An implicit formulation for incorporating different priors into a deformation model

Geometric Sciences in Action: from geometric statistics to shape analysis

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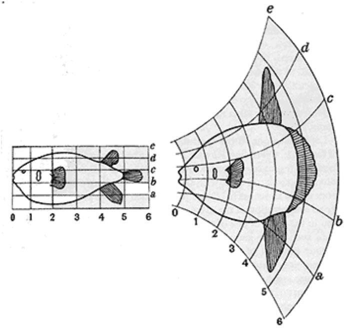
Joint work with Benjamin Charlier (Université de Montpellier), Stanley Durrleman (ICM, Paris), Adel Redjimi (Sorbonne Université), Alain Trouvé (ENS Paris-Saclay), Josua Sassen (ENS Paris-Saclay)
Work partially supported by the City of Paris.

Idea: characterizing the difference between two shapes thanks to the "best" diffeomorphism transforming one into the other.



D'Arcy Thompson (On Growth and Form, 1917)

Theorem

Let $v \in L^1([0,1], C_0^1(\mathbb{R}^d, \mathbb{R}^d))$*, then*

$$\begin{cases} \varphi^v_{t=0} & = & Id \\ \partial_t \varphi^v_t & = & v_t \circ \varphi^v_t \end{cases}$$

has a unique continuous solution called the flow of v*.*
For all t*,* φ^v_t *is a diffeomorphism.*

J. Glaunes (2005). Transport par difféomorphismes de points, de mesures et de courants pour la comparaison de formes et l'anatomie numérique.

Shape registration

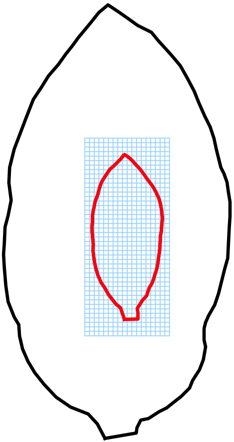
$$\min_{v \in U \subset L^1([0,1], C_0^1(\mathbb{R}^d, \mathbb{R}^d))} \left\{ \int_0^1 c(v_t) \mathrm{d}t + \lambda D(\varphi_{t=1}^v \cdot S, T) \right\}$$

Large deformation diffeomorphic metric mappings (LDDMM)[1] [2]

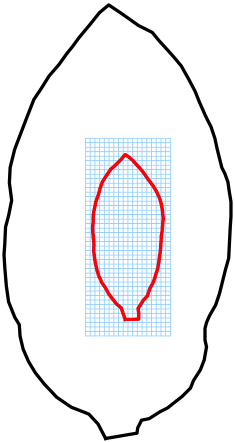
$$\min_{v \in L^2([0,1], V)} \left\{ \int_0^1 |v_t|_V^2 \mathrm{d}t + \lambda D(\varphi_{t=1}^v \cdot S, T) \right\}$$

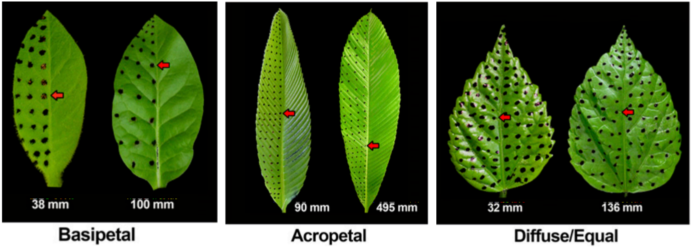
[1] Beg, M. F., Miller, M. I., Trouvé, A., Younes, L. (2005). Computing large deformation metric mappings via geodesic flows of diffeomorphisms. International journal of computer vision

[2] Arguillere, S., Trélat, E., Trouvé, A., Younes, L. (2015). Shape deformation analysis from the optimal control viewpoint. Journal de mathématiques pures et appliquées



Charon, N., Trouvé, A. (2013). The varifold representation of nonoriented shapes for diffeomorphic registration. SIAM Journal on Imaging Sciences, 6(4), 2547-2580.





Gupta, M. D., Nath, U. (2015). *Divergence in patterns of leaf growth polarity is associated with the expression divergence of miR396. The Plant Cell, tpc-15.*

Two main possibilities :

- Defining local field generators[3][4][5][6][7]
- Setting shape-dependent metric on the space of vector fields[8][9]

[3] S. Durrleman, M. Prastawa, G. Gerig, and S. Joshi. Optimal data-driven sparse parameterization of diffeomorphisms for population analysis. In Information Processing in Medical Imaging , 2011

[4] U. Grenander , A. Srivastava , S. Saini. A pattern-theoric characerization of biological growth. IEEE, 2007

[5] V. Arsigny, X. Pennec, N. Ayache. Polyrigid and Polyaffine Transformations: A Novel Geometrical Tool to Deal with Non-rigid Deformations – Application to the Registration of Histological Slices. Medical Image Analysis. 2005

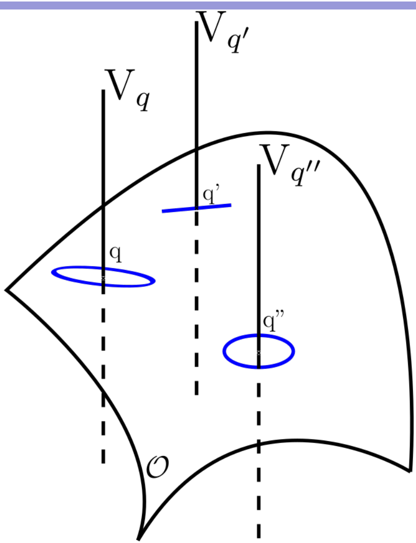
[6] L. Younes. Constrained diffeomorphic shape evolution. Foundations of Computational Mathematics, 2012.

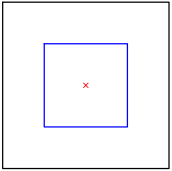
[7] Higher order momentum [S. Sommer M. Nielsen, F. Lauze, and X. Pennec. Higher-order momentum distributions and locally affine lddmm registration. SIAM Journal on Imaging Sciences, 2013]

[8] N. Charon and L. Younes. "Shape spaces: From geometry to biological plausibility. 2022

[9] D. N. Hsieh, S. Arguillère, N. Charon, M.I. Miller, L. Youne. A model for elastic evolution on foliated shapes. In International Conference on Information Processing in Medical Imaging. 2019.

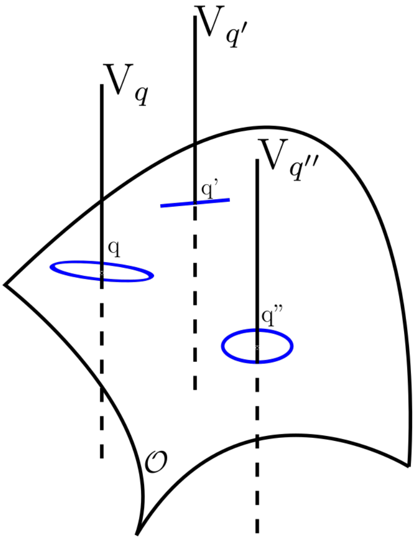
DEFORMATION MODULE





- Extend space of shape $q = (\tilde{q}, \theta)$
- $v_q : h \in H \longrightarrow v_{q,h} \in V_q$
- cost : $|v_{q,h}|^2 \leq Mc_q(h)$
- Combination:
  - $q = (\tilde{q}, \theta, \psi)$
  - $V_q = V_\theta + V_\psi$
- Trajectories *s.t.* $\exists v_t \in V_q$: $\dot{q}_t = (v_t \cdot \tilde{q}_t, v_t \cdot \theta_t, v_t \cdot \psi_t, \dots)$

Gris, B., Durrleman, S., Trouvé, A. (2018). A sub-riemannian modular framework for diffeomorphism-based analysis of shape ensembles. SIAM Journal on Imaging Sciences, 11(1), 802-833.

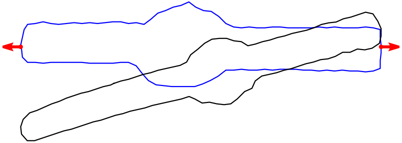


Modular registration

$$J(q,h) = \int c_q(h) + \lambda D(\varphi_{t=1}^{v_{q,h}} \cdot S, T)$$

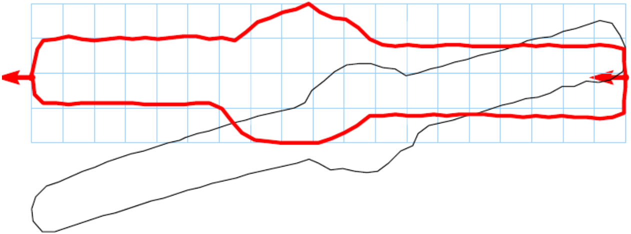
with $\dot{q}_t = v_{q_t,h_t} \cdot q_t$.

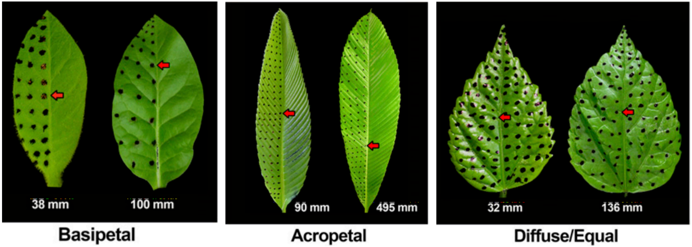
- Defining modules
- Minimizing J



Modules:

- Pose:
  - Global translation
  - Global rotation
- Strap lengths:
  - local translations with transported direction





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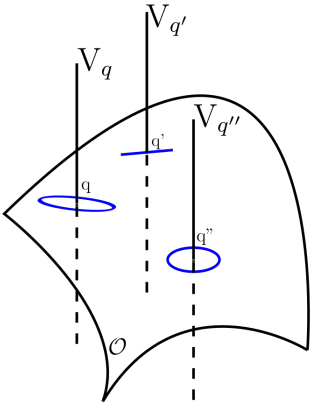


- Defining $v_q : H \mapsto V_q$ from properties to satisfy

$$v_{q,h} = \text{argmin}\{|v|_V^2 + \frac{1}{\nu}|S_q(v)S_q(v) - A_q(h)$$

- Evaluation operator S
- Observation operator A
- Model: S and A
- Explicit expression for $v_{q,h}$

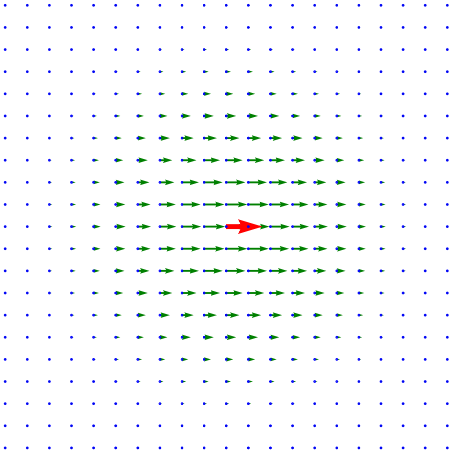
Lacroix, L., Charlier, B., Trouvé, A., Gris, B. (2021). *IMODAL: creating learnable user-defined deformation models. In Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition (pp. 12905-12913).*

$$v_{q,h} = \text{argmin}\{|v|_V^2 + \frac{1}{\nu}|S_q(v) - A_q(h)|^2\}$$

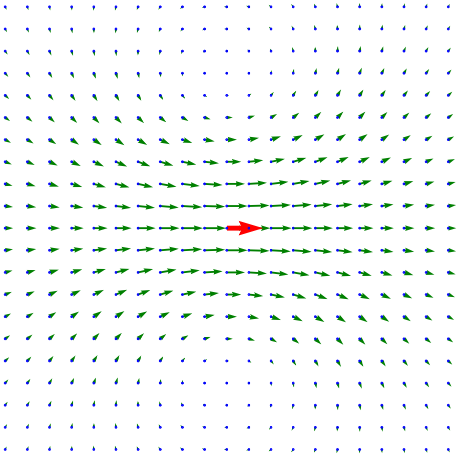
- $S_q(v) = (v(x_1), \dots, v(x_N))$, $q = (x_1, \dots, x_N) \in \mathcal{O} \doteq (\mathbb{R}^d)^N$
- $A_q(h) = (K_q + \nu Id)h(h_1, \dots, h_N)$,
  $h = (h_1, \dots, h_N) \in H \doteq (\mathbb{R}^d)^N$
- $v_{q,h} = \sum_i K(x_i, \cdot)h_i$
- $c_q(h) = (h \mid (K_q + \nu Id)h)$

$$v_{q,h} = \text{argmin}\{|v|_V^2 + \frac{1}{\nu}|S_q(v) - A_q(h)|^2\}$$

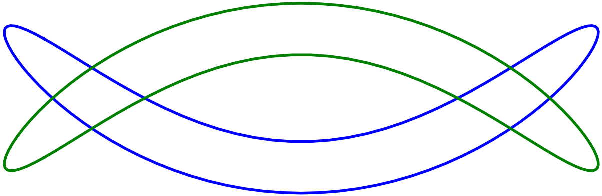
- $S_q(v) = \Big(v(x_1), \dots, v(x_N), \alpha div(v)(x_1), \dots, \alpha div(v)(x_N)\Big)$,
  $q = (x_1, \dots, x_N) \in \mathcal{O} \doteq (\mathbb{R}^d)^N$
- $A_q(h) = (h_1, \dots, h_N, 0, \dots, 0)$,
  $h = (h_1, \dots, h_N) \in H \doteq (\mathbb{R}^d)^N$
- $v_{q,h} = \text{argmin}\{|v|_V^2 + \frac{1}{\nu}\sum_i |v(x_i) - h_i|^2 + \frac{1}{\nu}\alpha^2 \sum_i |div(v)(x_i)|^2\}$

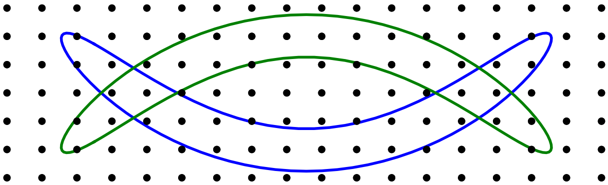


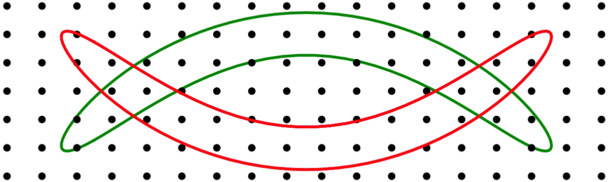
Local translation



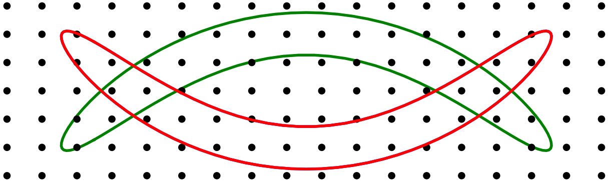
Local translation with divergence-free constraint



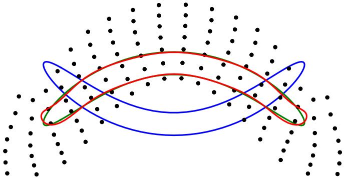




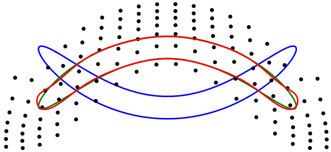
Divergence-free constraints



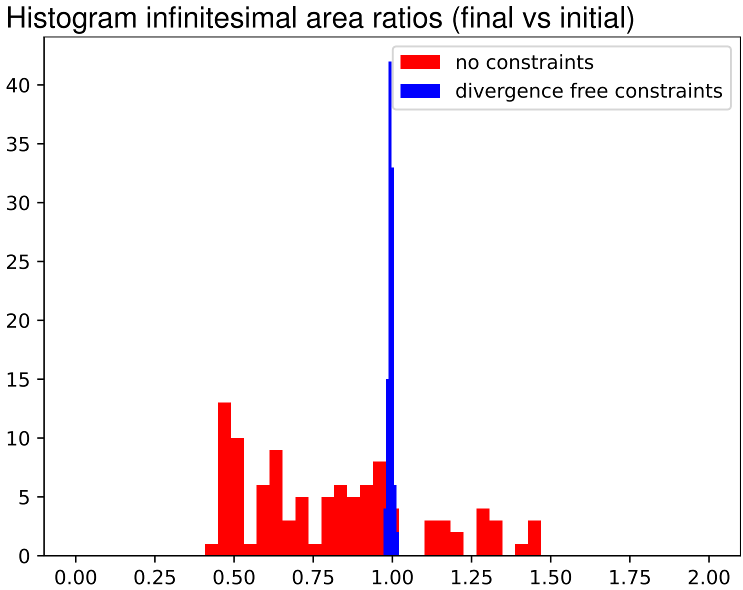
No constraints



Divergence-free constraints



No constraints


Histogram infinitesimal area ratios (final vs initial)
no constraints
divergence free constraints
0
5
10
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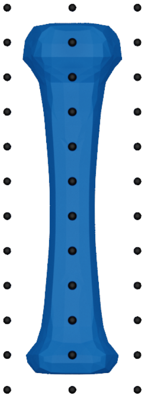
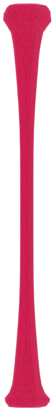
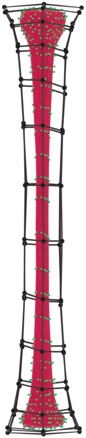
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

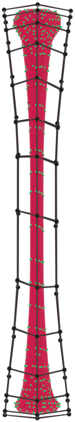




Target

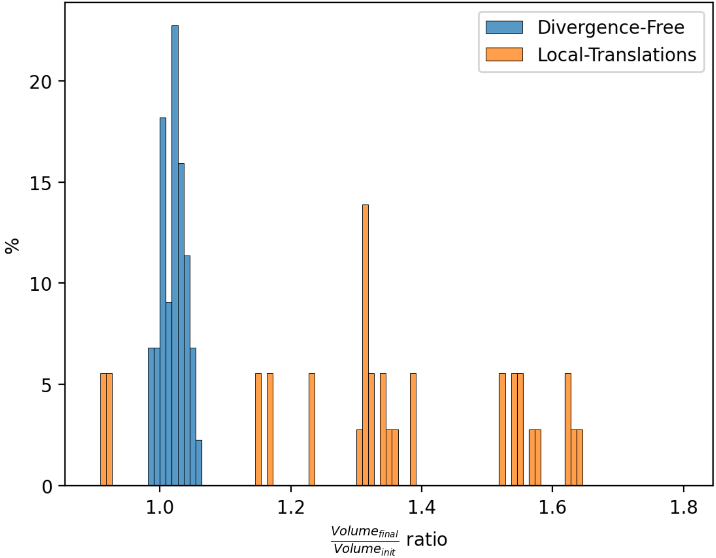


Divergence-free constraints



No constraints

Histogram infinitesimal area ratios (final vs initial)

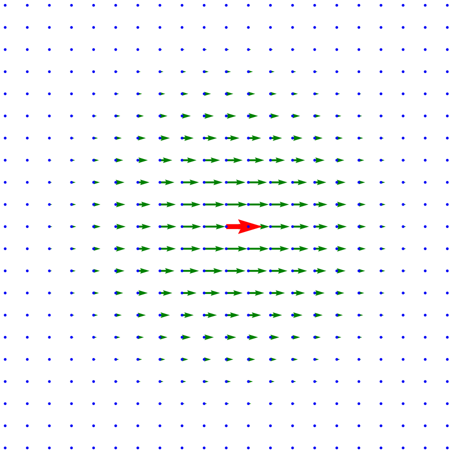


$$v_{q,h} = \operatorname{argmin}\{|v|_V^2 + \frac{1}{\nu}|S_q(v) - A_q(h)|^2\}$$

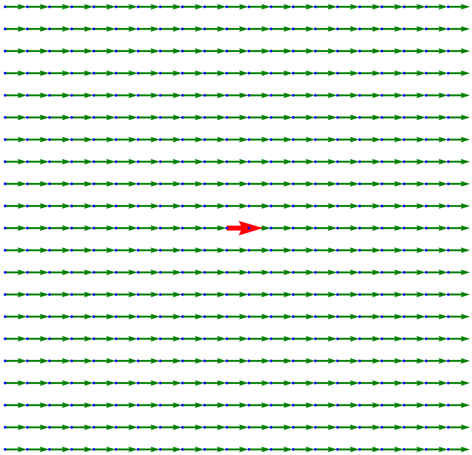
- $S_q(v) = \Big(v(x_1), \ldots, v(x_N), \alpha\epsilon(v)(x_1), \ldots, \alpha\epsilon(v)(x_N)\Big)$,
  $q = (x_1, \ldots, x_N) \in \mathcal{O} \doteq (\mathbb{R}^d)^N$

  $$\epsilon_x(v) = \frac{Dv(x) + Dv(x)^T}{2}$$

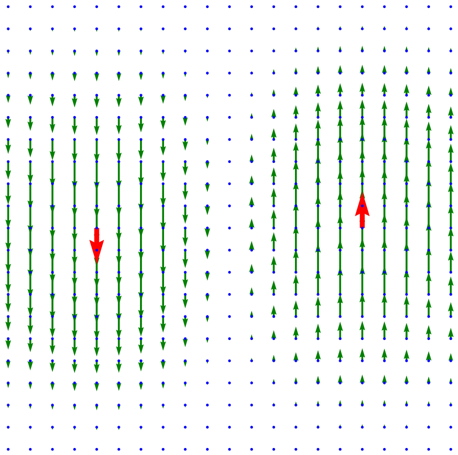
  $\longrightarrow$ Captures local metric changes induced by $Id + v$ around x
- $A_q(h) = (h_1, \ldots, h_N, 0, \ldots, 0)$,
  $h = (h_1, \ldots, h_N) \in H \doteq (\mathbb{R}^d)^N$
- $v_{q,h} = \operatorname{argmin}\{|v|_V^2 + \frac{1}{\nu}\sum_i |v(x_i) - h_i|^2 + \frac{1}{\nu}\alpha^2 \sum_i |\epsilon(v)(x_i)|^2\}$



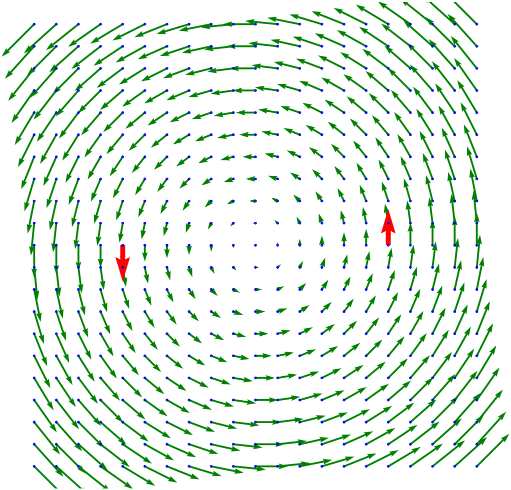
Local translation



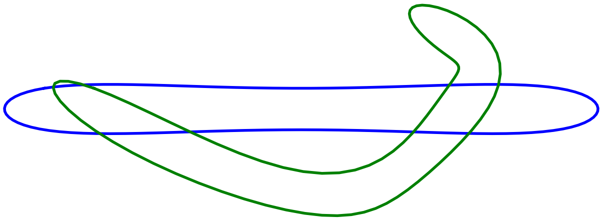
Local translation with ARAP constraint

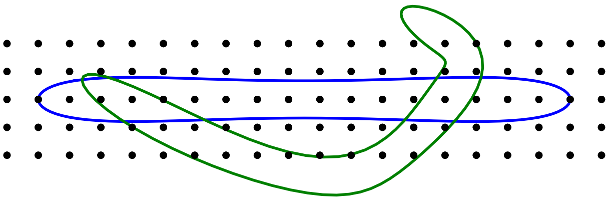


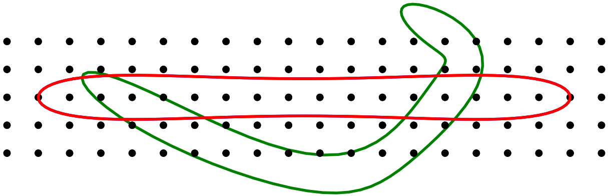
Sum of two local translations



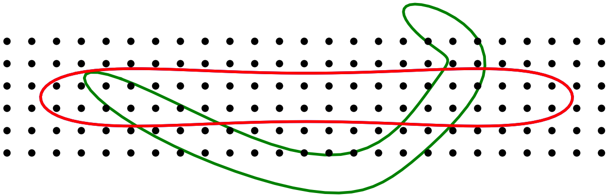
Sum of two local translations with ARAP constraint



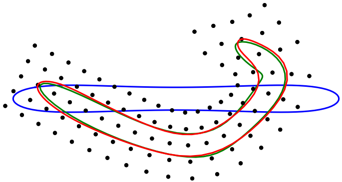




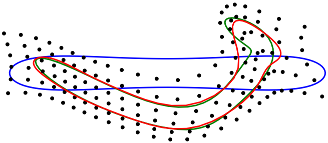
With ARAP constraints



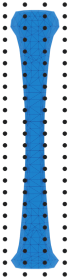
Without ARAP constraints



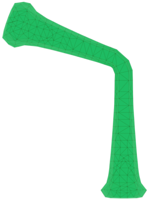
ARAP constraints



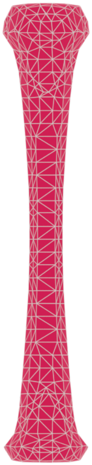
No constraints



Source



Target



- Many works on as-rigid-as-possible and divergence-free fields[10][11][12][13][14]
- One element of the 'Deformation modules toolbox'

[10] Alexa, M., Cohen-Or, D., Levin, D. (2023). As-rigid-as-possible shape interpolation. In Seminal Graphics Papers: Pushing the Boundaries, Volume 2 (pp. 165-172).

[11] Igarashi, T., Moscovich, T., Hughes, J. F. (2005). As-rigid-as-possible shape manipulation. ACM transactions on Graphics (TOG), 24(3), 1134-1141.

[12] Hartman, E., Pierson, E., Bauer, M., Charon, N., Daoudi, M. (2023). Bare-esa: A riemannian framework for unregistered human body shapes. In Proceedings of the IEEE/CVF International Conference on Computer Vision (pp. 14181-14191).

[13] Eisenberger, M., Lähner, Z., Cremers, D. (2019, August). Divergence-free shape correspondence by deformation. In Computer Graphics Forum (Vol. 38, No. 5, pp. 1-12).

[14] Micheli, M., Glaunes, J. A. (2013). Matrix-valued kernels for shape deformation analysis. arXiv preprint arXiv:1308.5739.

$$\zeta_q(h) = argmin\{|v|_V^2 + \frac{1}{\nu}|S_q(v) - A_q(h)|^2\}$$

- $\epsilon_x(v) = \frac{Dv(x)+Dv(x)^T}{2}$
- $S_q(v) = (\epsilon_{x_i}(v))_i$
- growth model operator $A_q : h \mapsto A_q(h)$

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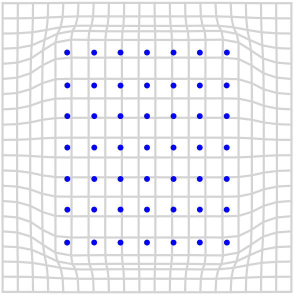
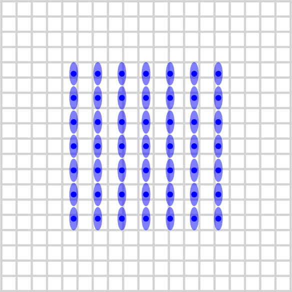
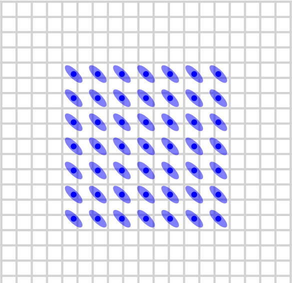
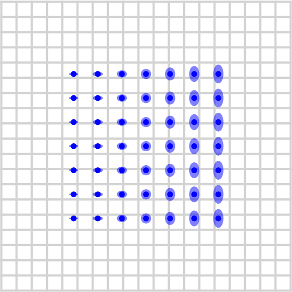
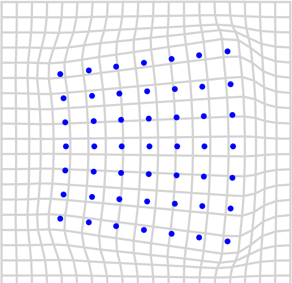
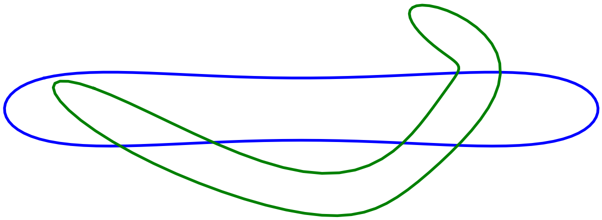


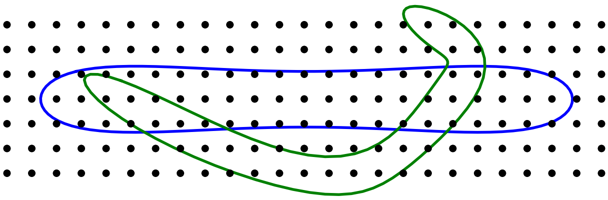


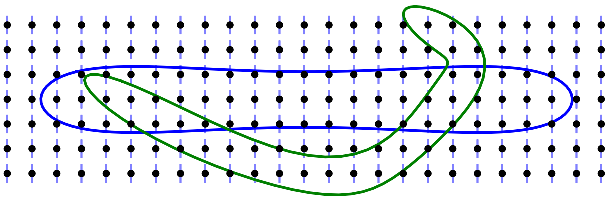


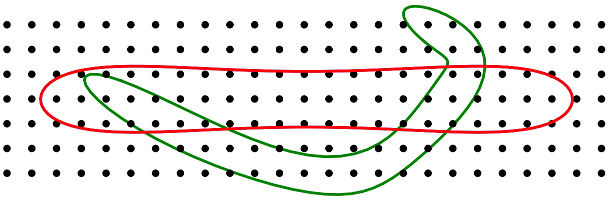


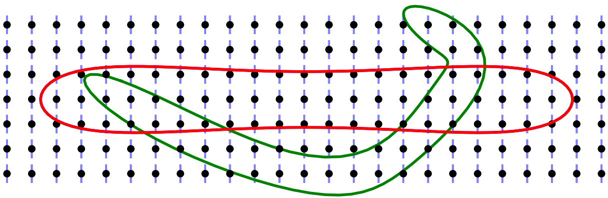


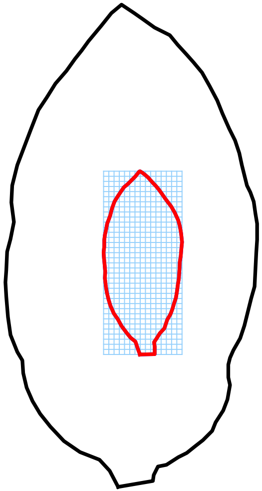


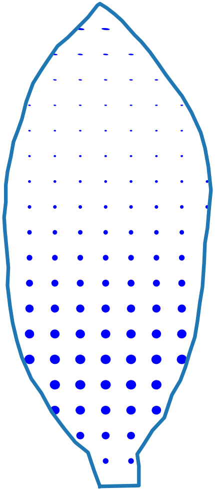


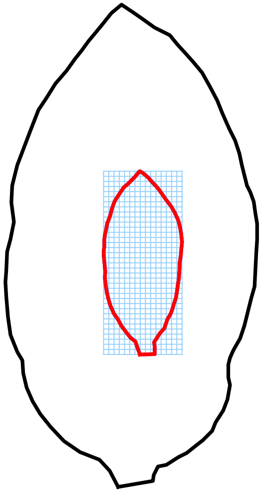












$$\zeta_q(h) = argmin\{|v|_V^2 + \frac{1}{\nu}|S_q(v) - A_q(h)|_q^2\}$$

- Morphoelasticity : growth of an elastic material[15] [16]
- $v = v_{growth} + v_{elastic}$
- $|S_q(v) - A_q(h)|^2 = |S_q(v_{elastic})|^2$
- Penalize $v_{elastic}$ with linear elastic energy
- Softness of material increases when σ or ν decrease

[15] A. Goriely. The mathematics and mechanics of biological growth, volume 45. Springer, 2017.
[16] N. Charon, L. Younes. Shape spaces: from geometry to biological plausibility. 2022.

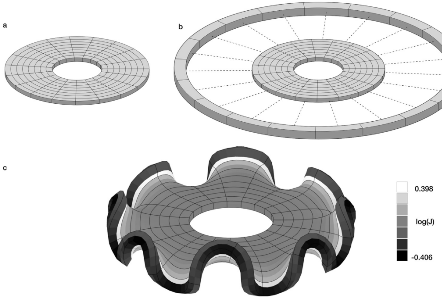


Fig. 3 – Buckling induced by growth. In this example, the outer rim of the ring grows to twice its original size. This leads to the formation of folds that minimise the elastic energy of the system. At equilibrium, a heterogeneous pattern of stress, appears with residual stress gradients from centre to periphery, in angular bands and through the thickness of the ring.

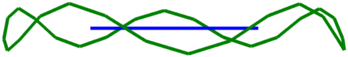
Foubet, O., Trejo, M., Toro, R. (2019). Mechanical morphogenesis and the development of neocortical organisation. Cortex, 118, 315-326.

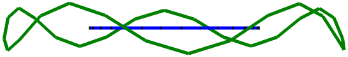
$$\zeta_q(h) = argmin\{|v|_V^2 + \frac{1}{\nu}|S_q(v) - A_q(h)|^2\}$$

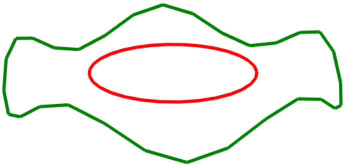
$$\zeta_{q^g}^g(h^g) = argmin\{|v|_V^2 + \frac{1}{\nu}|S_{q^g}(v) - A_{q^g}(h^g)|^2\}$$

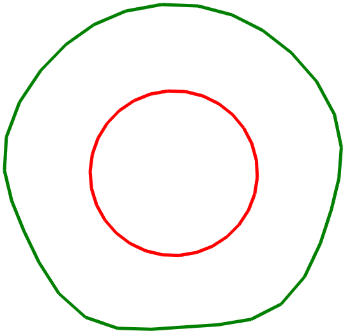
$$\zeta_q(h) = \zeta_q^g(h^g) + \sum_i T_i K(x_i, \cdot) h_i^t$$

$$c_q(h) = |\zeta_q(h)|_V^2 + |S_q(\zeta_q(h)) - A_q(h)|^2 + |h|$$







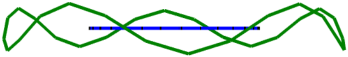


$$\zeta^g_{q^g}(h^g) = argmin\{|v|^2_V + \frac{1}{\nu}|S_{q^g}(v) - A_{q^g}(h^g)|^2\}$$

$$\zeta_q(h) = \zeta^g_q(h^g) + \sum_i T_i K(x_i, \cdot) h^t_i$$

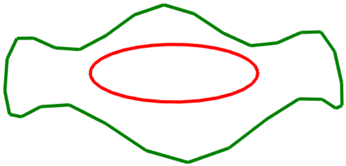
$$\zeta^{g}_{q^g}(h^g) = argmin\{|v|^2_V + \frac{1}{\nu}|S_{q^g}(v) - A_{q^g}(h^g)|^2\}$$

$$\zeta_q(h) = \zeta^{g}_{q}(h^g) + \sum_i T_i K(x_i, \cdot) h^t_i$$



$$\zeta_{q^g}^g(h^g) = argmin\{|v|_V^2 + \frac{1}{\nu}|S_{q^g}(v) - A_{q^g}(h^g)|^2\}$$

$$\zeta_q(h) = \zeta_q^g(h^g) + \sum_i T_i K(x_i, \cdot) h_i^t$$



- Incorporating structures in deformation models:
  - Deformation modules
  - Implicit formulation
  - Enrich the space of geometrical descriptors
- Future work
  - Stress from elastic energy, plasticity
  - Brain gyrification ? [17][18]
  - Formulation without inversion: adapted optimization scheme
- Source and documentation https://github.com/imodal

Thank you for your attention !
Questions ?

[17] Tallinen, T., Chung, J. Y., Rousseau, F., Girard, N., Lefèvre, J., Mahadevan, L. (2016). On the growth and form of cortical convolutions. Nature Physics, 12(6), 588-593.

[18] Foubet, O., Trejo, M., Toro, R. (2019). Mechanical morphogenesis and the development of neocortical organisation. Cortex, 118, 315-326.