GEOMETRY AND TOPOLOGY OF SPACES OF STRUCTURED MATRICES

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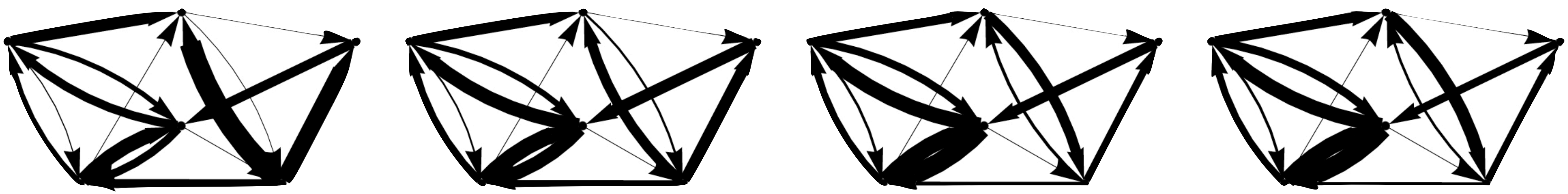
Joint work with Clayton Shonkwiler (Colorado State University)



Geometric Sciences in Action
CIRM
May 30, 2024




Spaces of Structured Matrices

We consider several types of structured matrices that arise in applications:

Unit norm, Tight frames: $\left\{ F = [f_1 \mid f_2 \mid \ldots \mid f_N] \in \mathbb{C}^{d\times N} \mid \|f_j\| = 1 \ \forall\, j \text{ and } FF^* = \frac{N}{d}\mathrm{I}_d \right\}$

Normal Matrices: $\left\{ A \in \mathbb{C}^{d\times d} \mid AA^* = A^*A \right\}$

Weighted Adjacency Matrices for Balanced Digraphs:

$$\left\{ A = (a_{ij})_{ij} \in \mathbb{R}^{d\times d}_{\geq 0} \mid \sum_i a_{ik} = \sum_j a_{kj} \ \forall\, k \right\}$$

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Main Idea:

Prove theorems about these spaces that are of interest in applied math/signal processing/data science, using tools from symplectic geometry.

Concepts from Symplectic Geometry

A symplectic manifold (M, ω) is an even-dimensional manifold M endowed with a closed, nondegenerate 2-form ω.

Let G be a Lie group with an action on M which preserves ω. A momentum map for this action is a smooth map

$$\mu : M \to \mathfrak{g}^* \approx \mathfrak{g}$$

which is equivariant with respect to the co-adjoint action $G \curvearrowright \mathfrak{g}^*$ and which satisfies

$$d_p\mu(X)(\xi) = \omega_p(Y_\xi|_p, X)$$

for $X \in T_pM$, $\xi \in \mathfrak{g}$, Y_ξ the associated infinitesimal vector field.

Given a Hamiltonian action $G \curvearrowright M$ with momentum map $\mu : M \to \mathfrak{g}^*$, the associated sympectic quotient is

$$M//G := \mu^{-1}(0)/G$$

If 0 is a regular value of μ and G acts freely on the fiber $\mu^{-1}(0)$, then $M//G$ has a canonical symplectic manifold structure.

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See Peter Michor's talk

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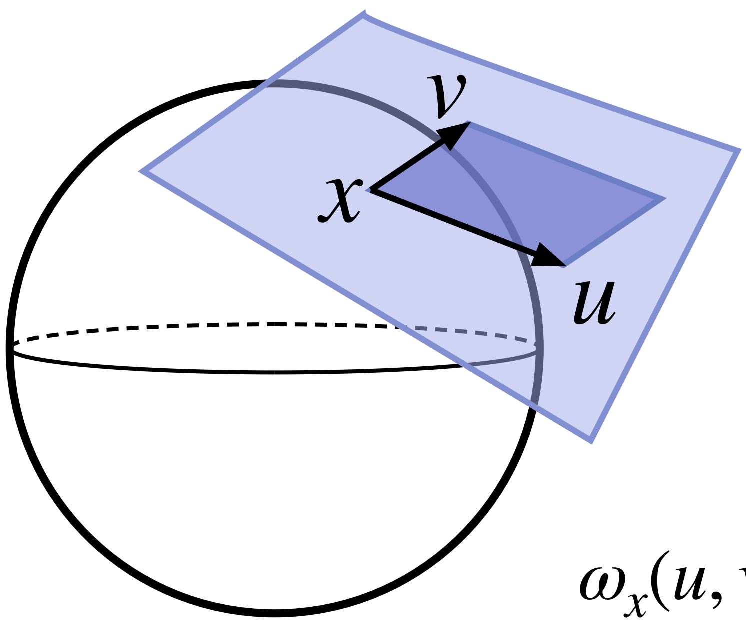
Intuition for Concepts from Symplectic Geometry

A symplectic manifold (M, ω) locally looks like $\left(\mathbb{C}^d, -\operatorname{Im}\langle \cdot, \cdot \rangle\right)$

Example.

(S^2, ω)



$\omega_x(u, v) = x \cdot (u \times v)$

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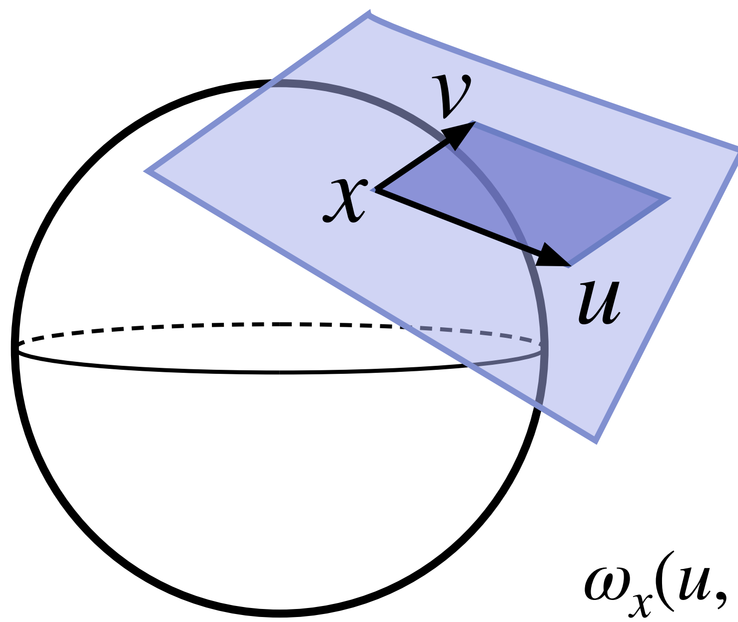

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A momentum map for an action $G \curvearrowright M$ is a smooth map

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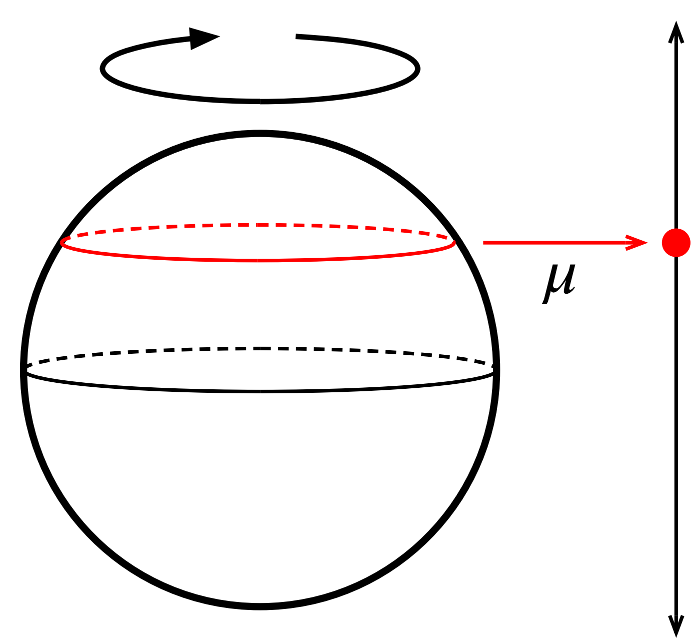
which encodes "conserved quantities" of the action.

Example.

$S^1 \curvearrowright S^2$ by rotation around z-axis

$\mu =$ height



Spaces of Frames

Unit Norm Tight Frames

An N-frame in $\mathbb{C}^d$ is a full rank matrix $F \in \mathbb{C}^{d \times N}$. The space of Unit norm, Tight frames is

$$\mathrm{UNTF}(d, N) = \left\{ F = [f_1 \mid f_2 \mid \ldots \mid f_N] \in \mathbb{C}^{d \times N} \mid \|f_j\| = 1 \; \forall \, j \text{ and } FF^* = \frac{N}{d} \mathrm{I}_d \right\}$$

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"unit norm"

"tight"

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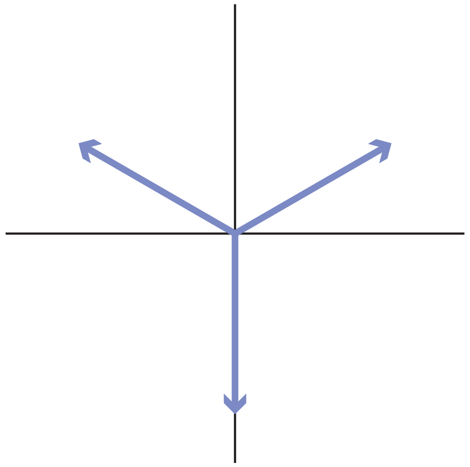
Frames are used to give redundant representations of signals $v \in \mathbb{C}^d$

$$\mathbb{C}^d \ni v \mapsto F^* v = \left(\langle v, f_j \rangle\right)_{j=1}^{N} \in \mathbb{C}^N$$



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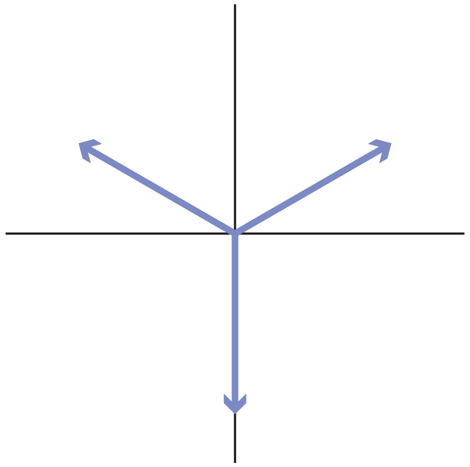

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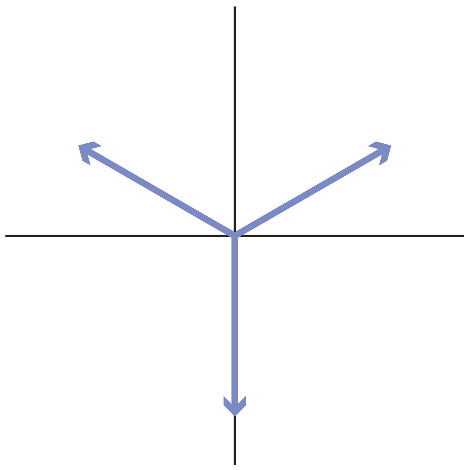

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Aside:

Why not represent signals via measurements w.r.t. an orthonormal basis?

Taking $N > d$ gives redundancy in measurements which is robust to noise or measurement loss!

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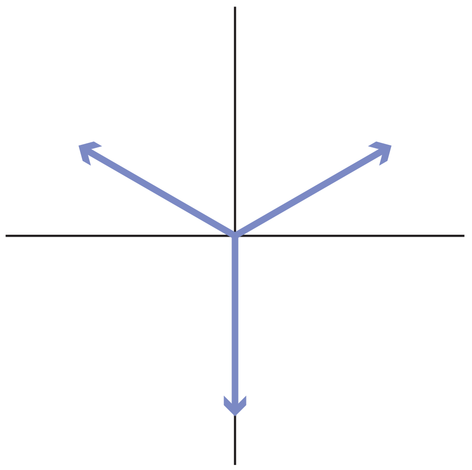

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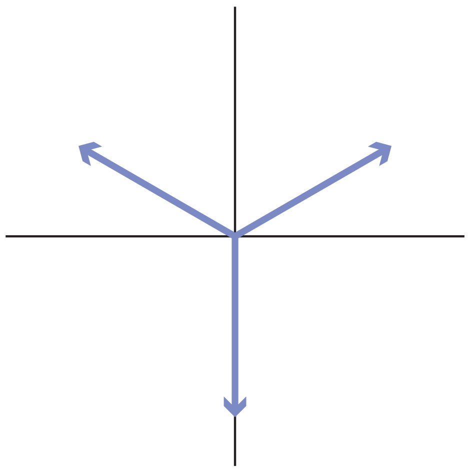

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"frame operator for F"

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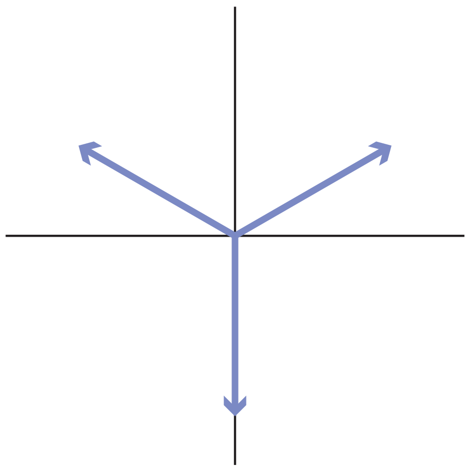

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Theorem (Casazza–Kovačevic, Goyal–Kovačevic–Kelner, Holmes–Paulsen). Among N-frames in $\mathbb{C}^d$, unit norm, tight frames give optimal reconstruction error under white noise or measurement erasures.

Unit norm, tight frames generalize orthonormal bases: $\mathrm{UNTF}(d,d) = \mathrm{U}(d)$

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The space of UNTFs

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Frame Homotopy Conjecture - Larson, '02: The space $\mathrm{UNTF}(d,N)$ is connected $\forall\, N \geq d \geq 1$.

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- $r = (r_1, \ldots, r_N) \in \mathbb{R}^N$ with $r_1 \geq r_2 \geq \ldots \geq r_N \geq 0$ is a collection of vector norms and
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frame operator can be tuned for “colored noise”

allows variable “measurement power”

Connectivity of Frame Spaces

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Proof Idea. Given $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0)$, the space

$$\left\{ F \in \mathbb{C}^{d\times N} \mid \mathrm{spec}(FF^*) = \lambda \right\} / \mathrm{U}(d)$$

has a natural symplectic structure (isomorphic to a complex flag manifold; Grassmannian if $\lambda = \mathbf{1}$).

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It has a Hamiltonian action by the torus $\mathrm{U}(1)^N$ (right multiplication) with momentum map

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Theorem (Atiyah '82). Level sets of momentum maps of torus actions are connected.

Connectivity of $\mathscr{F}(r, S)$, with $\mathrm{spec}(S) = \lambda$, follows easily from connectivity of $\mu^{-1}\left(-\frac{1}{2} \left(r_j^2 \right)_j \right)$.

Geometry of Frame Spaces

Theorem (N-Shonkwiler, '22). Given vectors of norms r and eigenvalues λ, the space

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is a smooth manifold $\Leftrightarrow$ $\nexists$ partitions $r = r' \sqcup r''$ and $\lambda = \lambda' \sqcup \lambda''$ with $r' \prec \lambda'$ and $r'' \prec \lambda''$.

For, $r = (r_1 \geq \cdots \geq r_N)$ and $\lambda = (\lambda_1 \geq \cdots \geq \lambda_d)$, write $r \prec \lambda$ if $\sum_{j=1}^{k} r_j \leq \sum_{j=1}^{k} \lambda_j \; \forall k = 1, \ldots, d$.

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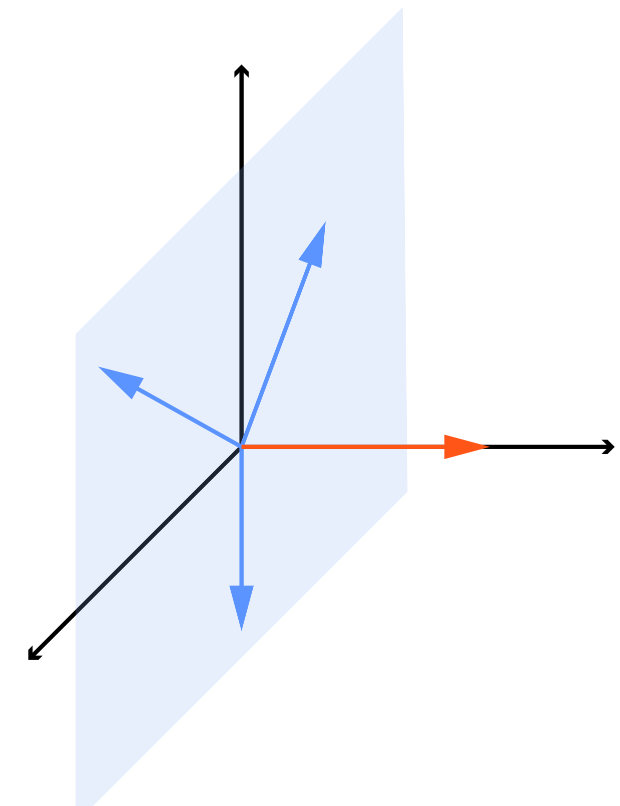
If it has singularities, they occur exactly at orthodecomposable frames, and singularities locally look like products of a quadratic cone and a manifold.

Description of singularities uses a result of Arms-Marsden-Moncrief '81.



Generalizes a result of Dykema-Strawn '06: The space $\mathrm{UNTF}(d, N)$ is a smooth manifold if d and N are relatively prime. Answers open questions of Cahill-Mixon-Strawn '17.

Random Matrix Theory Application

Rough idea of Compressed Sensing: “A random matrix $F \in \mathbb{C}^{d \times N}$ is good at compressing sparse vectors in $\mathbb{C}^N$, via $\mathbb{C}^N \ni v \mapsto Fv \in \mathbb{C}^d$, with high probability.”

Can the quantitative version of this statement be improved if we choose a random unit norm tight frame? Empirical evidence suggests that the answer is “yes”.

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A first step:

- We say a frame $F \in \mathbb{C}^{d \times N}$ is full spark if any choice of d columns is spanning.
- Question: What is the probability that a random UNTF is full spark?

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satisfies exactly one of three conditions:

- It is empty
- It is nonempty and contains only frames which are not full spark
- It is nonempty and full spark frames are a subset of full Hausdorff measure

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Proof Ingredients.

$$\left\{ F \in \mathbb{C}^{d \times N} \mid \|f_j\| = r_j \; \forall \, j \text{ and } \operatorname{spec}(FF^*) = \lambda \right\} / \left(\mathrm{U}(d) \times \mathrm{U}(1)^N \right)$$

is a symplectic manifold with Hamiltonian torus action whose momentum map takes the form

$[F] \mapsto (\mu_{jk})_{j,k}$, where μ_{jk} is the k^{th} eigenvalue of the partial frame operator $\displaystyle\sum_{\ell=1}^{j} f_\ell f_\ell^*$.

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The eigenvalues satisfy the Gelfand-Tsetlin pattern.

Defines a convex polytope whose Lebesgue measure can be used to compute Hausdorff measure on frame space (Duistermaat-Heckmann Theorem).

$$\begin{array}{ccccccccccc}
\lambda_1 & & \geq & & \lambda_2 & & \geq & & \lambda_3 & & \\
 & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & \rotatebox{-45}{$\geq$} & \\
 & & \mu_{31} & & \geq & & \mu_{32} & & \geq & & \mu_{33} \\
 & & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & \\
 & & & & \mu_{21} & & \geq & & \mu_{22} & & \\
 & & & & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & & \\
 & & & & & & \mu_{11} & & & &
\end{array}$$

$d = 3$ G-T pattern

Other Applications: Normal Matrices and Balancing Directed Graphs

Normal Matrices

A matrix $A \in \mathbb{C}^{d \times d}$ is normal if $AA^* = A^*A$.

Normal matrices the general setting for the Spectral Theorem

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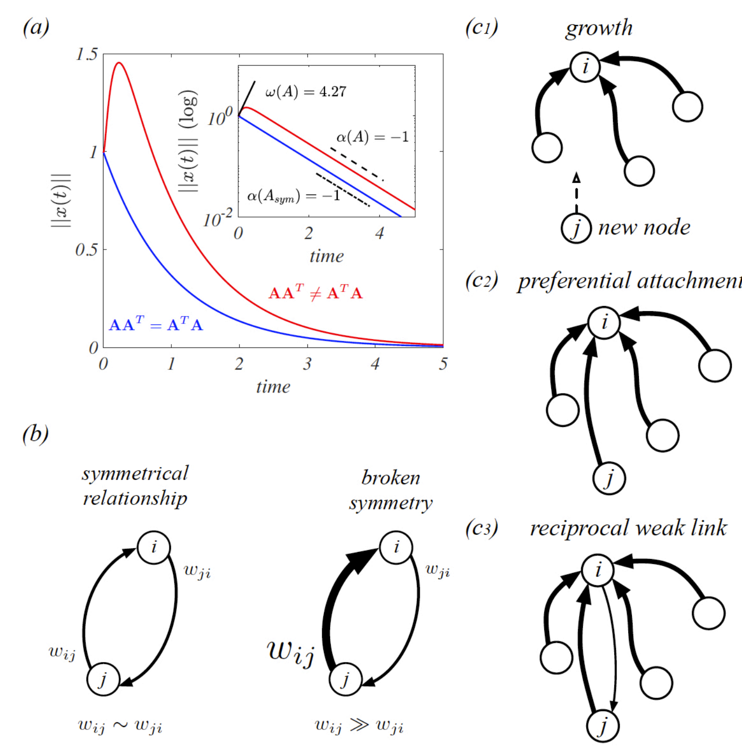
Normal matrices have spectra which are Lipschitz stable under perturbations [Bauer-Fike Theorem, 1960]

$\Rightarrow$ applications in control theory

Normality plays a role in dynamics on networks [Asllani-Carletti, 2018]

$\Rightarrow$ applications in mathematical biology

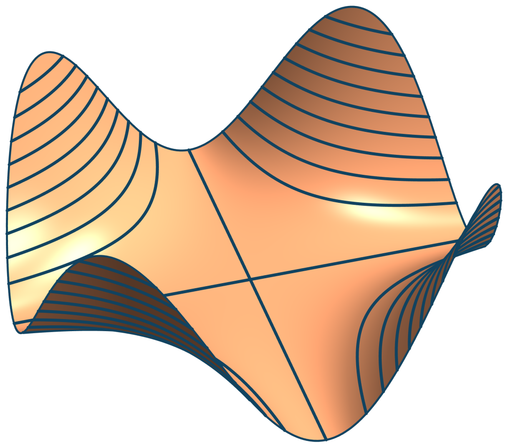


This motivates algorithms for finding the nearest normal matrix to a given $A \in \mathbb{C}^{d \times d}$.

Normal Matrices via Gradient Flow

A classical, natural measure of non-normality of a matrix A is $\mathrm{E}(A) := \|AA^* - A^*A\|_{\mathrm{Fro}}^2$.

The function $\mathrm{E} : \mathbb{C}^{d\times d} \to \mathbb{R}$ is not quasi-convex.



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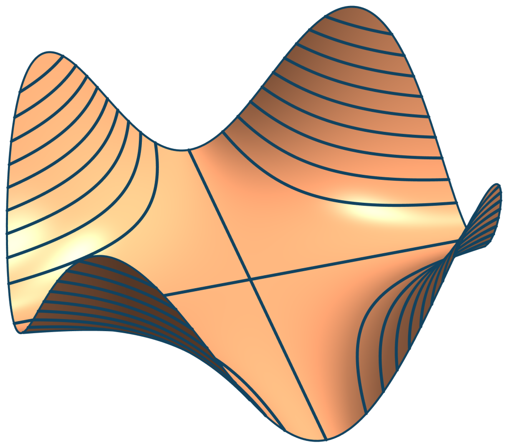

E is the norm squared of a momentum map for the action of $\mathrm{U}(d)$ on $\mathbb{C}^{d\times d}$ by conjugation. This class of functions has amazing gradient descent properties — see Kirwan '84, Lerman '05.

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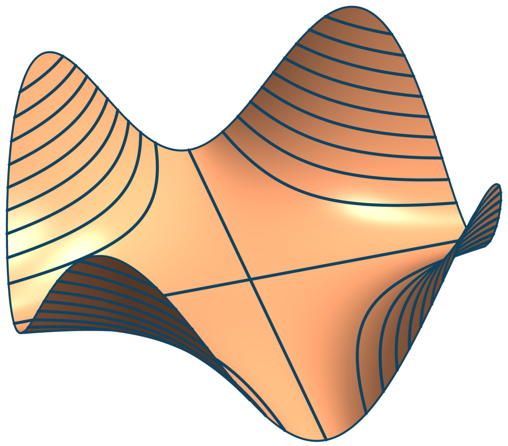
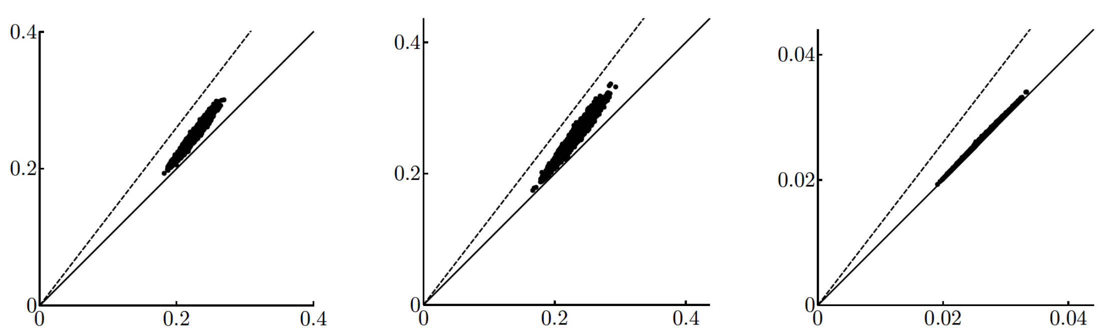


Let $A_0 \in \mathbb{C}^{d \times d}$ be an arbitrary matrix.

Theorem [N-Shonkwiler, '24]. Gradient descent of the functional $\mathrm{E} : A \mapsto \|AA^* - A^*A\|_{\mathrm{Fro}}^2$ converges to a normal matrix A_∞. If A_0 is real, then so is A_∞ and A_∞ has the same eigenvalues as A_0. Moreover, there exist $c, \epsilon > 0$ such that, if $\mathrm{E}(A_0) < \epsilon$ then $\|A_0 - A_\infty\|_{\mathrm{Fro}}^2 \leq c\sqrt{\mathrm{E}(A_0)}$.

This can be adapted to preserve total weight $\|A_0\|_{\mathrm{Fro}}^2$.



Topology of Unit Norm Normal Matrices

The space of normal matrices is contractible.

The space of unit norm normal matrices

$$\mathscr{U}\mathscr{N}_{\mathbb{F}}(d) = \left\{ A \in \mathbb{F}^{d \times d} \mid AA^* = A^*A \text{ and } \|A\|_{\mathrm{Fro}} = 1 \right\}, \ \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}$$

can have interesting topology.

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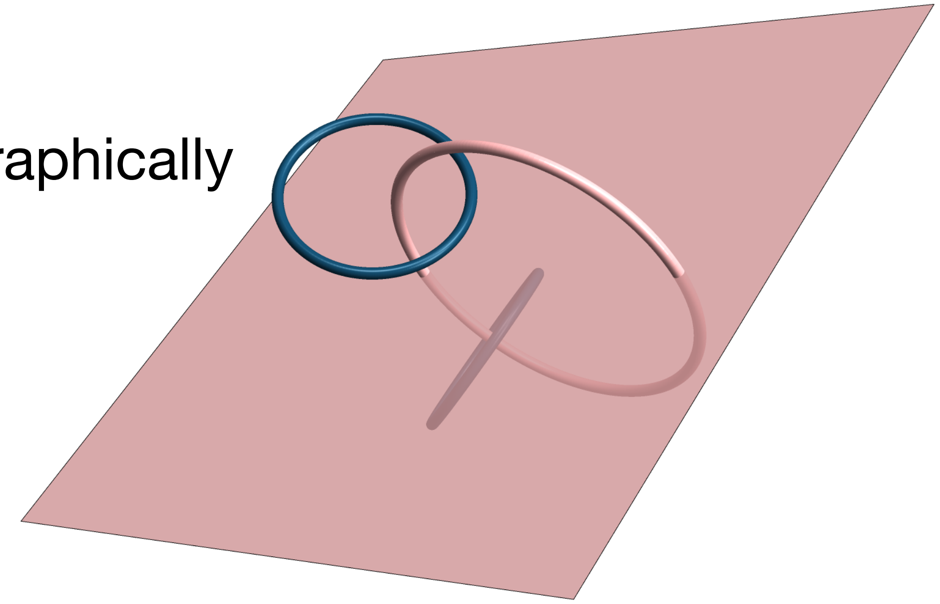
Example. $\{A \in \mathbb{R}^{2\times 2} \mid \|A\|_{\mathrm{Fro}} = 1\}$ stereographically projected to $\mathbb{R}^3$.

Image of unit norm nilpotent matrices in blue.

Image of $\mathscr{U}\mathscr{N}_{\mathbb{R}}(2)$ in pink.



Topology of Unit Norm Normal Matrices

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$$\mathcal{UN}_{\mathbb{F}}(d) = \left\{A \in \mathbb{F}^{d\times d} \mid AA^* = A^*A \text{ and } \|A\|_{\mathrm{Fro}} = 1\right\}, \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}$$

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Example. $\{A \in \mathbb{R}^{2\times 2} \mid \|A\|_{\mathrm{Fro}} = 1\}$ stereographically projected to $\mathbb{R}^3$.

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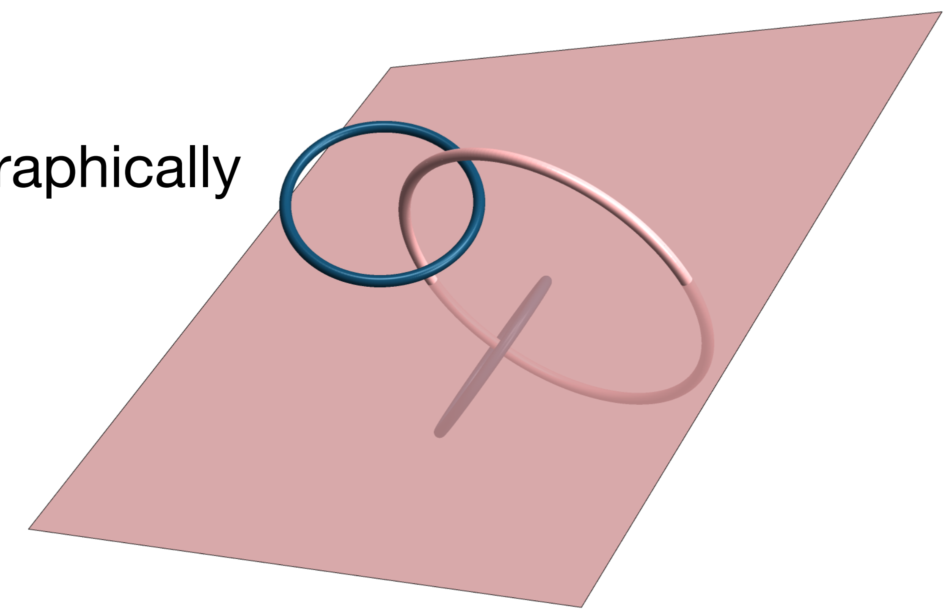


Theorem [N-Shonkwiler, '24].

- $\pi_k(\mathcal{UN}_{\mathbb{C}}(d))$ is trivial for all $k \leq 2d - 2$.
- $\pi_k(\mathcal{UN}_{\mathbb{R}}(d))$ is trivial for all $k \leq d - 2$.

Proof. $\mathcal{UN}_{\mathbb{F}}(d)$ is homotopy equivalent to $\{$non-nilpotent $d \times d$ matrices$\}$, via gradient descent of E.

The space of nilpotent matrices is a stratified space with high codimension strata. Use transversality.

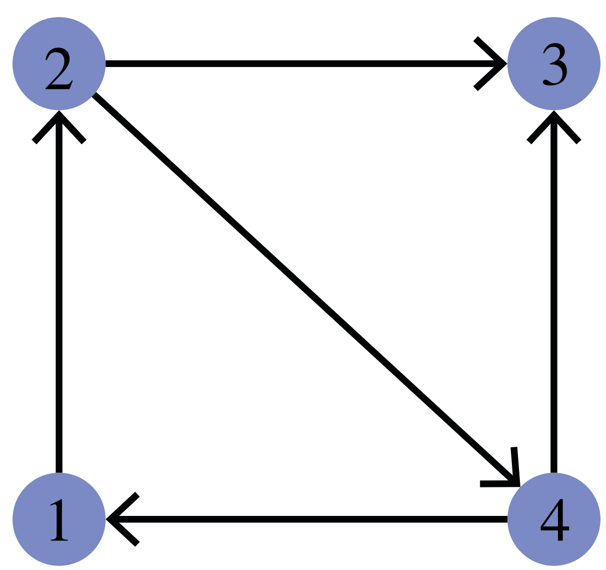
Balancing Digraphs

Let $A = (a_{ij})_{ij} \in \mathbb{R}^{d \times d}$ be the adjacency matrix of a weighted, directed graph.

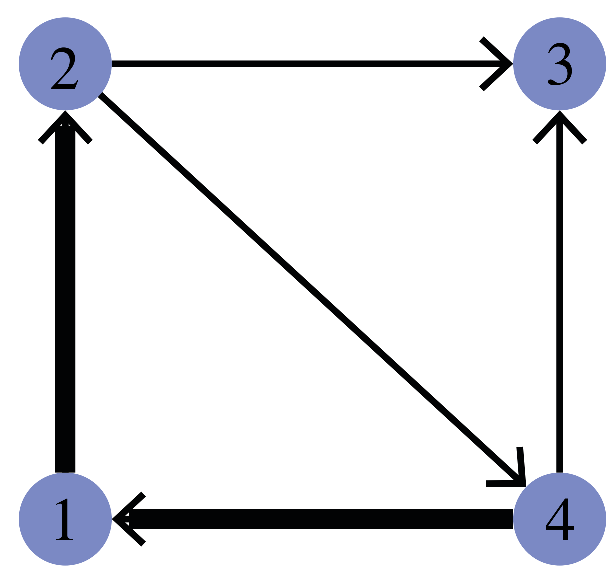
We say that the graph is balanced if

$$\sum_i a_{ik} = \sum_j a_{kj} \; \forall \; k$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



Unbalanced



$$\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix}$$

Balanced

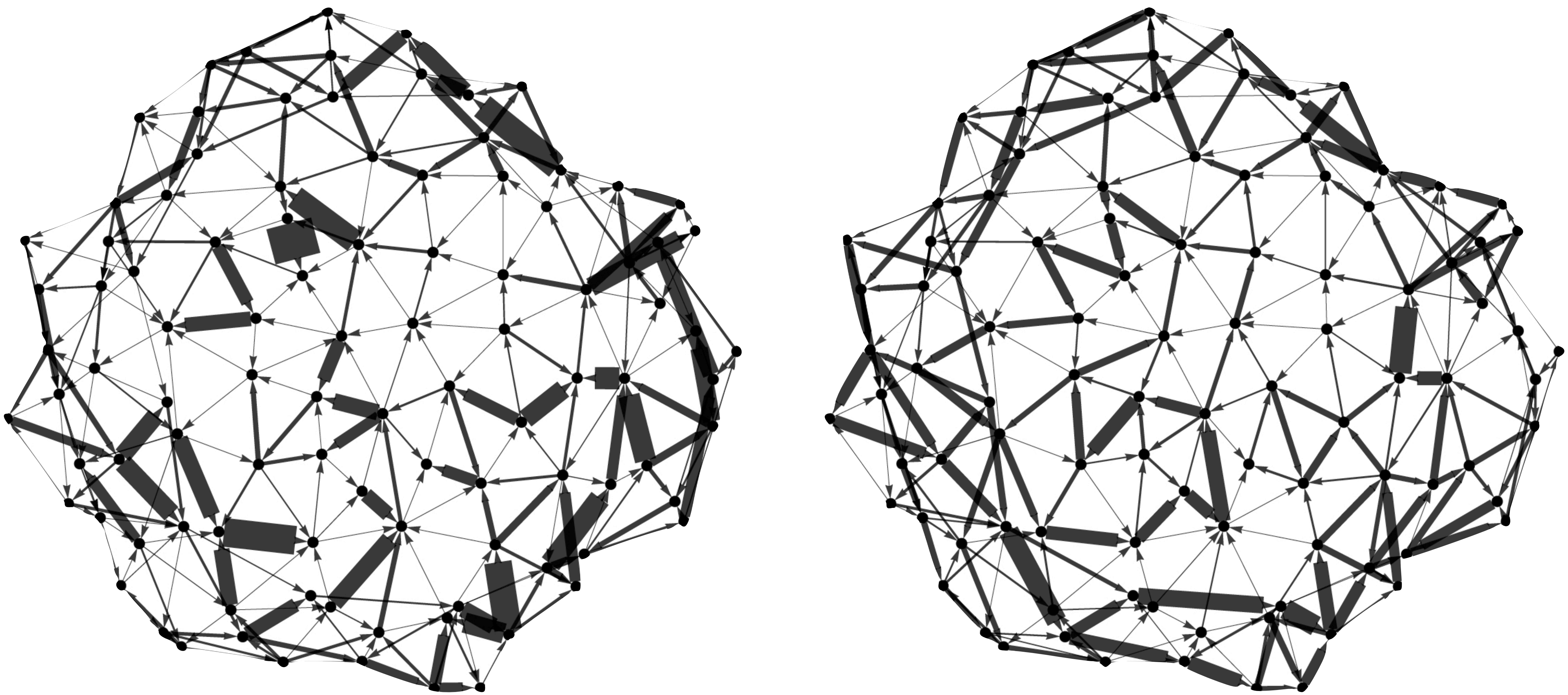
Balancing is necessary for, e.g., traffic flow problems [Hooi-Tong, 1970].

Balancing Graphs by Gradient Descent

Let $A_0 \in \mathbb{R}^{d \times d}$ be the entry-wise square of an adjacency matrix of a weighted digraph.

Theorem [N-Shonkwiler, '24]. Gradient descent of the functional $A \mapsto \|\mathrm{diag}(AA^* - A^*A)\|^2_{\mathrm{Frob}}$ converges to the entry-wise square of the adjacency matrix of a balanced digraph. It has the same eigenvalues and principal minors as A_0, and has zero entries whenever A_0 does.

This can be adapted to preserve total weight $\|A_0\|^2_{\mathrm{Frob}}$.



Thanks for Listening!

Open Questions:

- What about higher homotopy/(co)homology of $\mathscr{F}(r, S)$?
- What about the corresponding question for spaces of real frames?
- Can symplectic methods be applied to frames in infinite-dimensional Hilbert spaces?
- Can geometry of the Gelfand-Tsetlin polytope be used to get quantitative statements about compressed sensing properties of random frames?
- Can we efficiently generate random frames using Markov chain sampling in G-T polytope?

References:

Tom Needham and Clayton Shonkwiler, *Symplectic Geometry and Connectivity of Spaces of Frames*, Advances in Computational Mathematics, 2021.

Tom Needham and Clayton Shonkwiler, *Toric Symplectic Geometry and Full Spark Frames*, Applied and Computational Harmonic Analysis, 2022.

Tom Needham and Clayton Shonkwiler, *Geometric Approaches to Matrix Normalization and Graph Balancing*, arXiv preprint 2405.06190, 2024.

This research was supported by NSF DMS 2107808.