Boundary Value Problems for the Stationary Vlasov-Boltzmann-Poisson Equation

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Abstract

We investigate the well posedness of stationary Vlasov-Boltzmann equations both in the simpler case of linear problem with a space varying force field and a collisional integral satisfying the detailed balanced principle with a non-singular scattering function, and, the non-linear Vlasov-Poisson-Boltzmann system. For the former we obtain existence-uniqueness results for arbitrarily large integrable boundary data and justify further a priori estimates. For the later the boundary data needs to satisfy an entropy condition guaranteeing classical statistical equilibrium at the boundary. This stationary problem relates to the existence of phase transitions associated with slab geometries.

Key words. Stationary transport equations. Plasma physics models. Boltzmann-Poisson system.

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1 Introduction

The present study focus on the solvability of stationary boundary value problems associated to Boltzmann-Poisson systems. These problems are related to basic questions associated with existence of non classical statistical equilibrium states in bounded domains as well as stable phase transition problems in kinetic theory of plasmas. In addition, they also appear boundary and interface regions described by transition layers where a stationary kinetic equation is solved. A standard assumption is for the layer to have a slab symmetry, that is, the particle distribution is constant on surfaces parallel to the interface.

Previously to our work, rigorous analysis for the boundary value problem associated to these kinetic equations have been studied in the framework of boundary layers (the half space and Milne problem), and in the case of full space boundary prescribed data, as investigated in [13] where the proof appeals to comparison principles based on the fact that the boundary condition are comparable to equilibrium states that nullify the transport and collision terms. In such cases an absolute Maxwellian distributions can be used to derive pointwise a priori estimates and then it leads to existence results for incoming data comparable to Maxwellian distribution. The case of one sided boundary value problems for transport kinetic equations in the absence of forces, known as the half space problem, and its corresponding limiting behavior in a strong collisional regime and long time scaling linear, as the length of the transition layer is comparable to the reference collision frequency, known as the Milne problem, was initiated in [2] by means of spectral methods and semigroup theory.

For collisional plasma models, the force field, gradient of the electrostatic potential, is bounded along flat boundaries where the potential is either prescribed or is a solution of the corresponding mean field equation. In both cases, the force field will become a constant in the rescaled layer. In the case of a linear collisional integral satisfying the detailed balanced principle with a non-singular scattering function and weak force field forces, the rescaled force field vanishes and the corresponding half space and Milne problem was studied by [12], where in particular the asymptotic behavior exhibits a classical statistical equilibrium corresponding to a Maxwellian behavior, independent of the force field. In the case of strong force field regimes also a slab symmetry is obtained whenever the curvature of the interface is small compared to the reciprocal of the mean free path and the force field is normal to the interface. Consequently, the space coordinate reduces to, say, x the distance to the boundary or interface. After scaling it like $\frac{x}{\epsilon}$, where ϵ is the order magnitude of the mean free path, one has to solve a kinetic half-space problem where the corresponding stationary equation depends on the force field. Such problem has been solved by Ben Abdallah, Gamba and Klar in [4] in the case of both negative or positive constant force fields. These strong force field scalings are characterized by non-statistical equilibrium states $P = P(v)$, that is they are $L^1(\mathbb{R}_v)$ space homogeneous solution to the layer problem, with non-vanishing mean or first moment, which depend on the force field and on the Maxwellian in the kernel of the collision operator and the scattering function. This problem was treated by Trugman and Taylor [16] for the relaxation operator, and by Poupaud [14] for the general linear operator in higher dimensions. These problems related to non-equilibrium statistical states where also solved by comparisons methods provided the boundary data was comparable to these dominant state.

However the question associated to the full boundary value of connecting two states with just bounded incoming fluxes, in a slab geometry of an interface, with a varying force remains open. This fundamental problem we address in this paper is related to the existence of phase transitions for the Boltzmann-Poisson system of equations.

In this manuscript we first develop the study of existence of solutions to the linear stationary boundary value problem for the Vlasov-Boltzmann equation, with given force fields having bounded gradients, and incoming boundary data having bounded incoming fluxes (first moments). We use constructive techniques such as integrator along characteristics and the construction of monotone sequences with boundary control estimates. Both existence and uniqueness follows from the mass (order) preserving, non-expansive (dissipative) nature of the collisional form: these properties are the basis of the derivation of new sharp estimates for the stationary boundary value problem. Further, we also develop moments, L^∞ and entropy estimates. These estimates, which are interesting in themselves, allow us to consider the stationary boundary value problem for the self consistent Boltzmann-Poisson system; they not only control the regularity of self consistent force fields but also yield the existence of weak solutions by solving an auxiliary absorption perturbation problem and using compactness arguments.

1.1 Statement of problems

We consider the problem

$$v\partial_x f + E(x)\partial_v f = \kappa Q(f), \qquad x \in (0, L),\ v \in \mathbb{R} \tag{1.1}$$

where $\kappa > 0$ and Q is the linear collision operator (cf. [14, 4]) satisfying the detailed balance principle, defined by

$$Q(f)(x,v) = \int_{-\infty}^{+\infty} s(v,v')\{M_\theta(v)f(x,v') - M_\theta(v')f(x,v)\}\,\mathrm{d}v' = Q^+(f)(x,v) - \sigma(v)f(x,v) \tag{1.2}$$

where $Q^+(f)(x,v) = \int_{-\infty}^{+\infty} s(v,v')M_\theta(v)f(x,v')\,\mathrm{d}v'$ is the gain operator, $\sigma(v) = \int_{-\infty}^{+\infty} s(v,v')M_\theta(v')\,\mathrm{d}v'$ is the collision frequency and M_θ is the Maxwellian equilibrium

$$M_\theta(v) = \frac{e^{-v^2/2\theta}}{\sqrt{2\pi\theta}}, \quad v \in \mathbb{R}$$

with $\theta > 0$. The scattering function $s(v,v')$ satisfies

$$s(v,v') = s(v',v) \ \text{ and } \ 0 < s_0 \le s(v,v') \le s_1 < +\infty, \quad v, v' \in \mathbb{R}$$

for some constants $s_0, s_1 \in (0, +\infty)$. Notice that the collision frequency verifies

$$0 < s_0 \le \sigma(v) \le s_1 < +\infty, \quad v \in \mathbb{R}.$$

In particular, when the scattering function is constant we obtain the simple relaxation operator defined by

$$Q(f)(x,v) = s\{\langle f\rangle(x)\ M_\theta(v) - f(x,v)\}, \qquad \langle f\rangle(x) = \int_{-\infty}^{+\infty} f(x,v)\,\mathrm{d}v.$$

The equation has to be completed by incoming data

$$f(0,v)_{|v>0} = f_0(v), \qquad f(L,v)_{|v<0} = f_L(v). \tag{1.3}$$

This equation arises in the modeling of charge transport phenomena, with applications in semiconductor theory or plasma physics [11]. There, the force field E, which is space-varying, describes electric forces that influence the motion of the electrons. The problem has been investigated in [13]: the proof appeals to comparison principles based on the fact that

$$\mathcal{M}_{\theta,\Phi}(x,v) = \exp\Big(- \frac{v^2/2 + \Phi(x)}{\theta}\Big), \qquad E(x) = -\Phi'(x)$$

makes vanish both the transport operator $v\partial_x + E(x)\partial_v$ and the collision operator Q. Therefore the absolute Maxwellian distribution $\mathcal{M}_{\theta,\Phi}$ can be used to derive pointwise a priori estimates and we are led to existence results for incoming data comparable to $\mathcal{M}_{\theta,\Phi}$.

We also notice that more difficulties arise from the space variation of the force E or the variation of the scattering function. Indeed, in the specific case where E is constant and $s = 1$, it turns out that a solution of (1.1)-(1.2) is given by the explicit formula, see [16, 14]

$$P_E(v) = Z\ \frac{\kappa}{\sqrt{\pi}\ E}\ \exp\Big(- \frac{\kappa}{E} v + \frac{1}{2\theta}\big(\frac{\kappa}{E}\big)^2\Big) \int_{-(v-\kappa/E)/\sqrt{2\theta}}^{+\infty} e^{-u^2}\,\mathrm{d}u$$

where $Z > 0$ is a normalizing constant (so that the $v-$average of P_E is 1 when $Z = 1$). Again, this particular solution can be used with comparison principles to prove existence results for boundary value and Milne problems when considering incoming data compatible with P_E, see [4].

In this manuscript, we consider more general incoming data, only requiring some integrability conditions of moments and we treat the case of general linear collision operator.

We now state our main results. The first one is the existence and uniqueness of the stationary boundary value problem for the linear Vlasov-Boltzmann equation for a given electric field with bounded first derivative and boundary data whose incoming part has finite mean. The second main result is the existence of the stationary boundary value problem for the self-consistent Boltzmann-Poisson system, where now the incoming boundary data satisfies an entropy condition type. The results are stated as follows.

Theorem 1.1 (The linear Vlasov-Boltzmann equation) *Let $E \in W^{1,\infty}((0,L))$. Assume that the incoming data satisfy*

$$\int_0^{+\infty} v\ |f_0(v)|\,\mathrm{d}v = M_0 < \infty, \qquad \int_{-\infty}^0 (-v)\ |f_L(v)|\,\mathrm{d}v = M_L < \infty. \tag{1.4}$$

Then, there exists a unique (mild) solution $f(x,v) \in L^1((0,L)\times\mathbb{R})$ of (1.1)-(1.3); the outgoing traces are well-defined and satisfy

$$\int_{-\infty}^0 (-v)|f(0,v)|\,\mathrm{d}v < \infty, \qquad \int_0^\infty v|f(L,v)|\,\mathrm{d}v < \infty.$$

If $f_0, f_L \geq 0$, then $f \geq 0$ as well.

Our strategy of proof is different and allows to deal with arbitrary integrable data. The proof only relies on "energy" estimates and properties of the characteristics associated to the field $(v, E(x))$.

Remark 1.1 *Note that the boundary condition* (1.4) *is satisfied by both the classical equilibrium state* $\mathcal{M}_{\theta,\Phi}(v)$ *and the strong forced non-equilibrium distribution* $P_E(v)$ *above. This is a signature for the existence of strong phase transitions for the Vlasov-Boltzmann linear transport equation.*

Remark 1.2 *The notion of "mild" solution refers to a solution satisfying the equation integrated along the characteristics, as it will be detailed below. We do not detail the different notions of solutions for transport equations we use, referring instead to a textbook on this topics, like e.g. [8].*

The second main statement, which requires further moments boundedness and entropy estimates, ensures the existence of weak solutions to the Boltzmann-Poisson system

$$v\partial_x f + E(x)\partial_v f = \kappa Q(f), \qquad x \in (0,L), v \in \mathbb{R} \tag{1.5}$$

$$E(x) = -\Phi'(x), \qquad -\Phi''(x) = \langle f\rangle(x) - n_b(x), \qquad x \in (0,L) \tag{1.6}$$

with the boundary conditions

$$f(0,v)|_{v>0} = f_0(v), \qquad f(L,v)|_{v<0} = f_L(v) \tag{1.7}$$

$$\Phi(0) = \Phi_0, \qquad \Phi(L) = \Phi_L. \tag{1.8}$$

The function $n_b(x) \geq 0$, $n_b \in L^1((0,L))$ represents the concentration of a background population of ions. The following result of existence of weak solutions follows.

Theorem 1.2 (The self-consistent Boltzmann-Poisson system) *Assume that* f_0, f_L *are non-negative and satisfy the following boundary entropy condition*

$$\mathcal{H}_{_L} = \int_0^{+\infty} v\,\left(1 + \frac{v^2}{2} + |\ln f_0|\right) f_0\,\mathrm{d}v - \int_{-\infty}^0 v\left(1 + \frac{v^2}{2} + |\ln f_L|\right) f_L\,\mathrm{d}v < +\infty. \tag{1.9}$$

Then there is a weak solution (f, E) *for the Boltzmann-Poisson problem* (1.5)-(1.8) *satisfying* $E \in L^\infty((0,L)) \cap W^{1,1}((0,L))$ *and*

$$\int_0^L \int_{-\infty}^{+\infty} \left(|\ln f| + 1 + \frac{v^2}{2}\right) f(x,v)\,\mathrm{d}v\,\mathrm{d}x < +\infty$$

$$\int_0^{+\infty} v\left(|\ln f| + 1 + \frac{v^2}{2}\right) f(L,v)\,\mathrm{d}v - \int_{-\infty}^0 v\left(|\ln f| + 1 + \frac{v^2}{2}\right) f(0,v)\,\mathrm{d}v < +\infty$$

$$\int_0^L |\ln\langle f\rangle|\langle f\rangle\,\mathrm{d}x < +\infty.$$

The proof of Theorem 1.2 uses an α-perturbed linear kinetic transport problem coupled to (1.6)-(1.8) and its subsequent stability results letting the parameter α going to zero.

In the next section, we set up a few notation and recall basic properties of the characteristics. The proof of Theorem 1.1 is detailed in Section 3. Then, in order to prove the weak existence of the self-consistent system, we discuss in Section 4 entropy estimates and L^∞ bounds when dealing with entropy bounded incoming data and non-smooth force fields. Finally, we prove Theorem 1.2 in the last section.

2 Characteristics

It is convenient to introduce the potential $\Phi(x)$ such that $E(x) = -\Phi'(x)$, namely we set

$$\Phi(x) = \Phi_0 - \int_0^x E(y)\,\mathrm{d}y, \tag{2.10}$$

for some constant Φ_0. Since E belongs to $W^{1,\infty}((0,L))$, $\Phi \in C^1((0,L))$, with $\Phi'' \in L^\infty((0,L))$. We define the characteristic curves as the solutions of the ODEs system

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{X}(t;s,x,v) = \mathcal{V}(t;s,x,v), & \dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(t;s,x,v) = E(\mathcal{X}(t;s,x,v)) = -\Phi'(\mathcal{X}(t;s,x,v)), \\ \mathcal{X}(s;s,x,v) = x, & \mathcal{V}(s;s,x,v) = v. \end{cases} \tag{2.11}$$

The system is autonomous so that it suffices to consider the solution $(X(t;x,v), V(t;x,v))$ corresponding to the data (x,v) at time $s = 0$ and we get

$$(\mathcal{X},\mathcal{V})(t;s,x,v) = (X,V)(t-s;x,v).$$

In particular, $X(-t;x,v)$ (resp. $V(-t;x,v)$) gives the position (resp. the velocity) at time 0 of a particle that starts at time t from position x and velocity v. An important property is the local energy conservation which means, for any $t \in \mathbb{R}$,

$$\frac{V^2(t;x,v)}{2} + \Phi(X(t;x,v)) = \frac{v^2}{2} + \Phi(x). \tag{2.12}$$

For any $x \in [0,L]$ and $v \in \mathbb{R}$, we define the exit times

$$\begin{aligned} T_{\mathrm{out}}(x,v) &= \sup\big\{t \geq 0 \text{ such that for any } s \in (0,t),\ X(s;x,v) \in (0,L)\big\}, \\ T_{\mathrm{inc}}(x,v) &= \sup\big\{t \geq 0 \text{ such that for any } s \in (0,t),\ X(-s;x,v) \in (0,L)\big\}. \end{aligned}$$

Next, we introduce the critical velocities

$$V_0 = \left(2\big(\max_{x\in[0,L]} \Phi(x) - \Phi(0)\big)\right)^{1/2} \geq 0, \qquad V_L = -\left(2\big(\max_{x\in[0,L]} \Phi(x) - \Phi(L)\big)\right)^{1/2} \leq 0.$$

We shall need a few properties of the characteristics. Using the conservation of the total energy $\frac{v^2}{2} + \Phi(x)$ along the characteristics we first prove as in [6] (see also [10]):

Proposition 2.1 *Assume that $E \in W^{1,\infty}((0,L))$. Then, the following properties hold:*

i) For any $0 < v < V_0$ (resp. $V_L < v < 0$) there exist $x_\star \in (0,L)$ and $0 < s_\star \leq T_{\mathrm{out}}(0,v) \leq +\infty$ (resp. $0 < s_\star \leq T_{\mathrm{out}}(L,v) \leq +\infty$) such that, for any $0 < s < s_\star$,

$$0 < X(s;0,v) < x_\star, \qquad V(s;0,v) > 0, \qquad \lim_{s \nearrow s_\star} (X,V)(s;0,v) = (x_\star, 0)$$

(resp. $x_\star < X(s;L,v) < L$, $V(s;L,v) < 0$ and $\lim_{s\nearrow s_\star}(X,V)(s;L,v) = (x_\star,0)$).

ii) Under the assumptions of i), if moreover, $\Phi'(x_\star) \neq 0$, then $T_{\rm out}(0,v)$ (resp. $T_{\rm out}(L,v)$) is finite, $s_\star = T_{\rm out}(0,v)/2$ (resp. $s_\star = T_{\rm out}(L,v)/2$) and for any $0 < s < s_\star$, we get $(X,V)(s;0,v) = (X,-V)(2s_\star - s;0,v)$ (resp. $(X,V)(s;L,v) = (X,-V)(2s_\star - s;L,v)$). In particular, we have $X(T_{\rm out}(0,v);0,v) = 0$ (resp. $X(T_{\rm out}(L,v);L,v) = L$).

iii) For any $v > V_0$, $T_{\rm out}(0,v)$ is finite, and furthermore, $X(T_{\rm out}(0,v);0,v) = L$ and $V(s;0,v) > 0$ holds for any $0 < s < T_{\rm out}(0,v)$.

Similarly, for any $v < V_L$, $T_{\rm out}(L,v)$ is finite, and $X(T_{\rm out}(L,v);L,v) = 0$ and $V(s;L,v) < 0$ holds for any $0 < s < T_{\rm out}(L,v)$.

We shall use crucially the third property which tells us that, choosing large enough velocities, the characteristics cross the entire domain $(0,L)$.

Example 2.1 *If $E(x) \geq 0$ and $\Phi_0 = 0$, then the potential is non-increasing and non-positive. In this case, we obtain readily estimates on the exit time: for any $v > 0$, $L/\sqrt{v^2 - 2\Phi(L)} \leq T_{\rm out}(0,v) \leq L/v$ and for any $v < -\sqrt{2|\Phi(L)|}$, $L/|v| \leq T_{\rm out}(L,v) \leq L/\sqrt{v^2 + 2\Phi(L)}$.*

Example 2.2 *If $E(x)$ is non-decreasing, the potential is concave and the exit times are always finite, but for the initial data $(0,V_0)$ and (L,V_L).*

Example 2.3 *Let us set $E(x) = E_- - \Delta E x$ for some $E_-, \Delta E > 0$. Hence $\Phi(x) = \Delta E x^2/2 - E_- x$. The energy conservation can be recast as*

$$\left(\sqrt{\Delta E}X(t;x,v) - E_-/\sqrt{\Delta E}\right)^2 + V^2(t;x,v) = \left(\sqrt{\Delta E}x - E_-/\sqrt{\Delta E}\right)^2 + v^2$$

which means that $(\sqrt{\Delta E}X, V)$ describes the circle centered at $(E_-/\sqrt{\Delta E}, 0)$ with radius $\big(v^2 + (\sqrt{\Delta E}x - E_-/\sqrt{\Delta E})^2\big)^{1/2}$.

Next, we will make use of the following claim (see [5] for a more general result)

Lemma 2.1 *Let $E \in W^{1,\infty}((0,L))$. Then for any characteristic we have*

$$|V(s_1) - V(s_2)| \leq 2\left(2L\, \|E\|_\infty\right)^{1/2}, \qquad \text{for } \ T_{\rm inc} \leq s_1 \leq s_2 \leq T_{\rm out}.$$

Proof. Suppose that $V(s_1), V(s_2)$ have the same sign, for example $0 \leq V(s_1) \leq V(s_2)$. The conservation of the energy along the characteristics

$$\frac{1}{2}V(s_1)^2 + \Phi(X(s_1)) = \frac{1}{2}V(s_2)^2 + \Phi(X(s_2))$$

implies that $V(s_2) = \left(V(s_1)^2 + a\right)^{1/2}$ with $a = 2(\Phi(X(s_1)) - \Phi(X(s_2))) \in [0, 2L\|E\|_\infty]$. It is easily seen that

$$V(s_2) - V(s_1) = \frac{a}{\left(V(s_1)^2 + a\right)^{1/2} + V(s_1)} \leq \sqrt{a} \leq \left(2L\, \|E\|_\infty\right)^{1/2}.$$

If $V(s_1), V(s_2)$ have opposite signs, there exists $s_3 \in [s_1, s_2]$ such that $V(s_3) = 0$ and thus $|V(s_1) - V(s_2)| \leq |V(s_1) - V(s_3)| + |V(s_3) - V(s_2)|$ and we can apply the previous result to both terms of the right hand side. ∎

3 Existence-Uniqueness for Bounded first Moments of Incoming Data

The goal of this section is to prove Theorem 1.1. First of all, let us recall a few basic facts about the solutions of the stationary transport equation

$$v\partial_x f + E\partial_v f + \kappa\sigma(v) f = S \tag{3.13}$$

for a given $S \in L^1((0,L)\times\mathbb{R})$ and boundary condition (1.3). We recall that the incoming data are required to satisfy (1.4) and for further purposes it is convenient to introduce the quantity

$$\mathcal{J} = \int_0^{+\infty} v f_0(v)\,\mathrm{d}v - \int_{-\infty}^0 v f_L(v)\,\mathrm{d}v$$

which is positive when the incoming data are non-negative. We refer e.g. to [1, 15, 17] for details on the following classical results.

Proposition 3.1 *For any $\kappa > 0$, (3.13) admits a unique solution $f \in L^1((0,L)\times\mathbb{R})$ the outgoing traces of which verify $vf(0,v) \in L^1((-\infty,0))$ and $vf(L,v) \in L^1((0,+\infty))$. The solution is given by the following explicit formula*

$$\begin{aligned} f(x,v) &= f_{\mathrm{inc}}(X_e,V_e)\ e^{-\kappa\int_0^{T_{\mathrm{inc}}(x,v)}\sigma(V(-\tau;x,v))\,\mathrm{d}\tau} \\ &+ \int_0^{T_{\mathrm{inc}}(x,v)} e^{-\kappa\int_0^t \sigma(V(-\tau;x,v))\,\mathrm{d}\tau}\ S\big(X(-t;x,v),V(-t;x,v)\big)\,\mathrm{d}t \end{aligned} \tag{3.14}$$

where $f_{\mathrm{inc}}(X_e,V_e)$ stands for $f_0(V(-T_{\mathrm{inc}}(x,v);x,v))$ if $X(-T_{\mathrm{inc}}(x,v);x,v)=0$, $V(-T_{\mathrm{inc}}(x,v);x,v) > 0$ or $f_L(V(-T_{\mathrm{inc}}(x,v);x,v))$ if $X(-T_{\mathrm{inc}}(x,v);x,v)=L$, $V(-T_{\mathrm{inc}}(x,v);x,v)<0$.

Corollary 3.1 *The solution of (3.13) given by (3.14) verifies the following properties:*

i) If the data f_0, f_L and S are non-negative, then $f \geq 0$ as well.

ii) If f_0, f_L and S are bounded, then f is bounded too with

$$\|f\|_{L^\infty} \leq \max\Big\{\|f_0\|_{L^\infty((0,+\infty))}, \|f_L\|_{L^\infty((-\infty,0))}, \frac{1}{\kappa s_0}\|S\|_{L^\infty((0,L)\times\mathbb{R})}\Big\}.$$

iii) Assume that $vf_0 \in L^1((0,+\infty))$, $vf_L \in L^1((-\infty,0))$ and $S \in L^1((0,L)\times\mathbb{R})$. Then, we have

$$\begin{aligned} \kappa s_0\|f\|_{L^1((0,L)\times\mathbb{R})} &+ \|vf(0,\cdot)\|_{L^1((-\infty,0))} + \|vf(L,\cdot)\|_{L^1((0,+\infty))} \\ &\leq \|vf_0\|_{L^1((0,+\infty))} + \|vf_L\|_{L^1((-\infty,0))} + \|S\|_{L^1((0,L)\times\mathbb{R})}. \end{aligned}$$

Finally, we shall also need the following integrated formula

$$
\begin{aligned}
\int_0^L\!\!\int_{-\infty}^{+\infty} f\,\psi\,\mathrm{d}v\,\mathrm{d}x \;=\;& \int_0^{+\infty} v f_0(v)\left(\int_0^{T_{\rm out}(0,v)} e^{-\kappa\int_0^s \sigma(V(\tau;0,v))\,\mathrm{d}\tau}\ \psi(X(s;0,v),V(s;0,v))\,\mathrm{d}s\right)\mathrm{d}v \\
&- \int_{-\infty}^{0} v f_L(v)\left(\int_0^{T_{\rm out}(L,v)} e^{-\kappa\int_0^s \sigma(V(\tau;L,v))\,\mathrm{d}\tau}\ \psi(X(s;L,v),V(s;L,v))\,\mathrm{d}s\right)\mathrm{d}v \\
&+ \int_0^L\!\!\int_{-\infty}^{+\infty} S(x,v)\left(\int_0^{T_{\rm out}(x,v)} e^{-\kappa\int_0^s \sigma(V(\tau;x,v))\,\mathrm{d}\tau}\ \psi(X(s;x,v),V(s;x,v))\,\mathrm{d}s\right)\mathrm{d}v\,\mathrm{d}x,
\end{aligned}
\tag{3.15}
$$

which holds for any bounded test function ψ.

We show now that the proof of Theorem 1.1 relies on an iterative procedure with sufficient boundary control estimates.

Proof of Theorem 1.1: Notice that, by linearity, it suffices to deal with non-negative incoming data: $f_0, f_L \geq 0$. Then, the existence result follows by proving the convergence of the sequence $\left(f^{(n)}\right)_{n\in\mathbb{N}}$ constructed as follows:

- Set $f^{(0)} = 0$;
- Given $f^{(n)}$, define $f^{(n+1)}$ by using (3.14) with $S(x,v) = \kappa Q^+(f^{(n)})(x,v) = \kappa \left\langle f^{(n)}\right\rangle_s(x,v)\, M_\theta(v)$, where we denote $\left\langle g\right\rangle_s(x,v) = \int_{-\infty}^{+\infty} s(v,v')g(x,v')\,\mathrm{d}v'$.

Thus, the proof of Theorem 1.1 is a consequence of the following claims.

Lemma 3.1 *Let $f_0, f_L \geq 0$ satisfy* (1.4)*. The sequence $\left(f^{(n)}(x,v)\right)_{n\in\mathbb{N}}$ is non-decreasing:*

$$0 \leq f^{(n)}(x,v) \leq f^{(n+1)}(x,v) \leq \dots$$

and the outgoing traces satisfy the following estimate

$$\int_0^{\infty} v f^{(n)}(L,v)\,\mathrm{d}v + \int_{-\infty}^{0} (-v) f^{(n)}(0,v)\,\mathrm{d}v \leq M_0 + M_L. \tag{3.16}$$

Lemma 3.2 *Let $f_0, f_L \geq 0$ satisfy* (1.4)*. The sequence $\left(f^{(n)}\right)_{n\in\mathbb{N}}$ is bounded in $L^1((0,L)\times\mathbb{R})$.*

Let us postpone for a while the proof of these statements.

By combining Lemma 3.1 and 3.2 and appealing to the Beppo-Levi theorem we deduce that $f^{(n)}$ converges in $L^1((0,L)\times\mathbb{R})$ to

$$f(x,v) = \sup_{n\in\mathbb{N}} f^{(n)}(x,v)\,. \tag{3.17}$$

This is clearly enough to conclude by letting $n \to \infty$ that f is a solution of (1.1)-(1.3). So the existence follows.

To justify uniqueness, we repeat the tricky argument of [3, 4] which is a consequence of the non-expansive nature of the collision map, in the spirit of [7]. Indeed, for any non decreasing bounded function H we have by the symmetry of the scattering function

$$\begin{aligned}
\int_{-\infty}^{+\infty} Q(f) H\left(\frac{f}{M_\theta}\right) \,\mathrm{d}v &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(v,v')\{M_\theta(v) f(x,v') - M_\theta(v') f(x,v)\} H\left(\frac{f(x,v)}{M_\theta(v)}\right) \,\mathrm{d}v\mathrm{d}v' \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(v,v')\{M_\theta(v') f(x,v) - M_\theta(v) f(x,v')\} H\left(\frac{f(x,v')}{M_\theta(v')}\right) \,\mathrm{d}v\mathrm{d}v' \\
&= -\frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(v,v') M_\theta(v) M_\theta(v') \left(\frac{f(x,v)}{M_\theta(v)} - \frac{f(x,v')}{M_\theta(v')}\right) \\
&\times \left(H\left(\frac{f(x,v)}{M_\theta(v)}\right) - H\left(\frac{f(x,v')}{M_\theta(v')}\right)\right) \,\mathrm{d}v\mathrm{d}v' \\
&\leq 0. \qquad (3.18)
\end{aligned}$$

In particular, taking $H = \mathrm{sgn}$ we obtain

$$\int_{-\infty}^{+\infty} Q(f)(x,v)\mathrm{sgn}(f(x,v)) \,\mathrm{d}v \leq 0$$

with equality iff $\mathrm{sgn} f$ does not depend on v. Therefore, if f solves

$$v\partial_x f + E\partial_v f = \kappa Q(f)$$

with vanishing incoming data, we are led to

$$\int_0^{+\infty} v|f(L,v)| \,\mathrm{d}v + \int_{-\infty}^{0} (-v)|f(0,v)| \,\mathrm{d}v = \kappa \int_0^L \int_{-\infty}^{+\infty} Q(f)\mathrm{sgn}(f) \,\mathrm{d}v \,\mathrm{d}x \leq 0.$$

We deduce on the one hand that $f(0,v) = 0 = f(L,v)$ for a.e. $v \in \mathbb{R}$ and, on the other hand, that $\mathrm{sgn}(f(x,v))$ does not depend on v. It follows that

$$\begin{aligned}
(v\partial_x + E(x)\partial_v)|f| + \kappa\sigma(v)|f| &= \kappa M_\theta(v) \int_{-\infty}^{+\infty} s(v,w) f(x,w) \,\mathrm{d}w \ \mathrm{sgn}(f(x,v)) \\
&= \kappa M_\theta(v) \int_{-\infty}^{+\infty} s(v,w)|f(x,w)| \,\mathrm{d}w
\end{aligned}$$

with $|f|(0,v) = 0 = |f|(L,v)$ for a.e. $v \in \mathbb{R}$. Integrating along characteristics, see (3.14), we get for $\xi \in \{0, L\}$ and any $v \in \mathbb{R}$

$$0 = \int_0^{T_{\mathrm{inc}}(\xi,v)} \kappa e^{-\kappa\int_0^t \sigma(V(-\tau;\xi,v))\,\mathrm{d}\tau} \ \big\langle |f| \big\rangle_s (X(-t;\xi,v), V(-t;\xi,v)) \ M_\theta(V(-t;\xi,v)) \,\mathrm{d}t.$$

Using Proposition 3.1, we conclude that $|f(x,v)| = 0$ for a.e. $x \in (0,L)$, $v \in \mathbb{R}$.

So the proof of Theorem 1.1 is completed, provided the proofs of next two lemmas are completed as well ∎

Proof of Lemma 3.1. The monotonicity property is proved recursively by using formula (3.14); we also know that each term of the sequence belongs to $L^1((0,L)\times\mathbb{R})$. Next, integration of

$$\begin{cases} v\partial_x f^{(n)} + E\partial_v f^{(n)} + \kappa\sigma(v) f^{(n)} = \kappa\big\langle f^{(n-1)}\big\rangle_s(x,v) M_\theta(v) \\ f^{(n)}(0,v)_{|v>0} = f_0(v), \qquad f^{(n)}(L,v)_{|v<0} = f_L(v) \end{cases}$$

yields

$$\int_0^\infty v f^{(n)}(L,v)\,\mathrm{d}v + \int_{-\infty}^0 (-v) f^{(n)}(0,v)\,\mathrm{d}v = \int_0^\infty v f_0(v)\,\mathrm{d}v + \int_{-\infty}^0 (-v) f_L(v)\,\mathrm{d}v$$
$$+\kappa \int_0^L \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(v,v')\{M_\theta(v) f^{(n-1)}(x,v') - M_\theta(v') f^{(n)}(x,v)\}\,\mathrm{d}v'\mathrm{d}v\mathrm{d}x.$$

The last term is nothing but

$$\kappa \int_0^L \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(v,v') M_\theta(v')(f^{(n-1)}(x,v) - f^{(n)}(x,v))\,\mathrm{d}v'\mathrm{d}v\mathrm{d}x$$

which is non-positive as a consequence of the monotonicity of $f^{(n)}(x,v)$ and (3.16) holds. ∎

Proof of Lemma 3.2. Applying (3.15) with $\psi(x,v) = \sigma(v)$ yields

$$\begin{aligned} \int_0^L \int_{-\infty}^{+\infty} \sigma(v) f^{(n+1)}(x,v)\,\mathrm{d}v\,\mathrm{d}x \;&=\; \int_0^{+\infty} v f_0(v) \left(\int_0^{T_{\mathrm{out}}(0,v)} e^{-\kappa\int_0^s \sigma(V(\tau;0,v))\,\mathrm{d}\tau} \sigma(V(s;0,v))\,\mathrm{d}s\right)\mathrm{d}v \\ &- \int_{-\infty}^0 v f_L(v) \left(\int_0^{T_{\mathrm{out}}(L,v)} e^{-\kappa\int_0^s \sigma(V(\tau;L,v))\,\mathrm{d}\tau} \sigma(V(s;L,v))\,\mathrm{d}s\right)\mathrm{d}v \\ &+ \int_0^L \int_{-\infty}^{+\infty} \big\langle f^{(n)}\big\rangle_s(x,v)\; M_\theta(v) \\ &\times \left(\int_0^{T_{\mathrm{out}}(x,v)} \kappa e^{-\kappa\int_0^s \sigma(V(\tau;x,v))\,\mathrm{d}\tau} \sigma(V(s;x,v))\,\mathrm{d}s\right)\mathrm{d}v\,\mathrm{d}x. \end{aligned}$$

Let us denote, for any $(x,v)\in[0,L]\times\mathbb{R}$, the auxiliary function

$$h(x,v) = M_\theta(v)\int_0^{T_{\mathrm{out}}(x,v)} \kappa e^{-\kappa\int_0^s \sigma(V(\tau;x,v))\,\mathrm{d}\tau} \sigma(V(s;x,v))\,\mathrm{d}s\,. \tag{3.19}$$

Obviously we have for any $x\in[0,L]$

$$\int_0^{T_{\mathrm{out}}(x,v)} e^{-\kappa\int_0^s \sigma(V(\tau;x,v))\,\mathrm{d}\tau} \sigma(V(s;x,v))\,\mathrm{d}s \le \frac{1}{\kappa}$$

and by monotonicity we obtain

$$\int_0^L \int_{-\infty}^{+\infty} \sigma(v) f^{(n+1)}(x,v)\,\mathrm{d}v\,\mathrm{d}x \leq \frac{1}{\kappa}\,\mathcal{J} + \int_0^L \int_{-\infty}^{+\infty} \left\langle f^{(n+1)} \right\rangle_s (x,v)\ h(x,v)\ \mathrm{d}v\mathrm{d}x. \tag{3.20}$$

We now prove that there are $\beta \in (0,1)$, $R > 0$ such that

$$0 < h(x,v) \leq M_\theta(v) - \beta M_\theta(v) \mathbf{1}_{\{|v|>R\}}.$$

For $R > 0$, we write $h(x,v) = h(x,v)\mathbf{1}_{\{|v|\leq R\}} + h(x,v)\mathbf{1}_{\{|v|>R\}} = \underline{h}(x,v) + \overline{h}(x,v)$. We have

$$0 \leq \underline{h}(x,v) \leq M_\theta(v)\mathbf{1}_{\{|v|\leq R\}}.$$

For estimating $\overline{h}$ we use Lemma 2.1 which motivates a suitable choice of the parameter R. Indeed for any $(x,v) \in (0,L)\times\mathbb{R}$, and $s \in [0, T_{\mathrm{out}}(x,v)]$, when $|v| \geq 4\sqrt{2L(\|E\|_\infty + \alpha)}$ for some $\alpha > 0$,[1] we deduce from Lemma 2.1 that

$$|V(s;x,v)| \geq |v| - |V(s;x,v) - v| \geq |v| - 2\sqrt{2L\|E\|_\infty} \geq \frac{|v|}{2},$$

holds which in turn implies

$$T_{\mathrm{out}}(x,v) \leq \frac{2L}{|v|} \leq \sqrt{\frac{L}{8(\|E\|_\infty + \alpha)}}$$

for such v's. Therefore, choosing $R = 4\sqrt{2L(\|E\|_\infty + \alpha)}$, we obtain

$$\begin{aligned}
\overline{h}(x,v) &\leq \mathbf{1}_{\{|v|>4\sqrt{2L(\|E\|_\infty+\alpha)}\}} M_\theta(v) \int_0^{\sqrt{L/8(\|E\|_\infty+\alpha)}} \kappa e^{-\kappa \int_0^s \sigma(V(\tau;x,v))\,\mathrm{d}\tau} \sigma(V(s;x,v))\,\mathrm{d}s \\
&= \mathbf{1}_{\{|v|>4\sqrt{2L(\|E\|_\infty+\alpha)}\}} M_\theta(v) \big(1 - e^{-\kappa \int_0^{\sqrt{L/8(\|E\|_\infty+\alpha)}} \sigma(V(\tau;x,v))\,\mathrm{d}\tau}\big) \\
&\leq \mathbf{1}_{\{|v|>4\sqrt{2L(\|E\|_\infty+\alpha)}\}} M_\theta(v) \big(1 - e^{-\kappa \sqrt{L/8(\|E\|_\infty+\alpha)}\, s_1}\big)
\end{aligned}$$

for any $x \in [0,L]$. We conclude that

$$\begin{aligned}
h(x,v) &\leq \mathbf{1}_{\{|v|\leq 4\sqrt{2L(\|E\|_\infty+\alpha)}\}} M_\theta(v) + \mathbf{1}_{\{|v|>4\sqrt{2L(\|E\|_\infty+\alpha)}\}} M_\theta(v) \big(1 - e^{-\kappa \sqrt{L/8(\|E\|_\infty+\alpha)}\, s_1}\big) \\
&= M_\theta(v) - \beta M_\theta(v) \mathbf{1}_{\{|v|>4\sqrt{2L(\|E\|_\infty+\alpha)}\}}
\end{aligned}$$

[1] The parameter α is not particularly relevant: it is used to obtain formulae which make sense even when there is no electric field.

with $\beta = e^{-\kappa\sqrt{L/8(\|E\|_\infty+\alpha)}\, s_1} \in (0,1)$. Coming back to (3.20) we deduce that

$$\begin{aligned}\int_0^L \int_{-\infty}^{+\infty} \sigma(v) f^{(n+1)}(x,v)\,\mathrm{d}v\,\mathrm{d}x &\leq \frac{\mathcal{J}}{\kappa} + \int_0^L \int_{-\infty}^{+\infty} \langle f^{(n+1)}\rangle_s(x,v) M_\theta(v)\,\mathrm{d}v\,\mathrm{d}x \\ &- \beta \int_0^L \int_{-\infty}^{+\infty} \langle f^{(n+1)}\rangle_s(x,v) M_\theta(v) \mathbf{1}_{\{|v|>R\}}\,\mathrm{d}v\,\mathrm{d}x \\ &= \frac{\mathcal{J}}{\kappa} + \int_0^L \int_{-\infty}^{+\infty} \sigma(v) f^{(n+1)}(x,v)\,\mathrm{d}v\,\mathrm{d}x \\ &- \beta \int_0^L \int_{-\infty}^{+\infty} \langle f^{(n+1)}\rangle_s(x,v) M_\theta(v) \mathbf{1}_{\{|v|>R\}}\,\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

Finally one gets

$$\beta\; s_0 \int_0^L \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f^{(n+1)}(x,v') M_\theta(v) \mathbf{1}_{\{|v|>R\}}\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x \leq \frac{\mathcal{J}}{\kappa}$$

implying that

$$\int_0^L \int_{-\infty}^{+\infty} f^{(n+1)}(x,v')\,\mathrm{d}v'\,\mathrm{d}x \leq \frac{\mathcal{J}}{\kappa\gamma}$$

where

$$\gamma = s_0\; \beta \int_{|v|>R} M_\theta(v)\,\mathrm{d}v = s_0\; e^{-\kappa\sqrt{L/8(\|E\|_\infty+\alpha)}\, s_1} \int_{|v|>4\sqrt{2L(\|E\|_\infty+\alpha)}} M_\theta(v)\,\mathrm{d}v > 0.$$

As a matter of fact, observe that γ tends to 0 when κ, L or $\|E\|_\infty$ tends to $+\infty$. ∎

4 Further A Priori Estimates

4.1 Moments and L^∞ Estimates

Let us start with a basic estimate on the current.

Theorem 4.1 *Let $E \in W^{1,\infty}((0,L))$ and f_0, f_L satisfy* (1.4). *We denote by $f \in L^1((0,L)\times\mathbb{R})$ the unique (mild) solution of* (1.1)-(1.3). *Let us denote*

$$j_\pm(x) = \int_{\pm v>0} \pm v\; |f|(x,v)\,\mathrm{d}v \geq 0.$$

Then

i) The functions $j_\pm$ belong to $L^\infty((0,L))$ and satisfy

$$\|j_\pm\|_\infty \leq \int_0^{+\infty} v|f_0(v)|\,\mathrm{d}v - \int_{-\infty}^0 v|f_L(v)|\,\mathrm{d}v + \kappa s_1 \|f\|_{L^1((0,L)\times\mathbb{R})}.$$

ii) The total current $j = \int_{-\infty}^{+\infty} vf\,\mathrm{d}v$ is constant with respect to x and, assuming $f_0, f_L \geq 0$, it verifies $|j| \leq \mathcal{J}$. In particular we have

$$\int_0^{+\infty} vf(L,v)\,\mathrm{d}v - \int_{-\infty}^{0} vf(0,v)\,\mathrm{d}v = \mathcal{J}. \tag{4.21}$$

iii) If $f_0 \geq 0$ and $f_L = 0$ then $j \in [0, \mathcal{J}]$.

Proof. We prove i) by a duality argument. Denoting $v_\pm = \max(\pm v, 0)$, we apply (3.15) with $\psi(x,v) = v_\pm \zeta(x)$ where $\zeta \in L^1((0,L))$. We can suppose that $f \geq 0$ and pick $\zeta \geq 0$; we get

$$\begin{aligned}
\int_0^L j_\pm(x)\ \zeta(x)\,\mathrm{d}x \ &\leq\ \int_0^{+\infty} vf_0(v) \int_0^{T_{\mathrm{out}}(0,v)} V_\pm(s;0,v)\ \zeta(X(s;0,v))\,\mathrm{d}s\,\mathrm{d}v \\
&-\ \int_{-\infty}^{0} vf_L(v) \int_0^{T_{\mathrm{out}}(L,v)} V_\pm(s;L,v)\ \zeta(X(s;L,v))\,\mathrm{d}s\,\mathrm{d}v \\
&+\ \kappa \int_0^L \int_{-\infty}^{+\infty} \langle f \rangle_s(x,v)\ M_\theta(v) \int_0^{T_{\mathrm{out}}(x,v)} V_\pm(s,x,v)\ \zeta(X(s;x,v))\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned} \tag{4.22}$$

By using Proposition 2.1 it is easily seen that for any $(x,v) \in \big([0,L[\times(0,+\infty)\big) \cup \big(]0,L]\times(-\infty,0)\big)$ we have

$$\int_0^{T_{\mathrm{out}}(x,v)} V_\pm(s;x,v)\ \zeta(X(s;x,v))\,\mathrm{d}s \leq \int_0^L \zeta(u)\,\mathrm{d}u.$$

Taking into account that

$$\int_0^L \int_{-\infty}^{+\infty} \langle f \rangle_s(x,v) M_\theta(v)\ \mathrm{d}v\mathrm{d}x = \int_0^L \int_{-\infty}^{+\infty} \sigma(v) f(x,v)\ \mathrm{d}v\mathrm{d}x \leq s_1 \int_0^L \int_{-\infty}^{+\infty} f(x,v)\ \mathrm{d}v\mathrm{d}x$$

it follows that

$$0 \leq \int_0^L j_\pm(x)\ \zeta(x)\,\mathrm{d}x \leq \mathcal{J}\|\zeta\|_{L^1((0,L))} + \kappa s_1 \|f\|_{L^1((0,L)\times\mathbb{R})}\|\zeta\|_{L^1((0,L))}$$

which proves i).

Consequently, $j = \int_{-\infty}^{+\infty} vf\,\mathrm{d}v$ belongs to $L^\infty((0,L))$. Integrating (1.1) with respect to v tells us that $\frac{\mathrm{d}}{\mathrm{d}x} j = 0$ and integrating with respect to space leads to (4.21). When the data are non-negative, $f \geq 0$ we deduce that

$$-\mathcal{J} \leq \int_{-\infty}^{0} vf(0,v)\,\mathrm{d}v \leq j(0) = j = j(L) \leq \int_0^{+\infty} vf(L,v)\,\mathrm{d}v \leq \mathcal{J}$$

holds. When $f_L = 0$, the previous estimate becomes $0 \leq \int_0^{+\infty} vf(L,v)\,\mathrm{d}v = j(L) = j \leq \mathcal{J}$. ∎

Corollary 4.1 *Let $E \in W^{1,\infty}((0,L))$ and f_0, f_L satisfy* (1.4). *Let $f \in L^1((0,L)\times\mathbb{R})$ be the unique (mild) solution of* (1.1)-(1.3). *If, furthermore the incoming data are bounded, then the macroscopic density $\langle f\rangle(x)$ belongs to $L^\infty((0,L))$ and f belongs to $L^\infty((0,L)\times\mathbb{R})$.*

Proof. The existence proof works with an iterative argument looking at

$$\kappa f^{(n+1)} + v\partial_x f^{(n+1)} + E\partial_v f^{(n+1)} = \kappa\big\langle f^{(n)}\big\rangle_s(x,v)M_\theta(v)$$

with the incoming boundary conditions f_0, f_L. For the sake of simplicity, we suppose that $f_0, f_L \geq 0$ and work with non-negative quantities. The proof of Theorem 4.1 can be adapted to show that

$$\sup_{n\in\mathbb{N},\ x\in[0,L]} \int_{-\infty}^{+\infty} |v| f^{(n)}(x,v)\,\mathrm{d}v \leq \mathcal{K} < \infty$$

where the bound depends on L, κ, θ, $\|E\|_\infty$, s_0, s_1 and on the (weighted) L^1 norm of the data. Then, we compute

$$\big\langle f^{(n+1)}\big\rangle(x) = \int_{|v|\leq R} f^{(n+1)}\,\mathrm{d}v + \int_{|v|\geq R} f^{(n+1)}\,\mathrm{d}v \leq 2R\|f^{(n+1)}\|_\infty + \frac{1}{R}\int_{-\infty}^{+\infty} |v| f^{(n+1)}\,\mathrm{d}v.$$

Optimizing with respect to R yields

$$\big\langle f^{(n+1)}\big\rangle(x) \leq 2\sqrt{2\mathcal{K}\ \|f^{(n+1)}\|_\infty}.$$

Now, the formula along the characteristics leads to

$$\begin{aligned}
f^{(n+1)}(x,v) \ &\leq\ \|f_{\mathrm{inc}}\|_\infty\ e^{-\kappa\int_0^{T_{\mathrm{inc}}(x,v)}\sigma(V(-\tau;x,v))\,\mathrm{d}\tau}\\
&+\ \frac{1}{s_0}\int_0^{T_{\mathrm{inc}}(x,v)} e^{-\kappa\int_0^t\sigma(V(-\tau;x,v))\,\mathrm{d}\tau}\kappa(\sigma\big\langle f^{(n)}\big\rangle_s M_\theta)(X(-t;x,v),V(-t;x,v))\,\mathrm{d}t\\
&\leq\ \|f_{\mathrm{inc}}\|_\infty\ e^{-\kappa\int_0^{T_{\mathrm{inc}}(x,v)}\sigma(V(-\tau;x,v))\,\mathrm{d}\tau} + \frac{s_1}{s_0}\|\big\langle f^{(n)}\big\rangle\|_\infty\frac{1}{\sqrt{2\pi\theta}}(1-e^{-\kappa\int_0^{T_{\mathrm{inc}}(x,v)}\sigma(V(-\tau;x,v))\,\mathrm{d}\tau}).
\end{aligned}$$

Let $B>0$ such that $\|f_{\mathrm{inc}}\|_\infty \leq B$. Therefore if the induction hypothesis $\frac{s_1}{s_0}\frac{1}{\sqrt{2\pi\theta}}\|\big\langle f^n\big\rangle\|_\infty \leq B$ holds then we also have $f^{(n+1)} \leq B$.

We can now show that the set $\{f(x,v) \in L^\infty(0,L;L^1(\mathbb{R}))\ :\ \ 0 \leq \big\langle f\big\rangle \leq \frac{s_0}{s_1}\sqrt{2\pi\theta}\ B\}$ is left invariant by the recursion scheme.

Indeed, owing to the previous estimates, it is enough to find $B \geq \|f_{\mathrm{inc}}\|_\infty$ such that $2\sqrt{2\mathcal{K}B} \leq \frac{s_0}{s_1}\sqrt{2\pi\theta}\ B$, or equivalently

$$\left(\frac{s_1}{s_0}\right)^2\frac{4\mathcal{K}}{\pi\theta} \leq B.$$

Accordingly, choosing B large enough, depending on $\|f_{\mathrm{inc}}\|_\infty$, L, k, θ, $\|E\|_\infty$, s_0, s_1 we have for any $n\in\mathbb{N}$

$$\|f^{(n)}\|_{L^\infty((0,L)\times\mathbb{R})} \leq B, \qquad \|\big\langle f^{(n)}\big\rangle\|_{L^\infty((0,L))} \leq \frac{s_0}{s_1}\sqrt{2\pi\theta}B.$$

In particular, by Lemmas 3.1 and 3.2, every f^n solution constructed by the iterative monotone scheme is in the set $\{f(x,v) \in L^\infty(0,L;L^1(\mathbb{R})) :\ 0 \le \langle f \rangle \le \frac{s_0}{s_1}\sqrt{2\pi\theta}\ B\}$, and so the corresponding limiting function (3.17) belongs to this set as well. ∎

Next we discuss estimates on higher moments of traces.

Proposition 4.1 *Let the assumptions of Theorem 1.1 be fulfilled with $f_0, f_L \ge 0$. Suppose moreover that $v^2 f_0 \in L^1((0,+\infty))$ and $v^2 f_L(v) \in L^1((-\infty,0))$. Then, $v^2 f(x,v)$ belongs to $L^\infty(0,L;L^1(\mathbb{R}))$ and the outgoing traces fulfill $v^2 f(0,v) \in L^1((-\infty,0))$ and $v^2 f(L,v) \in L^1((0,+\infty))$.*

More generally, if $v^k f_0 \in L^1((0,+\infty))$ and $v^k f_L(v) \in L^1((-\infty,0))$ for some integer $k \ge 1$ then $v^k f(x,v)$ belongs to $L^\infty(0,L;L^1(\mathbb{R}))$ and $v^k f(0,v) \in L^1((-\infty,0))$ and $v^k f(L,v) \in L^1((0,+\infty))$.

Proof. Multiplying (1.1) by $v_+ = \max(0,v)$ we obtain

$$\partial_x \int_0^{+\infty} v^2 f \,\mathrm{d}v - E(x) \int_0^{+\infty} f \,\mathrm{d}v + \kappa \int_0^{+\infty} \sigma(v) v f \,\mathrm{d}v = \kappa \int_0^{+\infty} \langle f \rangle_s (x,v) v M_\theta(v) \,\mathrm{d}v.$$

Since $f \ge 0$, integration over $(0,x)$ yields

$$\begin{aligned} \int_0^{+\infty} v^2 f(x,v)\,\mathrm{d}v \ \ &\le \ \ \int_0^{+\infty} v^2 f_0(v)\,\mathrm{d}v + \|E\|_{L^\infty((0,L))} \int_0^L \int_0^{+\infty} f(y,v)\,\mathrm{d}v\,\mathrm{d}y \\ &\quad + \kappa s_1 \sqrt{\frac{\theta}{2\pi}} \int_0^L \int_{-\infty}^{+\infty} f(y,v)\,\mathrm{d}v\,\mathrm{d}y \end{aligned} \tag{4.23}$$

We conclude that $x \longmapsto \int_0^{+\infty} v^2 f(x,v)\,\mathrm{d}v$ belongs to $L^\infty((0,L))$ by using the estimates already obtained in Theorem 1.1. We proceed similarly by multiplying by $v_- = \max(0,-v)$ and integrating over (x,L) in order to estimate $\int_{-\infty}^0 v^2 f(x,v)\,\mathrm{d}v$. The generalization to higher moments is straightforward. ∎

4.2 Entropy Estimate

For any $f \in L^1((0,L)\times\mathbb{R})$, we can define the equilibrium state by

$$Z\mathcal{M}_{\theta,\Phi}(x,v) = Z \exp\Big(- \frac{v^2/2 + \Phi(x)}{\theta} \Big), \qquad Z \int_0^L \int_{-\infty}^{+\infty} \mathcal{M}_{\theta,\Phi}\,\mathrm{d}v\,\mathrm{d}x = \int_0^L \int_{-\infty}^{+\infty} f\,\mathrm{d}v\,\mathrm{d}x.$$

Then, we define in the following non-negative quantity (the relative entropy of f and $Z\mathcal{M}_{\theta,\Phi}$)

$$r(x,v) = f \ln\Big(\frac{f}{Z\mathcal{M}_{\theta,\Phi}}\Big) - f + Z\mathcal{M}_{\theta,\Phi} = f\Big(\ln f - \ln Z + \frac{v^2}{2\theta} + \frac{\Phi}{\theta}\Big) - f + Z\mathcal{M}_{\theta,\Phi} \ge 0. \tag{4.24}$$

We will use this function $r(x,v)$ to compute integral estimates that will be used to control entropy estimates of the mild solution of (1.1)-(1.3).

Proposition 4.2 *Assume that E belongs to $W^{1,\infty}((0,L))$ and let $f_0, f_L \geq 0$ satisfy the entropic boundary condition* (1.9)*, which we rewrite here*

$$\mathcal{H}_L = \int_0^{+\infty} v\ \Big(1 + \frac{v^2}{2} + |\ln f_0|\Big) f_0 \,\mathrm{d}v - \int_{-\infty}^0 v\Big(1 + \frac{v^2}{2} + |\ln f_L|\Big) f_L \,\mathrm{d}v < +\infty.$$

Let $f \in L^1((0,L)\times\mathbb{R})$ be the unique (mild) solution of (1.1)-(1.3)*; we denote $\langle f\rangle(x) = \int_{-\infty}^{+\infty} f(x,v)\,\mathrm{d}v$. Then there exists a constant C which depends on $\kappa, \theta, L, \|E\|_\infty$, s_0, s_1 such that*

$$\int_0^L \int_{-\infty}^{+\infty} \Big(\frac{v^2}{2} + |\ln f|\Big) f(x,v)\,\mathrm{d}v\,\mathrm{d}x \leq C(1 + \mathcal{H}_L) \tag{4.25}$$

$$\int_0^L \langle f\rangle |\ln \langle f\rangle|\,\mathrm{d}x \leq C(1 + \mathcal{H}_L) \tag{4.26}$$

$$\int_0^{+\infty} v\Big(\frac{v^2}{2} + |\ln f|\Big) f(L,v)\,\mathrm{d}v - \int_{-\infty}^0 v\Big(\frac{v^2}{2} + |\ln f|\Big) f(0,v)\,\mathrm{d}v \leq C(1 + \mathcal{H}_L). \tag{4.27}$$

Proof. Throughout the proof, C denotes a quantity which depends on $\kappa, \theta, L, \|E\|_\infty$, s_0, s_1, even if the value of the constant may vary from a line to another. We define the potential by (2.10) with $\Phi_0 = 0$. In particular we have $\|\Phi\|_\infty \leq L\|E\|_\infty$ and using the assumption (1.9) on the boundary data we get the boundary control

$$\int_0^{+\infty} vr(0,v)\,\mathrm{d}v - \int_{-\infty}^0 vr(L,v)\,\mathrm{d}v \leq C\mathcal{H}_L < +\infty\,. \tag{4.28}$$

In addition, if f is a mild solution of (1.1)-(1.3) and the function $r(x,v)$ is defined by (4.24), then

$$(\kappa\sigma(v) + v\partial_x + E\partial_v)r = \kappa\big\langle f\big\rangle_s(x,v)M_\theta(v)\ \ln\Big(\frac{f}{Z\mathcal{M}_{\theta,\Phi}}\Big) + \kappa\sigma(v)Z\mathcal{M}_{\theta,\Phi} - \kappa\sigma(v)f. \tag{4.29}$$

Next, we want to obtain integral estimates for (4.29). First, we rewrite the first term in the right hand side as follows

$$\begin{aligned}
\kappa\big\langle f\big\rangle_s M_\theta\ \ln\Big(\frac{f}{Z\mathcal{M}_{\theta,\Phi}}\Big) &= \kappa\big\langle f\big\rangle_s M_\theta\ \ln\Big(\frac{\sigma(v)f}{\langle f\rangle_s M_\theta}\ \frac{\langle f\rangle_s M_\theta}{\sigma(v)Z\mathcal{M}_{\theta,\Phi}}\Big) \\
&= \kappa\big\langle f\big\rangle_s M_\theta\Big[\ln\Big(\frac{\sigma(v)f}{\langle f\rangle_s M_\theta}\Big) - \frac{\sigma(v)f}{\langle f\rangle_s M_\theta} + 1\Big] \\
&+\ \kappa\sigma(v)f + \kappa\big\langle f\big\rangle_s M_\theta\Big(\ln\Big(\frac{\langle f\rangle_s}{\sigma(v)}\Big) + \frac{\Phi}{\theta} - \ln(\sqrt{2\pi\theta}Z) - 1\Big).
\end{aligned}$$

Therefore, identity (4.29) can be rewritten as

$$\begin{aligned}
(\kappa\sigma(v) + v\partial_x + E\partial_v)r &= \kappa\big\langle f\big\rangle_s M_\theta\Big[\ln\Big(\frac{\sigma(v)f}{\langle f\rangle_s M_\theta}\Big) - \frac{\sigma(v)f}{\langle f\rangle_s M_\theta} + 1\Big] \\
&\quad +\kappa\sigma(v)Z\mathcal{M}_{\theta,\Phi} + \kappa\big\langle f\big\rangle_s M_\theta\Big(\ln\Big(\frac{\langle f\rangle_s}{\sigma(v)}\Big) + \frac{\Phi}{\theta} - \ln(\sqrt{2\pi\theta}Z) - 1\Big)
\end{aligned} \tag{4.30}$$

where we observe that the first term in the right hand side is non-positive due to the concavity of the function $\tau \mapsto \ln(\tau)$. Next, we use integrated representation formula (3.15) with the test function $\sigma(v)$ to obtain

$$\begin{aligned}
\int_0^L \int_{-\infty}^{+\infty} \sigma(v) r(x,v)\,\mathrm{d}v\,\mathrm{d}x \quad \leq \quad & \int_0^{+\infty} v r(0,v) \int_0^{T_{\mathrm{out}}(0,v)} e^{-\kappa \int_0^s \sigma(V(\tau;0,v))\,\mathrm{d}\tau} \sigma(V(s;0,v))\,\mathrm{d}s\,\mathrm{d}v \\
& - \int_{-\infty}^{0} v r(L,v) \int_0^{T_{\mathrm{out}}(L,v)} e^{-\kappa \int_0^s \sigma(V(\tau;L,v))\,\mathrm{d}\tau} \sigma(V(s;Lv))\,\mathrm{d}s\,\mathrm{d}v \\
& + \int_0^L \int_{-\infty}^{+\infty} \langle f \rangle_s \Big[\frac{\Phi}{\theta} + \ln\Big(\frac{\langle f \rangle_s}{\sigma(v)}\Big) - \ln(\sqrt{2\pi\theta} Z) - 1\Big] h(x,v)\,\mathrm{d}v\,\mathrm{d}x \\
& + \int_0^L \int_{-\infty}^{+\infty} \sigma(v) \frac{Z\mathcal{M}_{\theta,\Phi}}{M_\theta} h(x,v)\,\mathrm{d}v\,\mathrm{d}x
\end{aligned} \tag{4.31}$$

where we got rid of the non-positive term having $\ln\Big(\frac{\sigma(v) f}{\langle f \rangle_s M_\theta}\Big) - \frac{\sigma(v) f}{\langle f \rangle_s M_\theta} + 1$ and we have used the same notation of function $h(x,v)$ defined in (3.19) for the proof of Lemma 3.2, where we have shown, for some $\beta \in (0,1)$, $R > 0$, the inequality

$$0 < h(x,v) \leq M_\theta(v) - \beta M_\theta(v) \mathbf{1}_{\{|v|>R\}}, \quad (x,v) \in [0,L] \times \mathbb{R}\,. \tag{4.32}$$

In order to control the third term of (4.31) we use Jensen's inequality

$$\langle f \rangle_s(x,v) \ln\Big(\frac{\langle f \rangle_s(x,v)}{\sigma(v)\sqrt{2\pi\theta}}\Big) \leq \int_{-\infty}^{+\infty} s(v,v') f(x,v') \Big(\frac{(v')^2}{2\theta} + \ln f(x,v')\Big)\,\mathrm{d}v'$$

which comes by considering the convex function $u \to u \ln\Big(\frac{u}{\sigma(v)\sqrt{2\pi\theta}}\Big)$, the measure $\frac{s(v,v')}{\sigma(v)} M_\theta(v')\,\mathrm{d}v'$ and the application $v' \to \frac{\sigma(v) f(x,v')}{M_\theta(v')}$. Based on the above inequality one gets

$$\langle f \rangle_s \Big[\frac{\Phi}{\theta} + \ln\Big(\frac{\langle f \rangle_s}{\sigma(v)}\Big) - \ln(\sqrt{2\pi\theta} Z) - 1\Big] \leq \int_{-\infty}^{+\infty} s(v,v') f(x,v') \Big(\frac{\Phi}{\theta} + \frac{(v')^2}{2\theta} + \ln f(x,v') - \ln Z - 1\Big)\,\mathrm{d}v'.$$

Notice also that the last term of (4.31) can be written

$$\begin{aligned}
\int_0^L \int_{-\infty}^{+\infty} \sigma(v) \frac{Z\mathcal{M}_{\theta,\Phi}}{M_\theta} h(x,v)\,\mathrm{d}v\,\mathrm{d}x \quad = \quad & \int_0^L \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} s(v,v') M_\theta(v') \frac{Z\mathcal{M}_{\theta,\Phi}(x,v')}{M_\theta(v')} h(x,v)\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x \\
= \quad & \int_0^L \int_{-\infty}^{+\infty} h(x,v) \int_{-\infty}^{+\infty} s(v,v') Z\mathcal{M}_{\theta,\Phi}(x,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\,.
\end{aligned}$$

Hence, collecting the above computations, the left hand side from (4.31) is bounded by

$$\begin{aligned}\int_0^L \int_{-\infty}^{+\infty} \sigma(v) r(x,v)\,\mathrm{d}v\,\mathrm{d}x \quad \le \quad & \frac{1}{\kappa}\Big(\int_0^{+\infty} v r(0,v)\,\mathrm{d}v - \int_{-\infty}^{0} v r(L,v)\,\mathrm{d}v\Big) \\ & + \int_0^L \int_{-\infty}^{+\infty} h(x,v) \int_{-\infty}^{+\infty} s(v,v') \\ & \times \Big[f\Big(\frac{\Phi}{\theta} + \frac{(v')^2}{2\theta} + \ln f - \ln Z - 1\Big) + Z\mathcal{M}_{\theta,\Phi}\Big](x,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x \\ & = \int_0^L \int_{-\infty}^{+\infty} h(x,v) \int_{-\infty}^{+\infty} s(v,v') r(x,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\,.\end{aligned}$$

As in the proof of Lemma 3.2, using (4.32) yields

$$\begin{aligned}\int_0^L \int_{-\infty}^{+\infty} \sigma(v) r(x,v)\,\mathrm{d}v\,\mathrm{d}x \quad \le \quad & \frac{1}{\kappa}\Big(\int_0^{+\infty} v r(0,v)\,\mathrm{d}v - \int_{-\infty}^{0} v r(L,v)\,\mathrm{d}v\Big) \\ & + \int_0^L \int_{-\infty}^{+\infty} M_\theta(v) \int_{-\infty}^{+\infty} s(v,v') r(x,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x \\ & -\beta \int_0^L \int_{-\infty}^{+\infty} M_\theta(v)\mathbf{1}_{\{|v|>R\}} \int_{-\infty}^{+\infty} s(v,v') r(x,v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x \\ & \le \frac{1}{\kappa}\Big(\int_0^{+\infty} v r(0,v)\,\mathrm{d}v - \int_{-\infty}^{0} v r(L,v)\,\mathrm{d}v\Big) \\ & + \int_0^L \int_{-\infty}^{+\infty} \sigma(v') r(x,v')\,\mathrm{d}v'\,\mathrm{d}x \\ & - s_0\beta \int_0^L \int_{-\infty}^{+\infty} r(x,v')\,\mathrm{d}v'\,\mathrm{d}x \int_{|v|>R} M_\theta(v)\,\mathrm{d}v\,.\end{aligned}$$

Then it is clear that

$$\int_0^L \int_{-\infty}^{+\infty} r(x,v)\,\mathrm{d}v\,\mathrm{d}x \le \frac{C}{\kappa\gamma}\mathcal{H}_{_L}\,. \tag{4.33}$$

Now, in order to show (4.25), we fist observe that the following inequality holds

$$\int_{-\infty}^{+\infty} g|\ln g|\,\mathrm{d}v \le \int_{-\infty}^{+\infty} g(\ln g + av^2)\,\mathrm{d}v + \frac{4}{e}\int_{-\infty}^{+\infty} e^{-av^2/4}\,\mathrm{d}v \ ;$$

then, combining with (4.33), (4.25) follows. It is, then, easy to see that (4.26) becomes a consequence of the Jensen's inequality

$$\langle f\rangle(x) \ln\Big(\frac{\langle f\rangle(x)}{\sqrt{2\pi\theta}}\Big) \le \int_{-\infty}^{+\infty} f(x,v)\Big(\ln f(x,v) + \frac{v^2}{2\theta}\Big)\,\mathrm{d}v\,.$$

Finally, we obtain (4.27) by integrating (4.29) when written in the equivalent form

$$(v\partial_x + E\partial_v) r = \kappa Q(f)\Big[\ln\Big(\frac{f}{M_\theta}\Big) + \frac{\Phi(x)}{\theta} - \ln(\sqrt{2\pi\theta}Z)\Big]$$

since

$$\int_0^L \int_{-\infty}^{+\infty} \kappa Q(f)\Big[\frac{\Phi(x)}{\theta} - \ln(\sqrt{2\pi\theta}Z)\Big]\,\mathrm{d}v\,\mathrm{d}x = 0$$

and by (3.18) applied with the function $H = \ln$ we know that

$$\int_0^L \int_{-\infty}^{+\infty} \kappa Q(f) \ln\Big(\frac{f}{M_\theta}\Big)\,\mathrm{d}v\,\mathrm{d}x \leq 0\,.$$

∎

4.3 Non-Smooth Electric Fields

Theorem 1.1 establishes the well-posedness of (1.1)-(1.3) when dealing with smooth electric fields $E \in W^{1,\infty}((0,L))$ and integrable data. Since all the estimates discussed above actually depend only on the L^∞ bound of E, we can extend this result to non-smooth electric fields, at the price of a slightly strengthened assumption on the boundary data.

Theorem 4.2 *Let $E \in L^\infty((0,L))$ and consider $f_0, f_L \geq 0$ such that* (1.9) *holds. Then there is a unique weak solution $f \in L^1((0,L)\times\mathbb{R})$, which also satisfies* (4.25)-(4.27)*, of* (1.1)-(1.3)*. Moreover we have*

$$\Big\| \int_{-\infty}^{+\infty} |v| f(\cdot,v)\,\mathrm{d}v \Big\|_\infty \leq 2\Big(\int_0^{+\infty} v f_0(v)\,\mathrm{d}v - \int_{-\infty}^0 v f_L(v)\,\mathrm{d}v\Big) + 2ks_1\|f\|_{L^1((0,L)\times\mathbb{R})}.$$

Proof. We establish first the existence part. To this end, we consider a sequence $\big(E_n\big)_{n\in\mathbb{N}}$ of smooth fields — for each n, $E_n \in W^{1,\infty}((0,L))$ — which converges a.e. towards $E \in L^\infty((0,L))$, with $\|E_n\|_\infty \leq \|E\|_\infty$. For any n, denote by f_n the unique mild solution of (1.1)- (1.3) associated to the field E_n. Theorem 4.1 and Proposition 4.2 tell us that

$$\Big\| \int_{-\infty}^{+\infty} |v| f_n(\cdot,v)\,\mathrm{d}v \Big\|_\infty \leq 2\Big(\int_0^{+\infty} v f_0(v)\,\mathrm{d}v - \int_{-\infty}^0 v f_L(v)\,\mathrm{d}v\Big) + 2ks_1\|f_n\|_{L^1((0,L)\times\mathbb{R})}$$

$$\sup_{n\in\mathbb{N}} \int_0^L \int_{-\infty}^{+\infty}\Big(1+\frac{v^2}{2}+|\ln f_n|\Big) f_n\,\mathrm{d}v\,\mathrm{d}x \leq C(1+\mathcal{H}_L), \qquad \sup_{n\in\mathbb{N}} \int_0^L \langle f_n\rangle |\ln\langle f_n\rangle|\,\mathrm{d}x \leq C(1+\mathcal{H}_L)$$

$$\sup_{n\in\mathbb{N}} \left(\int_0^{+\infty} v\Big(1+\frac{v^2}{2}+|\ln f_n|\Big) f_n(L,v)\,\mathrm{d}v + \int_{-\infty}^0 (-v)\Big(1+\frac{v^2}{2}+|\ln f_n|\Big) f_n(0,v)\,\mathrm{d}v\right) \leq C(1+\mathcal{H}_L)$$

where the constant C depends only on κ, θ, L, $\|E\|_\infty$, s_0, s_1. The Dunford-Pettis theorem, see e.g. [9], allows to consider a subsequence, still denoted with the index n, such that $f_n \rightharpoonup f$ weakly in $L^1((0,L)\times\mathbb{R})$ and $\langle f_n\rangle \rightharpoonup \langle f\rangle$ weakly in $L^1((0,L))$. Furthermore, the outgoing traces $v f_n(L,v)$ and $v f_n(0,v)$ converge weakly in $L^1((0,+\infty))$ and $L^1((-\infty,0))$ respectively. It can be easily seen that f remains non-negative and, in addition,

$$\int_0^L \int_{-\infty}^{+\infty}\Big(1+\frac{v^2}{2}+|\ln f|\Big) f\,\mathrm{d}v\,\mathrm{d}x \leq C(1+\mathcal{H}_L), \qquad \int_0^L \langle f\rangle |\ln\langle f\rangle|\,\mathrm{d}x \leq C(1+\mathcal{H}_L)$$

$$\int_0^{+\infty} v\Big(1+\frac{v^2}{2}+|\ln f|\Big) f(L,v)\,\mathrm{d}v + \int_{-\infty}^0 (-v)\Big(1+\frac{v^2}{2}+|\ln f|\Big) f(0,v)\,\mathrm{d}v \le C(1+\mathcal{H}_L)$$

$$\Big\| \int_{-\infty}^{+\infty} |v| f(\cdot,v)\,\mathrm{d}v \Big\|_\infty \le 2\Big(\int_0^{+\infty} v f_0(v)\,\mathrm{d}v - \int_{-\infty}^0 v f_L(v)\,\mathrm{d}v \Big) + 2ks_1 \|f\|_{L^1((0,L)\times\mathbb{R})}.$$

Since f_n is a weak solution of (1.1)-(1.3), we have

$$\begin{aligned}
&\int_0^L \int_{-\infty}^{+\infty} f_n\ (\kappa\sigma(v) - v\partial_x - E_n\partial_v)\psi\,\mathrm{d}v\,\mathrm{d}x + \int_0^{+\infty} v\ f_n\psi(L,v)\,\mathrm{d}v - \int_{-\infty}^0 v\ f_n\psi(0,v)\,\mathrm{d}v \\
&= \int_0^{+\infty} v\ f_0(v)\psi(0,v)\,\mathrm{d}v - \int_{-\infty}^0 v\ f_L(v)\psi(L,v)\,\mathrm{d}v + \kappa \int_0^L \int_{-\infty}^{+\infty} \big\langle f_n \big\rangle_s (x,v) M_\theta(v)\psi(x,v)\,\mathrm{d}v\,\mathrm{d}x,
\end{aligned}$$

for any $n \in \mathbb{N}$ and any trial function $\psi \in C_c^1([0,L]\times\mathbb{R})$. Combining the a.e. convergence of E_n with the weak convergence of f_n, we can pass to the limit $n \to \infty$ in this relation, which proves that f is indeed a solution, in the sense of distributions, of (1.1)-(1.3) with the field E and incoming traces f_0, f_L.

We discuss now the uniqueness of the weak solution. Notice that we can not apply exactly the same arguments as for the uniqueness of the mild solution since in this case the electric field is not smooth (the formulation by characteristics does not make sense anymore). Nevertheless, as in Section 3, if f solves $v\partial_x f + E(x)\partial_v f = kQ(f)$ with vanishing incoming data we are lead to

$$\int_0^{+\infty} v|f(L,v)|\,\mathrm{d}v + \int_{-\infty}^0 (-v)|f(0,v)|\,\mathrm{d}v = k\int_0^L \int_{-\infty}^{+\infty} Q(f)\ \mathrm{sgn}(f)\,\mathrm{d}v\,\mathrm{d}x \le 0.$$

Therefore $f(0,v) = 0 = f(L,v)$ a.e. $v \in \mathbb{R}$ and $\mathrm{sgn} f$ does not depend on v, implying that

$$v\partial_x|f| + E(x)\partial_v|f| = \kappa Q(|f|). \tag{4.34}$$

From now on the argument is different. We use the entropy computation

$$\begin{aligned}
(v\partial_x + E(x)\partial_v)\left[\left(\ln|f| + \frac{v^2}{2\theta} + \frac{\Phi(x)}{\theta}\right)|f|\right] &= kQ(|f|)\left(1+\ln|f| + \frac{v^2}{2\theta} + \frac{\Phi(x)}{\theta}\right) \\
&= kQ(|f|)\left(\ln\frac{|f|}{M_\theta} + \ln\frac{1}{\sqrt{2\pi\theta}} + 1 + \frac{\Phi(x)}{\theta}\right).
\end{aligned}$$

Since $f(0,\cdot) = f(L,\cdot) = 0$, after integration over $(0,L)\times\mathbb{R}$ we deduce that

$$\int_0^L \int_{-\infty}^{+\infty} Q(|f|)\ln\frac{|f|}{M_\theta}\,\mathrm{d}v\,\mathrm{d}x = 0$$

and therefore, since ln is strictly increasing, one gets by (3.18) that $|f(x)| = n(x)M_\theta(v)$ for some non-negative function n satisfying $n(0) = n(L) = 0$. By (4.34) we have $\frac{\mathrm{d}}{\mathrm{d}x}n - n(x)\frac{E(x)}{\theta} = 0$, $x \in (0,L)$ and finally $n = 0$ implying that $f = 0$. ∎

Remark 4.1 *It is easy to see that a similar existence result holds for incoming data in L^∞ by just using the estimates in Corollary 4.1.*

4.4 On a Perturbed Problem

We intend to investigate the coupling between the Boltzmann-Vlasov equation with the Poisson equation defining the electric field. In preparation of the fixed point procedure we have in mind, it is convenient to study the following perturbed equation

$$\alpha f + v\partial_x f + E(x)\partial_v f = \kappa Q(f), \qquad x \in (0, L), v \in \mathbb{R} \tag{4.35}$$

with the boundary conditions (1.3), where $\alpha > 0$ is a fixed parameter. In the sequel, the notation $\mathcal{C}$ stands for various constants depending only on $\kappa, \theta, L, s_0, s_1$ but not on α.

Proposition 4.3 *Let $E \in L^\infty((0, L))$ and consider $f_0, f_L \geq 0$ such that (1.9) holds. Then for any $\alpha > 0$ there is a unique weak solution f of (4.35), (1.2), (1.3) satisfying*

$$\int_0^L \int_{-\infty}^{+\infty} \left(|\ln f| + \frac{v^2}{2} + 1 \right) f(x, v) \,\mathrm{d}v \,\mathrm{d}x \leq \frac{\mathcal{C}}{\alpha}(1 + \mathcal{H}_{_L}(1 + \|E\|_\infty)) \tag{4.36}$$

$$\int_0^{+\infty} v \left(|\ln f| + \frac{v^2}{2} + 1 \right) f(L, v) \,\mathrm{d}v - \int_{-\infty}^0 v \left(|\ln f| + \frac{v^2}{2} + 1 \right) f(0, v) \,\mathrm{d}v \leq \mathcal{C}(1 + \mathcal{H}_{_L}(1 + \|E\|_\infty)) \tag{4.37}$$

$$\int_0^L |\ln\langle f\rangle|\langle f\rangle \,\mathrm{d}x \leq \frac{\mathcal{C}}{\alpha}(1 + \mathcal{H}_{_L}(1 + \|E\|_\infty)). \tag{4.38}$$

Proof. We only sketch the arguments, the details are left to the reader. The existence and uniqueness proofs are much simpler than those of Theorem 4.2 since we have for free the L^1 bound

$$\int_0^L \int_{-\infty}^{+\infty} f(x, v) \,\mathrm{d}v \,\mathrm{d}x \leq \frac{1}{\alpha} \left(\int_0^{+\infty} v f_0(v) \,\mathrm{d}v - \int_{-\infty}^0 v f_L(v) \,\mathrm{d}v \right). \tag{4.39}$$

We consider first mild solutions corresponding to smooth electric fields. Then we construct weak solutions by using weak L^1 compactness (cf. estimates (4.36)-(4.38). Since f is non-negative, the current $j = \int_{-\infty}^{+\infty} f \,\mathrm{d}v$ satisfies $\frac{\mathrm{d}}{\mathrm{d}x} j = -\alpha\langle f\rangle \leq 0$ implying that

$$\int_{-\infty}^0 v f_L(v) \,\mathrm{d}v \leq j(L) \leq j(x) \leq j(0) \leq \int_0^{+\infty} v f_0(v) \,\mathrm{d}v, \qquad \forall\, x \in (0, L).$$

Therefore the current is bounded uniformly with respect to $\alpha > 0$ and we have

$$\|j\|_\infty \leq \mathcal{J}. \tag{4.40}$$

At this stage also notice that the solution f satisfies entropy estimates. Indeed, with the notation $\Phi(x) = -\int_0^x E(y) \,\mathrm{d}y$ we have

$$\begin{aligned}
(\alpha + v\partial_x + E(x)\partial_v) \left[\left(\ln f + \frac{v^2}{2\theta} + \frac{\Phi(x)}{\theta} \right) f \right] &= kQ(f) \left(1 + \ln f + \frac{v^2}{2\theta} + \frac{\Phi(x)}{\theta} \right) - \alpha f \\
&\leq kQ(f) \left(\ln \frac{f}{M_\theta} + 1 + \frac{\Phi(x)}{\theta} - \ln \sqrt{2\pi\theta} \right).
\end{aligned}$$

The estimates (4.36) and (4.37) follow easily integrating over $(0, L) \times \mathbb{R}$ and using the inequalities

$$\int |\ln f| f \, \mathrm{d}\mu \leq \int (\ln f + av^2) f \, \mathrm{d}\mu + \frac{4}{e} \int e^{-\frac{a}{4}v^2} \, \mathrm{d}\mu$$

with the measures $\mathrm{d}v, v_\pm \, \mathrm{d}v$. The estimate (4.38) is a consequence of the Jensen inequality

$$\langle f \rangle \ln \frac{\langle f \rangle}{\sqrt{2\pi\theta}} \leq \int_{-\infty}^{+\infty} \left(\frac{v^2}{2\theta} + \ln f \right) f(x, v) \, \mathrm{d}v \, \mathrm{d}x.$$

∎

5 Boltzmann-Poisson System

In this section we construct weak solutions for the stationary boundary value problem for the Poisson-Boltzmann system (1.5)-(1.8), with with entropic boundary conditions given by (1.9). Without loss of generality, we assume that $\Phi_0 = 0$.

We first start by justifying the existence of weak solution for the perturbed problem (4.35) coupled to (1.6)-(1.8). We conclude by stability results, letting the parameter α going to zero.

Proposition 5.1 *Assume that* f_0, f_L *are non-negative and satisfy* (1.9). *Then for any* $\Phi_L \in \mathbb{R}, n_b \geq 0, \alpha > 0$ *there is a weak solution* (f_α, E_α) *of* (4.35) *coupled to* (1.6)-(1.8). *Furthermore, the following uniform in* α *estimates hold*

$$\sup_{0<\alpha\leq 1} \|E_\alpha\|_\infty < +\infty \tag{5.41}$$

$$\sup_{0<\alpha\leq 1} \int_0^L \int_{-\infty}^{+\infty} \left(|\ln f_\alpha| + 1 + \frac{v^2}{2} \right) f_\alpha(x, v) \, \mathrm{d}v \, \mathrm{d}x < +\infty \tag{5.42}$$

$$\sup_{0<\alpha\leq 1} \left(\int_0^{+\infty} v \left(|\ln f_\alpha| + 1 + \frac{v^2}{2} \right) f_\alpha(L, v) \, \mathrm{d}v - \int_{-\infty}^{0} v \left(|\ln f_\alpha| + 1 + \frac{v^2}{2} \right) f_\alpha(0, v) \, \mathrm{d}v \right) < +\infty \tag{5.43}$$

$$\sup_{0<\alpha\leq 1} \int_0^L |\ln \langle f_\alpha \rangle| \langle f_\alpha \rangle \, \mathrm{d}x < +\infty. \tag{5.44}$$

Proof. We proceed by a standard fixed point argument. For any bounded force field $E \in L^\infty((0, L))$ we denote by $\mathcal{F}E$ the unique solution of (1.6), (1.8), associated to the macroscopic density $\langle f_E \rangle$, where f_E is the unique weak solution of (4.35), (1.7). Combining (1.6), (1.8) to (4.39), it is easily seen that $\mathcal{F}E \in C([0, L])$ and

$$\begin{aligned} \|\mathcal{F}E\|_\infty &\leq \frac{|\Phi_L|}{L} + 2Ln_b + 2 \int_0^L \int_{-\infty}^{+\infty} f_E(x, v) \, \mathrm{d}v \, \mathrm{d}x \\ &\leq \frac{|\Phi_L|}{L} + 2Ln_b + 2\frac{\mathcal{J}}{\alpha} =: \mathcal{C}_\alpha. \end{aligned} \tag{5.45}$$

By (1.6) and (4.38) we have, for any E verifying $\|E\|_\infty \le \mathcal{C}_\alpha$,

$$\int_0^L \ln((\mathcal{F}E)' + n_b)((\mathcal{F}E)' + n_b)\,\mathrm{d}x \le \frac{\mathcal{C}}{\alpha}(1 + \mathcal{H}_{_L}(1 + \|E\|_\infty)) \le \frac{\mathcal{C}}{\alpha}(1 + \mathcal{H}_{_L}(1 + \mathcal{C}_\alpha)) =: \tilde{\mathcal{C}}_\alpha.$$

We deduce that the set $\mathcal{D}_\alpha$ of continuous fields $E \in C([0,L])$ satisfying

$$\|E\|_\infty \le \mathcal{C}_\alpha, \qquad E' \in L^1((0,L)), \qquad E' + n_b \ge 0, \qquad \int_0^L \ln(E' + n_b)(E' + n_b)\,\mathrm{d}x \le \tilde{\mathcal{C}}_\alpha$$

is left invariant by the application $\mathcal{F}$. Observe that $\mathcal{D}_\alpha$ is convex and compact with respect to the topology of $C([0,L])$ (the equi-continuity of $\mathcal{D}_\alpha$ follows by the equi-integrability of $\{E' + n_b : E \in \mathcal{D}_\alpha\}$). In order to apply the Schauder fixed point theorem it remains to check that $\mathcal{F}$ is continuous. Take $(E_n)_n \subset \mathcal{D}_\alpha$ a convergent sequence in $C([0,L])$ towards $E \in \mathcal{D}_\alpha$. We denote by f_n, f the weak solutions of the Boltzmann-Vlasov problems corresponding to E_n, E respectively. By standard arguments we deduce that $\lim_{n\to+\infty} f_n = f$ weakly in $L^1((0,L)\times\mathbb{R})$ (the uniqueness of the weak solution is crucial here), implying the pointwise convergence of $(\mathcal{F}E_n)_n$ towards $\mathcal{F}E$. Since $(\mathcal{F}E_n)_n$ belongs to $\mathcal{D}_\alpha$ which is a compact set of $C([0,L])$, finally we get $\lim_{n\to+\infty} \mathcal{F}E_n = \mathcal{F}E$ in $C([0,L])$. Therefore the application $\mathcal{F}$ has a fixed point $E_\alpha \in \mathcal{D}_\alpha$ meaning that $(f_\alpha = f_{E_\alpha}, E_\alpha)$ is a weak solution of (4.35)) with (1.6)-(1.8).

Next, we show uniform estimates with respect to the parameter α. First, multiplying (4.35) by v yields after integration

$$\alpha\langle v f_\alpha\rangle + \frac{\mathrm{d}}{\mathrm{d}x}\langle v^2 f_\alpha\rangle - E_\alpha(x)\langle f_\alpha\rangle = \kappa\int_{-\infty}^{+\infty}\langle f_\alpha\rangle_s M_\theta(v) v\,\mathrm{d}v - \kappa\langle v\sigma(v) f_\alpha\rangle. \tag{5.46}$$

It is convenient to consider the potentials $\phi, \phi_{\mathrm{ext}}$ such that

$$-\phi'' = \langle f_\alpha\rangle, \qquad x \in (0,L), \qquad \phi(0) = \phi(L) = 0$$

and

$$-\phi''_{\mathrm{ext}} = n_b, \qquad x \in (0,L), \qquad \phi_{\mathrm{ext}}(0) = 0, \;\; \phi_{\mathrm{ext}}(L) = -\Phi_L.$$

Thus, the electric field $E_\alpha = -\Phi'_\alpha$ can be written

$$E_\alpha = -\phi' + \phi'_{\mathrm{ext}} =: E - E_{\mathrm{ext}}.$$

Next, notice that E is non-decreasing and therefore $\sup_{x\in[0,L]}|E(x)| = \max\{|E(0)|, |E(L)|\}$. We deduce by (5.46) that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\langle v^2 f_\alpha\rangle - \frac{1}{2}|E_\alpha(x)|^2 + \Phi_\alpha(x) n_b\right\} = -\alpha\langle v f_\alpha\rangle - \kappa\langle\sigma(v) v f_\alpha\rangle + \kappa\int_{-\infty}^{+\infty}\langle f_\alpha\rangle_s M_\theta(v) v\,\mathrm{d}v\,. \tag{5.47}$$

Integration over (x_0, x) yields

$$\begin{aligned}\langle v^2 f_\alpha\rangle(x) - \frac{1}{2}|E(x) - E_{\mathrm{ext}}(x)|^2 + \Phi_\alpha(x) n_b \;&=\; \langle v^2 f_\alpha\rangle(x_0) - \frac{1}{2}|E(x_0) - E_{\mathrm{ext}}(x_0)|^2 + \Phi_\alpha(x_0) n_b \\ &\quad -\alpha\int_{x_0}^x \langle v f_\alpha\rangle\,\mathrm{d}y - \kappa\int_{x_0}^x\langle\sigma(v) v f_\alpha\rangle\,\mathrm{d}y \\ &\quad +\kappa\int_{x_0}^x\int_{-\infty}^{+\infty}\langle f_\alpha\rangle_s M_\theta(v) v\,\mathrm{d}v\,\mathrm{d}y\,. \end{aligned} \tag{5.48}$$

If $|E(0)| \leq |E(L)|$ we have $\sup_{x\in[0,L]} |E(x)|^2 = |E(L)|^2$ and by taking $x_0 = L$ in (5.48) we get

$$\begin{aligned}
\langle v^2 f_\alpha\rangle(x) \;&\leq\; \langle v^2 f_\alpha\rangle(L) + E(L)E_{\mathrm{ext}}(L) - E(x)E_{\mathrm{ext}}(x) - \frac{1}{2}|E_{\mathrm{ext}}(L)|^2 + \frac{1}{2}|E_{\mathrm{ext}}(x)|^2 \\
&\quad +(\Phi_\alpha(L) - \Phi_\alpha(x))n_b + \alpha \int_x^L \langle v f_\alpha\rangle \,\mathrm{d}y + \kappa \int_x^L \langle \sigma(v) v f_\alpha\rangle \,\mathrm{d}y \\
&\quad -\kappa \int_x^L \int_{-\infty}^{+\infty} \langle f_\alpha\rangle_s M_\theta(v) v \,\mathrm{d}v \,\mathrm{d}y\,.
\end{aligned}$$

Notice that we have

$$\begin{aligned}
\left| \int_x^L \int_{-\infty}^{+\infty} \langle f_\alpha\rangle_s M_\theta(v) v \,\mathrm{d}v \,\mathrm{d}y \right| \;&\leq\; \int_0^L \int_{-\infty}^{+\infty} s_1 \langle f_\alpha\rangle(y) M_\theta(v)|v| \,\mathrm{d}v \,\mathrm{d}y \\
&\leq C \int_0^L \langle f_\alpha\rangle(y) \,\mathrm{d}y \\
&\leq C(1 + \|E_\alpha\|_\infty) \qquad (5.49)
\end{aligned}$$

and that Theorem 4.2 implies

$$\begin{aligned}
\left| \int_x^L \langle \sigma(v) v f_\alpha\rangle \,\mathrm{d}y \right| \;&\leq\; s_1 \int_0^L \int_{-\infty}^{+\infty} |v| f_\alpha(y,v) \,\mathrm{d}v \,\mathrm{d}y \\
&\leq 2 s_1 L \left(\int_0^{+\infty} v f_0(v) \,\mathrm{d}v - \int_{-\infty}^0 v f_L(v) \,\mathrm{d}v \right) + 2\kappa s_1^2 L \|f_\alpha\|_{L^1((0,L)\times\mathbb{R})} \\
&\leq C(1 + \|E_\alpha\|_\infty)\,. \qquad (5.50)
\end{aligned}$$

Then, combining this last estimates with (4.37) and (4.40) yields

$$\sup_{0<\alpha\leq 1} \|\langle v^2 f_\alpha\rangle\|_\infty \leq \mathcal{C}(1 + \|E_\alpha\|_\infty). \qquad (5.51)$$

If $|E(L)| \leq |E(0)|$ we obtain a similar estimate by taking $x_0 = 0$ in (5.48). Consider now the point $x_1 \in (0, L)$ such that

$$-E_\alpha(x_1) = \Phi'_\alpha(x_1) = \frac{\Phi_\alpha(L) - \Phi_\alpha(0)}{L} = \frac{\Phi_L}{L}. \qquad (5.52)$$

Integrating (5.47) between x and x_1 and using (5.51), (5.52) together with (4.40), (5.49), (5.50) yields

$$\|E_\alpha\|_\infty^2 \leq \mathcal{C}(1 + \|E_\alpha\|_\infty),$$

which shows that the sequence $\left(\|E_\alpha\|_\infty\right)_{\alpha>0}$ remains in a bounded set, independently of the parameter α. Once we have obtained a uniform bound for $\|E_\alpha\|_\infty$, we then get the uniform in α estimates (5.42)-(5.44) as well (cf. Lemma 3.2, Proposition 4.2). ∎

Therefore, letting α tend to zero we obtain the existence of a weak solution for the problem (1.5)-(1.8) with the entropic boundary condition (1.9). In particular we complete the prove of Theorem 1.2 as follows.

Proof of Theorem 1.2: Take $(\alpha_n)_n$ a sequence of positive numbers converging to zero. Combining (5.41), (5.44) and the Poisson equation we deduce that $(E_n)_n$ is relatively compact in $C([0, L])$. Using (5.42), (5.43) we can also extract a subsequence of $(f_n)_n$ weakly compact in L^1. The proof follows easily by passing to the limit as $n \to +\infty$ in the weak formulations of $(f_{\alpha_n}, E_{\alpha_n})$ (see the proof of Theorem 4.2). ∎

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