ_r, $\phi_i \in L^p_{\mathbb{R}}(J)$, and α, $\beta \geq 0$ with $\alpha + \beta > 0$, as in [27].

We now pay particular attention to the Hilbertian case $p = 2$. Note that $\mathrm{tr}\, H^2_\nu$ is a closed subspace of $L^2(\mathbb{T})$ (see Prop. 1, Cor. 3), and that a (best approximation) Riesz projection operator $P_\nu : L^2(\mathbb{T}) \to \mathrm{tr}\, H^2_\nu$ has been introduced in Section 3.1 that generalizes the usual one. Using this we can formulate a solution to the extremal problem posed in Theorem 3 (more complicated expressions can be derived in the case $p \neq 2$, using the methods of [7, 11, 19]).

Proposition 2 *For $p = 2$, and if $F \notin \mathcal{C}_2$, the solution $g = g_*$ to the extremal problem (BEP) subject to $\|\operatorname{Re} g - \phi\|_{L^2(J)} \leq M$, $\phi \in L^2_{\mathbb{R}}(J)$, $M > 0$, is given by*

$$P_\nu \chi_I g + \gamma\, P_\nu \chi_J \operatorname{Re} g = P_\nu \chi_I F + \gamma\, P_\nu \chi_J \phi\,, \tag{7}$$

for the unique $\gamma > 0$ such that $\|\operatorname{Re} g - \phi\|_{L^2(J)} = M$.

Proof. The method of proof is variational, as in [3, 27], and indeed more general abstract results can be found in [18, 28]. We therefore give only a brief sketch of the argument. Let $\psi \in \operatorname{tr} H^2_{\nu|_I}$ lie in the real tangent space to $\mathcal{C}_2$ at g, that is,

$$\operatorname{Re}\langle g - \phi, \psi\rangle_{L^2(J)} = 0\,.$$

Then optimality implies that

$$\operatorname{Re}\langle g - F, \psi\rangle_{L^2(I)} = 0\,.$$

Thus $P_\nu \chi_I (g - F) = \gamma P_\nu \chi_J (\phi - \operatorname{Re} g)$ for some $\gamma \in \mathbb{R}$. Finally, a first-order perturbation argument shows that $\gamma > 0$. □

A discussion of the relations between the (Lagrange-type) parameter γ, the constraint M, and the error $e_* = \|F - g_*\|_{L^2(I)}$ in the "classical" case $\nu = 0$ can be found in [3], with further precise information obtained by spectral methods in [6]. All we require in the present situation is the easy observation that e_* is a strictly decreasing function of M, and as $M \to \infty$ we have $e_* \to 0$.

4 Families of solutions and density results

Let $\sigma(x, y) = \sigma_*(x, y) = 1/x$ be the particular conductivity involved in plasma equations, and the associated dilation coefficient $\nu = \nu_* = (1 - \sigma_*)/(1 + \sigma_*)$:

$$\nu_*(z, \bar{z}) = \frac{z + \bar{z} - 2}{z + \bar{z} + 2}\,.$$

We consider now equations (2) and (3) in the disc $\Omega = \mathbb{D}_0$:

$$\mathbb{D}_0 = \{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + y^2 < R^2\}, \qquad \text{where} \quad 0 < R < |x_0|,$$

with $\mathbb{T}_0$ its boundary (a circle). Let $\mathcal{R}$ be a rectangle containing $\mathbb{D}_0$: $\mathcal{R} = [a, b] \times [-c, c]$, $b > a > 0$ and $c > 0$, with (Lipschitz-continuous) boundary $\partial\mathcal{R}$. Its edges are denoted by $\Gamma_a = \{a\} \times [-c, c]$, $\Gamma_b = \{b\} \times [-c, c]$, $\Gamma_c = [a, b] \times \{c\}$ and $\Gamma_{-c} = [a, b] \times \{-c\}$.

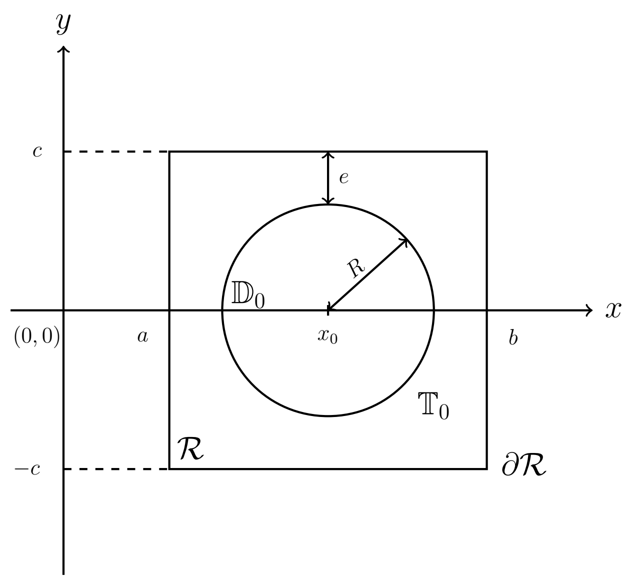

Fig.1 Geometrical setting in a poloidal section

Note that both $\sigma = \sigma_*$ and $\nu = \nu_*$ are smooth on $\mathcal{R}$ and that $\mathbb{D}_0$ is defined in order to have compact closure in $\mathcal{R}$.

4.1 Particular solutions of Bessel-exponential type

We write $u(x, y) = A(x)B(y)$ for a solution to (2) with separated variables and $v(x, y) = C(x)D(y)$ for its σ-harmonic conjugate function, a solution to (3) in $\mathcal{R}$. In view of the Cauchy–Riemann equations (4), we have

$$u_x = xv_y \qquad \text{or} \qquad A'B = xCD'$$

and

$$u_y = -xv_x \qquad \text{or} \qquad AB' = -xC'D.$$

The 2nd order equation for v gives

$$C''D + CD'' = -\frac{1}{x}C'D, \qquad \text{or} \qquad \frac{C''}{C} + \frac{1}{x}\frac{C'}{C} = -\frac{D''}{D} = -\lambda^2 \in \mathbb{R}, \qquad \text{say,}$$

since it is independent of both x and y and hence constant. This gives

$$D(y) = \exp(\pm\lambda y), \qquad \text{and} \qquad C'' + \frac{1}{x}C' + \lambda^2 C = 0.$$

(i) Assume $\lambda \neq 0$, $\lambda \in \mathbb{R}$.
There is a fundamental solution in terms of Bessel functions, namely,

$$C(x) = J_0(\lambda x) \qquad \text{or} \qquad C(x) = Y_0(\lambda x).$$

Our main references for the theory of Bessel functions are [34, Ch. III] and [35, Ch. XVII]. Also, we may take $B = D$ and then (since $J_0' = -J_1$ and $Y_0' = Y_1$) we have, correspondingly, $A(x) = -xC'(x)/\lambda$, i.e.,

$$A(x) = xJ_1(\lambda x) \qquad \text{or} \qquad A(x) = -xY_1(\lambda x).$$

In fact, we shall be able to construct a complete family of solutions without using the functions Y_n (see Section 4.2).

(ii) Now we assume that $\lambda \neq 0$, $\lambda = i\varrho \in i\mathbb{R}$.
Modified Bessel functions (Bessel functions of imaginary argument), namely

$$C(x) = I_0(\varrho x) \qquad \text{or} \qquad C(x) = K_0(\varrho x),$$

satisfy the equation

$$\frac{C''}{C} + \frac{1}{x}\frac{C'}{C} = \varrho^2 = -\lambda^2,$$

in conjunction with $D(y) = \cos(\varrho(y + y_0))$, say. Taking $B(y) = \sin(\varrho(y + y_0))$ again, and noting that $I_0' = I_1$ and $K_0' = -K_1$ [34], we have the corresponding solutions:

$$A(x) = -xI_1(\varrho x) \qquad \text{or} \qquad A(x) = xK_1(\varrho x).$$

(iii) Finally we look at the case $\lambda = 0$.
The dependence in x and y of each function allows us to write

$$\frac{A'}{xC} = \frac{D'}{B} = \rho \quad \text{and} \quad \frac{A}{xC'} = -\frac{D}{B'} = -\mu$$

and we have the following identities:

$$D'' = \lambda^2 D = \rho B' = \frac{\rho}{\mu} D\,,$$

so that $\lambda^2 = \dfrac{\rho}{\mu}$. Then $\lambda = 0$ implies that either $\rho = 0$ and $\mu \neq 0$, or $\mu = \infty$.
If $\rho = 0$ and $\mu \neq 0$, we have

$$D(y) = d,\ B(y) = \frac{d}{\mu}\, y + b,\ A(x) = a \quad \text{and} \quad C(x) = -\frac{a}{\mu}\, \ln(x) + c\,.$$

In the other case, $\mu = \infty$,

$$B(y) = b,\ D(y) = \rho b y + d,\ C(x) = c \quad \text{and} \quad A(x) = \frac{\rho c}{2}\, x^2 + a\,.$$

Thus, for every $N \in \mathbb{N}^*$, and for all sequences of real-valued coefficients (λ_n), (μ_n), (α_n), (β_n), (γ_n), (δ_n), $n = 1, \cdots, N$, and every $a_0, b_0, c_0 \in \mathbb{R}$, the following is a legitimate solution u to (2):

$$\begin{aligned} u(x,y) &= \sum_{n=1}^{N} xJ_1(\lambda_n x)[\alpha_n e^{\lambda_n y} + \beta_n e^{-\lambda_n y}] \\ &\quad + \sum_{n=1}^{N} x[\gamma_n I_1(\mu_n x) + \delta_n K_1(\mu_n x)] \sin(\mu_n(y + c)) + a_0 x^2 + b_0 y + c_0 \qquad (8) \\ &= u_\lambda(x,y) + u_\mu(x,y) + u_0(x,y), \quad \text{say.} \end{aligned}$$

The analogous expression for $\mathcal{H}_{\nu_*}u$ involves Bessel functions of order zero, as above, whence if u is given by (8), then $\mathcal{H}_{\nu_*}u = v - L\,v$ with the solution v to (3) and (4) given by:

$$\begin{aligned} v(x,y) &= \sum_{n=1}^{N} J_0(\lambda_n x)[\alpha_n e^{\lambda_n y} - \beta_n e^{-\lambda_n y}] \\ &\quad + \sum_{n=1}^{N} [-\gamma_n I_0(\mu_n x) + \delta_n K_0(\mu_n x)] \cos(\mu_n(y+c)) - b_0 \ln x + 2a_0 y \qquad (9) \\ &= v_\lambda(x,y) + v_\mu(x,y) + v_0(x,y)\,. \end{aligned}$$

It is convenient to choose λ_n to be the positive roots of $J_0(\lambda_n b) = 0$, in view of the fact that with this choice the functions $(\sqrt{x} J_0(\lambda_n x))_{n\geq 1}$ are a complete orthogonal system in $L^2((0,b); x\,dx)$. However, many other choices are possible.
An appropriate choice of μ_n is $\frac{n\pi}{2c}$, so that the expression $u_\mu(x,y)$ in (8) vanishes on the lines $y = \pm c$.

We then write $\mathcal{B}$ for the space of solutions u given by (8) with such (λ_n), (μ_n), $n = 1, \cdots, N$, and free parameters (α_n), (β_n), (γ_n), (δ_n), $a_0, b_0, c_0 \in \mathbb{R}$. Meanwhile, we introduce $\mathcal{F}$ as the corresponding (through (4)) space of solutions v given by (9).
Then, given L^2 data on the boundary of the rectangle $\mathcal{R} = [a,b] \times [-c,c]$, we may approximate it arbitrarily closely on the lines $y = \pm c$ by taking a finite sum and choosing (α_n) and (β_n) suitably. Then we may choose the coefficients (γ_n) and (δ_n) to obtain approximation on the lines $x = a$, $x = b$.

Remark 4 *Note that the Wronskian $I_1' K_1 - I_1 K_1'$ does not vanish, which implies that I_1/K_1 is monotonic, and so the vectors $(I_1(\mu_n a), K_1(\mu_n a))$ and $(I_1(\mu_n b), K_1(\mu_n b))$ in $\mathbb{R}^2$ are linearly independent.*

4.2 Density properties of $\mathcal{B}$ in $\mathcal{R}$

Proposition 3 *The family $\mathcal{B}$ is $W^{1,2}(\mathcal{R})$ dense in*

$$\mathcal{S}_{\mathcal{R}} = \left\{ u \in W^{1,2}_{\mathbb{R}}(\mathcal{R})\,;\, \mathrm{div}\left(\frac{1}{x}\nabla u\right) = 0 \right\}.$$

The proof relies on the following result that we first establish:

Lemma 1 *$\mathcal{B}_{|\partial\mathcal{R}}$ is $W^{1,2}$ dense in $W^{1,2}_{\mathbb{R}}(\partial\mathcal{R})$.*

Proof of Lemma 1. Recall that $\mathcal{B}_{|\partial\mathcal{R}}$ is a set of solutions of the conductivity equation

$$\mathrm{div}\left(\frac{1}{x}\nabla u\right) = 0\,, \qquad (10)$$

obtained by separating the variables on the rectangle $\mathcal{R} = [a,b] \times [-c,c]$, restricted to $\partial\mathcal{R}$, in $W^{1,2}_{\mathbb{R}}(\partial\mathcal{R})$.

The proof proceeds in three stages:

- we consider first the horizontal sides Γ_c and Γ_{-c} (Step 1), where Bessel functions of order 0 induce a complete famliy,
- then the vertical sides Γ_a and Γ_b (Step 2), with the usual Fourier basis,
- then we prove density for the complete boundary $\partial\mathcal{R}$ of the rectangle $\mathcal{R}$ (Step 3).

Step 1: density of $\mathcal{B}_{|\Gamma_c}$ in $W^{1,2}_{\mathbb{R}}(a,b)$.

Let $f \in W^{1,2}_{\mathbb{R}}(\partial\mathcal{R})$ so that $f(.,c) \in W^{1,2}_{\mathbb{R}}(a,b)$ denotes its restriction to Γ_c. Write $(\lambda_n)_{n\geq 1}$ for the strictly increasing sequence of roots [34, Ch. XV] of the equation $J_0(\lambda_n b) = 0$ which are all positive. Now we define a function ψ on $\mathcal{R}$ by $\psi(x,y) = \frac{1}{c\sqrt{x}}\partial_x f(x,c)\, y$. Its restriction to Γ_c still belongs to $L^2_{\mathbb{R}}(a,b)$ and, by [14, Thm. 2], $\left\{x^{1/2}J_0(\lambda_n x)\right\}$ forms a complete system in $L^2_{\mathbb{R}}(a,b)$. It follows that for every $\epsilon > 0$, there exist a sufficiently large integer N and scalars $(a_{n,c})_{n=1,\dots,N}$ such that

$$\left\|\psi(t,c) - \sum_{n=1}^{N} a_{n,c}\sqrt{t}J_0(\lambda_n t)\right\|_{L^2(a,b)} \leq \frac{\epsilon}{b-a}\sqrt{\frac{2}{b}}, \tag{11}$$

Then classical properties of the integral and the Cauchy–Schwarz inequality lead to

$$\frac{1}{\sqrt{b-x}}\left|\int_x^b \left(\partial_t f(t,c) - \sum_{n=1}^{N} a_{n,c}\, tJ_0(\lambda_n t)\right) dt\right| \leq \epsilon\sqrt{\frac{2}{b-a}}, \qquad x \in (a,b). \tag{12}$$

Using the identity $J_\nu(x) = x^{-\nu-1}\dfrac{d}{dx}(x^{\nu+1}J_{\nu+1}(x))$ from [34, p. 66] and calculating the integral in (12) allow us to write

$$(\xi_c(x,c) - f(x,c))^2 \leq 2\epsilon^2\,\frac{b-x}{b-a}\,, \tag{13}$$

where $\xi_c \in \mathcal{B}$ denotes the function defined on $\mathcal{R}$ by

$$\xi_c(x,y) = \sum_{n=1}^{N} \frac{a_{n,c}}{\lambda_n}\, xJ_1(\lambda_n x)\,\frac{e^{\lambda_n(y+c)} - e^{-\lambda_n(y+c)}}{e^{2\lambda_n c} - e^{-2\lambda_n c}} - (y+c)\,\frac{K_{N,c} - f(b,c)}{2c}\,, \tag{14}$$

and

$$K_{N,c} = \sum_{n=1}^{N} \frac{a_{n,c}}{\lambda_n}\, bJ_1(\lambda_n b).$$

Integrating (13) between a and b we have that $\|f(x,c) - \xi_c(x,c)\|_{L^2(a,b)} \leq \epsilon$. In conjunction with (11), this implies the density of $\mathcal{B}_{|\Gamma_c}$in $W^{1,2}_{\mathbb{R}}(a,b)$.

A similar calculation can be made on the side Γ_{-c} and we denote by ξ_{-c} the corresponding function. Since ξ_c vanishes on Γ_{-c} and ξ_{-c} vanishes on Γ_c, we conclude that if $f \in W^{1,2}_{\mathbb{R}}(\partial\mathcal{R})$, then $(\xi_c + \xi_{-c})$ approximates f on $\Gamma_c \cup \Gamma_{-c}$ in $W^{1,2}(a,b)$ norm.

The same principle will be used for the vertical sides Γ_a and Γ_b: construct an approximation on both, such that its contribution to the other side is arbitrarily small (or zero) in

$W^{1,2}$ norm.

Step 2: density of $\mathcal{B}_{|\Gamma_b}$ in $W^{1,2}_{\mathbb{R}}(-c,c)$.

Let $f \in W^{1,2}_{\mathbb{R}}(\partial\mathcal{R})$ so that $f(b,.) \in W^{1,2}_{\mathbb{R}}(-c,c)$ denotes its restriction to Γ_b. Hence $\partial_y f(b,y)$ can be expanded as a Fourier series using the family $\left\{1, \cos(\frac{n\pi}{2c}(y+c))\right\}_{n\in\mathbb{N}^*}$, which is complete in $L^2_{\mathbb{R}}(-c,c)$. Hence, for every $\epsilon > 0$, there exist an integer N and scalars $(a_{n,b})_{n=1,\dots,N}$ such that

$$\left\| \partial_t f(b,t) - \left(a_{0,b} + \sum_{n=1}^{N} a_{n,b} \cos\left(\frac{n\pi}{2c}(t+c)\right)\right)\right\|_{L^2(-c,c)} \leq \frac{\epsilon}{\sqrt{c}}. \tag{15}$$

Let us denote by $\mathcal{F}_N(b,.)$ the above partial sum of order N of the Fourier series expansion of $f'(b,.)$. As in Step 1, we have that

$$\frac{1}{\sqrt{y+c}} \left| \int_{-c}^{y} \left(\partial_t f(b,t) - \mathcal{F}_N(b,t)\right) dt \right| \leq \sqrt{2}\epsilon, \qquad y \in (-c,c). \tag{16}$$

Writing $\mu_n = \dfrac{n\pi}{2c}$, (16) can be calculated to give

$$\left| f(b,y) - f(b,-c) - \left(a_{0,b}\,(y+c) + \sum_{n=1}^{N} \frac{a_{n,b}}{\mu_n} \sin(\mu_n(y+c))\right)\right| \leq \sqrt{2(y+c)}\epsilon.$$

That is,

$$\left(f(b,y) - \xi_b(b,y)\right)^2 \leq 2(y+c)\epsilon^2, \tag{17}$$

where $\xi_b \in \mathcal{B}$ denotes the function defined on $\mathcal{R}$ by

$$\begin{aligned} \xi_b(x,y) &= \sum_{n=1}^{N} x\left[\gamma_{n,b} I_1(\mu_n x) + \delta_{n,b} K_1(\mu_n x)\right] \sin(\mu_n(y+c)) \\ &+ \frac{x^2-a^2}{b^2-a^2} f(b,-c) + (y+c)\,\frac{f(b,c)-f(b,-c)}{2c}, \end{aligned} \tag{18}$$

and

$$\begin{cases} \gamma_{n,b} = -\dfrac{a_{n,b}}{b\mu_n} \dfrac{K_1(\mu_n a)}{K_1(\mu_n b)I_1(\mu_n a) - K_1(\mu_n a)I_1(\mu_n b)}, \\[2ex] \delta_{n,b} = \dfrac{a_{n,b}}{b\mu_n} \dfrac{I_1(\mu_n a)}{K_1(\mu_n b)I_1(\mu_n a) - K_1(\mu_n a)I_1(\mu_n b)}. \end{cases}$$

By Remark 4, the expression of ξ_b is valid insofar as $K_1(\mu_n b)I_1(\mu_n a) - K_1(\mu_n a)I_1(\mu_n b) \neq 0$. Finally, integrating (17) between $-c$ and c, we have $\|f(b,y) - \xi_b(b,y)\|_{L^2(-c,c)} \leq c\epsilon\sqrt{2}$. In conjunction with (15), this implies the density of $\mathcal{B}_{|\Gamma_b}$ in $W^{1,2}_{\mathbb{R}}(-c,c)$.

As in the previous section, a similar result holds on the side Γ_a with a corresponding function ξ_a. The functions ξ_a and ξ_b are constructed so that if f is arbitrarily small on

Γ_c and Γ_{-c} then ξ_a is arbitrarily small on Γ_b and so does ξ_b on Γ_a. We conclude that if $f \in W^{1,2}_{\mathbb{R}}(\partial\mathcal{R})$ and is small enough, then $(\xi_a + \xi_b)$ is an approximation to f on $\Gamma_a \cup \Gamma_b$ in $W^{1,2}(-c, c)$ norm.

Step 3: density of $\mathcal{B}_{|\partial\mathcal{R}}$ in $W^{1,2}(\partial\mathcal{R})$.

Let $f \in W^{1,2}_{\mathbb{R}}(\partial\mathcal{R})$. From Step 1, there exist ξ_c and ξ_{-c} belonging to $\mathcal{B}$ such that on $\Gamma_c \cup \Gamma_{-c}$, we have $(f - \xi_c - \xi_{-c})$ arbitrarily small in $W^{1,2}(a, b)$ norm. We let $\mathcal{H}$ denote the function defined on $\partial\mathcal{R}$ such that $\mathcal{H} = (f - \xi_c - \xi_{-c})$. Since $\mathcal{H}$ still belongs to $W^{1,2}_{\mathbb{R}}(\partial\mathcal{R})$, there exist, from Step 2, $\xi_a \in \mathcal{B}$ and $\xi_b \in \mathcal{B}$ which approximate $\mathcal{H}$ in $W^{1,2}(-c, c)$ norm on the sides Γ_a and Γ_b respectively. Writing $I = \{a, b, c, -c\}$, we now show that

$$\|\mathcal{H} - (\xi_a + \xi_b)\|^2_{W^{1,2}(\partial\mathcal{R})} = \sum_{i\in I} \|\mathcal{H} - (\xi_a + \xi_b)\|^2_{W^{1,2}(\Gamma_i)}$$

can be made arbitrarily small, in other words $(\xi_c + \xi_{-c} + \xi_a + \xi_b) \in \mathcal{B}$ gives a good approximation to the function f in $W^{1,2}(\partial\mathcal{R})$ norm. For convenience, we only give the details for Γ_{-c}.
Let

$$A_{-c} = \|\mathcal{H} - (\xi_a + \xi_b)\|_{W^{1,2}(\Gamma_{-c})} = \|\mathcal{H}(x, -c) - \xi_a(x, -c) - \xi_b(x, -c)\|_{W^{1,2}(a,b)} .$$

Here, ξ_b is given by formula (18) with $\mathcal{H}$ in place of f, and ξ_a similarly. We thus obtain

$$A_{-c} \;=\; \|\mathcal{H}(x, -c) - \frac{x^2 - b^2}{a^2 - b^2}\mathcal{H}(a, -c) - \frac{x^2 - a^2}{b^2 - a^2}\mathcal{H}(b, -c)\|_{W^{1,2}(a,b)} .$$

Now, we remark that:

$$\begin{aligned}
\|\frac{x^2 - b^2}{a^2 - b^2}\mathcal{H}(a, -c)\|_{W^{1,2}(a,b)} &\leq |\mathcal{H}(a, -c)|\|\frac{x^2 - b^2}{a^2 - b^2}\|_{W^{1,\infty}(a,b)} \\
&\leq |\mathcal{H}(a, -c)| \left(1 + \frac{2b}{b^2 - a^2}\right) ,
\end{aligned}$$

and in the same way:

$$\|\frac{x^2 - a^2}{b^2 - a^2}\mathcal{H}(b, -c)\|_{W^{1,2}(a,b)} \leq |\mathcal{H}(b, -c)| \left(1 + \frac{2b}{b^2 - a^2}\right) .$$

Using the triangle inequality, it follows that:

$$A_{-c} \leq \|\mathcal{H}(x, -c)\|_{W^{1,2}(a,b)} + 2\left(1 + \frac{2b}{b^2 - a^2}\right) \times \max(|\mathcal{H}(a, -c)|, |\mathcal{H}(b, -c)|).$$

From (14), we have that $\xi_c(., -c) \equiv 0$, hence $\mathcal{H}(., -c) = f(., -c) - \xi_{-c}(., -c)$. Since ξ_{-c} is an approximation to f on Γ_{-c} in $W^{1,2}(a, b)$ norm from (13) in Step 1, thus also at the corner points $(a, -c)$ and $(b, -c)$, all the terms in the last inequality can be made arbitrarily small, thus also $A_{-c} = \|f - (\xi_c + \xi_{-c} + \xi_a + \xi_b)\|_{W^{1,2}(\Gamma_{-c})}$.
Similarly, $\|f - (\xi_c + \xi_{-c} + \xi_a + \xi_b)\|_{W^{1,2}(\partial\mathcal{R})} = \|\mathcal{H} - (\xi_a + \xi_b)\|_{W^{1,2}(\partial\mathcal{R})}$ can be made arbitrarily small. This ends the proof of Lemma 1. □

Proof of Proposition 3. Recall first that $\mathcal{R}$ is a simply-connected plane domain with Lipschitz boundary [23, p. 6]. Let $u \in \mathcal{S}_{\mathcal{R}}$. Since $\operatorname{tr} u \in W^{1/2,2}_{\mathbb{R}}(\partial\mathcal{R})$ [23, Thm 1.5.1.3], there exist $(\phi_n)_{n\in\mathbb{N}} \in W^{1,2}_{\mathbb{R}}(\partial\mathcal{R})$ such that

$$\|\operatorname{tr} u - \phi_n\|_{W^{1/2,2}(\partial\mathcal{R})} \to 0,$$

and by Lemma 1, we also have that for each $n \in \mathbb{N}$ there exists $b^{(n)} \in \mathcal{B}$ such that $\|\phi_n - b^{(n)}_{|\partial\mathcal{R}}\|_{W^{1,2}(\partial\mathcal{R})}$ can be made arbitrarily small. Since $W^{1,2}_{\mathbb{R}}(\partial\mathcal{R}) \subset W^{1/2,2}_{\mathbb{R}}(\partial\mathcal{R})$ (see [29]), and the injection is bounded (indeed, even compact) by [12, Thm 2.2] (see also [1, Par. 1.24]), we have the existence of a constant $c > 0$ such that

$$\|\phi_n - b^{(n)}_{|\partial\mathcal{R}}\|_{W^{1/2,2}(\partial\mathcal{R})} \leq c\|\phi_n - b^{(n)}_{|\partial\mathcal{R}}\|_{W^{1,2}(\partial\mathcal{R})} \to 0,$$

and, by the triangle inequality,

$$\|\operatorname{tr} u - b^{(n)}_{|\partial\mathcal{R}}\|_{W^{1/2,2}(\partial\mathcal{R})} \leq \|\operatorname{tr} u - \phi_n\|_{W^{1/2,2}(\partial\mathcal{R})} + \|\phi_n - b^{(n)}_{|\partial\mathcal{R}}\|_{W^{1/2,2}(\partial\mathcal{R})} \to 0.$$

Finally, by [12, Thm 4.4], the map from $W^{1/2,2}_{\mathbb{R}}(\partial\mathcal{R})$ to $W^{1,2}_{\mathbb{R}}(\mathcal{R})$ which to $\operatorname{tr} u$ associates $u \in \mathcal{S}_{\mathcal{R}}$ (unique solution to (10) associated to the given boundary values) is continuous, so that, for some $c' > 0$,

$$\|u - b^{(n)}\|_{W^{1,2}(\mathcal{R})} \leq c'\|\operatorname{tr} u - b^{(n)}_{|\partial\mathcal{R}}\|_{W^{1/2,2}(\partial\mathcal{R})} \to 0,$$

and thus $\mathcal{B}$ is dense in $\mathcal{S}_{\mathcal{R}}$. □

4.3 Density properties of $\mathcal{B}$ in $\mathbb{D}_0$

We show below that $L^2_{\mathbb{R}}(\mathbb{T}_0)$ functions may be approximated by functions belonging to $\mathcal{B}_{|\mathbb{T}_0}$ that are solutions to (2). For the particular dilation coefficient ν_*, this is a constructive view of the fact that $L^2_{\mathbb{R}}(\mathbb{T}_0) = \operatorname{Re}\operatorname{tr} H^2_{\nu_*}(\mathbb{D}_0)$ (see Theorem 1) and will be used in order to solve bounded extremal problems.
Recall that $H^2_{\nu_*}(\mathbb{D}_0)$ is defined accordingly to Section 2 and Remark 2, using for instance $\psi(z) = R\,z + x_0$ as a conformal map from $\mathbb{D}$ onto $\mathbb{D}_0$.

Proposition 4 *The restriction $\mathcal{B}_{|\mathbb{T}_0}$ to $\mathbb{T}_0$ of $\mathcal{B}$ is L^2 dense in $L^2_{\mathbb{R}}(\mathbb{T}_0) = \operatorname{Re}\operatorname{tr} H^2_{\nu_*}(\mathbb{D}_0)$.*

Proof. Let $\psi \in L^2_{\mathbb{R}}(\mathbb{T}_0)$ and $\varepsilon > 0$. There exists $\phi \in W^{1/2,2}_{\mathbb{R}}(\mathbb{T}_0)$ such that

$$\|\psi - \phi\|_{L^2(\mathbb{T}_0)} \leq \varepsilon\,.$$

However, Dirichlet-type results from [17] are to the effect that there exists a unique solution $u \in W^{1,2}_{\mathbb{R}}(\mathbb{D}_0)$ of (2) such that $\operatorname{tr} u = \phi$ on $\mathbb{T}_0$. Then, because of the $W^{1,2}(\mathcal{R})$ density of $\mathcal{B}$ in $\mathcal{S}_{\mathcal{R}}$, see Proposition 3, an approximation result of Browder [16, Thm 5] asserts that restrictions to $\mathbb{D}_0$ of functions in $\mathcal{B}$ still form a dense subset of $W^{1,2}(\mathbb{D}_0)$, the involved operator and domain being sufficiently smooth. This ensures that there exists $b \in \mathcal{B}$ such that

$$\|u - b\|_{W^{1,2}(\mathbb{D}_0)} \leq \varepsilon\,,$$

whence also $\|\phi - b\|_{L^2(\mathbb{T}_0)} \leq c\,\varepsilon$, for some $c > 0$, from [15, Ch. 9] (we are taking the restrictions of b to the disc or the circle as appropriate). Finally, we get that

$$\|\psi - b\|_{L^2(\mathbb{T}_0)} \leq (1 + c)\,\varepsilon\,.$$

This implies that $\mathcal{B}_{|_{\mathbb{T}_0}}$ is dense in $L^2_{\mathbb{R}}(\mathbb{T}_0)$. Further, the result that

$$L^2_{\mathbb{R}}(\mathbb{T}_0) = \operatorname{Re} \operatorname{tr} H^2_{\nu_*}(\mathbb{D}_0) \tag{19}$$

follows from Theorem 1. □

Remark 5 *Results of Lax [26] and Malgrange [30], assert similar Runge-like density properties, namely that real solutions of the partial differential equation on a disc can be approximated locally uniformly by solutions that extend to a large simply-connected domain.*

We then obtain:

Corollary 4 *(i)* $\mathcal{B}^{\nu_*} = \{b + i\mathcal{H}_{\nu_*} b + i\, c\,,\ b \in \mathcal{B}\ ,\ c \in \mathbb{R}\}$ *is* $L^2(\mathbb{T}_0)$ *dense in* $\operatorname{tr} H^2_{\nu_*}(\mathbb{D}_0)$.
(ii) $\mathcal{B} + i\mathcal{H}_{\nu_*}\mathcal{B} + i\mathbb{R} = \{b_1 + i\mathcal{H}_{\nu_*} b_2 + i\, c\,,\ b_1\,,\, b_2 \in \mathcal{B}\,,\ c \in \mathbb{R}\}$ *is dense in* $L^2(\mathbb{T}_0)$.

Proof. (i) Let $f \in H^2_{\nu_*}(\mathbb{D}_0)$ and $\varepsilon > 0$. So $u = \operatorname{Re}(\operatorname{tr} f) \in L^2_{\mathbb{R}}(\mathbb{T}_0)$, and Proposition 4 ensures that there exists $b \in \mathcal{B}$ such that

$$\|u - b\|_{L^2(\mathbb{T}_0)} < \varepsilon.$$

Now, as $\mathcal{H}_{\nu_*}$ is continuous from $L^2_{\mathbb{R}}(\mathbb{T}_0)$ to $L^2_{\mathbb{R}}(\mathbb{T}_0)$ (from Corollary 1), we have

$$\|\mathcal{H}_{\nu_*}(u) - \mathcal{H}_{\nu_*}(b)\|_{L^2(\mathbb{T}_0)} = \|\mathcal{H}_{\nu_*}(u - b)\|_{L^2(\mathbb{T}_0)} < \varepsilon.$$

Finally, using Corollary 2, we get that $\mathcal{H}_{\nu_*}(u) = v + c = \operatorname{Im}(\operatorname{tr} f) + c$, for some $c \in \mathbb{R}$ and point (i) follows $(\operatorname{tr} H^2_{\nu_*}(\mathbb{D}_0) = L^2_{\mathbb{R}}(\mathbb{T}_0) + i\mathcal{H}_{\nu_*}(L^2_{\mathbb{R}}(\mathbb{T}_0)) + i\mathbb{R})$.
Point (ii) may be obtained from the decomposition $L^2(\mathbb{T}_0) = L^2_{\mathbb{R}}(\mathbb{T}_0) + iL^2_{\mathbb{R}}(\mathbb{T}_0)$, using that $\mathcal{H}_{\nu_*}\mathcal{B} + \mathbb{R}$ is dense in $L^2_{\mathbb{R}}(\mathbb{T}_0)$, from point (i). □

Remark 6 *From point* (i) *of the above corollary, for* $K \subset \mathbb{T}_0$*, such that* $\mathbb{T}_0 \setminus K$ *has positive Lebesgue measure, we get that* $\mathcal{B}^{\nu_*}_{|_K}$ *is* $L^2(K)$ *dense in* $[\operatorname{tr} H^2_{\nu_*}(\mathbb{D}_0)]_{|_K}$*, whence in* $L^2(K)$*, in view of Theorem 2.*

5 Constructive aspects

In order to compute solutions to (BEP) in $\mathbb{D}_0$ using Proposition 2 (see also Remark 2), we first need to find approximations in $\mathcal{B}$ and $\mathcal{H}_{\nu_*}\mathcal{B}$ to given data (u, v) on the circle $\mathbb{T}_0$ with centre $(x_0, 0)$ and radius R, contained in the rectangle $\mathcal{R}$. From now on, we denote by $e > 0$ the minimal distance between $\mathbb{T}_0$ and $\mathcal{R}$ (see Fig.1).
In this section, we take advantage of the properties established for $\mathcal{B}$ in Section 4. Indeed, recall that given $b \in \mathcal{B}$, with a set of coefficients on the family of Bessel-exponential functions (8) that generate $\mathcal{B}$, then $\mathcal{H}_{\nu_*} b$ has the same coefficients on the family (9) of Bessel-exponentials that generate $\mathcal{H}_{\nu_*}\mathcal{B}$ and $\mathcal{F}$.

5.1 Approximation algorithms

In order to determine numerically P_{ν_*} for given boundary data u and v in $L^2_{\mathbb{R}}(\mathbb{T}_0)$, it is sufficient to compute an L^2-approximation of $\mathcal{H}_{\nu_*}u$ and $\mathcal{H}_{-\nu_*}v$, according to the formulas in Section 3.1. To proceed, recall first that $\mathcal{B}$ is the space of solutions to (2) with separated variables and $\sigma = \sigma_*$, while $\mathcal{F}$ is the corresponding space of solutions to (3), (4), see the expressions (8), (9).

That $\operatorname{tr}\mathcal{F}$ is dense in $L^2_{\mathbb{R}}(\mathbb{T}_0)$ follows from Corollary 4. The issue now consists in writing the expansions of u on $\mathcal{B}$ and of v on $\mathcal{F}$ and then in computing both $\mathcal{H}_{\nu_*}u$ and $\mathcal{H}_{-\nu_*}v$ by the following rule (see (8), (9)):

$$\mathcal{H}_{\nu_*}\mathcal{B} = \mathcal{F} - L(\mathcal{F}) \qquad \text{and} \qquad \mathcal{H}_{-\nu_*}\mathcal{F} = -\mathcal{B} - L(-\mathcal{B}).$$

Here the averaging operator L is considered on $\mathbb{T}_0$:

$$\forall\, f \in L^2_{\mathbb{R}}(\mathbb{T}_0), \quad Lf = \frac{1}{2\pi}\int_0^{2\pi} f(x_0 + R\cos\theta, R\sin\theta)\; d\theta.$$

We remark that there is no connection between the coefficients resulting from the decomposition of u in $\mathcal{B}$ and those of v in $\mathcal{F}$ except when $u + iv \in \operatorname{tr} H^2_{\nu_*}(\mathbb{D}_0)$. Of course, only a finite number N of functions in $\mathcal{B}$ and $\mathcal{F}$ will be used for computations. More particularly, the same number n of elementary functions of each kind is involved, whence $N = 4n + 3$ for $\mathcal{B}$, from (8), while $N = 4n + 2$ for $\mathcal{F}$, using (9).

Let $(b_p)_{p=1,\dots,N}$ be the functions of one of these finite systems, whose union is complete in $L^2_{\mathbb{R}}(\mathbb{T}_0)$ (by Proposition 4). In $\mathcal{B}$, choose for instance, for $p = 1, \cdots n$:

$$\begin{cases} b_{4p-3}(x,y) = xJ_1(\lambda_p x)e^{\lambda_p y}, & b_{4p-2}(x,y) = xJ_1(\lambda_p x)e^{-\lambda_p y}, \\ b_{4p-1}(x,y) = xI_1(\mu_p x)\sin(\mu_p(y+c)), & b_{4p}(x,y) = xK_1(\mu_p x)\sin(\mu_p(y+c)), \end{cases}$$

and $b_{N-2}(x,y) = x^2$, $b_{N-1}(x,y) = y$, $b_N(x,y) = 1$.

Then an optimal $L^2(\mathbb{T}_0)$ approximation of a function $\psi \in L^2(\mathbb{T}_0)$ in the span of $(b_p)_{p=1,\dots,N}$ is obtained by solving in $(k_m)_m$ the *normal equations*:

$$\langle \sum_{m=1}^{N} k_m b_m - \psi, b_p\rangle_{L^2(\mathbb{T}_0)} = 0, \quad \text{or} \quad \sum_{m=1}^{N} k_m \langle b_m, b_p\rangle_{L^2(\mathbb{T}_0)} = \langle \psi, b_p\rangle_{L^2(\mathbb{T}_0)}, \quad n = 1, \dots, N.$$

Note that functions in the family $\mathcal{B}$ (or $\mathcal{F}$) are mutually non-orthogonal on $\mathbb{T}_0$. As always in this case, the stiffness matrix $(\langle b_m, b_p\rangle_{L^2(\mathbb{T}_0)})$, $m, p = 1, \dots, N$, appearing in the resolution of the normal equations is ill-conditioned. A first approach to decreasing the conditioning number and thereby to improve the accuracy of the approximation is to average the basis functions after subtracting their mean value. Hence, we consider the families:

$$\tilde{\mathcal{B}} = \left\{ \frac{b - L(b)}{\|b - L(b)\|_{L^2(\mathbb{T}_0)}}; b \in \mathcal{B} \right\} \quad \text{and} \quad \tilde{\mathcal{F}} = \left\{ \frac{f - L(f)}{\|f - L(f)\|_{L^2(\mathbb{T}_0)}}; f \in \mathcal{F} \right\}$$

and approximations are made for $u - Lu$ and $v - Lv$. Then a representation of the algorithm is:

$$
\begin{tikzcd}[column sep=small]
 & \begin{array}{c} u \in L^2_{\mathbb{R}}(\mathbb{T}_0) \\ v \in L^2_{\mathbb{R}}(\mathbb{T}_0) \end{array} \arrow[dl] \arrow[dr] & \\
U = u - Lu \arrow[d] & & V = v - Lv \arrow[d] \\
\tilde{U} = \sum_{i=1}^{N} \alpha_i \tilde{b}_i, \quad \tilde{b}_i \in \tilde{\mathcal{B}} \arrow[d] & & \tilde{V} = \sum_{i=1}^{N} \beta_i \tilde{f}_i, \quad \tilde{f}_i \in \tilde{\mathcal{F}} \arrow[d] \\
\mathcal{H}_{\nu_*} \tilde{U} = \sum_{i=1}^{N} \alpha_i \mathcal{H}_{\nu_*} \tilde{b}_i = \mathcal{H}_{\nu_*} \tilde{u} \arrow[dr] & & \mathcal{H}_{-\nu_*} \tilde{V} = \sum_{i=1}^{N} \beta_i \mathcal{H}_{-\nu_*} \tilde{f}_i = \mathcal{H}_{-\nu_*} \tilde{v} \arrow[dl] \\
 & \tilde{P}_{\nu_*} \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u - \mathcal{H}_{-\nu_*} \tilde{v} \\ \mathcal{H}_{\nu_*} \tilde{u} + v \end{pmatrix} + \frac{1}{2} \begin{pmatrix} Lu \\ Lv \end{pmatrix} &
\end{tikzcd}
$$

where $\tilde{u}$, $\tilde{v}$, $\tilde{U}$, $\tilde{V}$, $\mathcal{H}_{\nu_*}\tilde{u}$, $\mathcal{H}_{-\nu_*}\tilde{v}$ represent the approximations of order N in $\mathcal{B}$ and $\mathcal{F}$ of the corresponding functions. As a consequence, the algorithm leads us to define the operator $\tilde{P}_{\nu_*}$ which is a $L^2(\mathbb{T}_0)$ approximation of P_{ν_*}. For given data u and v, the error between $\tilde{P}_{\nu_*}(u,v)$ and $P_{\nu_*}(u,v)$ only stems from those between $\mathcal{H}_{\nu_*}\tilde{u}$ and $\mathcal{H}_{\nu_*}u$ on the one hand, and between $\mathcal{H}_{-\nu_*}\tilde{v}$ and $\mathcal{H}_{-\nu_*}v$ on the other hand.

5.2 Numerical computations

In this section, we present several numerical results obtained with the software Matlab (R2008b). These are still preliminar and mainly illustrate the approximation of functions in tr $H^2_{\nu_*}(\mathbb{D}_0)$ by functions in the families $\tilde{\mathcal{B}}$, $\tilde{\mathcal{F}}$ and of the projection operator P_{ν_*}. Further numerical work is to be done, in particular towards the resolution of the extremal problem (BEP). All the simulations are computed with the specific choice of parameters $(x_0, R, e) = (5, 2, 1)$ and with a discretization of $\mathbb{T}_0$ by 62 points ($p = 1, \cdots, 62$). Moreover, we specify that the integrations are performed with the *quad* method which is a recursive adaptive Simpson quadrature method (see [31]).
The first simulations consist in approximating a function $u \in L^2_{\mathbb{R}}(\mathbb{T}_0)$, or more exactly $u - Lu$, with $N = 4n + 2$ functions of the family $\tilde{\mathcal{B}}$. In the first case (Fig. 2, 3) u is defined on $\mathbb{T}_0$ by $(x, y) \mapsto u(x, y) = xy$ and in the second case (Fig. 4, 5) by $(x, y) \mapsto u(x, y) = x \ln(x + y)$. Recall that on $\mathbb{T}_0$, $x = x_0 + R\cos\theta$ and $y = R\sin\theta$. In both situations, we show the behaviour with respect to $n \in \{1, \cdots, 6\}$ of the error between $U = u - Lu$ and its approximant $\tilde{U}$: $\sum_p (U - \tilde{U})^2 / \sum_p U^2$ (Fig. 3, 5).

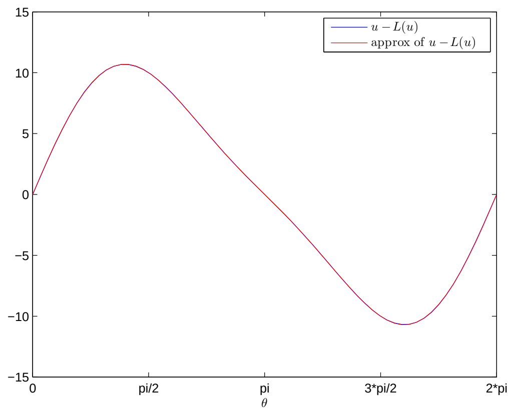

Fig.2 Approximation of $u = xy$
for $n = 3$

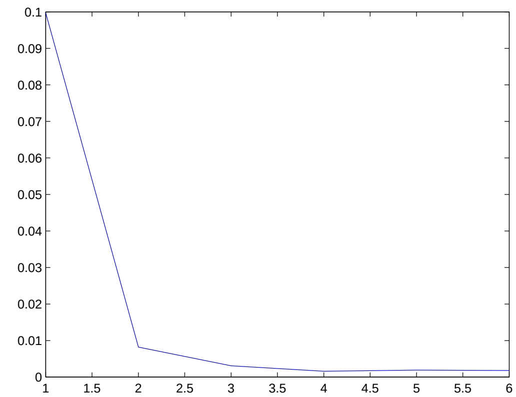

Fig.3 Error between U *and* $\tilde{U}$
when $u = xy$

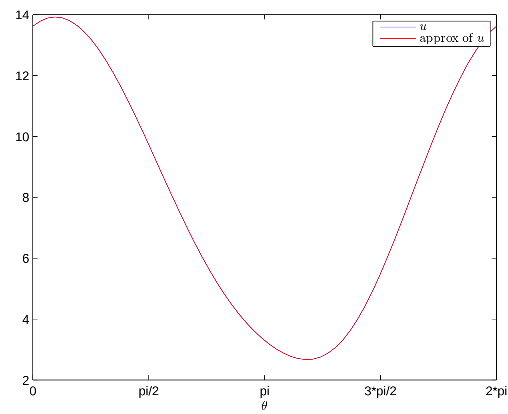

Fig.4 Approximation of $u = x\ln(x+y)$
for $n = 3$

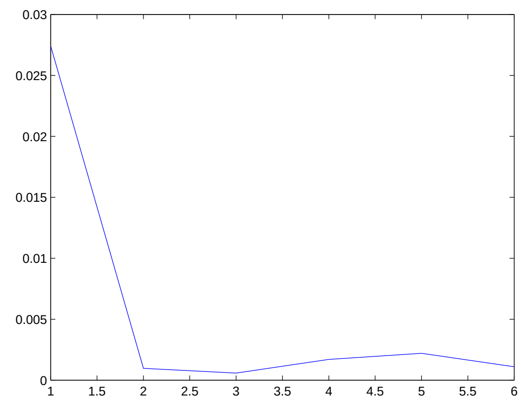

Fig.5 Error between U *and* $\tilde{U}$
when $u = x\ln(x+y)$

In both cases (Fig. 2, 4), the fact that $n = 3$ implies that the approximation is composed of $N = 4n + 2 = 14$ different elementary functions of the system $\tilde{\mathcal{B}}$. The graphs of the associated errors (Fig. 3, 5) show that the approximation keeps a high accuracy as the approximation order N increases. We just mention that it is of the same quality when computed with the system $\tilde{\mathcal{F}}$.

From these approximations of functions belonging to $L^2_{\mathbb{R}}(\mathbb{T}_0)$, we now compute the operator $\tilde{P}_{\nu_*}$ in order to verify that it matches the projection from $L^2(\mathbb{T}_0)$ onto $\operatorname{tr} H^2_{\nu_*}(\mathbb{D}_0)$. Let f be the function defined on $\mathbb{T}_0$ as

$$f : (x, y) \mapsto u(x, y) + iv(x, y) = x^2 + i(2y + 7).$$

It is an easy computation to see that $f \in \operatorname{tr} H^2_{\nu_*}(\mathbb{D}_0)$. Then $P_{\nu_*}(f) = f$. Related computations are plotted in Fig. 6, 7:

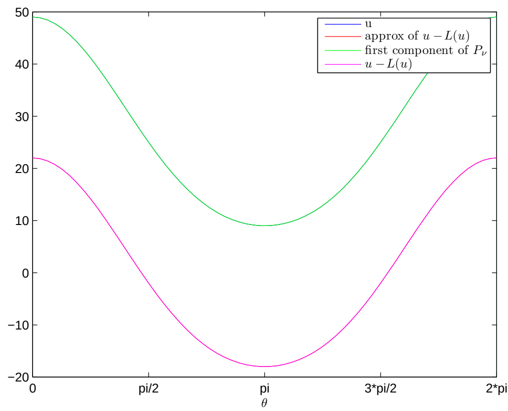

Fig.6 Approximation of $u = x^2$ *and* $P_{\nu_*}(u)$ *for* $n = 1$

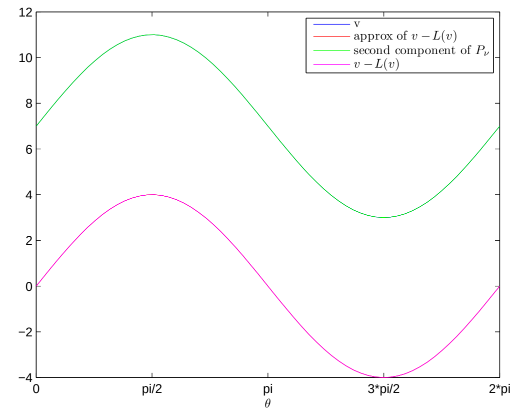

Fig.7 Approximation of $v = 2y + 7$ *and* $P_{\nu_*}(v)$ *for* $n = 1$

The error between U and $\tilde{U}$ is approximately equal to 10^{-15}, and between V and $\tilde{V}$, to 10^{-7}. A single elementary function is needed ($n = 1$) in $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{F}}$ (u and v already belong to $\mathcal{B}$ and $\mathcal{F}$, respectively). Then the relation $\tilde{P}_{\nu_*}(f) \simeq f = P_{\nu_*}(f)$ is verified.

We now test the accuracy of $\tilde{P}_{\nu_*}$ on functions which still belong to $trH^2_{\nu_*}(\mathbb{D}_0)$ but no longer to the families $\mathcal{B}$ and $\mathcal{F}$. Indeed, it is possible to find homogeneous polynomials u and v satisfying (4). With deg $u = k$ and deg $v = k - 1$ (for $k \geq 2$) the unique solution is to take a constant multiple of the pair:

$$\begin{aligned} u_k(x, y) &= x^2 y^{k-2} - \frac{(k-2)(k-3)}{2.4} x^4 y^{k-4} + \frac{(k-2)(k-3)(k-4)(k-5)}{2.4.4.6} x^6 y^{k-6} - \cdots \\ v_{k-1}(x, y) &= \frac{2}{(k-1)} y^{k-1} - \frac{k-2}{2} x^2 y^{k-3} + \frac{(k-2)(k-3)(k-4)}{2.4.4} x^4 y^{k-5} - \cdots \end{aligned}$$

where each sum has finitely-many non-zero terms. We denote by $\mathcal{G}$ the corresponding family of polynomials $f_{k,k-1} = u_k + iv_{k-1}$. The first test (Fig. 8-11) is computed with the function $f_{3,2}$ defined on $\mathbb{T}_0$ by:

$$f_{3,2} : (x, y) \mapsto u_3(x, y) + iv_2(x, y) = x^2 y + i(y^2 - \frac{x^2}{2})\,.$$

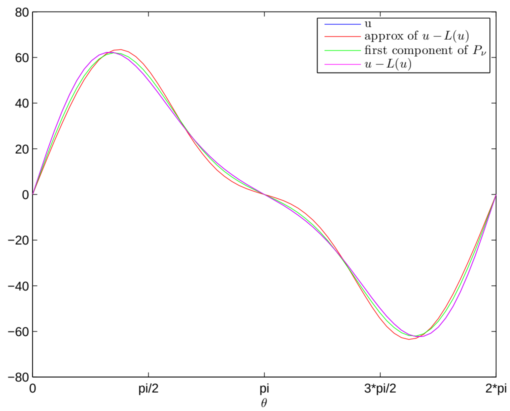

Fig.8 Approximation of u_3 and $P_{\nu_*}(u_3)$ for $n=1$

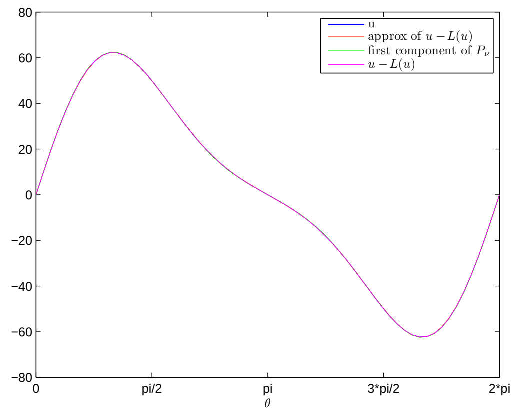

Fig.9 Approximation of u_3 and $P_{\nu_*}(u_3)$ for $n=3$

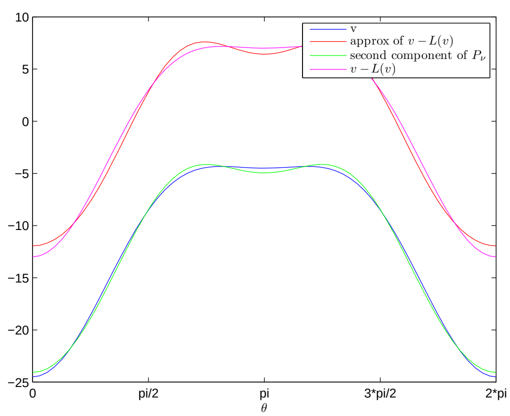

Fig.10 Approximation of v_2 and $P_{\nu_*}(v_2)$ for $n=1$

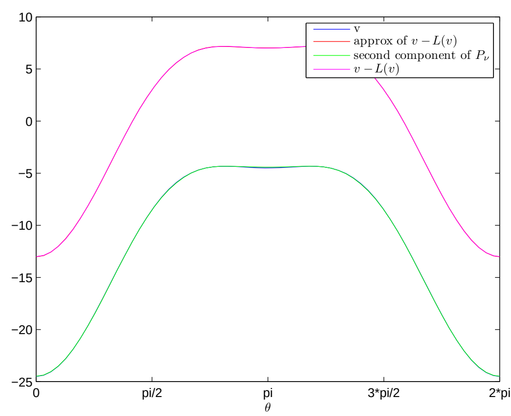

Fig.11 Approximation of v_2 and $P_{\nu_*}(v_2)$ for $n=3$

The error between U and $\tilde{U}$ is approximately equal to 10^{-2}, and between V and $\tilde{V}$, to 10^{-3} when $n=3$. On this example (Fig. 9, 11), it can be noticed that the operator P_{ν_*} is well approximated with a small number of functions in the families $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{F}}$.

Moreover this remark still holds when the degree of the polynomials u and v increases. The following simulation (Fig. 12-15) is obtained with the function $f_{5,4} \in \mathcal{G}$ defined on $\mathbb{T}_0$ by:

$$f_{5,4} : (x,y) \mapsto u_5(x,y) + iv_4(x,y) = x^2y^3 - \frac{3}{4}x^4y + i(\frac{y^4}{2} - \frac{3}{2}x^2y^2 + \frac{3}{16}x^4)\,.$$

This time the error between U and $\tilde{U}$ is approximately equal to 10^{-2}, and between V and $\tilde{V}$, to 10^{-3} when $n=5$. Even if the error increases compared with the last simulation, the operator P_{ν_*} remains well approximated as can be observed in Fig. 13, 15. It appears that, as the degree k of the polynomials u_k and v_k increases, a higher number N of functions in the families $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{F}}$ is needed in order to get a good approximation. More detailed error estimates would be the subject of a future investigation, also for more general data.

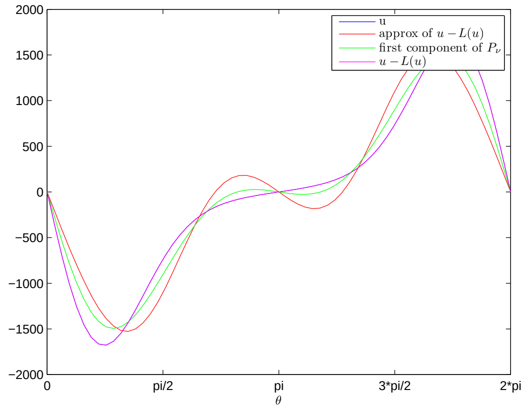

Fig.12 Approximation of u_5 *and* $P_{\nu_*}(u_5)$ *for* $n = 1$

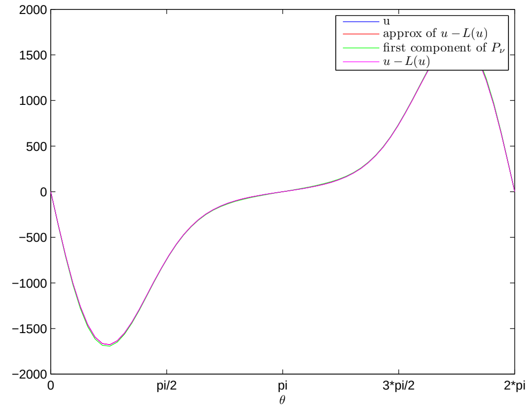

Fig.13 Approximation of u_5 *and* $P_{\nu_*}(u_5)$ *for* $n = 5$

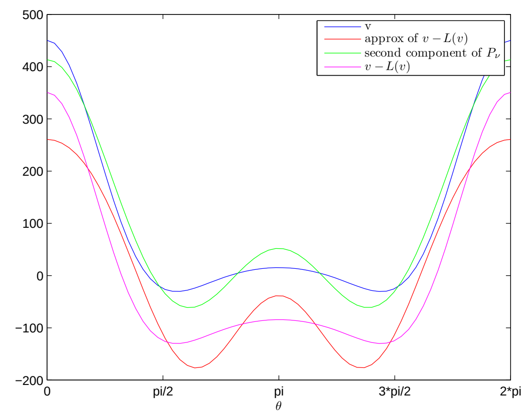

Fig.14 Approximation of v_4 *and* $P_{\nu_*}(v_4)$ *for* $n = 1$

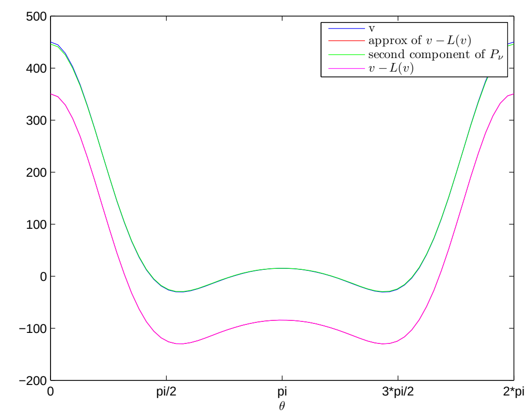

Fig.15 Approximation of v_4 *and* $P_{\nu_*}(v_4)$ *for* $n = 5$

6 Conclusion

Although preliminary numerical results are encouraging, a deep study of the constructive aspects of the presented approximation issues is still to be undertaken. The use of alternative basis functions may be considered (for example, the family $\mathcal{G}$ of polynomials that satisfy the Cauchy–Riemann equations, although it is not clear whether it is dense).
Concerning the link with plasma modelling in tokamaks, Hardy spaces H^p_ν of annular domains are already under study. As in the classical case, they can be expressed as direct sums of Hardy classes of related discs. It would also be of interest to solve such approximation problems in the L^∞ norm (in this case the classical version of the problem was analysed in [8]); finally, one may also hope to consider “mixed” L^2–L^∞ problems as in [10].

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