Geometric Sciences in Action : from geometric statistics to shape analysis (CIRM)

Experimental presentation

Part I. IA vs. humans : what shape space for the mind ?

Part II. Partial matchings of curves and surfaces.

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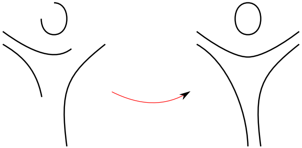
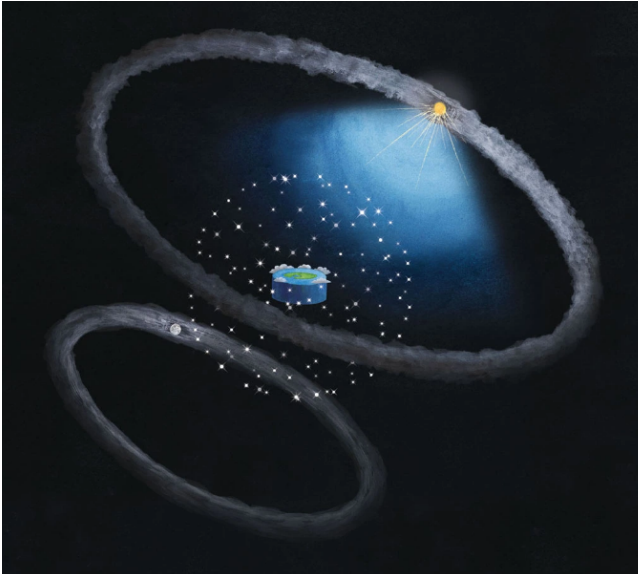
MAP5, Université Paris Cité

Part II is j.w. Pierre-Louis Antonsanti and Joan Glaunès (MAP5)



May 2024

Anaximander



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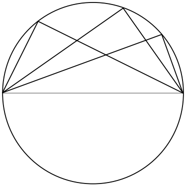
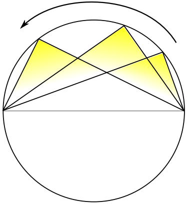
Speculations on Anaximander's intuitions

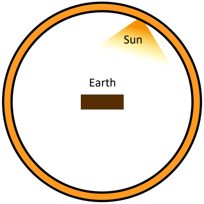
Speculations on Anaximander's intuitions

Thales

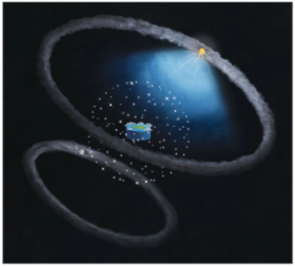


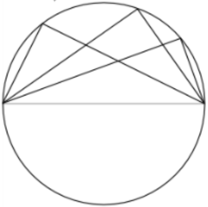


Anaximander's cosmos



Speculations on Anaximander's intuitions





From Leibniz to Gödel (and Turing, von Neumann, Shannon, etc.)

If controversies were to arise, there would be **no more need of disputation** between two philosophers than between two accountants. For it would suffice to take their pens in their hands, to sit down to their desks, and to say to each other (...), 'Let us calculate.'

Leibniz, *Nova methodus pro maximis et minimis*
in Acta Eruditorum, 1684

Über formal unentscheidbare Sätze der Principia Mathematica etc.

„0" . . . 1	„∨" . . . 7	„(" . . . 11
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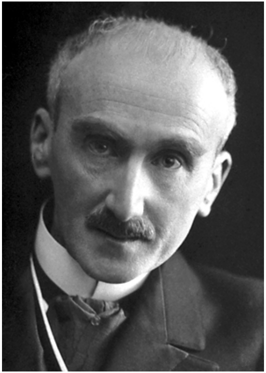
Gödel, *On Formally Undecidable Propositions of Principia Mathematica and Related Systems*, 1931

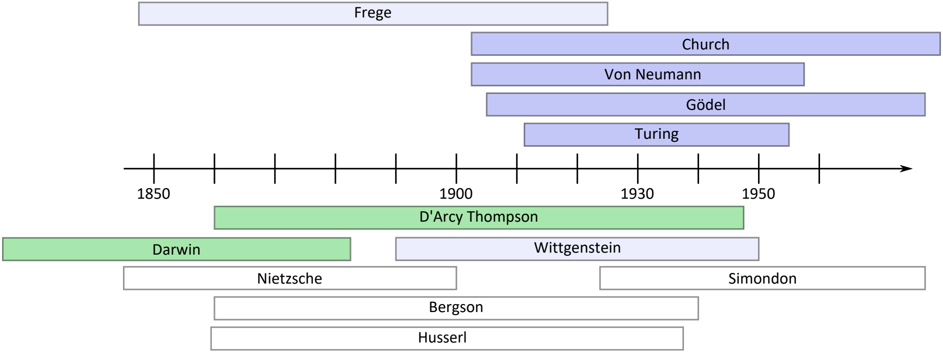
Bergson (1859-1941)



Individuation, retentions, interpretation

- primary retentions : perceptions retained within the stream of consciousness.
- secondary retentions : memories
- tertiary retentions : collective memory retained within practices and technical objects.

Nietzsche : living beings and their confrontation with the world

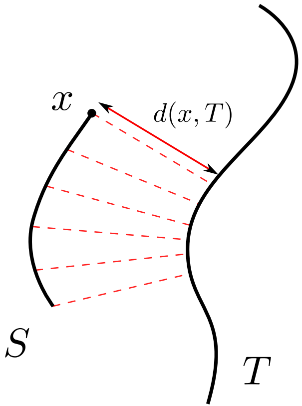


Outline

What shape space for the mind ?

Partial matchings

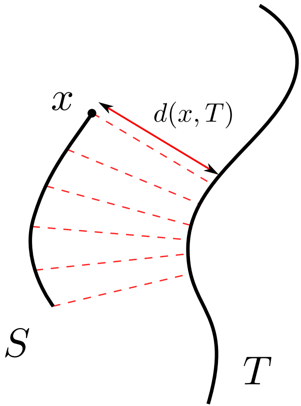
A mathematical digestion



$$||\mu_S - \mu_T||^2_{W'}$$

Nearest neighbor projection vs. Varifold metrics

A mathematical digestion



$$||\mu_S - \mu_T||^2_{W'}$$

$$\mu_S, \mu_T \in W'$$

$$\omega_S, \omega_T \in W$$

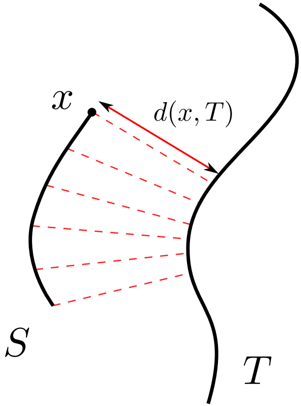
$$\langle \mu_S, \mu_T \rangle_{W'} =$$

$$\int_S \omega_T(\vec{\mathbf{x}})dx$$

where $\vec{\mathbf{x}} = (x, \vec{t}_S(x))$

Nearest neighbor projection vs. Varifold metrics

Heuristic



Like a nearest-neighbor projection formula

$$\text{data term}(S,T) = \int_S d(x,T)dx$$

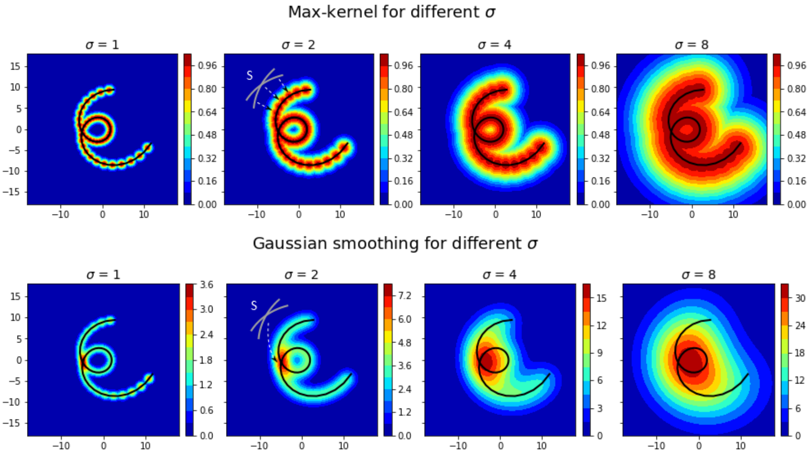
we aim to define a term of the type

$$\text{data term}(S,T) = \int_S 1 - \text{similarity}(x,T)dx$$

Kernel-Chamfer map

Max-kernel map :

$$\mathrm{kmax}(x, T) = \max_{y \in T} k(x, y) = \max_{y \in T} \exp^{-\frac{\|x-y\|^2_{\mathbb{R}^n}}{\sigma^2}} = \exp^{-\frac{\min_{y \in T} \|x-y\|^2_{\mathbb{R}^n}}{\sigma^2}} .$$

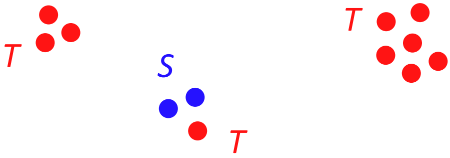


Weighted kernel-Chamfer maps

If $x \in S$ has a local mass strictly greater than that of $y \in T$, i.e.

$$\omega_S(x) > \omega_T(y),$$

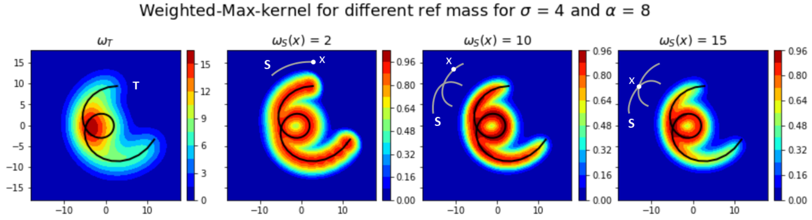
then x should not be attracted by y.



We thus replace $\text{kmax}(x, T) = \max_{y \in T} k(x, y)$ by

$$\text{kmax}(x, T) = \max_{y \in T} k(x, y) \min\left(1, \frac{\omega_T(y)}{\omega_S(x)}\right) .$$

Weighted kernel-Chamfer maps



$$\mathrm{kmax}(x, T) = \max_{y \in T} k(x, y) \min\left(1, \frac{\omega_T(y)}{\omega_S(x)}\right) .$$

From point clouds to varifolds

$$\mathrm{kmax}(x, T) = \max_{y \in T} k(x, y) \min\left(1, \frac{\omega_T(y)}{\omega_S(x)}\right)$$

characterizes the proximity of a point x to the shape T potentially penalized by a weight criterion.

Likewise,

$$\mathrm{kmax}(\vec{\mathbf{x}}, T) = \max_{y \in T} k(\vec{\mathbf{x}}, \vec{\mathbf{y}}) \min\left(1, \frac{\omega_T(\vec{\mathbf{y}})}{\omega_S(\vec{\mathbf{x}})}\right) .$$

characterizes the proximity of a pair $\vec{\mathbf{x}} = (x, \vec{t}_S(x))$ of a point x and a tangent/normal vector $\vec{t}_S(x)$ to the shape T. The weight criterion also integrates the tangential data.

Partial matching problem

A deformation model for partial matching can be defined by

$$\min_{\Phi} \quad \text{cost}(\Phi) + \text{data term}(\Phi(S), T)$$

with

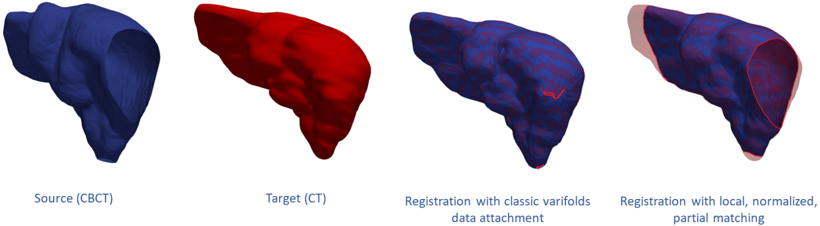
$$\text{data term}(S, T) = \frac{1}{vol(S)} \int_S \Big(1 - \text{kmax}(\vec{\mathbf{x}}, T)\Big)\, dx\,,$$

where $\vec{\mathbf{x}} = (x, \vec{t}_S(x))$ and

$$\text{kmax}(\vec{\mathbf{x}}, T) = \underset{y \in T}{max}\; k(\vec{\mathbf{x}}, \vec{\mathbf{y}}) \min\left(1, \frac{\omega_T(\vec{\mathbf{y}})}{\omega_S(\vec{\mathbf{x}})}\right)\,.$$

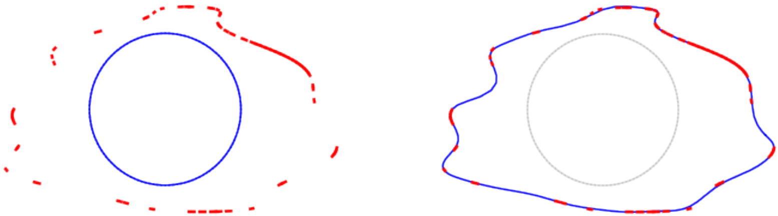
The constant 1 appears under the assumption that $k(y, y) = 1$ for all y.

Surface partial matching



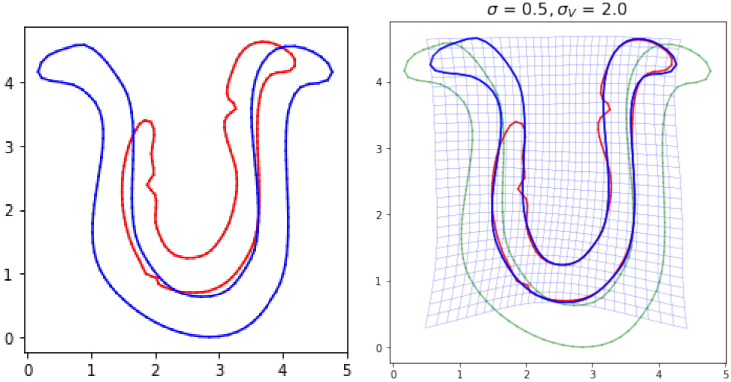
Covering partial matching

Covering matching with locally heterogeneous sampling of the target

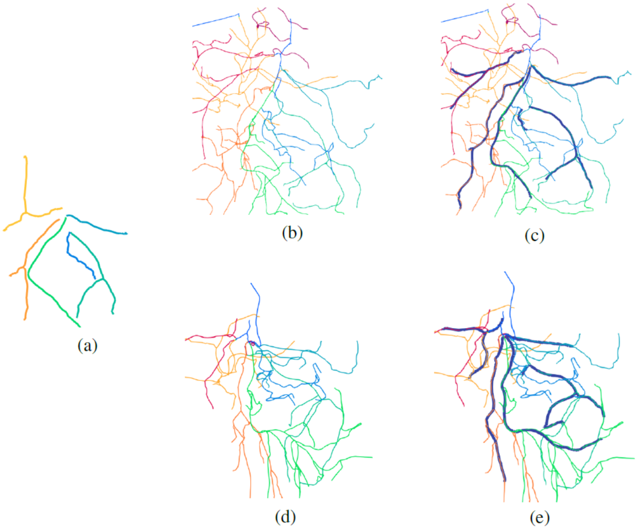


$$\text{data term}(S, T) = \frac{1}{vol(T)} \int_T \left(1 - \text{kmax}(\vec{\mathbf{y}}, S)\right) dy$$

Partial matching on jaws

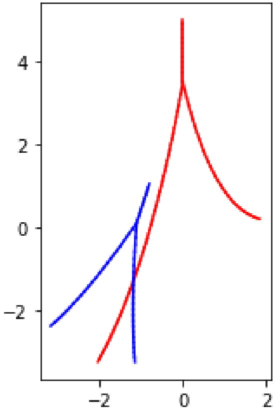


Partial matching on vascular trees



a) Atlas, b,d) Targets, c,e) Partial registration

Exploring the limits of the method



First results

Settings :

- LDDMM :

$$\gamma \,\text{cost}(\Phi) + \text{data term}(\Phi(S), T)$$

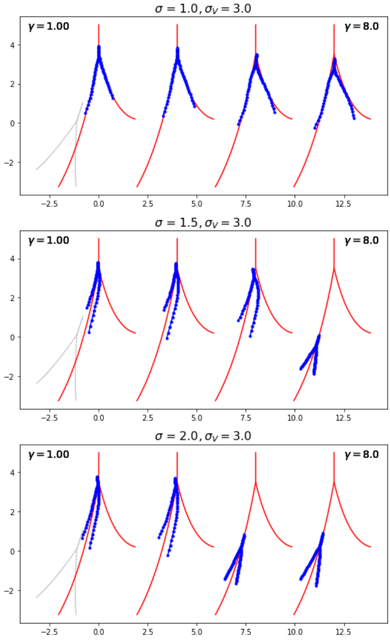
where cost depends on a Gaussian kernel of scale σ_V.

- data term$(S, T) =$

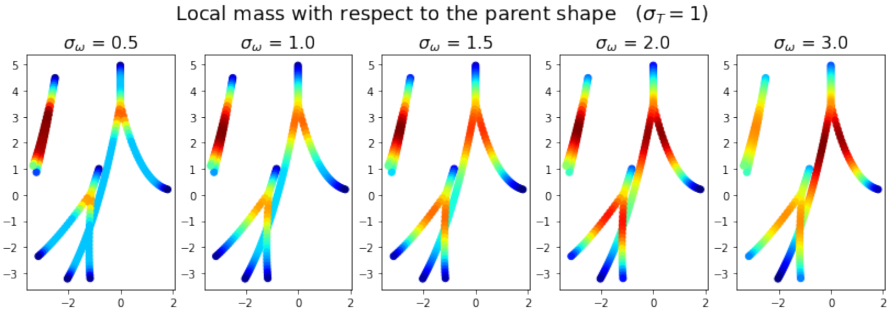
$$\frac{1}{vol(S)} \int_S \Big(1 - \text{kmax}(\vec{x}, T)\Big)\, dx \;, \text{ with}$$

$$\text{kmax}(\vec{x}, T) = \max_{y \in T} k(\vec{x}, \vec{y}) \min\left(1, \frac{\omega_T(\vec{y})}{\omega_S(\vec{x})}\right) .$$

Main kernel :
k_{pos} : Cauchy kernel of scale σ
k_{or} : Gaussian kernel of scale $\sigma_T = 1$
Local mass estimator :
k_{pos} : Gaussian kernel of scale $\sigma_\omega = \sigma$
k_{or} : Gaussian kernel of scale $\sigma_T = 1$



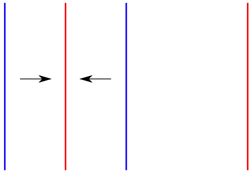
What's the signal ?



The local mass $x \in S \mapsto \omega_S(\vec{\mathbf{x}})$ on the source shape,
$y \in T \mapsto \omega_T(\vec{\mathbf{y}})$ on the target shape,
and $x \in \tilde{S} \mapsto \omega_{\tilde{S}}(\vec{\mathbf{x}})$ on a crushed version of the source shape.

(i) For a large σ, ω_S and ω_T are not discriminating enough to characterize the local mass variation because ω_T is inflated by the entire mass of T .
(ii) The smaller the scale, the less the mass criterion anticipates collusion : it cannot see if two separated branches are converging to the same target branch.

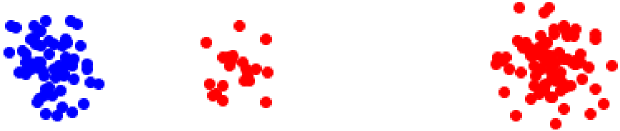
Multi-scale mass criterion



$$\min\left(1, \frac{\omega_T(y)}{\omega_S(x)}, \frac{\omega_T(y)}{\omega_S(x)}, \frac{\omega_T(y)}{\omega_S(x)}\right)$$

e.g. with $\sigma_\omega = [0.5, 1, 2]\,\sigma$.

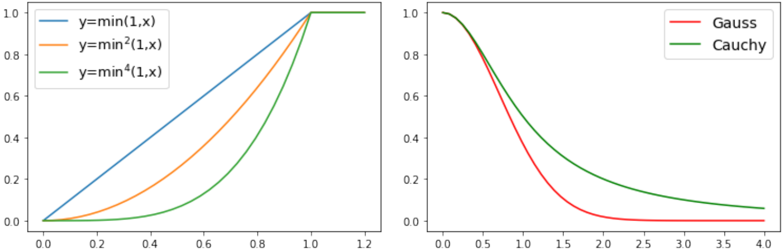
About the mass criterion



About the mass criterion

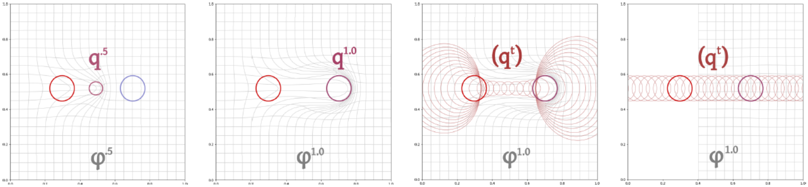
$$\text{data term}(S,T) = \frac{1}{vol(S)} \int_S 1 - \max_{y \in T} k(x,y) \min\left(1, \frac{\omega_T(y)}{\omega_S(x)}\right) dx$$

Alternative options : consider $\min^2$, $\min^3$ or more restrictive criteria.



It may also prevent $\omega_S(x)$ to decrease so that $\min\left(1, \frac{\omega_T(y)}{\omega_S(x)}\right)$ tends to 1.

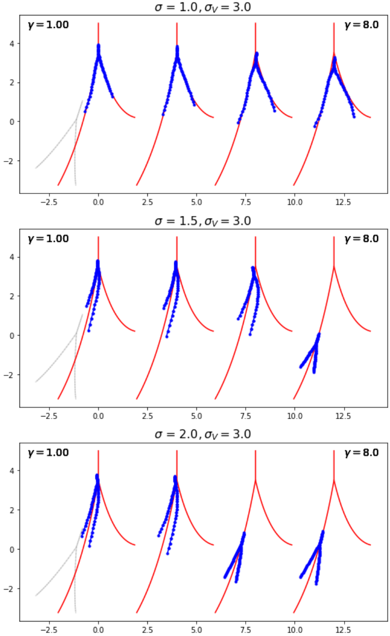
Normalized LDDMM



(a) $t = .5$. (b) $t = 1.0$. (c) $t \in [-1, 2]$. (d) Normalized.

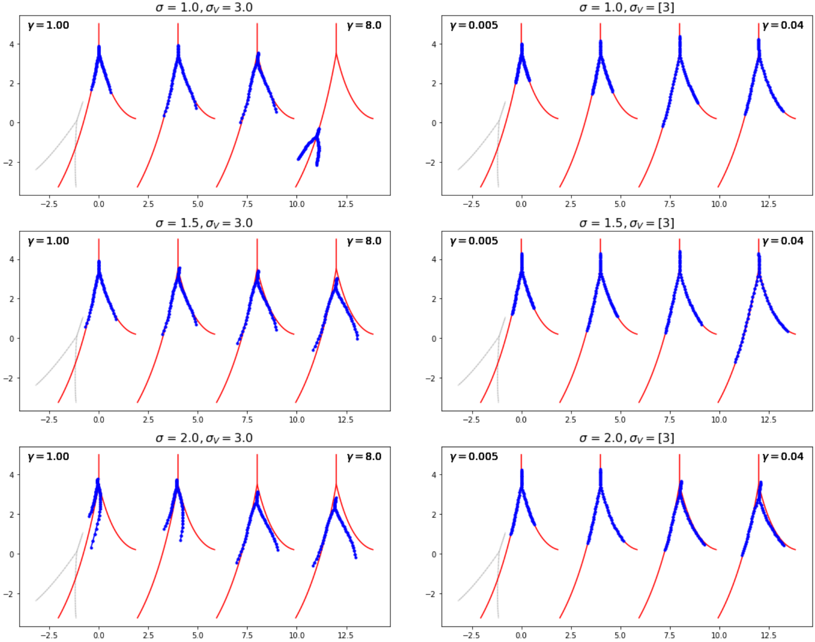
Work from Feydy and Trouvé (PhD thesis 2020)

Recall of the first results



Left : LDDMM. Right : normalized LDDMM

Multi-scale mass criterion : $\sigma_\omega = [0.5\sigma, 1\sigma, 2\sigma]$, $\sigma_T = 1$



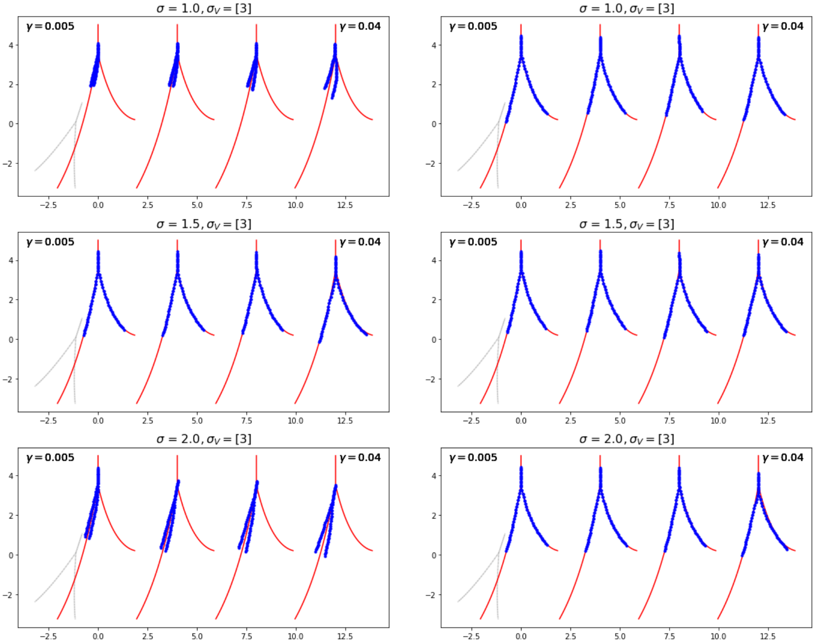
Additionnal constraint

The following regularization term tends to prevent the shrinking and stretching of the source shape :

$$R_{int}(S, \Phi(S)) = \int_S \left(1 - \left| d_x \phi_{|\tau_x S} \right| \right)^2 dx$$

Regularization (R_{int}, $\gamma_R = 1$) and normalized LDDMM

Left : $\sigma_\omega = \sigma$; Right : multi-scale mass criterion : $\sigma_\omega = [0.5\sigma, 1\sigma, 2\sigma]$



References for the first part

- Carlo Rovelli, *Anaximandre de Milet, ou la naissance de la pensée scientifique*, 2009
- Giuseppe Longo, *Le cauchemar de Prométhée : les sciences et leurs limites*, 2023
- Bergson, *Essai sur les données immédiates de la conscience* (*Time and Free Will*), 1889
- Individuation : Simondon, Bernard Stiegler.
- Nietzsche : Barbara Stiegler, Michèle Cohen-Halimi

Special thanks to Aurélien Galateau for the long discussions.

References on partial matchings with varifolds

Hopefully, a preprint soon...

- *Partial Matching in the Space of Varifolds*, P-L. Antonsanti, J. Glaunès, T. Benseghir, V. Jugnon, I. Kaltenmark (2021).
- *Diffeomorphic Registration with Density Changes for the Analysis of Imbalanced Shapes*, H. Hsieh and N. Charon (2021).
  Weight metamorphosis of varifolds and the LDDMM-Fisher-Rao metric, H. Hsieh and N. Charon (2022).
- *A new variational model for shape graph registration with partial matching constraints*, Yashil Sukurdeep, Martin Bauer, and Nicolas Charon (2022)
  Elastic shape analysis of surfaces with second-order Sobolev metrics : a comprehensive numerical framework, E. Hartman, Y. Sukurdeep, E. Klassen, N. Charon, and M. Bauer (2022)

Merci pour votre attention !

IPMI formula

$$\text{data term}(S, T) = \int_S g\left(\omega_S(\vec{\mathbf{x}}) - \int_T \min\nolimits_\epsilon\left(1, \frac{\omega_S(\vec{\mathbf{x}})}{\omega_T(\vec{\mathbf{y}})}\right) k(\vec{\mathbf{x}}, \vec{\mathbf{y}}) dy\right) dx\,.$$

where $g : \mathbb{R} \to \mathbb{R}, r \mapsto \max(0, r)^2$.