Fluids, diffeomorphisms, and shapes

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Table of contents

Arnold's setting for the Euler equation

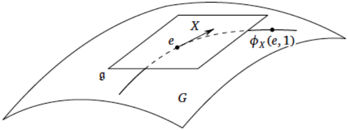
M — a Riemannian manifold with volume form μ
v — velocity field of an inviscid incompressible fluid filling M
The classical *Euler equation* (1757) on v:

$$\partial_t v + \nabla_v v = -\nabla p\,.$$

Here $\operatorname{div} v = 0$ and v is tangent to ∂M.
$\nabla_v v$ is the Riemannian covariant derivative.

Theorem (Arnold 1966)

The Euler equation is the geodesic flow on the group $G = \mathrm{Diff}_\mu(M)$ of volume-preserving diffeomorphisms w.r.t. the right-invariant L^2-metric $E(v) = \frac{1}{2}\int_M (v, v)\,\mu$ (fluid's kinetic energy).



Application: Other groups and energies

Group	Metric	Equation
$SO(3)$	$\langle \omega, A\omega \rangle$	Euler top
$E(3) = SO(3) \ltimes \mathbb{R}^3$	quadratic forms	Kirchhoff equation for a body in a fluid
$SO(n)$	Manakov's metrics	n-dimensional top
$\mathrm{Diff}(S^1)$	L^2	Hopf (or, inviscid Burgers) equation
$\mathrm{Diff}(S^1)$	$\dot{H}^{1/2}$	Constantin-Lax-Majda-type equation
Virasoro	L^2	KdV equation
Virasoro	H^1	Camassa–Holm equation
Virasoro	$\dot{H}^1$	Hunter–Saxton (or Dym) equation
$\mathrm{Diff}_\mu(M)$	L^2	Euler ideal fluid
$\mathrm{Diff}_\mu(M)$	H^1	averaged Euler flow
$\mathrm{Symp}_\omega(M)$	L^2	symplectic fluid
Diff(M)	L^2	EPDiff equation
$\mathrm{Diff}_\mu(M) \ltimes \mathrm{Vect}_\mu(M))$	$L^2 \oplus L^2$	magnetohydrodynamics
$C^\infty(S^1, SO(3))$	H^{-1}	Heisenberg magnetic chain

Remark These are Hamiltonian systems on $\mathfrak{g}^*$ with the quadratic Hamiltonian=kinetic energy for the Lie-Poisson bracket.

There are suitable functional-analytic settings of Sobolev (H^s for $s > 1 + n/2$) and tame Fréchet (C^∞) spaces.

Exterior geometry of $\mathrm{Diff}_\mu(M) \subset \mathrm{Diff}(M)$

$\mathrm{Dens}(M)$ — the *space of smooth density functions* ("probability densities") on M:

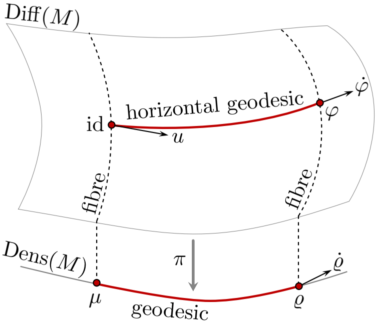
$$\mathrm{Dens}(M) = \{\rho \in C^\infty(M) \mid \rho > 0, \int_M \rho\mu = 1\}.$$

Note: $\mathrm{Dens}(M) = \mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$, the space of (left) cosets of $\mathrm{Diff}_\mu(M)$, with the projection $\pi : \mathrm{Diff}(M) \to \mathrm{Dens}(M)$.

For a density $\varrho := \rho\mu$ the fiber is $\pi^{-1}(\varrho) = \{\varphi \in \mathrm{Diff}(M) \mid \varphi_*\mu = \varrho\}$.

Define an *L^2-metric* on $\mathrm{Diff}(M)$ by $G_\varphi(\dot\varphi, \dot\varphi) = \int_M |\dot\varphi|^2_\varphi \mu$.

It is flat for a flat M.



The Euler geodesic property for a flat M

Let a flow $(t,x) \mapsto g(t,x)$ be defined by its velocity field $v(t,x)$:

$$\partial_t g(t,x) = v(t, g(t,x)),\ \ g(0,x) = x.$$

The chain rule immediately gives the acceleration

$$\partial^2_{tt} g(t,x) = (\partial_t v + \nabla_v v)(t, g(t,x)).$$

Geodesics on $\mathrm{Diff}(M)$ are straight lines, $\partial^2_{tt} g(t,x) = 0$, which is equivalent to the *Burgers equation*

$$\partial_t v + \nabla_v v = 0.$$

The Euler equation $\partial_t v + \nabla_v v = -\nabla p$ is equivalent to

$$\partial^2_{tt} g(t,x) = -(\nabla p)(t, g(t,x)),$$

which means that the acceleration $\partial^2_{tt} g \perp_{L^2} \mathrm{Diff}_\mu(M)$.

Hence the flow $g(t,.)$ is a geodesic on the submanifold $\mathrm{Diff}_\mu(M) \subset \mathrm{Diff}(M)$ for the L^2-metric.

Kantorovich-Wasserstein L^2-metric

Theorem (Otto 2000)

The left coset projection π is a Riemannian submersion with respect to the L^2-metric on $\mathrm{Diff}(M)$ and the Kantorovich-Wasserstein metric on $\mathrm{Dens}(M)$.

Definition of the Kantorovich-Wasserstein L^2-metric

The *KW distance* between $\mu, \nu \in \mathrm{Dens}(M)$:

$$\mathrm{Wass}^2(\mu,\nu) := \inf\{\int_M \mathrm{dist}_M^2(x,\varphi(x))\,\mu \mid \varphi_*\mu = \nu\}\,.$$

The corresponding *Riemannian metric* on $\mathrm{Dens}(M)$:

$$\bar{G}_\rho(\dot\rho,\dot\rho) = \int_M |\nabla\theta|^2 \rho\mu, \quad \text{for } \dot\rho + \mathrm{div}(\rho\nabla\theta) = 0,$$

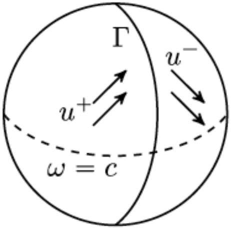
where $\dot\rho \in C_0^\infty(M)$ is a tangent vector to $\mathrm{Dens}(M)$ at the point $\rho\mu$.

The motion of vortex sheets

Flows with vortex sheets have jump discontinuities in the velocity (the velocity tangential component jumps, while the normal component is continuous).

The Euler equations for a fluid flow discontinuous along a vortex sheet $\Gamma \subset M$ are

$$\begin{cases} \partial_t u^+ + \nabla_{u^+} u^+ = -\nabla p^+, \\ \partial_t u^- + \nabla_{u^-} u^- = -\nabla p^-, \\ \partial_t \Gamma = u_{normal} \end{cases}$$



where $u = \chi_\Gamma^+ u^+ + \chi_\Gamma^- u^-$ is the fluid velocity, $\operatorname{div} u^\pm = 0$, u_{normal} is the normal to Γ component of u, while the pressure p satisfies $p^+|_\Gamma = p^-|_\Gamma$.

Vortex sheets as geodesics

Consider the space $\mathrm{VS}(M)$ of vortex sheets (of a given topological type) in M, i.e. the space of hypersurfaces which bound fixed volume in M. Define the following (weak) metric on $\mathrm{VS}(M)$. A tangent vector to a point $\Gamma \in \mathrm{VS}(M)$ can be regarded as a vector field v attached at the vortex sheet $\Gamma \subset M$ and normal to it. Then its square length is set to be

$$\langle\langle v, v\rangle\rangle_{\mathrm{vs}} := \inf \left\{ \langle u, u\rangle_{L^2(M)} \mid \mathrm{div}\, u = 0 \text{ and } (u, \nu)\,\nu = v \text{ on } \Gamma \right\}$$

where $\langle u, u\rangle_{L^2} := \int_M (u, u)\,\mu$ is the squared L^2-norm of a vector field u on M, and ν is the unit normal field to Γ.

Theorem (Loeschcke-Otto 2012)

Geodesics with respect to the metric $\langle\langle\ ,\ \rangle\rangle_{\mathrm{vs}}$ on the space $\mathrm{VS}(M)$ describe the motion of vortex sheets in an incompressible flow which is globally potential outside of the vortex sheet (i.e. $u^{\pm} = \nabla f^{\pm}$).

Question: How to unify Arnold's and Loeschcke-Otto's geodesic approaches?

Note: The metric $\langle\langle\ ,\ \rangle\rangle_{\mathrm{vs}}$ makes $\mathrm{VS}(M)$ into an interesting shape space!

Heuristics for the space of vortex sheets

Look at the same submersion picture with the following replacements: the projection

$$\mathrm{Diff}(M) \to \mathrm{Dens}(M), \quad \text{change to}$$

$$\mathrm{Diff}_\mu(M) \to \mathrm{VS}(M),$$

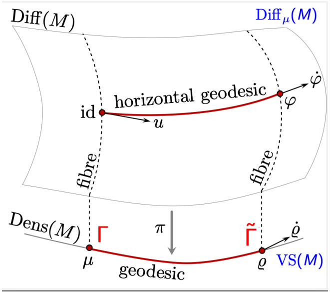
the fiber $\mathrm{Diff}_\mu(M)$ change to $\mathrm{Diff}_{\mu,\Gamma}(M) = \{\varphi \in \mathrm{Diff}_\mu(M) \mid \varphi(\Gamma) = \Gamma\}$.

"Theorem": The projection $\pi : \mathrm{Diff}_\mu(M) \to \mathrm{VS}(M)$ is a Riemannian submersion of the L^2-metric on $\mathrm{Diff}(M)$ to the metric $\langle\!\langle\ ,\ \rangle\!\rangle_{\mathrm{vs}}$ on $\mathrm{VS}(M)$.

"Proof": This is the definition of $\langle\!\langle\ ,\ \rangle\!\rangle_{\mathrm{vs}}$.

"Corollary": Arnold implies Loeschcke-Otto, as horizontal(=potential) geodesics project to geodesics on the base $\mathrm{VS}(M)$.

Problem: Fibers are L^2-dense in $\mathrm{Diff}_\mu(M)$.



Instead...

Instead...

From fluids to multiphase fluids...

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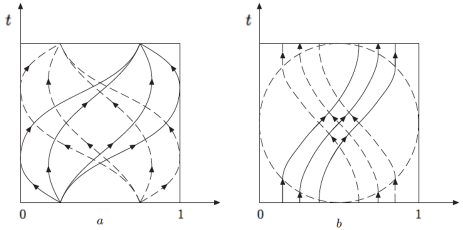
From fluids to multiphase fluids...

From groups to groupoids...

Multiphase flows and generalized flows

A multiphase fluid consists of several (or continuum of) fractions that can freely penetrate through each other without resistance, but are constrained by the conservation of the total volume form.

Example: *homogenized vortex sheets* or *generalized flows* by Y.Brenier.



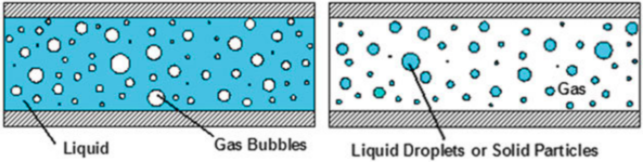
Trajectories of particles in one-dimensional generalized flows for
(a) continuum of phases for the flip of the interval [0, 1] and
(b) a two-phase flow for the interval-exchange map $[0, 1/2] \leftrightarrow [1/2, 1]$.
While a shortest curve on $\mathrm{Diff}_\mu(M)$ does not always exist (A.Shnirelman), it does in the class of generalized flows (Y.Brenier).

The Euler equation for multiphase flows

Multiphase flows on a mfd M are governed by the following equations:

$$\begin{cases} \partial_t u_j + \nabla_{u_j} u_j = -\nabla p\,, \\ \partial_t \mu_j + \operatorname{div}(\mu_j u_j) = 0\,. \end{cases}$$

Here $\mu_1, \ldots, \mu_n \in C^\infty(M)$ are mass densities of n phases of the fluid subject to the condition $\sum_{j=1}^n \mu_j = \mathrm{vol}_M$, the vector fields $u_1, \ldots, u_n \in \mathrm{vect}(M)$ are the corresponding fluid velocities, and the pressure $p \in C^\infty(M)$ is common for all phases.



Dispersed two-phase flows: gas bubbles (or liquid droplets) dispersed in a liquid or solid particles or droplets dispersed in gas.

Lie groupoids for multiphase fluids

What is the group-type structure behind such fluids? This is the "multiphase groupoid" $G \rightrightarrows B$, a pair of sets with two maps to base B, called the source and target maps, and a partial operation $(g, h) \mapsto gh$ on G defined for all pairs $g, h \in G$ such that $src(g) = trg(h)$, satisfying certain properties. For multiphase fluids base $B = \mathrm{MDens}(M)$ is

$$\mathrm{MDens}(M) = \{\bar{\mu} := (\mu_1, ..., \mu_n) \mid \mu_i \in \mathrm{Dens}(M),\ \mu_j > 0,\ \sum_j \mu_j = \mathrm{vol}_M\},$$

with $\int_M \mu_j = c_j$ for some fixed constants $c_j \in \mathbb{R}$.

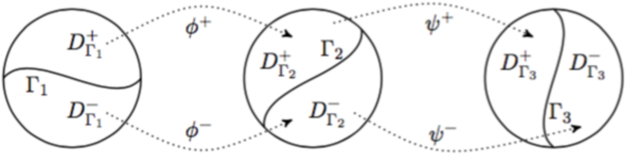
The Lie groupoid $G = \mathrm{MDiff}(M)$ consists of n-tuples of diffeomorphisms of M preserving the incompressibility property of multiphase densities, i.e. the set of tuples $(\bar{\phi}\,; \bar{\mu}, \bar{\mu}') := (\phi_1, ..., \phi_n; \mu_1, ..., \mu_n, \mu_1', ..., \mu_n')$ where $\bar{\phi}_* \bar{\mu} = \bar{\mu}'$ component-wisely.

The composition is given by composition of diffeomorphisms:

$$(\bar{\psi}\,; \bar{\mu}', \bar{\mu}'')(\bar{\phi}\,; \bar{\mu}, \bar{\mu}') := (\bar{\psi}\bar{\phi}\,; \bar{\mu}, \bar{\mu}'').$$

Example: Classical vortex sheets

Classical vortex sheets are a particular case of multiphase fluids where the densities are indicator functions of the connected components separated by a hypersurface in M. Then the multiphase Lie groupoid becomes the groupoid of volume-preserving diffeomorphisms of M discontinuous along a hypersurface. Its elements are quadruples $(\Gamma_1, \Gamma_2, \phi^+, \phi^-)$, where Γ_1, Γ_2 are hypersurfaces (vortex sheets) in M confining the same total volume, while $\phi^\pm \colon D^\pm_{\Gamma_1} \to D^\pm_{\Gamma_2}$ are volume preserving diffeomorphisms between connected components of $M \setminus \Gamma_i$. The multiplication of the quadruples is given by the natural composition of discontinuous diffeomorphisms:



Lie algebroid for multiphase fluids

What is the space of infinitesimal objects?

The corresponding Lie algebroid $\mathrm{Mvect}(M)$ is the space of possible velocities of the multiphase fluid. It is a vector bundle over $\mathrm{MDens}(M)$ where the fiber of $\mathrm{Mvect}(M)$ over a multiphase density $\bar{\mu} \in \mathrm{MDens}(M)$ is the space of *multiphase vector fields* on M "divergence-free" with respect to the multiphase volume form $\bar{\mu}$, i.e. vector fields of the form $\bar{u} := (u_1, ..., u_n)$, where $u_i \in \mathrm{Vect}(M)$ are such that $\sum_j \mathcal{L}_{u_j} \mu_j = 0$.

Example

a) The case $n = 1$ gives an incompressible fluid in M.
b) The case of indicator densities $\mu^\pm$ on $D_\Gamma^\pm$ corresponds to classical vortex sheets. Note that the velocity fields on $D_\Gamma^\pm$ have the same normal component on Γ ("impermeability" of Γ).

The multiphase Euler equation as a geodesic flow

Theorem (A.Izosimov-B.K.)

The Euler equations

$$\begin{cases} \partial_t u_j + \nabla_{u_j} u_j = -\nabla p\,, \\ \partial_t \mu_j + \mathrm{div}(\mu_j u_j) = 0\,. \end{cases}$$

for a multiphase fluid flow are groupoid Euler-Arnold equations corresponding to the L^2-metric on the algebroid $\mathrm{MVect}(M)$. Equivalently, these Euler equations are a geodesic equation for the right-invariant L^2-metric on (source fibers of) the Lie groupoid $\mathrm{MDiff}(M)$ of multiphase volume-preserving diffeomorphisms.

For the case of a flat space M the geodesic nature of homogenized vortex sheets was established by C.Loeschcke (2012).
The standard Euler hydrodynamical is a particular case of the above equations with only one phase, $n = 1$.

Hamiltonian framework and continuum of phases

Furthermore, these equations allow a Hamiltonian framework, an analogue of the Hamiltonian property of the Euler-Arnold equation on the dual to a Lie algebra with respect to the Lie-Poisson structure:

Theorem

The Euler equations for a multiphase flow written on the dual $\mathrm{MVect}(M)^*$ of the algebroid are Hamiltonian with respect to the natural Poisson structure on the dual algebroid and the Hamiltonian function given by the L^2 kinetic energy.

Generalized flows

The above extends *mutatis mutandis* to a “continuous” index i, i.e. to multiphase flows where phases (fractions of the fluid) are enumerated by a continuous parameter $a \in A$ in a measure space A. This provides the geodesic and Hamiltonian frameworks for generalized flows of Y.Brenier.
Generalized flows satisfy equations

$$\begin{cases} \partial_t(\mu_a u_a) + \operatorname{div}(\mu_a u_a \otimes u_a) + \mu_a \nabla p = 0\,, \\ \partial_t \mu_a + \operatorname{div}(\mu_a u_a) = 0\,, \end{cases}$$

on the fraction velocities $u_a \in \mathrm{Vect}(M)$, along with the constraint $\int_A \mu_a\, da = 1$ on the fraction densities $\mu_a \in C^\infty(M)$. The pressure function $p \in C^\infty(M)$ is common for all fractions.

Remark. An equivalent form is $\partial_t u_a + \nabla_{u_a} u_a = -\nabla p$, while condition $\operatorname{div}(\int_A \mu_a u_a da) = 0$ is an analog of $\operatorname{div} u = 0$ for the classical Euler equation.

Multiphase Hodge decomposition

Given a multiphase density $\bar\mu = (\mu_1, ..., \mu_n)$ we introduce the (weighted) L^2 inner product: on a Riemannian M:

$$\langle \bar u, \bar u \rangle_{\bar\mu} := \int_M \sum_i (u_i, u_i)\mu_i \,.$$

Recall that the multifield $\bar u = (u_1, ..., u_n)$ is $\bar\mu$-div-free if $L_{\bar u}\bar\mu = 0$.

Theorem

There is a generalized Hodge decomposition: given a multi-density $\bar\mu$, any multiphase vector field $\bar v$ admits a unique L^2-orthogonal decomposition $\bar v = \bar u \oplus_{L^2} \bar\nabla f$, where $f \in C^\infty(M)$, $\bar\nabla f := (\nabla f, ..., \nabla f)$, and $\bar u$ is $\bar\mu$-div-free.

Proof. Find f as a solution of the Poisson equation $\Delta_{\bar\mu} f = \mathrm{div}_{\bar\mu} \bar v$ (i.e. $\sum_i \mathrm{div}_{\mu_i} \nabla f = \sum_i \mathrm{div}_{\mu_i} v_i$). Then $\bar u := \bar v - \bar\nabla f$ is $\bar\mu$-div-free and $\bar\nabla f \perp_{L^2} \bar u$, since
$\langle \bar\nabla f, \bar u \rangle_{\bar\mu} = \int_M \sum_i (\nabla f, u_i)\mu_i = \int_M \sum_i f(\mathrm{div}_{\mu_i} u_i)\mu_i = \int_M f\, L_{\bar u}\bar\mu = 0.$

Computation of geodesics

Adapt the above computation above to multiphase fluids. Let a multi-flow $(t, x_i) \mapsto g_i(t, x_i)$ be defined by its velocity field $v_i(t, x_i)$:

$$\partial_t g_i(t, x_i) = v_i(t, g_i(t, x_i)),\ \ g_i(0, x_i) = x_i.$$

The same chain rule gives the acceleration

$$\partial^2_{tt}\bar{g}(t, \bar{x}) = (\partial_t \bar{v} + \nabla_{\bar{v}}\bar{v})(t, \bar{g}(t, \bar{x})).$$

Again, the geodesics on $\mathrm{Diff}^{\times n}(M)$ are straight lines, and $\partial^2_{tt}\bar{g}(t, \bar{x}) = 0$ is equivalent to the multi-Burgers equation $\partial_t \bar{v} + \nabla_{\bar{v}}\bar{v} = 0$.

Now the multi-phase Euler equation $\partial_t \bar{v} + \nabla_{\bar{v}}\bar{v} = -\bar{\nabla} p$ is equivalent to

$$\partial^2_{tt}\bar{g}(t, \bar{x}) = -(\bar{\nabla} p)(t, \bar{g}(t, \bar{x})),$$

or the orthogonality of the acceleration $\partial^2_{tt}\bar{g} \perp_{L^2} \mathrm{MDiff}_{\bar{\mu}}(M)$ in the multiphase Hodge decomposition. Hence the flow $\bar{g}(t, .)$ is a geodesic on the submanifold $\mathrm{MDiff}_{\mu}(M) \subset \mathrm{Diff}^{\times n}(M)$.

Vorticity metric on vortex sheets

Recall the metric on $\mathrm{VS}(M)$. A tangent vector to a point $\Gamma \in \mathrm{VS}(M)$ is a vector field v attached and normal to the vortex sheet $\Gamma \subset M$. Then

$$\langle\langle v, v \rangle\rangle_{\mathrm{vs}} := \inf \left\{ \int_M (u, u)\,\mu \mid \mathrm{div}\, u = 0 \text{ and } (u, \nu)\,\nu = v \text{ on } \Gamma \right\}$$

where is a vector field u on M, and ν is the unit normal field to Γ.

Theorem

Consider the vortex sheet algebroid $\mathrm{DVect}(M) \to \mathrm{VS}(M)$, equipped with the L^2-metric. Then vortex sheets in potential flows evolve along geodesics of a metric $\langle\langle v, v \rangle\rangle_{\mathrm{vs}}$ on $\mathrm{VS}(M)$ obtained as the projection of the L^2-metric on the algebroid $\mathrm{DVect}(M)$ to the base $\mathrm{VS}(M)$, the shape space of diffeomorphic hypersurfaces bounding the same volume.

Explicitly, since the infimum of $\int_M (u, u)\,\mu$ is attained on gradient vector fields $u^{\pm} = \nabla f^{\pm}$,...

The vorticity metric explicitly is ...

Corollary

The vorticity metric $\langle\!\langle v, v\rangle\!\rangle_{\mathrm{vs}}$ *on* $\mathrm{VS}(M)$ *has the following explicit expression via the solution of the Neumann problem:*

$$\langle\!\langle v, v\rangle\!\rangle_{\mathrm{vs}} = \int_{D_\Gamma^+} (\nabla f^+, \nabla f^+)\,\mu + \int_{D_\Gamma^-} (\nabla f^-, \nabla f^-)\,\mu\,,$$

where $\Delta f^\pm = 0$ *in* $D_\Gamma^\pm$ *and the normal component of* $\nabla f^\pm$ *at* Γ *is* v. *Equivalently,*

$$\langle\!\langle v, v\rangle\!\rangle_{\mathrm{vs}} = \langle(\mathrm{NtD}^+ + \mathrm{NtD}^-)v, v\rangle_{L^2(\Gamma)}\,,$$

where $\mathrm{NtD}^\pm$ *are the Neumann-to-Dirichlet operators on the domains* $D_\Gamma^\pm$.

Properties of the vorticity metric

Remark 1. This vorticity metric $\langle\langle v, v\rangle\rangle_{\mathrm{vs}}$ is non-local in terms of v and it is $H^{-1/2}$-like (and unlike various H^s-types with $s \geq 0$).

Indeed, the reconstruction of a harmonic potential from the normal derivative data requires an integration of the fundamental solution in M against the boundary data over Γ, hence nonlocality. The Neumann-to-Dirichlet operators $\mathrm{NtD}^{\pm}$ have order -1 as pseudo-differential operators on the boundary Γ, hence the metric $\langle\langle v, v\rangle\rangle_{\mathrm{vs}}$ is $H^{-1/2}$-like. A simpler version is $\langle\langle v, v\rangle\rangle'_{\mathrm{vs}} := \langle \mathrm{NtD}^{+} v, v\rangle_{L^2(\Gamma)}$.

Remark 2. Regarding shapes $\Gamma = \partial D_\Gamma$ as measures μ_Γ supported on $D_\Gamma := D_\Gamma^{+} \subset M$ one can define the Wasserstein distance between the shapes. Then

$$\mathrm{Wass}(\mu_\Gamma, \mu_{\tilde\Gamma}) \leq \mathrm{Dist}_{\mathrm{vs}}(\Gamma, \tilde\Gamma)\,.$$

Indeed, in both cases one takes the L^2-norm of the vector fields moving the shape/mass, but in the Wasserstein distance one minimizes over all, not necessarily volume-preserving, diffeomorphisms of M.

Open questions: Multiphase fluids and beyond

0) Study properties of this vorticity metric: curvatures, relation to instability of vortex sheets, relation to water waves, etc.

1) Describe the groupoid geometry of barotropic multiphase fluids.

2) Vector densities are usually described by an n-tuple of densities on which an n-tuple of diffeomorphisms acts, and there are coefficients for mass exchanges between the components. Is the groupoid framework a natural setting for optimal transport of vector densities?

3) Develop the H^1 geometry of multi-phase fluids or vector-valued information geometry.

4) Vector Madelung and multiphase fluids; their symplectic, Kähler, and momentum map properties in the groupoid setting.

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THANK YOU!