SIAM J. APPL. MATH.
Vol. 64, No. 5, pp. 1526–1549

LOW-FIELD LIMIT FOR A NONLINEAR DISCRETE DRIFT-DIFFUSION MODEL ARISING IN SEMICONDUCTOR SUPERLATTICES THEORY*

THIERRY GOUDON†, OSCAR SÁNCHEZ‡, JUAN SOLER‡, AND LUIS L. BONILLA§

Abstract. Charge transport in semiconductor superlattices can be described through a discrete drift-diffusion model. In this model, we identify some small parameter $h > 0$, related to the ratio between the length of a superlattice period and the observation length scale. Specifically, we investigate a regime where the length of the superlattice period is small while the doping profile is low. In the limit $h \to 0$, we are led to a nonlinear drift-diffusion model, coupled to the Poisson equation.

Key words. semiconductor supperlattices, drift-diffusion models

AMS subject classifications. 35Q99, 35K55, 82C70

DOI. 10.1137/S003613990241730X

1. Introduction.

A semiconductor superlattice (SL) is a periodic array of layers of two different semiconductors whose lateral dimension is much larger than the length of one period. These devices exhibit nonlinear charge transport phenomena due to the existence of electric field domains. Depending on the charge density (produced by doping or irradiating the SL) and on the applied voltage, different qualitative responses of the current can be obtained, as shown, e.g., by Bonilla [6]. In experiments at intermediate values of the charge density, stationary responses and self-sustained oscillations are observed depending on the values of the voltage. There exist solutions corresponding to low voltages which are stationary and typically develop low electric fields. It is very important to understand the time evolution of the solutions toward these stationary profiles (see [5] where relocation experiments are studied).

Electronic transport in such semiconductor devices can be described by a discrete drift-diffusion model. Details on the modeling will be given in section 2, following the works by Aguado, Platero, Moscoso, and Bonilla [1] and Bonilla, Platero, and Sánchez [4] and review papers by Bonilla [6] and Wacker [15]. The model consists of the Poisson equation coupled to charge continuity equations for the electron density n and average electric field F at each SL period. Tunneling currents across barriers are approximated by a discrete drift-diffusion (DDD) law, whose coefficients are themselves field dependent. We aim at investigating asymptotics regimes for this model. To this end, we shall write the equations in dimensionless form. Hence, we are able to identify some small parameter—denoted $h > 0$ in what follows—by means of physically relevant dimensionless parameters of the DDD system. Having set up this DDD system, we prove that the solutions converge, in an appropriate weak setting, to solutions of a continuous drift-diffusion-Poisson problem with field-dependent mobilities,

*Received by the editors November 5, 2002; accepted for publication (in revised form) November 3, 2003; published electronically June 22, 2004. This research was partially supported by the EU financed network IHP-HPRN-CT-2002-00282 and by MCYT (Spain), Proyecto BFM2002–00831.
http://www.siam.org/journals/siap/64-5/41730.html

†CNRS-UMR8524, Université des Sciences et Technologies Lille 1, Cité scientifique, 59655 Villeneuve d'Ascq cedex, France (thierry.goudon@univ-lille1.fr).

‡Departamento de Matematica Aplicada, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain (ossanche@ugr.es, jsoler@ugr.es).

§Departamento de Matemáticas, Escuela Politécnica Superior, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Spain, and Unidad Asociada al Instituto de Ciencia de Materiales (CSIC), 28049 Cantoblanco, Spain (bonilla@ing.uc3m.es).

as the parameter h tends to 0. The limit equation reads

$$\left\{\begin{array}{ll} \partial_t n + \partial_x J(F,n) = 0 & \text{in } (0,T)\times[-X,X], \\ J(F,n) = v(F)n - D(F)\partial_x n\,, & \\ \partial_x F = n - N_D & \text{in } (0,T)\times[-X,X]. \end{array}\right. \tag{1.1}$$

These equations are completed by bias, boundary, and initial conditions

$$\left\{\begin{array}{ll} \displaystyle\int_{-X}^{X} F\,dx = V & \text{on } (0,T), \\ J(F,n)(X) = W^{(f)}(F)n(X) & \text{on } (0,T), \\ J(F,n)(-X) = (j^{(e)}(F) - W^{(b)}(F)n)(-X) & \text{on } (0,T), \\ n(t=0,x) = n^0(x) & \text{on } [-X,X]. \end{array}\right.$$

The techniques employed to prove the convergence are based on a priori estimates and compactness properties. At this point, we have to remark that the solutions to the DDD model are stepwise constant functions, which forces us to consider the solutions in the framework of the bounded variation (BV) spaces.

Our main result admits a reversal lecture. Since the dimensionless system of the DDD model coincides with a finite difference discretization of (1.1), the analysis proposed can be read as a convergence analysis for numerical approximations of that drift-diffusion continuous equation. In this direction our problem is related with other works on approximation of field-dependent mobilities. System (1.1) is a monopolar one-dimensional version of the drift-diffusion system analyzed by Gajewski and Gröger [8] and Jerome [10]. Our DDD system is simpler than the general version studied in those works, where different boundary conditions are considered. Here, we deal with the time-dependent problem, while, to the best of our knowledge, the previous analyses are devoted to approximate steady state solutions.

This paper is organized as follows. In section 2 we present in detail the DDD model, recalling some aspects of its derivation. We also justify its well-posedness. In section 3 we perform the dimension analysis of the system. Then, we derive the dimensionless equations for which the analysis is actually performed and we state precisely our main convergence result for the DDD model. Actually, our analysis applies either when considering a Dirichlet boundary condition for the electric field or with the bias condition. Section 4 is devoted to the crucial a priori estimates satisfied by the solution. For the sake of simplicity, we start with the Dirichlet boundary condition. Then, in section 5 we use these estimates to show the convergence to the continuous model, through compactness arguments. Finally, section 6 sketches the slight adaptation of the proof to treat the DDD system endowed with the bias condition. The paper ends with two appendices: in the first one we deal with a technical auxiliary result, and the latter investigates uniqueness of the limit system.

2. Discrete drift-diffusion model. Since the two semiconductors constituting the SL have different energy gaps, the conduction band of an SL can be viewed as a periodic array of potential wells and barriers, of widths w and d, respectively, with $\ell = d + w$ the length of one period. We assume that scattering times are shorter than escape times from quantum wells, the latter being shorter than dielectric relaxation times. In such a weakly coupled semiconductor SL, the dominant mechanism of charge transport is sequential resonant tunneling. In the simplest situation, the center of each quantum well is n-doped and the thermal energy is large compared to the energy of the lowest miniband. Then, a description of charge transport in such devices has been proposed through a DDD model; see [1, 4, 6, 15]. This model has been extended by taking into account stochastic effects by Bonilla, Sánchez, and Soler [5], in comparison with the experimental results of Rogozia et al. [12].

In such a modeling, we consider an array of $2N+1$ consecutive cells, which are well-barrier pairs, labeled by the index $i \in \{-N, \dots, +N\}$. The barrier separating the injecting contact from the first well of the SL is considered as the $(-N-1)$th barrier, while the barrier of the Nth SL period separates the Nth well from the collector. The model assumes that the electrons are singularly concentrated on a two-dimensional region located in the center of the quantum well. The unknowns are the two-dimensional electron density n_i (number of electrons per unit area of the SL cross section at the center of the ith well) and the average electric field F_i in each cell. These quantities are related through the following discrete Poisson equation:

$$F_i - F_{i-1} = \frac{e}{\overline{\varepsilon}}(n_i - N_D^w), \quad i \in \{-N, \dots, N\}. \tag{2.1}$$

In (2.1), N_D^w stands for the two-dimensional doping in the wells, assumed to be constant, while $\overline{\varepsilon}$ is the average permittivity in the SL and $(-e)$ stands for the electron charge. Notice that the set of relations (2.1) involves as an additional unknown the electric field F_{-N-1} at the injecting contact. On the other hand, denoting by $eJ_{i\to i+1}$ the tunneling current density through the barrier separating the cells $\#i$ and $\#(i+1)$, the density in the ith cell satisfies the following charge continuity equation:

$$\frac{dn_i}{dt} = J_{i-1\to i} - J_{i\to i+1}, \quad i \in \{-N, \dots, N\}. \tag{2.2}$$

Consequently, differentiating (2.1) and using (2.2), we notice that the quantity

$$\frac{\overline{\varepsilon}}{e}\frac{dF_i}{dt} + J_{i\to i+1} = J(t), \quad i \in \{-N-1, \dots, N\} \tag{2.3}$$

does not depend on the considered cell. This is the so-called Ampère's law, where $eJ(t)$ stands for the total current density through the SL which does not depend on the index i.

Then, the model is completed by a constitutive law which defines the current density $eJ_{i\to i+1}$ by means of the (n_k, F_k)'s. The tunneling current density depends on the electrochemical potentials at cells $\#i$ and $\#(i+1)$ and on the average electric field F_i; see [6, 15]. The electrochemical potentials that "drive" the tunneling current (a nonzero current is a consequence of unequal electrochemical potentials at cells $\#i$ and $\#(i+1)$) are functions of the electron densities and therefore, according to [6, 15], we may consider that the tunneling current $eJ_{i\to i+1}$ depends on n_i, n_{i+1} and F_i. First-principles calculations of $eJ_{i\to i+1}$ are at best sketchy. In the literature, formulas have been derived from quantum kinetic equations for Green's functions (see [15], assuming constant electric field across the SL, simplified hopping Hamiltonians, and scattering) and from the transfer Hamiltonian formalism as in [1, 4, 6] (a many-body version of the WKB method originally proposed by Bardeen [3]). At high (room) temperature, all these formulas imply that the tunneling current is given by the difference of a drift term and a diffusion term as follows:

$$J_{i\to i+1} = \frac{n_i v(F_i)}{\ell} - \frac{D(F_i)(n_{i+1} - n_i)}{\ell^2}, \quad i \in \{-N, \dots, N-1\}. \tag{2.4}$$

The drift velocity and the diffusion coefficient are defined through functions v and D of the electric field, which depend on the physical properties of the material used in the SL; see [6] for more details. The special nature of the three-dimensional emitter and collector layers (different from the essentially two-dimensional quantum wells that form the SL) is considered in the calculation of the boundary tunneling current. By using the transfer Hamiltonian formalism, the following approximate formulas can be derived [4]:

$$J_{-N-1\to -N} = j^{(e)}(F_{-N-1}) - \frac{n_{-N}W^{(b)}(F_{-N-1})}{\ell}, \tag{2.5}$$

$$J_{N\to N+1} = \frac{n_N W^{(f)}(F_N)}{\ell}. \tag{2.6}$$

These equations involve the emitter current density $ej^{(e)}$, the emitter backward velocity $W^{(b)}$, and the collector forward velocity $W^{(f)}$, which are given functions of the electric field. All the coefficients $v, D, W^{(b)}, W^{(f)}, j^{(e)}$ are supposed to be nonnegative and satisfy some regularity properties. Typical graphs for these functions can be found in [5].

We remark that one equation is still missing since we have one more unknown than we have equations. There are several ways to close the system. The simplest way is to assume that the electric field at the emitter is prescribed as

$$F_{-N-1}(t) = F_{-}(t), \tag{2.7}$$

the right-hand side being a given function $F_{-} : \mathbb{R}^{+} \to \mathbb{R}$. This Dirichlet boundary condition has been proposed when the number of periods considered in the SL is high enough (infinite supperlattice). Therefore, this condition is well adapted to our work since we shall deal with an asympotic problem where the number of cells goes to infinity.

However, from a physical viewpoint, it is certainly more realistic to complete the system by using the so-called voltage bias condition: the total voltage across the SL,

$$\ell \sum_{i=-N}^{N} F_i = V, \tag{2.8}$$

remains equal to a given quantity V. In what follows we essentially deal with the Dirichlet-like boundary condition (2.7) for the electric field. We will come back to the voltage bias condition (2.8) at the end of the paper.

Relations (2.1), (2.2), and (2.7) form a closed system of equations for n_i and F_i with $i \in \{-N, \dots, N\}$, referred to in what follows as the DDD model. We remark that the electric field in the cell $\#i$ can be expressed as a function of the incoming field F_{-} and the density in the previous cells as follows:

$$F_i(t) = F_{-}(t) + \frac{e}{\varepsilon} \sum_{j=-N}^{i} (n_j(t) - N_D^w), \quad i \in \{-N, \dots, N\} \quad \forall t \in [0, T]. \tag{2.9}$$

Consequently, we can rewrite the initial value problem associated to the DDD model in terms of the densities

$$\frac{d\vec{n}}{dt} = g(t, \vec{n}(t)), \qquad \vec{n}(0) = \vec{n}^0, \tag{2.10}$$

where $\vec{n}(t) = (n_{-N}, \dots, n_N)^T \in \mathbb{R}^{2N+1}$, $g : \mathbb{R}^{2N+1} \to \mathbb{R}^{2N+1}$ is a smooth function, and $\vec{n}^0 \in \mathbb{R}^{2N+1}$ is the initial condition.

THEOREM 2.1. *Let $n_i^0 \geq 0$ for $i \in \{-N, \dots, N\}$ be the initial data for the DDD system. Let F_- be a C^1 function of time. Also let $v, D, W^{(b,f)}, j^{(e)}$ be C^1 nonnegative functions. Then, there exists a unique global solution associated with the initial value problem* (2.10)*. The solution verifies $n_i(t) \geq 0$ for all $i \in \{-N, \dots, N\}$, $t \geq 0$.*

Proof. Local existence and uniqueness follow by a direct application of the Cauchy–Lipschitz theorem for ODE, since the function g inherits the regularity properties of the coefficients. The estimates proved in section 4, especially in Lemma 4.1, also provide a uniform bound on the solution which prevents a finite time blowup. Consequently, the solution is globally defined. There remains only to justify the nonnegativeness of the solution. To this end, it is convenient to rewrite (2.2) as a difference between a gain term and a loss term as follows:

$$\frac{dn_i}{dt} = \begin{cases} \frac{v(F_{i-1})}{\ell} n_{i-1} + \frac{D(F_i)}{\ell^2} n_{i+1} + \frac{D(F_{i-1})}{\ell^2} n_{i-1} - \left(\frac{v(F_i)}{\ell} + \frac{D(F_i)}{\ell^2} + \frac{D(F_{i-1})}{\ell^2} \right) n_i \\ \qquad\qquad \text{for } i \in \{-N+1, \dots, N-1\}, \\ \frac{D(F_{-N})}{\ell^2} n_{-N+1} + j^{(e)}(F_{-N-1}) - \left(\frac{v(F_{-N})}{\ell} + \frac{D(F_{-N})}{\ell^2} + \frac{W^{(b)}(F_{-N-1})}{\ell} \right) n_{-N} \\ \qquad\qquad \text{for } i = -N, \\ \frac{v(F_{N-1})}{\ell} n_{N-1} + \frac{D(F_{N-1})}{\ell^2} n_{N-1} - \left(\frac{D(F_{N-1})}{\ell^2} + \frac{W^{(f)}(F_N)}{\ell} \right) n_N \\ \qquad\qquad \text{for } i = N. \end{cases}$$

Let $t \geq 0$ such that $n_i(t) \geq 0$ for any $i \in \{-N, \dots, N\}$. Suppose $n_j(t) = 0$ for some $j \in \{-N, \dots, N\}$. Thus, we notice that its time derivative $\frac{dn_j}{dt}(t)$ is nonnegative and, hence, we deduce the nonnegative character of the solution along the time evolution. ∎

3. Dimensionless equations. The aim of this section is to write the system in dimensionless form. Hence, we will identify some dimensionless physical parameters. Next, we appropriately order these parameters in terms of a quantity $h > 0$ intended to tend to 0. Studying the limit $h \to 0$ we obtain a nonlinear continuous drift-diffusion model, as described in the introduction. This approach relating discrete to continuous models is reminiscent of hydrodynamic limits in kinetic theory (see [9]). Actually, it has been used for models of phase transition, for example, in [7].

Let us introduce time and length units, respectively, denoted by $\mathcal{T}$ and $\mathcal{L}$. They correspond to observation scales. We also need characteristic values for the electron density and for the electric field, respectively, denoted by $\mathcal{N}$ and $\mathcal{F}$. For instance, it is quite natural to define $\mathcal{N}$ from the doping profile N_D^w and $\mathcal{F}$ from the emitter field F_-. Then, using the convention that overlined quantities are dimensionless, we set

$$\begin{cases} \mathcal{N}\ \overline{n_i}(\overline{t}) = n_i(\mathcal{T}\overline{t}), & \mathcal{N}\ \overline{N_D} = N_D^w, \\ \mathcal{F}\ \overline{F_i}(\overline{t}) = F_i(\mathcal{T}\overline{t}), & \mathcal{F}\ \overline{F_-}(\overline{t}) = F_-(\mathcal{T}\overline{t}), \\ \dfrac{\mathcal{L}}{\mathcal{T}}\ \overline{v}(\overline{F}) = v(\mathcal{F}\overline{F}), & \dfrac{\mathcal{L}}{\mathcal{T}}\ \overline{W^{(b,f)}}(\overline{F}) = W^{(b,f)}(\mathcal{F}\overline{F}), \\ \dfrac{\mathcal{L}^2}{\mathcal{T}}\ \overline{D}(\overline{F}) = D(\mathcal{F}\overline{F}), & \dfrac{\overline{\varepsilon}\mathcal{F}}{e}\ \dfrac{1}{\mathcal{T}}\ \overline{j^{(e)}}(\overline{F}) = j^{(e)}(\mathcal{F}\overline{F}). \end{cases}$$

Note that the emitter current density has been scaled with respect to the density $\frac{\overline{\varepsilon}}{e}\mathcal{F}$ instead of with respect to $\mathcal{N}$ (the other choice is also possible; the proof adapts immediately and the emitter current density disappears as $h \to 0$ in that case).

Therefore, we are led to the continuity equations in the following dimensionless form:

$$\frac{d\overline{n}_i}{d\bar{t}} = \frac{\mathcal{L}}{\ell}\Big(\overline{v}(\overline{F}_{i-1})\overline{n}_{i-1} - \overline{v}(\overline{F}_i)\overline{n}_i - \frac{\mathcal{L}}{\ell}\overline{D}(\overline{F}_{i-1})(\overline{n}_{i-1} - \overline{n}_i) + \frac{\mathcal{L}}{\ell}\overline{D}(\overline{F}_i)(\overline{n}_i - \overline{n}_{i+1})\Big)$$

for $i \in \{-N+1, \dots, N-1\}$ and

$$\begin{aligned} \frac{d\overline{n}_{-N}}{d\bar{t}} &= \frac{\mathcal{L}}{\ell}\Big(\frac{\ell}{\mathcal{L}}\frac{\bar{\varepsilon}\mathcal{F}}{e\mathcal{N}}\bar{j}^{(e)}(\overline{F}_{-N-1}) - \overline{n}_{-N}\overline{W}^{(b)}(\overline{F}_{-N-1}) \\ &\qquad -\overline{v}(\overline{F}_{-N})\overline{n}_{-N} - \frac{\mathcal{L}}{\ell}\overline{D}(\overline{F}_{-N})(\overline{n}_{-N} - \overline{n}_{-N+1})\Big), \\ \frac{d\overline{n}_N}{d\bar{t}} &= \frac{\mathcal{L}}{\ell}\Big(\overline{v}(\overline{F}_{N-1})\overline{n}_{N-1} + \frac{\mathcal{L}}{\ell}\overline{D}(\overline{F}_{N-1})(\overline{n}_{N-1} - \overline{n}_N) - \overline{n}_{-N}\overline{W}^{(f)}(\overline{F}_N)\Big). \end{aligned}$$

On the other hand, the Poisson equation reads

$$\frac{\bar{\varepsilon}\mathcal{F}}{e\mathcal{N}}\,(\overline{F}_i - \overline{F}_{i-1}) = (\overline{n}_i - \overline{N}_D)$$

for $i \in \{-N, \dots, N\}$.

In these expressions, we identify two dimensionless parameters

$$\alpha = \frac{\bar{\varepsilon}\mathcal{F}}{e\mathcal{N}}, \quad \beta = \frac{\mathcal{L}}{\ell}.$$

Roughly speaking, we go from the discrete equations to a continuous description by interpreting the difference between consecutive cells as differential quotients. It means that we shall consider the situation

$$\alpha = \beta = \frac{1}{h} \gg 1,$$

where h is a positive quantity intended to tend to 0. (Actually, we might suppose, with some obvious adaptations in the proofs, that $\alpha = \frac{1}{h} \gg 1$, and $\frac{\alpha}{\beta}$ has a finite positive limit.) Coming back to the physical meaning, the ordering for β means that the size of the cells is small compared to the observation length scale $\ell \ll \mathcal{L}$, while the ordering for α is an assumption about the data; the doping profile N_D^w is small compared to the density $\frac{\bar{\varepsilon}}{e}F_-$ associated with the electric field at the injecting contact ($\mathcal{N} \ll \frac{\bar{\varepsilon}}{e}\mathcal{F}$). Furthermore, we shall assume that the total length of the SL is given and is equal to $2X$, so that the number of cells in the SL also should be appropriately rescaled. Namely, the number of cells is defined in terms of the parameter $h > 0$ by

$$N^h = X/h.$$

The limit performed in this paper is motivated by the comparison between the profiles of the drift velocity and of the diffusion coefficient. Figure 3.1 shows these profiles for a 9nm/4nm GaAs/AlAs SL at $5K$, while the inset picture enlarges these coefficients in the low-field range. In this region the diffusion coefficient is larger than the drift velocity, which is close to zero; i.e., $v(F) \ll D(F)/\ell$ holds. This implies that the diffusion coefficient is large (order h^{-2}) in comparison to the drift velocity (order h^{-1}). Accordingly, we call this asymptotic approach low-field limit. A continuum limit also can be performed in a regime in which $v(F) \approx D(F)/\ell$; this high-field regime will be investigated in a forthcoming work. A complementary interpretation of our asymptotic analysis can be given in terms of a parameter (the so-called Lorentzian half-width) defining the Lorentzian functions involved in the expression of the coefficients of the DDD model; see [4]. The smaller the Lorentzian half-width, the lower the field.

A stationary solution for the DDD model can be obtained in the low-field range as shown by the dotted line in Figure 3.1 (right). In this experiment, we have applied

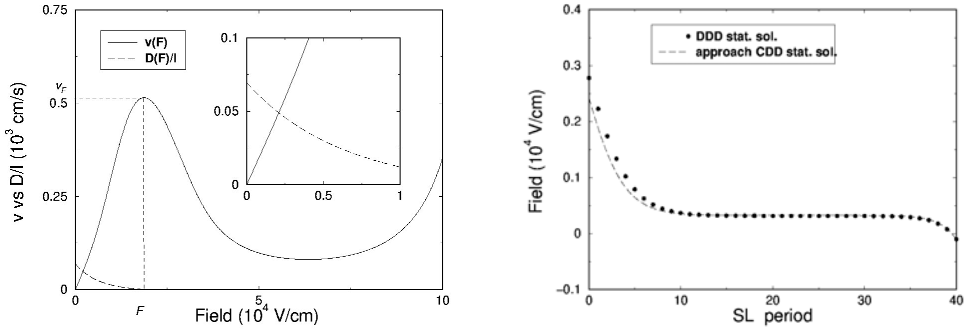

FIG. 3.1. *Left: Drift velocity versus diffusion coefficient for a* 9*nm*/4*nm GaAs/AlAs SL. Inset: Detailed low-field range. Right: Electric field distribution for a stationary solution of* 40 *periods* 9*nm*/4*nm GaAs/AlAs SL (dots) and numerical approach to the continuum drift-diffusion model (dashed line) with bias constraint* $V = 0.52$ V.

voltage 0.52 V, $N_D^w = 0.05 \times 10^{11}$ cm^{-2} and contact doping $N_D = 0.2 \times 10^{18}$ cm^{-3}. The Lorentzian half-width involved in the computation of the coefficients of the DDD model is 1 eV and the other parameters are equal to those used in [5]. Thus, $\mathcal{N} = N_D^w$, $\mathcal{F} \approx 0.3\, 10^4$ V/cm, $v \approx 0.001\, 10^3$ cm/s, and $D/\ell \approx 0.076\, 10^3$ cm/s. This leads to the values $\alpha = 3.29$ and $\beta = 7.67$; a sequence of values of the same order can be obtained (low-field limit) by modifying V.

Let us summarize the low-field problem we are interested in as follows. We drop the overlines and emphasize the dependence of the solution (n, F) with respect to the parameter h by a superscript. Hence, we consider the system

$$\frac{dn_i^h}{dt} = \frac{1}{h}(J_{i-1\to i}^h - J_{i\to i+1}^h), \quad i \in \{-N^h, \dots, N^h\}, \tag{3.1}$$

coupled to

$$F_i^h - F_{i-1}^h = h(n_i^h - N_D), \quad i \in \{-N^h, \dots, N^h\}, \tag{3.2}$$

with $F_{-N^h-1}^h = F_-$ given. Note that, coming back to (2.9), we also have

$$F_i^h(t) = F_-(t) + h \sum_{j=-N^h}^{i} (n_j^h(t) - N_D)\,, \quad i \in \{-N^h, \dots, N^h\}. \tag{3.3}$$

Here, we used the following definition for the tunneling currents:

$$\begin{cases} J_{i\to i+1}^h = n_i^h v_i^h - \dfrac{1}{h} D(F_i^h)(n_{i+1}^h - n_i^h), \quad i \in \{-N^h, \dots, N^h - 1\}, \\ J_{-N^h-1\to -N^h}^h = j^{(e)}(F_-) - n_{-N^h}^h W^{(b)}(F_-), \\ J_{N^h\to N^h+1}^h = n_{N^h} W^{(f)}(F_{N^h}^h). \end{cases}$$

The idea is to investigate the limit as $h \to 0$.

To this end, we set $I = (-X, +X) = (-N^h h, N^h h)$ and we associate to the unknowns $(n_{-N^h}^h, \dots, n_{N^h-1}^h) \in \mathbb{R}^{2N^h}$ and $(F_{-N^h}^h, \dots, F_{N^h-1}^h) \in \mathbb{R}^{2N^h}$, the stepwise constant functions $n^h(t,x)$ and $F^h(t,x)$ defined almost everywhere on $[0,\infty)\times I$ by saying

$$n^h(t,x) = n_i^h(t), \quad F^h(t,x) = F_i^h(t), \quad ih < x < (i+1)h, \quad i \in \{-N^h, \dots, N^h - 1\}.$$

Note that it is not relevant to define these functions on the negligible set of points $\{ih,\, i \in \{-N^h, \dots, N^h\}\}$; note also that $F_-, n_{N^h}^h, F_{N_h}^h$ seem to play no role in these definitions. However, they will be used in the definition of traces in the limit $h \to 0$. As a consequence of these definitions, we shall use that sums of n_i^h or F_i^h can be considered as integrals: for example, for any function $\psi : \mathbb{R} \to \mathbb{R}$ we have

$$\int_{-X}^{+X} \psi(n^h)\, dx = h \sum_{i=-N^h}^{N^h-1} \psi(n_i^h)\,,$$

because n_i^h is constant on $ih < x < (i+1)h$. Then, passing to a continuous variable, it is tempting to interpret finite differences as differential quotients. Following this rough idea, we formally guess that the limiting problem corresponding to $h \to 0$ consists of the following nonlinear drift-diffusion equation:

$$(3.4) \qquad \begin{cases} \partial_t n + \partial_x J(F,n) = 0 & \text{in } (0,T)\times I,\\ J(F,n) = v(F)n - D(F)\partial_x n, & \\ \partial_x F = n - N_D & \text{in } (0,T)\times I,\\ F(-X) = F_- & \text{on } (0,T),\\ J(F,n)(X) = W^{(f)}(F)n(X) & \text{on } (0,T),\\ J(F,n)(-X) = (j^{(e)}(F) - W^{(b)}(F)n)(-X) & \text{on } (0,T),\\ n(t=0,x) = n^0(x) & \text{on } I. \end{cases}$$

Thus, the main result of the paper is the following.

THEOREM 3.1. *Let $v, D, W^{(b,f)}, j^{(e)} : \mathbb{R} \to \mathbb{R}$ be continuous and nonnegative functions. Suppose that $D(F) > 0$ and $W^{(b,f)}(F) > 0$ for any $F \in \mathbb{R}$. Let $F_- \in \mathrm{C}^1(\mathbb{R}^+)$. Let $n^{h,0} = (n_{-N^h}^{h,0}, \dots, n_{N^h}^{h,0}) \in \mathbb{R}^{2N^h+1}$ be the initial data for the rescaled problem. We suppose that $n_i^{h,0} \geq 0$ satisfy*

$$(3.5) \qquad \sup_{h>0} \left(h \sum_{i=-N^h}^{N^h} |n_i^{h,0}|^2 \right) \leq C_0 < \infty.$$

Let (n^h, F^h) be the associated solution of (3.1), (3.2). Then, up to a subsequence, we have

$$\begin{cases} n^h \to n & \textit{strongly in } \mathrm{L}^2((0,T)\times I) \textit{ and in } \mathrm{C}^0([0,T];\mathrm{L}^2(I) - \textit{weak}),\\ F^h \to F & \textit{uniformly in } [0,T]\times \overline{I}. \end{cases}$$

Furthermore, the limits satisfy $n \in \mathrm{L}^2(0,T;\mathrm{H}^1(I))$, $F \in \mathrm{C}^0([0,T]\times\overline{I})$ and solve the nonlinear problem (3.4) in the sense that

$$\frac{d}{dt}\int_{-X}^{X} n\phi\, dx = \int_{-X}^{X} J(F,n)\phi'\, dx + W^{(f)}(F)n\phi(X) + (j^{(e)}(F) - W^{(b)}(F)n)\phi(-X)$$

holds in $\mathcal{D}'(0,T)$ for any test function $\phi \in \mathrm{C}^\infty(\overline{I})$, coupled to the Poisson equation

$$\partial_x F = n - N_D, \quad F(-X) = F_-$$

also considered in the sense of the distributions.

This kind of nonlinear parabolic equation, coupled to the Poisson equation, has been investigated by Liang [11]. Actually, in [11] the diffusion coefficient is constant and the boundary conditions are slightly different. In the convergence proof, we need only to assume the continuity of the coefficients; however, using locally Lipschitz properties of the coefficients, we can prove the uniqueness of solution for (3.4); see Appendix B. Consequently, assuming the convergence of the initial data, in Theorem 3.1 the entire sequence converges.

A stationary solution (continuous line) for the continuous drift-diffusion model (CDD) has been obtained in Figure 3.1 (right). The corresponding stationary solution for the DDD model, with the same voltage, can be seen now as a numerical approach to that of the CDD model with $h \in [\frac{1}{7.67}, \frac{1}{3.29}]$. We can observe that the agreement between the solutions to the discrete model and the continuous one is better at the inner periods, where the low-field hypothesis plays a determinant role. The difference between both profiles in the emitter region comes from the fact that the simulations have been done under bias constraint.

4. A priori estimates. This section is devoted to the derivation of the crucial estimates on the solutions (n^h, F^h) that will lead us to rigorously perform the limit $h \to 0$. We assume that the initial data $n_i^{h,0} \geq 0$ satisfies (3.5). This implies that the $\mathrm{L}^1[-X, X]$-norm is bounded as follows:

$$h \sum_{i=-N^h}^{N^h} n_i^{h,0} \leq \left(h \sum_{i=-N^h}^{N^h} |n_i^{h,0}|^2 \right)^{1/2} \sqrt{(2N^h+1)h}$$

is bounded independently of $h \in (0,1)$. We recall that

$$\left\{ \begin{array}{l} D, W^{(b,f)}, j^{(e)}, v \in \mathrm{C}^0(\mathbb{R}), \\ v(F) \geq 0, \quad j^{(e)} \geq 0, \\ W^{(b,f)}(F) > 0, \quad D(F) > 0. \end{array} \right. \tag{4.1}$$

We split our argument into several steps. We shall use the convention that C_T stands for a constant possibly depending on T and on the data $F_-, j^{(e)}, W^{(b,f)}$, or on the estimates (3.5), but not on h. Also, we denote as usual by $\mathcal{M}(I)$ the set of Radon measures on the open interval I. Elements of $\mathcal{M}(I)$ identify with distributions Φ on I satisfying $|\langle \Phi, \varphi \rangle| \leq C \|\varphi\|_{L^\infty(I)}$ for all $\varphi \in C_c^\infty(I)$ for some $C > 0$ being independent of the support of the test function (see, e.g., [13]). As usual we denote by $\mathrm{BV}(I)$ the set of bounded variation functions, i.e., functions which are in $\mathrm{L}^1(I)$ and such that their distributional derivative belongs to $\mathcal{M}(I)$.

LEMMA 4.1 (L^1 estimate on the density). *The sequence n^h is bounded in $\mathrm{L}^\infty(0, T; \mathrm{L}^1(I))$.*

Proof. Summing up the equations in (3.1), we obtain

$$\begin{aligned} h \frac{d}{dt} \sum_{i=-N^h}^{N^h} n_i^h &= \sum_{i=-N^h}^{N^h} (J_{i-1\to i}^h - J_{i\to i+1}^h) = J_{-N^h-1\to -N^h}^h - J_{N^h\to N^h+1}^h \\ &= j^{(e)}(F_-) - n_{-N^h}^h W^{(b)}(F_-) - n_{N^h}^h W^{(f)}(F_{N^h}^h). \end{aligned}$$

Therefore, integrating with respect to time and using $n_i^h \geq 0$ and $W^{(b,f)} \geq 0$, we find

$$\begin{aligned}(4.2)\qquad & h\sum_{i=-N^h}^{N^h} n_i^h(t) + \int_0^t n_{-N^h}^h W^{(b)}(F_-(s))\,ds + \int_0^t n_{N^h}^h W^{(f)}(F_{N^h}^h)(s)\,ds \\ & = h\sum_{i=-N^h}^{N^h} n_i^{h,0} + \int_0^t j^{(e)}(F_-(s))\,ds \leq C_0 + \|j^{(e)}(F_-)\|_{L^1(0,T)} \leq C_T\,,\end{aligned}$$

which concludes the proof. □

LEMMA 4.2 (estimates on the electric field). *The sequence F^h is bounded in* $\mathrm{L}^\infty((0,T)\times I)$ *and in* $\mathrm{L}^\infty(0,T;\mathrm{BV}(I))$, *while $F_{N_h}^h$ is bounded in* $\mathrm{L}^\infty(0,T)$.

Proof. We combine the estimate in Lemma 4.1 with the identity (3.3) to yield

$$\begin{aligned}|F_i^h(t)| &= \left|F_-(t) + h\sum_{j=-N^h}^{i}(n_j^h(t) - N_D)\right| \\ &\leq |F_-(t)| + h\sum_{j=-N^h}^{i} n_j^h(t) + h(i+N^h+1)N_D \\ &\leq |F_-(t)| + h\sum_{j=-N^h}^{N^h} n_j^h(t) + (2X+h)N_D \leq C_T\,,\end{aligned}$$

which proves that F^h is bounded in $L^\infty((0,T)\times I)$ and implies the estimate on $F_{N^h}^h$.

Next, let $\phi \in \mathrm{C}_0^\infty(I)$ be a test function. We have

$$\begin{aligned}\langle \partial_x F^h, \phi\rangle &= -\int_{-X}^{X} F^h(t,x)\phi'(x)\,dx = -\sum_{i=-N^h}^{N^h-1} F_i^h \int_{ih}^{(i+1)h}\phi'(x)\,dx \\ &= \sum_{i=-N^h}^{N^h-1} F_i^h(\phi(ih) - \phi((i+1)h)) \\ &= \sum_{i=-N^h}^{N^h}\Big((F_i^h - F_{i-1}^h)\ \phi(ih)\Big) + F_{-N^h-1}^h\phi(-N^h h) - F_{N^h}^h\phi(N^h h) \\ &= h\sum_{i=-N^h}^{N^h}\Big((n_i^h - N_D)\ \phi(ih)\Big) + F_-\phi(-X) - F_{N^h}^h\phi(X),\end{aligned}$$

where we have used (3.2). Hence, by using the above bounds we deduce that the following estimate,

$$|\langle \partial_x F^h, \phi\rangle| \leq \|\phi\|_{\mathrm{L}^\infty(I)}\left(h\sum_{i=-N^h}^{N^h} n_i^h + (2X+h)N_D\right) \leq \|\phi\|_{\mathrm{L}^\infty(I)}\ C_T\,,$$

holds. This proves that $\partial_x F^h$ is bounded in $\mathrm{L}^\infty(0,T;\mathcal{M}(I))$. □

Remark 4.3. Since the functions $W^{(b,f)}$ and D are continuous and positive in $\mathbb{R}$, the uniform bound on F_i^h guarantees that

$$\begin{cases} \inf_{h>0, i\in\{-N^h,\ldots,N^h\}, 0\le t\le T} D(F_i^h(t)) \ge \delta > 0, \\ \inf_{h>0, 0\le t\le T} W^{(f)}(F_{N^h}^h(t)) \ge \delta > 0, \\ \inf_{0\le t\le T} W^{(b)}(F_-(t)) \ge \delta > 0 \end{cases}$$

for some $\delta > 0$. Coming back to (4.2), we deduce that the boundary terms $n_{\pm N^h}$ are bounded in $\mathrm{L}^1(0,T)$. Similarly, there exists $0 < M < \infty$ such that

$$\begin{cases} \sup_{h>0, i\in\{-N^h,\ldots,N^h\}, 0\le t\le T} |D(F_i^h)| \le M, \\ \sup_{h>0, i\in\{-N^h,\ldots,N^h\}, 0\le t\le T} |v(F_i^h)| \le M, \\ \sup_{h>0, 0\le t\le T} |W^{(f)}(F_{N_h}^h)| \le M, \\ \sup_{h>0, 0\le t\le T} |W^{(b)}(F_-^h)| \le M, \\ \sup_{0\le t\le T} |j^{(e)}(F_-)| \le M. \end{cases}$$

LEMMA 4.4 (L^2 estimate on the density). *The sequence n^h is bounded in $\mathrm{L}^\infty(0,T;$ $\mathrm{L}^2(I))$. The "boundary terms" $n_{\pm N^h}^h$ are bounded in $\mathrm{L}^2(0,T)$. Moreover, we have*

$$\int_0^T \sum_{i=-N^h}^{N^h-1} \frac{|n_{i+1}^h - n_i^h|^2}{h}\, ds \le C_T\,.$$

Proof. Multiplying (3.1) by n_i^h and summing over i, we obtain

$$\begin{aligned} \frac{h}{2}\frac{d}{dt}\sum_{i=-N^h}^{N^h} |n_i^h|^2 &= \sum_{i=-N^h}^{N^h} (J_{i-1\to i}^h - J_{i\to i+1}^h) n_i^h \\ &= \sum_{i=-N^h}^{N^h-1} J_{i\to i+1}^h (n_{i+1}^h - n_i^h) + J_{-N^h-1\to -N^h}^h n_{-N^h}^h - J_{N^h\to N^h+1}^h n_{N^h}^h \\ &= \sum_{i=-N^h}^{N^h-1} \left(n_i^h v(F_i^h) - \frac{1}{h} D(F_i^h)(n_{i+1}^h - n_i^h) \right) (n_{i+1}^h - n_i^h) \\ &\qquad + j^{(e)}(F_-) n_{-N^h}^h - |n_{-N^h}^h|^2 W^{(b)}(F_-) - |n_{N^h}|^2 W^{(f)}(F_{N^h}^h). \end{aligned}$$

By using Remark 4.3, we deduce the inequality

$$\frac{h}{2}\sum_{i=-N^h}^{N^h}|n_i^h(t)|^2+\delta\int_0^t\left(\sum_{i=-N^h}^{N^h-1}\frac{|n_{i+1}^h-n_i^h|^2}{h}+|n_{-N^h}^h|^2+|n_{N^h}|^2\right)ds$$
$$\leq\frac{h}{2}\sum_{i=-N^h}^{N^h}|n_i^h(0)|^2+M\int_0^t\left(\sum_{i=-N^h}^{N^h-1}n_i^h|n_{i+1}^h-n_i^h|+n_{-N^h}^h\right)ds\,.$$

Now, by using the Young inequality we estimate

$$\int_0^t\sum_{i=-N^h}^{N^h-1}n_i^h|n_{i+1}^h-n_i^h|\,ds\leq\frac{2Mh}{\delta}\int_0^t\sum_{i=-N^h}^{N^h}|n_i^h|^2\,ds+\frac{\delta}{2M}\int_0^t\sum_{i=-N^h}^{N^h-1}\frac{|n_{i+1}^h-n_i^h|^2}{h}\,ds.$$

It follows that

$$\frac{h}{2}\sum_{i=-N^h}^{N^h}|n_i^h(t)|^2+\frac{\delta}{2}\int_0^t\sum_{i=-N^h}^{N^h-1}\frac{|n_{i+1}^h-n_i^h|^2}{h}\,ds+\delta\int_0^t(|n_{-N^h}^h|^2+|n_{N^h}|^2)\,ds$$
$$\leq\frac{h}{2}\sum_{i=-N^h}^{N^h}|n_i^h(0)|^2+\frac{2M^2}{\delta}\int_0^t\left(h\sum_{i=-N^h}^{N^h}|n_i^h|^2\right)ds+M\int_0^t n_{-N^h}^h\,ds.$$

We conclude the proof by applying the Gronwall inequality and by taking into account that $n_{-N^h}^h$ is bounded in $\mathrm{L}^1(0,T)$ (see Remark 4.3). ∎

In order to study the limit in boundary terms we consider the next statement.

LEMMA 4.5 (H^1 estimate of the electric field at the boundary). *The sequence $F_{N^h}^h$ is bounded in* $\mathrm{H}^1(0,T)$.

Proof. We have proved that $F_{N^h}^h$ is bounded in $\mathrm{L}^\infty(0,T)$. There remains to bound its time derivative in $\mathrm{L}^2(0,T)$. This is a consequence of (3.3) together with the estimates in Lemma 4.2 and 4.4. Indeed, we get (see the argument given in Lemma 4.1)

$$\begin{aligned}\left|\frac{d}{dt}F_{N^h}^h(t)\right| &= \left|\frac{d}{dt}F_-+\frac{d}{dt}\left(h\sum_{i=-N^h}^{N^h}(n_i^h-N_D)\right)\right|\\ &= \left|\frac{d}{dt}F_-+j^{(e)}(F_-)-n_{-N^h}^hW^{(b)}(F_-)-n_{N^h}^hW^{(f)}(F_{N^h}^h)\right|\\ &\leq \left\|\frac{d}{dt}F_-\right\|_{\mathrm{L}^\infty(0,T)}+M(1+n_{-N^h}^h+n_{N^h}^h).\end{aligned}$$

By Lemma 4.4 the right-hand side is bounded in $\mathrm{L}^2(0,T)$, which ends the proof. ∎

LEMMA 4.6 (BV estimate on the density). *The sequence n^h is bounded in* $\mathrm{L}^2(0,T;\mathrm{BV}(I))$.

Proof. Once the L^2 estimate on n^h is known, we derive some bounds for $\partial_x n^h$.

Consider $\phi \in \mathrm{C}_0^\infty(I)$. We have

$$
\begin{aligned}
|\langle \partial_x n^h, \phi\rangle| &= \left| -\int_{-X}^{X} n^h \phi' \, dx \right| = \left| -\sum_{i=-N^h}^{N^h-1} n_i^h \int_{ih}^{(i+1)h} \phi' \, dx \right| \\
&= \left| -\sum_{i=-N^h}^{N^h-1} n_i^h \, (\phi((i+1)h) - \phi(ih)) \right| \\
&= \left| \sum_{i=-N^h+1}^{N^h} (n_i^h - n_{i-1}^h) \, \phi(ih) + n_{-N^h}^h \phi(-N^h h) - n^h N^h \phi(N^h h) \right| \\
&\le \left(h \sum_{i=-N^h+1}^{N^h} |\phi(ih)|^2 \right)^{1/2} \left(\frac{1}{h} \sum_{i=-N^h+1}^{N^h} |n_i^h - n_{i-1}^h|^2 \right)^{1/2} \\
&\le \|\phi\|_{\mathrm{L}^\infty(I)} \, (2X)^{1/2} \left(\sum_{i=-N^h}^{N^h-1} \frac{|n_{i+1}^h - n_i^h|^2}{h} \right)^{1/2} . \qquad (4.3)
\end{aligned}
$$

Lemma 4.4 implies that the $\mathrm{L}^2(0,T)$-norm of the right-hand side of (4.3) is bounded uniformly with respect to h. Hence, we conclude that $\partial_x n^h$ is in $\mathrm{L}^2(0,T;\mathcal{M}(I))$. ∎

LEMMA 4.7 (estimate on the time derivative). *The sequences $\partial_t n^h$ and $\partial_t F^h$ are bounded in $\mathrm{L}^2(0,T;\mathcal{M}(I)) + \mathrm{L}^2(0,T;\mathrm{W}^{-1,1}(I))$ and in $\mathrm{L}^2(0,T;\mathcal{M}(I))$, respectively.*

Proof. Let $\phi \in \mathrm{C}_0^\infty(I)$ and denote

$$
\Gamma_i^h = \int_{ih}^{(i+1)h} \phi(x) \, dx
$$

for $i \in \{-N^h, \dots, N^h - 1\}$. Since the support of ϕ is included in I, we can extend Γ_i^h by 0 for $i \ge N^h$. We shall use the following basic estimates:

$$
\begin{cases}
|\Gamma_i^h| \le h \, \|\phi\|_{\mathrm{L}^\infty(I)}, \\
|\Gamma_{i+1}^h - \Gamma_i^h| \le h^2 \, C \|\phi'\|_{\mathrm{L}^\infty(I)}.
\end{cases}
$$

Now we estimate the time derivative of the electric field by using the Ampère equations (2.3). We have

$$
\begin{aligned}
\langle \partial_t F^h, \phi\rangle &= \sum_{i=-N^h}^{N^h-1} \frac{d}{dt} F_i^h \int_{ih}^{(i+1)h} \phi(x) \, dx \\
&= J^h \sum_{i=-N^h}^{N^h-1} \Gamma_i^h - \sum_{i=-N^h}^{N^h-1} J_{i\to i+1}^h \Gamma_i^h = I_1 + I_2, \qquad (4.4)
\end{aligned}
$$

where $J^h(t)$ stands for the total current density, which is defined by the $(-N^h-1)$th Ampère equation,

$$
J^h(t) = \frac{d}{dt} F_- + J_{-N^h-1\to -N^h}^h = \frac{d}{dt} F_- + j^{(e)}(F_-) - W^{(b)}(F_-) n_{-N^h}^h .
$$

By Lemma 4.4, this quantity is bounded in $\mathrm{L}^2(0,T)$. Therefore, the first term of the right-hand side of (4.4) is bounded by

$$|I_1| \le \|\varphi\|_{\mathrm{L}^\infty(I)}\ 2hN^h\ |J^h| = \|\varphi\|_{\mathrm{L}^\infty(I)}\ 2X\ |J^h|,$$

which belongs to a bounded set in $\mathrm{L}^2(0,T)$. Next, I_2 is estimated as follows:

$$\begin{aligned}
|I_2| &\le \left|\sum_{i=-N^h}^{N^h-1} n_i^h v(F_i^h)\ \Gamma_i^h\right| + \left|\sum_{i=-N^h}^{N^h-1} \frac{1}{h} D(F_i^h)(n_i^h - n_{i+1}^h)\ \Gamma_i^h\right| \\
&\le M\ \|\phi\|_{\mathrm{L}^\infty(I)}\ h \left(\sum_{i=-N^h}^{N^h-1} n_i^h + \sum_{i=-N^h}^{N^h-1} \frac{|n_i^h - n_{i+1}^h|}{h}\right) \\
&\le M\ \|\phi\|_{\mathrm{L}^\infty(I)} \left(h\sum_{i=-N^h}^{N^h-1} n_i^h + \sqrt{2hN^h \sum_{i=-N^h}^{N^h-1} \frac{|n_i^h - n_{i+1}^h|^2}{h}}\right).
\end{aligned}$$

We conclude that $\partial_t F^h$ is bounded in $\mathrm{L}^2(0,T;\mathcal{M}^1(I))$.

Similarly, we deal with the time derivative of n^h. We have

$$\begin{aligned}
&|\langle \partial_t n^h, \varphi\rangle| = \left|\sum_{i=-N^h}^{N^h-1} \frac{dn_i^h}{dt} \int_{ih}^{(i+1)h} \phi(x)\,dx\right| = \left|\frac{1}{h}\sum_{i=-N^h}^{N^h-1} (J_{i-1\to i}^h - J_{i\to i+1}^h)\ \Gamma_i^h\right| \\
&= \frac{1}{h}\left|\sum_{i=-N^h}^{N^h-1} J_{i\to i+1}^h\ (\Gamma_{i+1}^h - \Gamma_i^h) + J_{-N^h-1\to -N^h}^h \Gamma_{-N^h}^h - J_{N^h-1\to N^h}^h \Gamma_{N^h}^h\right| \\
&\le \frac{1}{h}\left|\sum_{i=-N^h}^{N^h-1} v(F_i^h) n_i^h\ (\Gamma_{i+1}^h - \Gamma_i^h)\right| + \frac{1}{h^2}\left|\sum_{i=-N^h}^{N^h-1} D(F_i^h)(n_i^h - n_{i+1}^h)(\Gamma_{i+1}^h - \Gamma_i^h)\right| \\
&\qquad + \frac{1}{h}|j^{(e)}(F_-) - n_{-N^h}^h W^{(b)}(F_-)|\ |\Gamma_{-N^h}^h| \\
&\le C\ h^2\ \|\phi'\|_{\mathrm{L}^\infty(I)} \left(\frac{M}{h}\sum_{i=-N^h}^{N^h-1} n_i^h + \frac{M}{h^2}\sum_{i=-N^h}^{N^h-1} |n_i^h - n_{i+1}^h|\right) \\
&\qquad + h\ \|\phi\|_{\mathrm{L}^\infty(I)}\ \frac{M}{h}(1 + n_{-N^h}^h) \\
&\le C\ \|\phi'\|_{\mathrm{L}^\infty(I)} \left(h\sum_{i=-N^h}^{N^h-1} n_i^h + \sqrt{2hN^h \sum_{i=-N^h}^{N^h-1} \frac{|n_i^h - n_{i+1}^h|^2}{h}}\right) \\
&\qquad + \|\phi\|_{\mathrm{L}^\infty(I)}\ M(1 + n_{-N^h}^h),
\end{aligned}$$

which proves the estimate on $\partial_t n^h$. ∎

5. Continuous model. Let us combine the estimates discussed in the previous section with the following classical compactness result (see, e.g., [2], [14]).

PROPOSITION 5.1. *Consider Banach spaces B, X, and Y. We suppose that $X \subset B \subset Y$, the first embedding being compact. Let $\mathcal{C}$ be a bounded set in $\mathrm{L}^p(0,T;X)$, $1 \le p \le \infty$. Assume that $\partial_t \mathcal{C} = \{\partial_t f,\ f \in \mathcal{C}\}$ is a bounded set in $\mathrm{L}^r(0,T;Y)$. Then, $\mathcal{C}$ is relatively compact in $\mathrm{L}^p(0,T;B)$ if $1 \le p < \infty$ and $r \ge 1$, or in $\mathrm{C}^0([0,T];B)$ if $p = \infty$ and $r > 1$.*

Hence, from the previous estimates we have, possibly at the cost of extracting subsequences, that

$$(5.1)\quad \begin{cases} n^h \to n & \text{strongly in } \mathrm{L}^2((0,T)\times I) \text{ and in } \mathrm{C}^0([0,T]; L^2(I) - weak), \\ \partial_x n^h \rightharpoonup \partial_x n & \text{weakly}-* \text{ in } \mathrm{L}^2(0,T;\mathcal{M}(I)), \\ F^h \to F & \text{strongly in } \mathrm{C}^0([0,T]; L^p(I)) \text{ for any } 1 \le p < \infty, \end{cases}$$

as h goes to 0. Notice in particular that the convergence of traces in time makes sense and

$$n^h(t,x)_{|t=0} = n^{h,0}(x) \rightharpoonup n^0(x) = n(t,x)_{|t=0} \text{ weakly in } \mathrm{L}^2(I)$$

holds, with $n^{h,0}(x) = n_i^h$ for $ih < x < (i+1)h$, $i \in \{-N^h, \ldots, N^h - 1\}$. In other words, we recover the initial condition in the limit $h \to 0$. Finally, we can also guarantee from Lemmas 4.4 and 4.5 the following properties:

$$(5.2)\quad \begin{cases} n^h_{\pm N^h} \rightharpoonup n_\pm & \text{weakly in } \mathrm{L}^2(0,T), \\ F^h_{N^h} \to F_+ & \text{uniformly in } \mathrm{C}^0([0,T]). \end{cases}$$

We first get the continuous Poisson equation.

PROPOSITION 5.2. *The electric field limit F and the density limit n satisfy the continuous Poisson equation*

$$\partial_x F = n - N_D, \quad F_{|x=-X} = F_-$$

in a weak sense.

Remark 5.3. The Poisson relation with $n \in \mathrm{L}^2((0,T)\times I)$ implies, by the Sobolev embedding, that F is in $\mathrm{L}^2(0,T;\mathrm{C}^0(\bar{I}))$ so that the traces of F are well defined.

Proof. Let $\phi \in C^\infty(\bar{I})$ and $\phi_i^h = \phi(ih)$ for $i \in \{-N^h, \ldots, N^h\}$. We denote by ϕ^h the associated stepwise constant function. For the sake of simplicity it will be convenient to also introduce the stepwise constant function $\nabla^h(\phi)(x) = \frac{\phi_{i+1}^h - \phi_i^h}{h}$ for $x \in (ih, (i+1)h)$. Multiplying (3.2) by ϕ_i^h, we get

$$\begin{aligned} h \sum_{i=-N^h}^{N^h} \frac{F_i^h - F_{i-1}^h}{h}\, \phi_i^h &= h \sum_{i=-N^h}^{N^h} (n_i^h - N_D)\, \phi_i^h \\ &= \int_{-X}^{X} (n^h - N_D)\, \phi^h \, dx + h\, (n^h_{N^h} - N_D)\, \phi(X) \\ &= h \sum_{i=-N^h}^{N^h - 1} F_i^h\, \frac{\phi_i^h - \phi_{i+1}^h}{h} + F^h_{N^h}\phi(X) - F_-\phi(-X) \\ &= -\int_{-X}^{X} F^h\, \nabla^h(\phi)\, dx + F^h_{N^h}\phi(X) - F_-\phi(-X). \end{aligned}$$

Since $\nabla^h(\phi)$ converges uniformly to $\phi'(x)$ on $\overline{I}$, we have

$$\int_{-X}^{X} (n^h - N_D)\ \phi^h\, dx \to -\int_{-X}^{X} F\ \phi'(x)\, dx + F_+\phi(X) - F_-\phi(-X)$$

as $h \to 0$.

We conclude that $\partial_x F = n - N_D \in \mathrm{L}^2((0,T)\times I)$ and, by the Sobolev embedding, F lies in $\mathrm{L}^2(0,T;\mathrm{C}^0(\overline{I}))$ and the traces of F are well defined and are given by $F(t,\pm X) = F_\pm(t)$. □

Let us now show that the limit n is more regular than n^h is. In fact, we will prove that $n \in \mathrm{L}^2(0,T;\mathrm{H}^1(I))$, which guarantees that $n \in \mathrm{L}^2(0,T;\mathrm{C}^0(\overline{I}))$ due to the Sobolev embedding, so that the traces of the limit n with respect to the space variable are also well defined.

PROPOSITION 5.4. *The density limit n of n^h belongs to* $\mathrm{L}^2(0,T;\mathrm{H}^1(I))$.

Proof. Let $\phi \in \mathrm{C}^\infty_c(I)$. We have seen in the proof of Lemma 4.6 that the estimate

$$\|\langle \partial_x n^h, \phi\rangle\|_{\mathrm{L}^2(0,T)} \le C_T \left(h \sum_{i=-N^h+1}^{N^h} |\phi(ih)|^2 \right)^{1/2} = C_T\ \|\phi^h\|_{\mathrm{L}^2(I)}$$

holds. We also readily check that ϕ^h tends to ϕ in $\mathrm{L}^2(I)$. Hence, letting $h \to 0$ leads to

$$\|\langle \partial_x n, \phi\rangle\|_{\mathrm{L}^2(0,T)} \le C_T\ \|\phi\|_{\mathrm{L}^2(I)}.$$

By a density argument the estimate can be extended for any function $\phi \in \mathrm{L}^2(I)$. We conclude that $\partial_x n \in L^2((0,T)\times I)$. □

Convergence properties stronger than (5.1) will be necessary due to the nonlinear term. The idea is that the estimate in Lemma 4.4 is close to an $\mathrm{L}^2(0,T;\mathrm{H}^1(I))$ estimate on n^h. To this end we introduce the following $\mathbb{P}_1$ approximation: for $x \in (ih,(i+1)h)$, $i \in \{-N^h,\dots,N^h-1\}$, we set

$$\text{(5.3)} \qquad \begin{cases} m^h(t,x) = \dfrac{n^h_{i+1} - n^h_i}{h}\ (x - ih) + n^h_i, \\ G^h(t,x) = \dfrac{F^h_{i+1} - F^h_i}{h}\ (x - ih) + F^h_i. \end{cases}$$

Then, the sequences (m^h, G^h) are close to the original quantities (n^h, F^h) and enjoy better compactness properties as shown in the following lemma.

LEMMA 5.5. *The following estimates are verified:*

$$\begin{cases} \|n^h - m^h\|_{\mathrm{L}^2((0,T)\times I)} \le C_T\ h, \\ \|F^h - G^h\|_{\mathrm{L}^\infty((0,T)\times I)} \le C_T\ \sqrt{h}. \end{cases}$$

Furthermore, $\left(m^h\right)_{h>0}$ is relatively compact in $\mathrm{L}^2(0,T;\mathrm{C}^0(\overline{I}))$ and $\left(G^h\right)_{h>0}$ is relatively compact in $\mathrm{C}^0([0,T]\times\overline{I})$.

Proof. By taking into account the definition of the $\mathbb{P}_1$ approximations, we have

$$m^h(t,x) - n^h(t,x) = \frac{n^h_{i+1} - n^h_i}{h}\ (x - ih)$$

in the interval $(ih,(i+1)h)$, $i \in \{-N^h,\dots,N^h-1\}$. Hence, by using Lemma 4.4 we get

$$\begin{aligned}\|m^h - n^h\|^2_{\mathrm{L}^2((0,T)\times I)} &= \int_0^T \sum_{i=-N^h}^{N^h-1} \left|\frac{n^h_{i+1}-n^h_i}{h}\right|^2 \int_{ih}^{(i+1)h} (x-ih)^2\,dx\,ds \\ &= \frac{h^2}{3}\int_0^T \sum_{i=-N^h}^{N^h-1} \frac{|n^h_{i+1}-n^h_i|^2}{h}\,ds \le C_T\ h^2\,.\end{aligned}$$

On the other hand, (3.2) yields

$$\begin{aligned}|G^h(t,x)-F^h(t,x)| &= \left|\frac{F^h_{i+1}-F^h_i}{h}\ (x-ih)\right| \\ &= |n^h_{i+1}-N_D|\ (x-ih) \le |n^h_{i+1}-N_D|\ h\end{aligned}$$

for $x \in (ih,(i+1)h)$, $i \in \{-N^h,\dots,N^h-1\}$. Therefore, Lemma 4.4 allows us to control this quantity as follows:

$$\begin{aligned}|G^h(t,x)-F^h(t,x)| &\le \sqrt{h}\,\sqrt{h}\ (n^h_{i+1}+N_D) \\ &\le \sqrt{h}\left(\left(h|n^h_{i+1}|^2\right)^{1/2} + \sqrt{h}N_D\right) \\ &\le \sqrt{h}\left(\left(h\sum_{j=-N^h}^{N^h}|n^h_j|^2\right)^{1/2} + \sqrt{h}N_D\right) \le C_T\ \sqrt{h}.\end{aligned}$$

This proves the first part of the result.

Note that m^h and G^h are bounded in $\mathrm{L}^2(0,T;\mathrm{H}^1(I))$ and $\mathrm{L}^\infty(0,T;\mathrm{H}^1(I))$, respectively. Indeed, we have $\partial_x m^h = (n^h_{i+1}-n^h_i)/h$ on $(ih,(i+1)h)$, and the bound for $\partial_x m^h$ in L^2 follows directly from Lemma 4.4. For the approximate electric field we have $\partial_x G^h = (F^h_{i+1}-F^h_i)/h = n^h_{i+1}-N_D$, so that

$$\begin{aligned}\|\partial_x G^h\|^2_{\mathrm{L}^2(I)} &= \sum_{i=-N^h}^{N^h}|n^h_{i+1}-N_D|^2\int_{ih}^{(i+1)h}dx \le 2\sum_{i=-N^h}^{N^h}\left(|n^h_{i+1}|^2+N_D^2\right)h \\ &\le 2\left(h\sum_{i=-N^h}^{N^h}|n^h_{i+1}|^2+(2X+h)N_D^2\right) \le C_T.\end{aligned}$$

Hence, to justify the compactness properties there remains to obtain some estimates on the time derivatives. We check that (see Appendix A)

$$\begin{aligned}&\partial_t(G^h-F^h) \quad \text{is bounded in } \mathrm{L}^2(0,T;\mathcal{M}(I)),\\ &\partial_t(m^h-n^h) \quad \text{is bounded in } \mathrm{L}^2(0,T;\mathcal{M}(I))+\mathrm{L}^2(0,T;\mathrm{W}^{-1,1}(I)).\end{aligned} \tag{5.4}$$

Then, combining this information with Lemma 4.7, we deduce the asserted compactness by application of Proposition 5.1. ◻

As a consequence of the compactness property, and by identifying limits, we can assure that

$$\begin{cases} G^h \to F & \text{uniformly on } [0,T]\times\overline{I},\\ m^h \to n & \text{strongly in } \mathrm{L}^2(0,T;C^0(\overline{I})),\\ \partial_x m^h \rightharpoonup \partial_x n & \text{weakly in } \mathrm{L}^2((0,T)\times I).\end{cases} \tag{5.5}$$

Since G^h is $\sqrt{h}$-close to F^h in the L^∞-norm, we can improve the convergence in (5.1). Actually, we have

$$F^h \to F \text{ uniformly on } [0,T] \times \overline{I}. \tag{5.6}$$

Notice also in (5.5) that the traces are well defined and the following convergences

$$\begin{cases} m^h(\pm X) = n^h_{\pm N^h} \to n(\pm X) = n_\pm & \text{strongly in } \mathrm{L}^2(0,T), \\ G^h(\pm X) = F^h_{\pm N^h} \to F(\pm X) = F_\pm & \text{strongly in } \mathrm{L}^2(0,T), \end{cases}$$

hold. In particular, the traces of n at $\pm X$ can be identified with the limits $n_\pm$, respectively, which were defined in (5.2).

In order to pass to the limit in the equation, we write a discrete weak formulation. Let $\phi \in \mathrm{C}^\infty(\overline{I})$. We denote $\phi^h_i = \phi(ih)$, and ϕ^h stands for the associated piecewise constant approximation. Then, we get

(5.7)
$$\begin{aligned} h \sum_{i=-N^h}^{N^h} \frac{d}{dt} n^h_i \; \phi^h_i &= \sum_{i=-N^h}^{N^h} (J^h_{i-1\to i} - J^h_{i\to i+1}) \; \phi^h_i \\ &= \sum_{i=-N^h}^{N^h-1} J^h_{i\to i+1} \; (\phi^h_{i+1} - \phi^h_i) - J^h_{N^h\to N^h+1} \phi^h_{N^h} + J^h_{-N^h-1\to -N^h} \phi^h_{-N^h} \\ &= \sum_{i=-N^h}^{N^h-1} v(F^h_i) n^h_i \; (\phi^h_{i+1} - \phi^h_i) - \sum_{i=-N^h}^{N^h-1} D(F^h_i) \frac{1}{h} \; (n^h_{i+1} - n^h_i) \; (\phi^h_{i+1} - \phi^h_i) \\ &\quad - W^{(f)}(F^h_{N^h}) n^h_{N^h} \phi^h_{N^h} + (j^{(e)}(F_-) - W^{(b)}(F_-) n^h_{-N^h}) \phi^h_{-N^h}. \end{aligned}$$

Let us rewrite the discrete sums as integrals as follows:

$$\begin{aligned} &\frac{d}{dt} \int_{-X}^{X} n^h \; \phi^h \, dx + h \frac{d}{dt} n^h_{N^h} \; \phi(X) \\ &\quad = \int_{-X}^{X} v(F^h) n^h \; \nabla^h \phi \, dx - \int_{-X}^{X} D(F^h) \partial_x m^h \; \nabla^h \phi \, dx \\ &\qquad - W^{(f)}(F^h_{N^h}) n^h_{N^h} \phi(X) + (j^{(e)}(F_-) - W^{(b)}(F_-) n^h_{-N^h}) \phi(-X), \end{aligned} \tag{5.8}$$

following the notation $\nabla^h \phi(x) = (\phi^h_{i+1} - \phi^h_i)/h$, for $x \in (ih, (i+1)h)$. We can now pass to the limit $h \to 0$.

We check that $\phi^h \to \phi$ and $\nabla^h \phi \to \phi'$ uniformly on $\overline{I}$. Let us pass to the limit in each term of (5.8). Taking into account that $n^h \to n$ in $\mathrm{C}^0([0,T]; \mathrm{L}^2(I) - weak)$, we have $\int_{-X}^{X} n^h \phi^h \, dx \to \int_{-X}^{X} n \phi \, dx$ in $\mathrm{C}^0([0,T])$. Since $n^h_{N^h}$ is bounded in $\mathrm{L}^2(0,T)$, the second term in the left-hand side of (5.8) vanishes as $h \to 0$ in $\mathcal{D}'(0,T)$. Next, by using (5.6), $v(F^h)\nabla^h \phi \to v(F)\phi'$ and $D(F^h)\nabla^h \phi \to D(F)\phi'$ uniformly on $[0,T] \times \overline{I}$. To do that we combine the strong convergence $n^h \to n$ and the weak convergence $\partial_x m^h \to \partial_x n$ in $\mathrm{L}^2((0,T) \times I)$ so that the integrals in the right-hand side of (5.8) tend to

$$\int_X^X v(F) n \; \phi', dx - \int_X^X D(F) \partial_x n \; \phi' \, dx$$

as $h \to 0$ in $\mathcal{D}'(0,T)$. Finally, for the boundary terms we combine the convergence properties in (5.2) to find as the limit as $h \to 0$ the expression

$$-W^{(f)}(F) n \phi(X) + \Big(j^{(e)}(F) - W^{(b)}(F) n \Big) \phi(-X) \, .$$

Therefore, letting $h \to 0$ in (5.8), we have

$$\frac{d}{dt}\int_X^X n\ \phi, dx = \int_X^X v(F)n\ \phi', dx - \int_X^X D(F)\partial_x n\phi'\,dx$$
$$+W^{(f)}(F)n\phi(X) + \left(j^{(e)}(F) - W^{(b)}(F)n\right)\phi(-X)$$

in $\mathcal{D}'(0,T)$. This ends the proof of Theorem 3.1.

Remark 5.6. The proof adapts readily if instead of assuming a constant doping density N_D, we deal with a sequence $\{N^h_{D,i},\ i \in \{-N^h,\dots,N^h\}\}$ verifying

$$h\sum_{i=-N^h}^{N^h} |N^h_{D,i}|^2 < \infty.$$

Accordingly, we obtain in the continuous limit a (possibly nonconstant) $\mathrm{L}^2(-X,X)$ doping density.

6. The bias constraint. In this section we reconsider the bias condition (2.8) as an alternative to the prescription of the emitter electric field (2.7). The arguments are exactly those of the previous section and we point out only the main differences in the proof. In rescaled form the condition is

$$h\sum_{i=-N^h}^{N^h} F_i^h = V, \tag{6.1}$$

which is added to the system (3.1), (3.2). This scaling means that the ratio $\frac{\mathcal{LF}}{\mathcal{V}}$ has order 1, $\mathcal{V}$ being a characteristic value for the total voltage. Of course, the L^1 estimate in Lemma 4.1 still holds, provided that $j^{(e)}$ is a bounded function. Then, the key point in the previous analysis is to establish a uniform estimate (with respect to h) on the electric field $F^h_{-N^h-1}$.

LEMMA 6.1. *The quantity $F^h_{-N^h-1}$ is bounded in $\mathrm{L}^\infty((0,T))$.*

Proof. Let us sum the relations (3.3). We get

$$\begin{aligned} h\sum_{i=-N^h}^{N^h} F_i^h &= V = h\sum_{i=-N^h}^{N^h}\left(F^h_{-N^h-1} + h\sum_{j=-N^h}^{i}(n^h_j - N_D)\right)\\ &= (2N^h+1)h\ F^h_{-N^h-1} + h^2\sum_{j=-N^h}^{N^h}\left((n^h_j - N_D)\sum_{i=j}^{N^h} 1\right)\\ &= (2N^h+1)h\ F^h_{-N^h-1} + h^2\sum_{j=-N^h}^{N^h}(n^h_j - N_D)(N^h - j + 1).\end{aligned}$$

Consequently, the electric field at the emitter is given by

$$F^h_{-N^h-1} = \frac{V}{(2N^h+1)h} - \frac{h}{2N^h+1}\sum_{j=-N^h}^{N^h}(n^h_j - N_D)(N^h - j + 1). \tag{6.2}$$

It follows that

$$
\begin{aligned}
|F^h_{-N^h-1}| &\le \frac{|V|}{(2N^h+1)h} + \frac{h^2}{(2N^h+1)h} \sum_{j=-N^h}^{N^h} |n_j^h - N_D|\ |N^h - j + 1| \\
&\le \frac{|V|}{2X} + \frac{h}{(2N^h+1)h} \left(h \sum_{j=-N^h}^{N^h} n_j^h + (2N^h+1) h N_D \right) (2N^h+1) \\
&\le \frac{|V|}{2X} + h \sum_{j=-N^h}^{N^h} n_j^h + (2X+h) N_D.
\end{aligned}
$$

This leads to the estimate of $F^h_{-N^h-1}$ in $\mathrm{L}^\infty((0,T))$. ∎

Once we have this estimate, we can justify the bounds in Lemmas 4.2 and 4.4. We also need some control on the time derivative of $F^h_{-N^h-1}$.

LEMMA 6.2. *The quantity* $F^h_{-N^h-1}$ *is bounded in* $\mathrm{H}^1((0,T))$.

Proof. Differentiating (6.2), we find

$$
\begin{aligned}
\frac{d}{dt} F^h_{-N^h-1} &= \frac{h}{2N^h+1} \sum_{i=-N^h}^{N^h} \left(\sum_{j=-N^h}^{i} \frac{d}{dt} n_j^h \right) \\
&= \frac{1}{2N^h+1} \sum_{i=-N^h}^{N^h} \left(\sum_{j=-N^h}^{i} (J^h_{j-1\to j} - J^h_{j\to j+1}) \right) \\
&= \frac{1}{2N^h+1} \sum_{i=-N^h}^{N^h} (J^h_{-N^h-1\to -N^h} - J^h_{i\to i+1}) \\
&= J^h_{-N^h-1\to -N^h} - \frac{1}{2N^h+1}\ J_{N^h\to N^h+1} \\
&\quad + \frac{1}{2N^h+1} \sum_{i=-N^h}^{N^h-1} \left(v(F_i^h) n_i^h - D(F_i^h) \frac{n_{i+1}^h - n_i^h}{h} \right).
\end{aligned}
$$

Using the bounds of Lemmas 6.1 and 4.2, we can bound $v(F_i^h)$, $D(F_i^h)$, $j^{(e)}(F^h_{-N^h-1})$, $W^{(b)}(F^h_{-N^h-1})$, and $W^{(f)}(F^h_{N^h})$ by some constant $0 < M < \infty$. Hence, we deduce that

$$
\begin{aligned}
\left| \frac{d}{dt} F^h_{-N^h-1} \right| &\le M(1 + n^h_{-N^h} + n^h_{N^h}) \\
&\quad + \frac{M}{(2N^h+1)h} \left(h \sum_{i=-N^h}^{N^h} n_i^h + \sum_{i=-N^h}^{N^h} |n_{i+1}^h - n_i^h| \right) \\
&\le M(1 + n^h_{-N^h} + n^h_{N^h}) + \frac{M}{2X}\ h \sum_{i=-N^h}^{N^h} n_i^h \\
&\quad + \frac{M}{\sqrt{2X}} \left(\sum_{i=-N^h}^{N^h} \frac{|n_{i+1}^h - n_i^h|^2}{h} \right)^{1/2}.
\end{aligned}
$$

We conclude by applying the estimates of Lemma 4.4. ∎

By using these estimates, we can reproduce mutatis mutandis the arguments of the previous section. We conclude with the following result.

THEOREM 6.3. *Assume that $j^{(e)}$ is a bounded function. Then, the conclusions of Theorem* 3.1 *are still valid by replacing the condition* (2.7) *by* (6.1). *Accordingly, in the limit problem the electric field satisfies the Poisson equation $\partial_x F = n - N_D$ coupled to the constraint $\int_{-X}^{X} F\,dx = V$.*

Appendix A. Proof of (5.4). We write $m^h = \nu^h + n^h$, $G^h = \Phi^h + F^h$. Recall that ν^h, Φ^h are defined on $(0,T) \times (ih,(i+1)h)$, $i \in \{-N^h, \dots, N^h - 1\}$, by

$$\nu^h(t,x) = \frac{1}{h}\,(n^h_{i+1} - n^h_i), \qquad \Phi^h(t,x) = \frac{1}{h}\,(F^h_{i+1} - F^h_i) = n^h_{i+1} - N_D\,,$$

where we have used (3.2) in the second relation. As in the proof of Lemma 4.7, we consider a test function $\phi \in \mathrm{C}^\infty_0(I)$ and set $\Gamma^h_i = \int_{ih}^{(i+1)h} (x - ih)\phi(x)\,dx$, which verifies $|\Gamma^h_i| \le \|\phi\|_{\mathrm{L}^\infty(I)} h^2/2$. We have

$$\begin{aligned}
\langle \partial_t \Phi^h, \phi\rangle &= \sum_{i=-N^h}^{N^h-1} \frac{dn^h_{i+1}}{dt} \int_{ih}(i+1)h (x - ih)\phi(x)\,dx \\
&= \sum_{i=-N^h}^{N^h-1} \frac{1}{h}\,(J^h_{i\to i+1} - J^h_{i+1\to i+2})\,\Gamma^h_i \\
&= \sum_{i=-N^h}^{N^h-1} J^h_{i\to i+1}\,\frac{1}{h}\,(\Gamma^h_i - \Gamma^h_{i-1}) - \frac{1}{h} J^h_{N^h\to N^h+1}\Gamma^h_{N^h-1}\,.
\end{aligned}$$

We can bound this expression as follows:

$$\begin{aligned}
|\langle \partial_t \Phi^h, \phi\rangle| &\le \|\phi\|_{\mathrm{L}^\infty(I)} h \left(\sum_{i=-N^h}^{N^h-1} \left| v(F^h_i) n^h_i + \frac{1}{h} D(F^h_i)(n^h_{i+1} - n^h_i)\right| \right) \\
&\quad + \|\phi\|_{\mathrm{L}^\infty(I)} h\, |W^{(f)}(F^h_{N^h}) n^h_{N^h}| \\
&\le \|\phi\|_{\mathrm{L}^\infty(I)}\; M \left(h \sum_{i=-N^h}^{N^h-1} n^h_i + \left(\sum_{i=-N^h}^{N^h-1} \frac{|n^h_{i+1} - n^h_i|^2}{h} \right)^{1/2} \sqrt{2X} + n^h_{N^h} \right).
\end{aligned}$$

Thus, from Lemma 4.4 we deduce that $\partial_t \Phi^h$ is bounded in $\mathrm{L}^2(0,T;\mathcal{M}(I))$.

We proceed with ν^h in a similar way. Indeed, we can write

$$\begin{aligned}
\langle \partial_t \nu^h, \phi\rangle &= \frac{1}{h} \sum_{i=-N^h}^{N^h-1} \left(\frac{dn^h_{i+1}}{dt} - \frac{dn^h_i}{dt} \right) \int_{ih}(i+1)h (x - ih)\phi(x)\,dx \\
&= \frac{1}{h^2} \sum_{i=-N^h}^{N^h-1} (-J^h_{i+1\to i+2} + 2J^h_{i\to i+1} - J^h_{i-1\to i})\,\Gamma^h_i \\
&= \frac{1}{h^2} \sum_{i=-N^h}^{N^h-1} J^h_{i\to i+1}\,(-\Gamma^h_{i+1} + 2\Gamma^h_i - \Gamma^h_{i-1}) \\
&\quad - \frac{1}{h^2} J^h_{N^h\to N^h+1}\Gamma^h_{N^h-1} - \frac{1}{h^2} J^h_{-N^h-1\to -N^h}\Gamma^h_{-N^h}\,.
\end{aligned} \tag{A.1}$$

The boundary terms in (A.1) are bounded by

$$M(1+n^h_{-N^h}+n^h_{N^h})\|\phi\|_{\mathrm{L}^\infty(I)},$$

which belongs to a bounded set of $\mathrm{L}^2(0,T)$. Next, we have the bound

$$\frac{1}{h^2}|-\Gamma^h_{i+1}+2\Gamma^h_i-\Gamma^h_{i-1}|\leq C\|\phi'\|_{\mathrm{L}^\infty(I)}\ h.$$

Therefore, the sum in the right-hand side of (A.1) can be estimated by

$$\begin{aligned}C\|\phi'\|_{\mathrm{L}^\infty(I)}\ h\sum_{i=-N^h}^{N^h-1}|J^h_{i\to i+1}| \quad &\leq \quad CM\|\phi'\|_{\mathrm{L}^\infty(I)}\left(h\sum_{i=-N^h}^{N^h-1}n^h_i\right.\\ &\left.+\left(\sum_{i=-N^h}^{N^h-1}\frac{|n^h_{i+1}-n^h_i|^2}{h}\right)^{1/2}\sqrt{2X}\right),\end{aligned}$$

as we did in the previous proof for Φ^h. We conclude that $\partial_t\nu^h$ is bounded in $\mathrm{L}^2(0,T;\mathcal{M}(I))+\mathrm{L}^2(0,T;\mathrm{W}^{-1,1}(I))$. This ends the proof of (5.4). ∎

Appendix B. Uniqueness for the limit problem. In this section, we show the uniqueness of the solution of (3.4). Let us consider two solutions (n_1,F_1) and (n_2,F_2) of (3.4) with $n_i\in\mathrm{C}^0([0,T];\mathrm{L}^2(I))\cap\mathrm{L}^2(0,T;\mathrm{H}^1(I))$. For the difference, we have

$$\partial_t(n_1-n_2)+\partial_xJ(F_1,n_1-n_2)+\partial_x\Big((v(F_1)-v(F_2))n_2-(D(F_1)-D(F_2))\partial_xn_2\Big)=0,$$

where $J(F,n)=v(F)n-D(F)\partial_xn$. The boundary conditions read

$$\begin{cases}J(F_1,n_1-n_2)(X)=W^{(f)}(F_1)(n_1-n_2)+(W^{(f)}(F_1)-W^{(f)}(F_2))n_2,\\ J(F_1,n_1-n_2)(X)=j^{(e)}(F_1)-j^{(e)}(F_2)-W^{(b)}(F_1)(n_1-n_2)\\ \qquad\qquad\qquad\qquad\qquad\qquad -(W^{(b)}(F_1)-W^{(b)}(F_2))n_2.\end{cases}$$

Thus, we are left with only the task of evaluating

$$\begin{aligned}&\frac{d}{dt}\int_{-X}^X\frac{|n_1-n_2|^2}{2}\,dx+\int_{-X}^XD(F_1)|\partial_x(n_1-n_2)|^2\,dx\\ &=\int_{-X}^Xv(F_1)(n_1-n_2)\partial_x(n_1-n_2)\,dx+\int_{-X}^X(v(F_1)-v(F_2))n_2\partial_x(n_1-n_2)\,dx\\ &\quad-\int_{-X}^X(D(F_1)-D(F_2))\partial_xn_2\partial_x(n_1-n_2)\,dx\\ &\quad+J(F_1,n_1-n_2)(n_1-n_2)(-X)-J(F_1,n_1-n_2)(n_1-n_2)(X).\end{aligned}\tag{B.1}$$

Denote by $A,B,C,D,$ and E the five terms in the right-hand side of (B.1). Recall that F_i belongs to L^∞, so that the coefficients are lying in a bounded set. Also denote by Λ a Lipschitz constant for the functions $v,D,j^{(e)}$ and $W^{(b,f)}$ in the range of values of F_1 and F_2. Let $\nu>0$ be a parameter to be specified later on. By using the Cauchy–Schwarz and Young inequalities, we can estimate

$$|A|\leq C_\nu\int_{-X}^X|n_1-n_2|^2\,dx+\nu\int_{-X}^X|\partial_x(n_1-n_2)|^2\,dx.$$

Next, we have

$$\begin{aligned} |B| &\le \Lambda \|F_1 - F_2\|_{\mathrm{L}^\infty(I)} \int_{-X}^{X} |n_2|\ |\partial_x(n_1 - n_2)|\, dx \\ &\le C_\nu \Lambda^2 \int_{-X}^{X} |n_2|^2\, dx\, \|F_1 - F_2\|^2_{\mathrm{L}^\infty(I)} + \nu \int_{-X}^{X} |\partial_x(n_1 - n_2)|^2\, dx. \end{aligned}$$

The Poisson equations yield to

$$(F_1 - F_2)(t,x) = F_{-,1} - F_{-,2} + \int_{-X}^{x} (n_1 - n_2)(t,y)\, dy,$$

which provides the bound

$$\|F_1 - F_2\|^2_{\mathrm{L}^\infty(I)} \le 2|F_{-,1} - F_{-,2}|^2 + 4X \int_{-X}^{X} |n_1 - n_2|^2\, dx.$$

Hence, we get (changing the value of C_ν)

$$|B| \le C_\nu \int_{-X}^{X} |n_2|^2\, dx \left(|F_{-,1} - F_{-,2}|^2 + \int_{-X}^{X} |n_1 - n_2|^2\, dx \right) + \nu \int_{-X}^{X} |\partial_x(n_1 - n_2)|^2\, dx.$$

A similar reasoning for C leads to

$$|C| \le C_\nu \int_{-X}^{X} |\partial_x n_2|^2 dx \left(|F_{-,1} - F_{-,2}|^2 + \int_{-X}^{X} |n_1 - n_2|^2 dx \right) + \nu \int_{-X}^{X} |\partial_x(n_1 - n_2)|^2 dx.$$

For the boundary terms, we get rid of the terms $-W^{(b,f)}(F_1)|n_1 - n_2|^2$ which are nonnegative and get

$$D + E \le \Lambda\Big((1 + n_2)\ |F_1 - F_2|\ |n_1 - n_2|(-X) + n_2\ |F_1 - F_2|\ |n_1 - n_2|(+X) \Big).$$

Then, we use the Sobolev embedding to control the traces of $n_1 - n_2$ with the H^1-norm. Finally, we obtain

$$\begin{aligned} D + E &\le C_\nu (1 + |n_2(-X)|^2 + |n_2(X)|^2) \left(|F_{-,1} - F_{-,2}|^2 + \int_{-X}^{X} |n_1 - n_2|^2 \right) \\ &\quad + \nu \left(\int_{-X}^{X} |n_1 - n_2|^2\, dx + \int_{-X}^{X} |\partial_x(n_1 - n_2)|^2\, dx \right). \end{aligned}$$

Having disposed of these preliminaries, we recall that $D(F_1)$ is bounded from below by some $\delta > 0$. Then, we put all the pieces together and choose $\nu = \nu(\delta)$ appropriately so that we finally find

$$\begin{aligned} &\frac{d}{dt} \int_{-X}^{X} |n_1 - n_2|^2\, dx + \frac{\delta}{2} \int_{-X}^{X} |\partial_x(n_1 - n_2)|^2\, dx \\ &\quad \le f(t) \int_{-X}^{X} |n_1 - n_2|^2\, dx + g(t)|F_{-,1} - F_{-,2}|^2, \end{aligned}$$

where the nonnegative functions $f, g \in \mathrm{L}^1(0,T)$ depend on Λ, δ and $\int_{-X}^{X}(n_2^2 + |\partial_x n_2|^2)\,dx$. The Gronwall lemma provides the inequality

$$\int_{-X}^{X} |n_1 - n_2|^2(t,x)\,dx \leq e^{\int_0^t f(s)\,ds} \left(\int_{-X}^{X} |n_1 - n_2|^2(0,x)\,dx + \int_0^t g(s) |F_{-,1} - F_{-,2}|^2(s)\,ds \right).$$

This proves the continuity of the solution with respect to the data and, consequently, the uniqueness of the solution. We skip the adaptation of the proof to the bias condition.

REFERENCES

[1] R. Aguado, G. Platero, M. Moscoso, and L. L. Bonilla, *Microscopic model for sequential tunneling in semiconductor multiple quantum wells*, Phys. Rev. B, 55 (1997), R 16053–16056.

[2] J. P. Aubin, *Un théorème de compacité*, C. R. Acad. Sci. Paris Sér. I Math., 256 (1963), pp. 5042–5044.

[3] J. Bardeen, *Tunneling from a many-particle point of view*, Phys. Rev. Lett., 6 (1961), pp. 57–59.

[4] L. L. Bonilla, G. Platero, and D. Sánchez, *Microscopic derivation of transport coefficient and boundary conditions in discrete drift-diffusion models of weakly coupled superlattices*, Phys. Rev. B, 62 (2000), pp. 2786–2796.

[5] L. L. Bonilla, O. Sánchez, and J. Soler, *Nonlinear stochastic discrete drift-diffusion theory of charge fluctuations and domain relocation times in semiconductor superlattices*, Phys. Rev. B, 65 (2002), pp. 195308/1–8.

[6] L. L. Bonilla, *Theory of nonlinear charge transport, wave propagation and self-oscillations in semiconductor superlattices*, J. Phys. Condens. Matter, 14 (2002), R341–R381.

[7] J.-F. Collet, T. Goudon, F. Poupaud, and A. Vasseur, *The Beker–Döring system and its Lifshitz–Slyozov limit*, SIAM J. Appl. Math., 62 (2002), pp. 1488–1500.

[8] H. Gajewski and K. Gröger, *On the basic equations for carrier transport in semiconductors*, J. Math. Anal. Appl., 113 (1986), pp. 12-35.

[9] F. Golse, *From kinetic to macroscopic models in kinetic equations and asymptotic theory*, in Kinetic Equations and Asymptotic Theory, Ser. Appl. Math. 4, B. Perthame and L. Desvillettes, eds., Gauthiers-Villars, Paris, 2000, pp. 41–126.

[10] J. W. Jerome, *The approximation problem for drift-diffusion systems*, SIAM Rev., 37 (1995), pp. 552–572.

[11] J. Liang, *On a nonlinear integrodifferential drift-diffusion semiconductor model*, SIAM J. Math. Anal., 25 (1994), pp. 1375–1392.

[12] M. Rogozia, S. W. Teitsworth, H. T. Grahn, and K. H. Ploog, *Statistics of the domain-boundary relocation time in semiconductor superlattices*, Phys. Rev. B, 64 (2001), 041308(R).

[13] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw–Hill, New York, 1987.

[14] J. Simon, *Compact sets in the space* $\mathrm{L}^p(0,T;B)$, Ann. Mat. Pura Appl. 4, 146 (1987), pp. 65–96.

[15] A. Wacker, *Semiconductor superlattices: A model system for nonlinear transport*, Phys. Rep. 357 (2002), pp. 1–111.