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Research Statement and Description of Scientific Work

My main interests in mathematics are crystallographic symmetries related to root systems, representations of Lie algebras and invariant theory of multiplicative group actions. Furthermore, my research revolves around computations over non-commutative algebras as well as applications in optimization, integration and differential equations.

The question that I work on is essentially how symmetries of reflection groups can be exploited in algebraic and numerical algorithms to reduce complexity, improve robustness and optimize computational resources. On one hand, this requires to understand the framework of crystallographic symmetries, which involves root systems, Weyl groups, representation theory, Lie groups and Lie/Cartan/Borel algebras. On the other hand, it is required to understand the motivation and background for existing algorithms in order to improve them. I have mainly worked with algorithms which are based on results from real algebraic geometry and moment theory.

My contribution to this question so far spans over two articles. We consider the problem of optimizing a trigonometric polynomial with crystallographic symmetry. Such a function can be written as a classical multivariate polynomial with so called generalized Chebyshev polynomials. This reduces the complexity of the problem immensely, but transforms the region of optimization. Hence, the first thing to check is that the new region is still relatively nicely behaved. Indeed, together with Evelyne Hubert and Cordian Riener, we have shown that the new region is a basic compact semi-algebraic set and given an explicit formula. This means that the trigonometric optimization problem can be treated with tools from classical polynomial optimization.

The second article concerning crystallographic symmetries targets the optimization of trigonometric polynomials in practice. As explained above, this can be understood as a polynomial optimization problem, but with a very explicit and specific structure. We adapt the algorithm of Lasserre’s moment and sum of squares hierarchies to be more efficient with generalized Chebyshev polynomials. The idea remains that to exploit positivity certificates from real algebraic geometry, but with an adapted notion of degree. As an application, we consider a problem in algebraic combinatorics, where symmetric trigonometric polynomials naturally arises. With our technique we can give new bounds for chromatic numbers of geometric graphs and reprove known results easily, which were difficult to show before.

Another question that interests me is how to solve polynomial and algebraic systems over algebras of, for example, operators, Hopf, Iwahori-Hecke or Askey-Wilson. This setting is usually non-commutative and may even be non-associative. Computational problems can still be approached with Gröbner bases and Gröbner-Shirshov bases, mirroring Buchberger’s classical algorithm.

Contributing to this question, I have written an article with Viktor Levandovskyy and Karim Abou Zeid on the implementation of strong Gröbner bases in the non-commutative setting with coefficients not in fields, but in principal ideal domains. This addresses a long time myth that the theory is just as in the field case, which is not true.

The contributions are listed below in chronological order.

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1) Computing Free Non-commutative Groebner Bases over Z with Singular:Letterplace
(with Viktor Levandovskyy and Karim Abou Zeid)
https://www.sciencedirect.com/science/article/abs/pii/S0747717122000736?via%3Dihub
https://hal.archives-ouvertes.fr/hal-03085431

This paper is an extension of an ISSAC conference paper about computations of Groebner(-Shirshov) bases over free associative algebras. We present all the needed proofs in details, add a part on the direct treatment of the ring Z/mZ as well as new examples and applications to e.g. Iwahori-Hecke algebras. The extension of Groebner bases concept from polynomial algebras over fields to polynomial rings over rings allows to tackle numerous applications, both of theoretical and of practical importance. Groebner and Groebner-Shirshov bases can be defined for various non-commutative and even non-associative algebraic structures. We study the case of associative rings and aim at free algebras over principal ideal rings. We concentrate ourselves on the case of commutative coefficient rings without zero divisors (i.e. a domain). Even working over Z allows one to do computations, which can be treated as universal for fields of arbitrary characteristic. By using the systematic approach, we revisit the theory and present the algorithms in the implementable form. We show drastic differences in the behavior of Groebner bases between free algebras and algebras, close to commutative. Even the process of the formation of critical pairs has to be reengineered, together with the implementing the criteria for their quick discarding. We present an implementation of algorithms in the Singular subsystem called Letterplace, which internally uses Letterplace techniques (and Letterplace Groebner bases), due to La Scala and Levandovskyy.

2) Polynomial description for the T-Orbit Spaces of Multiplicative Actions
(with Evelyne Hubert and Cordian Riener)
https://hal.archives-ouvertes.fr/hal-03590007

A finite group with an integer representation has a multiplicative action on the ring of Laurent polynomials, which is induced by a nonlinear action on the algebraic torus. We restrict this action to the compact torus and study the structure of the associated orbit space as the image of the fundamental invariants. For Weyl groups associated to crystallographic root systems of types A, B, C and D, this image is a compact basic semi-algebraic set and we present the defining polynomial inequalities explicitly. We show a correspondence between orbits and symmetric polynomial systems with solutions in the compact torus. A characterization of the orbit space is given as the positivity-locus of a Hermite matrix polynomial. The resulting domain is the region of orthogonality for two families of generalized Chebyshev polynomials. These polynomials are defined via the weights and characters of the associated Lie algebra and are connected to applications of Lie theory and Fourier analysis.

3) Optimization of trigonometric polynomials with crystallographic symmetry and applications
(with Evelyne Hubert, Philippe Moustrou and Cordian Riener)
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https://hal.archives-ouvertes.fr/hal-03768067

This work studies the problem of optimizing a trigonometric polynomial with crystallographic symmetry. Optimization of trigonometric polynomials has been subject to many recent works, but a theory for exploiting symmetries has hardly been developed or generalized. We consider the action of a crystallographic group, also known as a Weyl group, on the exponents. By rewriting an invariant trigonometric polynomial as a classical polynomial in fundamental invariants, the region of optimization is transformed into the orbit space of the multiplicative Weyl group action. This orbit space is the image of fundamental invariants and, for the Weyl groups of types A, B, C, D and G, it is a compact

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basic semi-algebraic set. Our rewriting approach transforms the trigonometric optimization problem into a classical polynomial optimization problem. We adapt and implement Lasserre’s hierarchy in the basis of generalized Chebyshev polynomials with a new notion of degree. The optimal value of the original trigonometric optimization problem is then approximated through solutions of semi-definite programs, which are solved numerically. With our approach, we provide an alternative to known strategies for trigonometric optimization. Where other techniques utilize sums of Hermitian squares or generalizations of Lasserre's hierarchy to the complex setting, we exploit symmetry first and then apply techniques from classical polynomial optimization. This reduces the size of arising semi-definite relaxations and can also improve the quality of the approximation. As an application, we study the problem of computing lower bounds for the chromatic number of geometric distance graphs. Given a norm and a set of vertices, this problem asks to find the minimal number of colors needed to paint the vertices, so that no two of those of distance 1 between them have the same color. The spectral bound was generalized from finite to infinite graphs to deal with such problems. This bound involves the optimization of a trigonometric polynomial. We focus on norms which are given by polytopes with crystallographic symmetry. Then the problem can be tackled with the techniques developed in this thesis. We give several bounds through analytical and numerical computations.