On a Fokker-Planck equation arising in population dynamics

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Abstract

This paper is devoted to a Fokker-Planck equation arising in population dynamics. For general non negative initial data $u_0 \in L^1(\Omega)$, we prove the existence of a weak non negative and mass-preserving solution belonging to $L^q(0,T;W^{1,q}(\Omega))$ for all $1 \le q < \frac{4}{3}$.

1. Introduction

In [10], Jager and Segel proposed an integro-differential equation of Boltzmann type as a model describing the evolution of certain properties in populations of social organisms. In such a model, encounters between individuals produce some change on a character of these individuals called dominance and represented by a variable $x \in (0,1)$. The Boltzmann like collision operator describes this dynamic of interaction according to various physical rules. The model of Jager and Segel is studied and extended to various realistic situations in biology by Bellomo et al. in [1], [2], [3]. Moreover, reproducing ideas well known in the context of gas dynamics (see [8] and the references therein), when assuming that encounters between organisms only produce small changes in the state of the individuals, the following Fokker-Planck model is also derived in [10]

$$\begin{cases} \partial_t u = \partial_x\{M(u;t,x)u + \partial_x(D(u;t,x)u)\} \;\; in \;(0,T)\times(0,1) \\ u_{|t=0} = u_0 \;\; in \;(0,1) \end{cases} \tag{1.1}$$

with a null flux boundary condition

$$M(u)u + \partial_x(D(u)u) = 0 \;\; on \; x \in \{0,1\}. \tag{1.2}$$

Equation (1.1) is nonlinear since the coefficients M and D depends on the unknown u as follows

$$\begin{cases} M(u;t,x) = \int_0^1 m(x,y)u(t,y)dy \\ D(u;t,x) = \int_0^1 d(x,y)u(t,y)dy \end{cases} \tag{1.3}$$

where m and d are regular functions defined on $(0,1)\times(0,1)$. The unknown $u(t,x)$ has the meaning of a density of individuals having, at the time $t \geq 0$, the character called "dominance" $x \in (0,1)$. The boundary condition (1.2) is related to the fact that one expects the following preservation of the total density

$$\int_0^1 u(t,x)dx = \int_0^1 u_0(x)dx. \tag{1.4}$$

The interested reader may find discussions on models describing cell growth close to those studied here, in particular linearized version of (1.1-1.3) where the coefficients N and D do not depend on the unknown u, in [19], [9] ; we also mention the recent works [12], [11].

The difficulty of the problem is two-fold. First, non linearities are given by global terms, involving the values of the unknown in all space, and not in a pointwise setting. Next, it is physically natural to consider non negative initial data u_0 in L^1, without additional regularity. The approach we adopt is rather close to that introduced by Boccardo and Gallouet in [5] to deal with elliptic and parabolic equations with integrable initial data and source term. Consult also for recent developments on these questions [7], [15], [4]. Note that one dimensional (linear) parabolic problems with a L^1 source term are easily solved by classical variational techniques, [7] ; however, these techniques break down, even in the simple monodimensional case, when initial data lie in L^1. The quoted papers deal with equations involving classical Leray-Lions operators : possible nonlinearities concern the derivative of the unknown, and, moreover, these operators present monotonicity property with respect to the gradient variable essential to the proofs of existence (and uniqueness in some appropriate sense...) of solutions. In [13], existence and uniqueness of renormalized solution is proved for a linear parabolic equation containing a first order term with a given free-divergence coefficient.

In this paper, we establish the existence of a non negative and "mass-preserving" weak solution of (1.1-1.3) for general non negative and integrable data. Let us denote $\Omega = (0,1)$ and $Q = (0,T)\times\Omega$. In the sequel, we write simply $u = u(t,x)$, $M(u) = M(u;t,x)$, $D(u) = D(u;t,x)$ when no confusion can arise and we adopt the notation $N(u)$ to designate $M(u) + \partial_x D(u)$. Our main result is the following

Theorem 1 *Let $u_0 \in L^1(\Omega)$ be non negative. We assume that*

$$\begin{cases} 0 < \underline{\delta} \leq d(x,y) \leq \overline{\delta} \\ \mid m(x,y) \mid + \mid \partial_x d(x,y) \mid \leq C_{m,d} \end{cases} \tag{1.5}$$

holds. Let $M(u), D(u)$ be defined by (1.3). Then, there exists a non negative solution $u \in L^q(0,T;W^{1,q}(\Omega))$, with $1 \leq q < \frac{4}{3}$, of (1.1-1.3), in the following weak sense

$$\int_0^t \int_\Omega u\partial_t\phi dxds + \int_\Omega u_0\phi(0,\cdot)dx$$

$$= \int_\Omega u\phi(t,\cdot)dx + \int_0^t \int_\Omega (N(u)u + D(u)\partial_x u)\partial_x \phi dx ds \qquad (1.6)$$

for $\phi \in C^0(0,T;W^{1,q'}(\Omega))$ *with* $\partial_t\phi \in C^0(0,T;L^{q'}(\Omega))$. *Moreover,* u *satisfies (1.4).*

Remark 1 Note that all terms in (1.6) make sense since, by definition (1.3), the coefficients $N(u)$ and $D(u)$ belong to $L^\infty(Q)$ as soon as u lies in $L^\infty(0,T;L^1(\Omega))$. Furthermore, we shall show that the function of the time variable $U : t \longmapsto \int_\Omega u\phi dx$ is continuous on $(0,T)$.

Remark 2 It is worth pointing out the gain in regularity of the obtained solution, while the data only lie in L^1. This fact is not surprising in view of the parabolic nature of the equation, but is still lacking in the theory of the more complicated Fokker-Planck equations of gas dynamics.

Remark 3 We restrict ourselves to the case of the one dimensional problem with its physical interpretation due to [10] ; note however that a part of our results extends to greater dimensions where, for instance, equation (1.1-1.3) arises when we treat numerically the Fokker-Planck equation of gas dynamics, restricting the velocities to bounded values.

This work is divided into three steps. In Section 2, we are concerned with problem (1.1-1.3) assuming that the data are regular, say $u_0 \in L^2(\Omega)$. In Section 3, we discuss essential estimates, depending only on the L^1 norm of the data, on the obtained solutions. Finally, in Section 4, we pass to the physically natural framework of integrable data, proving the existence of a weak solution.

2. Problem with L^2 data

This Section is devoted to (1.1-1.3) with regular data. First, we recall some well known facts about the linear problem, with given coefficients M and D and $u_0 \in L^2(\Omega)$. Next, still for regular data, we use the Schauder fixed point theorem to solve the nonlinear problem. For such data, existence and uniqueness of a non negative solution are proved.

2.1. *Linear problem with L^2 data*

We are concerned with equation (1.1-1.2) where the coefficients M and D are given and assumed to satisfy

$$\left\{\begin{array}{l} 0 < \underline{d} \leq D(t,x) \leq \overline{d} \\ \sup\limits_{t,x} \mid \partial_x D(t,x) \mid + \sup\limits_{t,x} \mid M(t,x) \mid \leq C_{M,D}. \end{array}\right. \tag{2.1}$$

To make our exposition self-contained and to fix some notation, we briefly proceed with the study of this classical problem.

Let u be a solution of (1.1-1.2) and let $\lambda \geq 0$ to be choosen later. We set $v(t,x) = e^{-\lambda t} u(t,x)$. Then, for a regular test function ϕ defined on Ω, v satisfies

$$\frac{d}{dt}\int_\Omega v\phi dx = -\lambda \int_\Omega v\phi dx - \int_\Omega N v \partial_x \phi dx - \int_\Omega D \partial_x v \partial_x \phi dx \tag{2.2}$$

where N stands for $M + \partial_x D$. We denote by $-a(t,v,\phi)$ the right hand side of (2.2). Basic properties of $a(t,\cdot,\cdot)$ are obtained as consequences of (2.1). Indeed, on the one hand, we have

$$\begin{aligned} \mid a(t,v,\phi) \mid &\leq \lambda \parallel v \parallel_{L^2(\Omega)} \parallel \phi \parallel_{L^2(\Omega)} + C_{M,D} \parallel v \parallel_{L^2(\Omega)} \parallel \partial_x \phi \parallel_{L^2(\Omega)} \\ &\quad + \overline{d} \parallel \partial_x v \parallel_{L^2(\Omega)} \parallel \partial_x \phi \parallel_{L^2(\Omega)} \\ &\leq (\lambda + C_{M,D} + \overline{d}) \parallel v \parallel_{H^1(\Omega)} \parallel \phi \parallel_{H^1(\Omega)} \end{aligned} \tag{2.3}$$

and, on the other hand

$$a(t,\phi,\phi) = \lambda \int_\Omega \phi^2 dx + \int_\Omega N \phi \partial_x \phi dx + \int_\Omega D (\partial_x \phi)^2 dx. \tag{2.4}$$

Thus, the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\mid \int_\Omega N\phi\partial_x\phi dx \mid &\leq C_{M,D} \parallel \phi \parallel_{L^2(\Omega)} \parallel \partial_x\phi \parallel_{L^2(\Omega)} \\
&\leq \frac{C_{M,D}}{2}\left(\frac{\parallel \phi \parallel^2_{L^2(\Omega)}}{\mu} + \mu \parallel \partial_x\phi \parallel^2_{L^2(\Omega)}\right)
\end{aligned}$$

where $\mu > 0$ is to be determined. One deduces that

$$\begin{aligned}
a(t,\phi,\phi) \geq \underline{d} \parallel \partial_x\phi \parallel^2_{L^2(\Omega)} \\
+\lambda \parallel \phi \parallel^2_{L^2(\Omega)} - \frac{C_{M,D}}{2\mu} \parallel \phi \parallel^2_{L^2(\Omega)} - \frac{\mu C_{M,D}}{2} \parallel \partial_x\phi \parallel^2_{L^2(\Omega)}
\end{aligned}$$

$$\geq (\underline{d} - \frac{\mu C_{M,D}}{2}) \parallel \phi \parallel^2_{H^1(\Omega)} + (\lambda - \underline{d} - \frac{C_{M,D}}{2\mu}) \parallel \phi \parallel^2_{L^2(\Omega)}. \tag{2.5}$$

Let us fix μ so that

$$\underline{d} - \frac{\mu C_{M,D}}{2} \geq \frac{d}{2} > 0$$

holds and, then, we choose λ satisfying

$$\lambda - \underline{d} - \frac{C_{M,D}}{2\mu} = \Lambda \geq 0.$$

In all what follows, we denote by λ a constant, depending on $\underline{d}, \overline{d}$ and $C_{M,D}$, which satisfies this last inequality. Finally, the bilinear form $a(t,\cdot,\cdot)$ defined on $H^1(\Omega) \times H^1(\Omega)$ satisfies

$$\begin{cases} \mid a(t,v,\phi) \mid \leq C \parallel v \parallel_{H^1(\Omega)} \parallel \phi \parallel_{H^1(\Omega)} \\ a(t,\phi,\phi) \geq \frac{d}{2} \parallel \phi \parallel^2_{H^1(\Omega)} + \Lambda \parallel \phi \parallel^2_{L^2(\Omega)} \geq \frac{d}{2} \parallel \phi \parallel^2_{H^1(\Omega)}. \end{cases} \tag{2.6}$$

These properties allows us to apply general results in [14] (Theorem 4.1, p. 257), [17] and we obtain

Proposition 1 *We assume that (2.1) is fulfilled. Let $u_0 \in L^2(\Omega)$. Then, there exists a unique $v \in C^0(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ with $\partial_t v \in L^2(0,T;(H^1(\Omega))')$ satisfying, for all $\phi \in H^1(\Omega)$*

$$\begin{cases} \langle \partial_t v, \phi \rangle_{(H^1)',H^1} + a(t,v,\phi) = 0 \\ v_{|t=0} = u_0. \end{cases} \tag{2.7}$$

Since we shall consider non negative and integrable data, the following corollary is useful.

Corollary 1 *Let $u_0 \geq 0$ in $L^2(\Omega)$ and v be the solution of (2.7) given by Proposition 1. We set $u(t,x) = e^{\lambda t} v(t,x)$. Then, u is non negative and satisfies*

$$\int_\Omega u dx = \int_\Omega u_0 dx \tag{2.8}$$

Proof. Let us denote $u_- = \max(0, -u)$. One recalls, see [6] or [18] (Part 3, p. 2) that for $u \in H^1(\Omega)$, we have $u_- \in H^1(\Omega)$ and

$$\partial_x(u_-) = \begin{cases} \partial_x u \ \ if \ u < 0 \\ 0 \ \ if \ u \geq 0. \end{cases}$$

Plugging $\phi = v_-$ in (2.7), it follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (v_-)^2 dx = -a(t, v_-, v_-) \leq 0$$

by (2.6). Since the datum u_0 is non negative, we deduce that $v_- = 0$. Moreover, taking the test function $\phi = 1$, we get $\frac{d}{dt}\int_\Omega v dx = -\lambda \int_\Omega v dx$. Hence, it follows that $\int_\Omega v dx = e^{-\lambda t}\int_\Omega u_0 dx$ which clearly leads to the conservation law (2.8). □

2.2. *Fixed point step*

This section is devoted to the nonlinear problem (1.1-1.3), with coefficients M, D depending on the unknown u as in (1.3). However, we still consider a non negative initial datum u_0 belonging to $L^2(\Omega)$. Let us introduce the following convex bounded subset in $L^\infty(0,T; L^2(\Omega))$

$$\mathcal{C} = \{g \in L^\infty(0,T; L^2(\Omega)), g \geq 0, \ \int_\Omega g dx = \int_\Omega u_0 dx$$

$$and \ \| g(t,\cdot) \|_{L^2(\Omega)} \leq e^{\lambda t} \| u_0 \|_{L^2(\Omega)} \}. \tag{2.9}$$

We denote by $\mathcal{T}(u_0, \cdot)$ the mapping

$$\mathcal{T}(u_0, \cdot) : g \in \mathcal{C} \longmapsto u \tag{2.10}$$

where u is a solution of (1.1-1.2) with coefficients $M = M(g)$ and $D = D(g)$. We shall show that this mapping has a fixed point in $\mathcal{C}$, assuming that the datum u_0 lies in L^2.

>From now on, the regular functions m and d, involved in (1.3), are required to satisfy (1.5). Therefore, for g given in $\mathcal{C}$, the coefficients $M(g), D(g)$ satisfy (2.1) with $\underline{d} = \underline{\delta} \int_\Omega u_0 dx$, $\overline{d} = \overline{\delta} \int_\Omega u_0 dx$ and $C_{M,D} = C_{m,d} \int_\Omega u_0 dx$. Note, however, that the technical restriction (1.5) seems unphysical in view of most of the models proposed in [10].

Proposition 2 *Let $u_0 \geq 0$ in $L^2(\Omega)$. We assume that (1.5) holds. Then, there exists a unique solution u of (1.1-1.3) satisfying*

$$\| u \|_{L^\infty(0,T;L^2(\Omega))} \leq C_{u_0}, \ \| u \|_{L^2(0,T;H^1(\Omega))} \leq C_{u_0}$$

where C_{u_0} depends on (1.5) and on $\| u_0 \|_{L^2(\Omega)}$.

Proof. Keeping the notation of Section 2, we set $v(t, x) = e^{-\lambda t}\mathcal{T}(u_0, g)$. First of all, we remark that, by Proposition 1 and Corollary 1, $\mathcal{T}(u_0, \cdot)$ is well defined on $\mathcal{C}$ and, obviously,

$$\mathcal{T}(u_0, \mathcal{C}) \subset \mathcal{C}. \tag{2.11}$$

Indeed, choosing $\phi = v$ as test function in (2.7), we get

$$\frac{1}{2}\frac{d}{dt} \| v \|_{L^2(\Omega)} + \frac{\underline{d}}{2} \| v \|^2_{H^1(\Omega)} \leq \frac{1}{2}\frac{d}{dt}\int_\Omega v^2 dx + a(t, v, v) = 0 \tag{2.12}$$

by (2.6). Thus, v is bounded in $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$, the bound depending on the norm of u_0 in L^2. First, we show the uniqueness and next, we will turn to the existence part of Proposition 2 by applying the Schauder fixed point theorem.

Uniqueness.

For $i \in \{1,2\}$, let $v_i \in \mathcal{C}$ satisfy for $\phi \in H^1(\Omega)$,

$$\langle \partial_t v_i, \phi \rangle = -a_i(t, v_i, \phi), v_{i,|t=0} = u_0$$

where $a_i(t,\cdot,\cdot)$ stands for the right hand side of (2.2) with coefficients $M = M(u_i)$, $D = D(u_i)$; precisely

$$a_i(t, v_i, \phi) = \lambda \int_\Omega v_i \phi dx + \int_\Omega \left(N(e^{\lambda t} v_i) v_i + D(e^{\lambda t} v_i) \partial_x v_i \right) \partial_x \phi dx.$$

We still denote $N = M + \partial_x D$ and we set $w = v_2 - v_1$. Take $\phi = w$ as test function in the equation satisfied by v_i. Then, substracting the obtained relations gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega | w |^2 \, dx + a_2(t,w,w) = \int_\Omega (N(e^{\lambda t} w) v_1 \partial_x w + D(e^{\lambda t} w) \partial_x v_1 \partial_x w) dx. \tag{2.13}$$

By using (2.6) and Young's inequality, we estimate as follows

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega | w |^2 \, dx + \frac{d}{2} \| w \|^2_{H^1(\Omega)} \leq & \| N(e^{\lambda t} w) \|_{L^\infty(\Omega)} \| v_1 \|_{L^2(\Omega)} \| \partial_x w \|_{L^2(\Omega)} \\ & + \| D(e^{\lambda t} w) \|_{L^\infty(\Omega)} \| \partial_x v_1 \|_{L^2(\Omega)} \| \partial_x w \|_{L^2(\Omega)} \\ \leq & \frac{\underline{d}}{4} \| w \|^2_{H^1(\Omega)} + \frac{2}{\underline{d}} \Big(\| N(e^{\lambda t} w) \|^2_{L^\infty(\Omega)} \| v_1 \|^2_{L^2(\Omega)} \\ & + \| D(e^{\lambda t} w) \|^2_{L^\infty(\Omega)} \| \partial_x v_1 \|^2_{L^2(\Omega)} \Big). \end{aligned} \tag{2.14}$$

However, by the definition (1.3) of N and D, we note that

$$\| N(e^{\lambda t} w) \|_{L^\infty(\Omega)} \leq c e^{\lambda t} \| w \|_{L^2(\Omega)}, \| D(e^{\lambda t} w) \|_{L^\infty(\Omega)} \leq c e^{\lambda t} \| w \|_{L^2(\Omega)} \tag{2.15}$$

holds a.e. $(0,T)$, where c depends on (1.5). It follows from (2.14) and (2.15) that $z(t) = \| w(t,\cdot) \|^2_{L^2(\Omega)}$ satisfies

$$z'(t) \le b(t) z(t) \tag{2.16}$$

where $b(t)$ stands for $\frac{2ce^{\lambda t}}{\underline{d}} \| v_1(t,\cdot) \|^2_{H^1(\Omega)}$. We conclude by Gronwall's lemma that $z(t) = 0$, i.e. $v_1 = v_2$.

Existence.

We have seen that $\mathcal{T}(u_0, \mathcal{C}) \subset \mathcal{C}$ and $v(t,x) = e^{-\lambda t}\mathcal{T}(u_0, g)$ for $g \in \mathcal{C}$ is bounded in $L^2(0,T; H^1(\Omega))$. In view of the equation satisfied by v, we also note that $\partial_t v$ is bounded in $L^2(0,T;(H^1(\Omega))')$. Hence, classical compactness lemma [16], [18] implies that v belongs to a compact set in $L^2(0,T;L^2(\Omega))$. Let g_n be a sequence in $\mathcal{C}$. Extracting a subsequence if necessary, $v_n = e^{-\lambda t}\mathcal{T}(u_0, g_n)$ converges in $L^2(Q)$. It remains to prove that $\mathcal{T}(u_0,\cdot)$ is continuous in $L^2(Q)$. Let $g_n \longrightarrow g$ in $L^2(Q)$, with $g_n \in \mathcal{C}$. Then, it is clear that the coefficients $N_n = N(g_n)$, $D_n = D(g_n)$ converge to $N = N(g), D = D(g)$ respectively in $L^2(0,T;L^\infty(\Omega))$ since

$$\| N_n - N \|_{L^\infty(\Omega)} \le c \, \| g - g_n \|_{L^2(\Omega)},$$

$$\| D_n - D \|_{L^\infty(\Omega)} \le c \, \| g - g_n \|_{L^2(\Omega)}$$

holds a.e. $(0,T)$ with c depending on (1.5). We consider the associated sequence $v_n = e^{-\lambda t}\mathcal{T}(u_0, g_n)$ and $v = e^{-\lambda t}\mathcal{T}(u_0, g)$. We follow the uniqueness proof to obtain

$$\| (v_n - v)(t,\cdot) \|^2_{L^2(\Omega)} \le \frac{2c}{\underline{d}} \int_0^t \| (g_n - g)(s) \|^2_{L^2(\Omega)} \| v(s) \|^2_{H^1(\Omega)} \, ds. \tag{2.17}$$

Possibly at the cost of extracting a subsequence, we can assume that a.e. $(0,T)$, $g_n(t,\cdot) \to g(t,\cdot)$ in $L^2(\Omega)$. Then, applying Lebesgue's theorem, we can assert from (2.17), that v_n converges to v for a.e. t in

$L^2(\Omega)$ and, thus, in $L^2(Q)$. Since v is the unique solution of the linear problem associated to g, the whole sequence v_n actually converges to v in $L^2(Q)$. We finally apply the Schauder fixed point theorem to deduce the existence of a fixed point $u = \mathcal{T}(u_0, u)$ in $\mathcal{C}$. $\square$

3. Estimates

We establish in this Section a set of estimates only depending on the L^1 norm of the initial data u_0 on the solutions obtained in Proposition 2. We follow arguments in [5] to prove the following claim.

Lemma 1 *We assume that (1.5) holds. Let v be as in Proposition 2. We set $B_n = \{(t,x) \in Q, n \leq\mid v(t,x) \mid\leq n+1\}$. Then, one has*

$$\int_{B_n} \mid \partial_x v \mid^2 dxdt \leq C_0 + C_1 \int_Q v^2 dxdt \tag{3.1}$$

where C_0, C_1 only depend on $\parallel u_0 \parallel_{L^1(\Omega)}$ and $\underline{\delta}$.

Before we detail the proof, let us give the statement of the following fundamental consequence of Lemma 1.

Proposition 3 *Let $v \in L^2(0,T;H^1(\Omega))$ satisfy (3.1) and assume that $\sup\limits_{t\in(0,T)} \parallel v(t,\cdot) \parallel_{L^1(\Omega)} \leq \kappa$. Let $1 \leq q < \frac{4}{3}$ and $\frac{q}{q-1} < s$. Then, there exists a constant K which depends on κ, C_0, C_1 such that*

$$\left\{ \begin{array}{l} \displaystyle\int_Q \mid \partial_x v \mid^q dxdt \leq K, \\ \displaystyle\int_Q \mid v \mid^2 dxdt \leq K, \\ \parallel v \parallel_{L^q(0,T;L^s(\Omega))} \leq K. \end{array} \right. \tag{3.2}$$

Remark 4 The case $C_1 = 0$, with a space dimension greater than 1, appears in [5]. Taking into account the additional L^2 term, it seems that our arguments break down when considering greater dimensions since we are led to contradictory conditions on q.

Proof of Lemma 1. Let us denote by χ_{B_n} the characteristic function of the set B_n and by $\chi_{|v|>n}$ the characteristic function of the set $\{(t,x) \in Q,\ such\ that\ \mid v(t,x) \mid > n\}$. As in [5], we introduce the following test function

$$\phi_n(v) = \begin{cases} +1 \ \ if\ v \geq n+1, \\ v-n \ \ if\ n \leq v \leq n+1, \\ 0 \ \ if\ \ \mid v \mid \leq n, \\ v+n \ \ if\ \ -n-1 \leq v \leq -n, \\ -1 \ \ if\ v \leq -n-1. \end{cases}$$

We set

$$\Psi_n(v) = \begin{cases} v-n-\frac{1}{2} \ \ if\ v \geq n+1, \\ \frac{v^2}{2} - nv + \frac{n^2}{2} \ \ if\ n \leq v \leq n+1, \\ 0 \ \ if\ \ \mid v \mid \leq n, \\ \frac{v^2}{2} + nv + \frac{n^2}{2} \ \ if\ \ -n-1 \leq v \leq -n, \\ -v-n-\frac{1}{2} \ \ if\ v \leq -n-1. \end{cases}$$

Let $v \in H^1(\Omega)$. Since ϕ_n is Lipschitzian, $\phi_n(v) \in H^1(\Omega)$ and [18], [6]

$$\partial_x(\phi_n(v)) = \begin{cases} \partial_x v \ \ if\ (t,x) \in B_n, \\ 0 \ \ otherwise. \end{cases}$$

Hence, we get

$$\frac{d}{dt}\int_\Omega \Psi_n(v)dx + a(t,v,\phi_n(v)) = 0.$$

Then, assumption (1.5) leads to

$$\frac{d}{dt}\int_\Omega \Psi_n(v)dx + \underline{d}\int_\Omega \mid \partial_x v \mid^2 \chi_{B_n} dx$$

$$+\lambda\int_\Omega (v^2 - n \mid v \mid)\chi_{B_n} dx + \lambda\int_\Omega \mid v \mid \chi_{|v|>n+1} dx$$

$$\leq \frac{C_{M,D}}{2\mu}\int_\Omega v^2\chi_{B_n} dx + \frac{\mu C_{M,D}}{2}\int_\Omega \mid \partial_x v \mid^2 \chi_{B_n} dx.$$

Remarking that the fourth term in the left hand side is non negative, we proceed as in Section 2 to obtain

$$\frac{d}{dt}\int_\Omega \Psi_n(v)dx + \frac{\underline{d}}{2}\int_\Omega \mid \partial_x v \mid^2 \chi_{B_n} dx + \Lambda\int_\Omega v^2\chi_{B_n} dx$$

$$\leq \lambda \int_\Omega n \mid v \mid \chi_{B_n} dx \leq \lambda \int_\Omega v^2 dx.$$

Since $0 \leq \Psi_n(s) \leq s$, integrating with respect to t achieves the proof of Lemma 1. □

Proof of Proposition 3. We seek an estimate in the space $L^q(0, T, W^{1,q}(\Omega))$ where $1 \leq q < \frac{4}{3}$. To this end, we write

$$\int_0^T \int_\Omega \mid \partial_x v \mid^q dxdt = \sum_{n=0}^{N-1} \int_{B_n} \mid \partial_x v \mid^q dxdt + \sum_{n=N}^{\infty} \int_{B_n} \mid \partial_x v \mid^q dxdt \quad (3.3)$$

where N is an integer to be suitably determined. In what follows, we simply note C for various constants depending on $\kappa, C_0, C_1, q, T, \mid \Omega \mid$,... neglecting the fact that the value of C may change from a relation to another. Similarly, writing $C(N)$ or $\delta(N)$, we emphasize the dependence with respect to N with the meaning that $C(N)$ is large and $\delta(N)$ is small as N goes to ∞.

By using Holder's inequality (with exponents $\frac{2}{q} > 1$ and $\frac{2}{2-q}$) and (3.1), one has

$$\int_{B_n} \mid \partial_x v \mid^q dxdt \leq \left(\int_{B_n} \mid \partial_x v \mid^2 dxdt \right)^{\frac{q}{2}} \mid B_n \mid^{\frac{2-q}{2}}$$

$$\leq (C_0^{\frac{q}{2}} + C_1^{\frac{q}{2}} \parallel v \parallel^q_{L^2((0,T)\times\Omega)}) \mid B_n \mid^{\frac{2-q}{2}}. \quad (3.4)$$

Thus, we get

$$\sum_{n=0}^{N-1} \int_{B_n} \mid \partial_x v \mid^q dxdt \leq C(N)(1 + \parallel v \parallel^q_{L^2((0,T)\times\Omega)}). \quad (3.5)$$

Let $r \geq 0$. It is obvious that

$$\mid B_n \mid \leq \frac{1}{n^r} \int_{B_n} \mid v \mid^r dxdt$$

and it follows that

$$\sum_{n=N}^{\infty}\int_{B_n} | \partial_x v |^q \, dxdt \le (C_0^{\frac{q}{2}} + C_1^{\frac{q}{2}} \parallel v \parallel^q_{L^2((0,T)\times\Omega)}) \sum_{n=N}^{\infty} \left(\frac{1}{n^r}\int_{B_n} | v |^r \, dxdt\right)^{\frac{2-q}{2}}$$

$$\le (C_0^{\frac{q}{2}} + C_1^{\frac{q}{2}} \parallel v \parallel^q_{L^2((0,T)\times\Omega)}) \parallel v \parallel^{r\frac{2-q}{2}}_{L^r((0,T)\times\Omega)} \left(\sum_{n=N}^{\infty} \frac{1}{n^{r\frac{2-q}{q}}}\right)^{\frac{q}{2}} \qquad (3.6)$$

by using Holder's inequality. Note that the serie which appears in the right hand side of (3.6) converges as soon as $r\frac{2-q}{q} > 1$. In particular, assuming $1 \le q < \frac{4}{3}$, we may choose $r = 2$. In this case, let us write (3.6) as follows

$$\sum_{n=N}^{\infty}\int_{B_n} | \partial_x v |^q \, dxdt \le \delta(N)(\parallel v \parallel^2_{L^2((0,T)\times\Omega)} + \parallel v \parallel^{2-q}_{L^2((0,T)\times\Omega)}) \qquad (3.7)$$

where $\delta(N)$ may be made arbitrarily small by choosing N large enough. Then, our next aim is a bound on the L^2 norm of v.

Let $s > 2$. We set $r = 2 = \theta + (1-\theta)s$. One gets

$$\int_{\Omega} | v |^r \, dx \le \left(\int_{\Omega} | v | \, dx\right)^{\theta} \left(\int_{\Omega} | v |^s \, dx\right)^{1-\theta} \le \kappa^{\theta}\left(\int_{\Omega} v^s dx\right)^{\frac{r-1}{s-1}} . \qquad (3.8)$$

Assume now $s \ge \frac{q}{q-1} > 2 > q$. Integration of (3.8) with respect to t gives

$$\parallel v \parallel^2_{L^2((0,T)\times\Omega)} \le C \parallel v \parallel^q_{L^q(0,T;L^s(\Omega))} . \qquad (3.9)$$

Since $W^{1,1}(\Omega)$ embeds continuously in $L^s(\Omega)$, we are led to

$$\parallel v \parallel^q_{L^q(0,T;L^s(\Omega))} \le C \int_0^T \left(\int_{\Omega} (| v | + | \partial_x v |) dx\right)^q dt$$

$$\le C \left(1 + \int_Q | \partial_x v |^q \, dxdt\right) . \qquad (3.10)$$

Combining (3.3) with (3.5) and (3.7) yields

$$\int_Q |\, \partial_x v \,|^q \, dxdt \leq C(N)\left(1+ \| \, v \, \|^q_{L^2((0,T)\times\Omega)}\right)$$

$$+\delta(N)\left(\| \, v \, \|^2_{L^2((0,T)\times\Omega)} + \| \, v \, \|^{2-q}_{L^2((0,T)\times\Omega)}\right) \qquad (3.11)$$

so that, by (3.9), (3.10) becomes

$$\| \, v \, \|^q_{L^q(0,T;L^s(\Omega))} \leq C(N)\left(1+ \| \, v \, \|^{\frac{q^2}{2}}_{L^q(0,T;L^s\Omega))}\right)$$

$$+\delta(N)\left(\| \, v \, \|^q_{L^q(0,T;L^s\Omega))} + \| \, v \, \|^{(2-q)\frac{q}{2}}_{L^q(0,T;L^s\Omega))}\right). \qquad (3.12)$$

Therefore, by using the Young inequality, it follows that

$$\| \, v \, \|^q_{L^q(0,T;L^s\Omega))} \leq C(N) + (\frac{1}{2} + 2\delta(N)) \, \| \, v \, \|^q_{L^q(0,T;L^s\Omega))} \qquad (3.13)$$

holds. We may choose N large enough so that $\frac{1}{2} - 2\delta(N) > 0$ which gives the last bound of (3.2). The other bounds of Proposition 3 are deduced as consequences of (3.9), (3.11) and (3.13). □

4. End of proof of Theorem 1

In this section we achieve the proof of Theorem 1. Let $u_0 \geq 0$, with $u_0 \in L^1(\Omega)$ be an initial datum for problem (1.1-1.3). We approach this datum by regular functions

$$\begin{cases} u_{0,\varepsilon} \in C_0^\infty(\Omega), \ u_{0,\varepsilon} \geq 0 \\ \| \, u_{0,\varepsilon} \, \|_{L^1(\Omega)} \leq \| \, u_0 \, \|_{L^1(\Omega)} \ . \end{cases} \qquad (4.1)$$

Proposition 2 ensures existence and uniqueness of a solution u_ε of (1.1-1.3) with $u_{0,\varepsilon}$ as initial datum. Moreover, Proposition 3 provides some bounds on u_ε, depending only on $\| \, u_0 \, \|_{L^1}$. Summarizing, we construct an associated sequence

$$u_\varepsilon \text{ bounded in } L^q(0,T;W^{1,q}(\Omega)), \ 1 \leq q < \frac{4}{3}. \qquad (4.2)$$

Furthermore, in view of the equation satisfied by u_ε,

$$\partial_t u_\varepsilon \textit{ is bounded in } L^q(0,T;L^q(\Omega)) + L^q(0,T;W^{-1,q}(\Omega)). \tag{4.3}$$

Therefore, by applying compactness lemma in [16], one deduces that u_ε is compact in $L^q((0,T)\times\Omega)$. Hence, possibly at the cost of extracting a subsequence, we can assume that

$$\begin{cases} u_\varepsilon \longrightarrow u \ \textit{ strongly in } L^q((0,T)\times\Omega) \textit{ and a.e. } t,x, \\ 0 \le u_\varepsilon \le h(t,x) \in L^q((0,T)\times\Omega) \ \textit{ a.e. } t,x, \\ \partial_x u_\varepsilon \rightharpoonup \partial_x u \ \textit{ weakly in } L^q((0,T)\times\Omega). \end{cases} \tag{4.4}$$

Next, let us claim that we can obtain, by construction, additional properties on the sequence $u_\varepsilon = e^{\lambda t} v_\varepsilon$. The following statement is derived from Proposition 3.

Lemma 2 *We have*

$$\lim_{k\to\infty}\{\sup_{\varepsilon,t}\int_{v_\varepsilon>k} v_\varepsilon(t,x)dx\} = 0. \tag{4.5}$$

Proof. Proposition 3 shows that, for all ε, n,

$$\int_{B_n} \mid \partial_x v_\varepsilon \mid^2 dxdt \le K. \tag{4.6}$$

We introduce the following truncation

$$T_k(s) = \begin{cases} s \ \ if \ \mid s \mid \le k, \\ k \ \ if \ s > k, \\ -k \ \ if s < -k, \end{cases} \tag{4.7}$$

and we define the primitive of T_k which vanishes on 0 as follows

$$S_k(s) = \begin{cases} \frac{s^2}{2} \ \ if \ \mid s \mid \le k, \\ k \mid s \mid -\frac{k^2}{2} \ \ if \ \mid s \mid > k. \end{cases} \tag{4.8}$$

Let $k \in \mathbb{N}$ fixed. By (4.6), $(T_k(v_\varepsilon))_\varepsilon$ is bounded in $L^2(0,T;H^1(\Omega))$ since, clearly, $\mid T_k(v_\varepsilon) \mid \le k$ and

$$\int_Q \mid \partial_x(T_k v_\varepsilon) \mid^2 dxdt = \sum_{n=0}^{k-1}\int_{B_n} \mid \partial_x v_\varepsilon \mid^2 dxdt \le kK. \tag{4.9}$$

Thus, by using the continuity of the functions T_k and (4.4), we deduce that the following convergences

$$\left\{ \begin{array}{l} T_k(v_\varepsilon) \to T_k(v) \ \ \textit{strongly in } L^q((0,T)\times\Omega) \textit{ and a.e. } t,x, \\ \partial_x(T_k(v_\varepsilon)) \rightharpoonup \partial_x(T_k(v)) \ \ \textit{weakly in } L^2((0,T)\times\Omega) \end{array} \right. \tag{4.10}$$

hold as ε goes to 0. Note in particular that $T_k(v)$ belongs to $L^2(0,T;H^1(\Omega))$.

Let $\chi_{_{v_\varepsilon<k}}$ stand for the characteristic function of the set

$$\mathcal{B}_{\varepsilon,k} = \{(t,x) \textit{ such that } 0 \le v_\varepsilon(t,x) \le k\}.$$

By plugging $\phi = T_k(v_\varepsilon) \in L^2(0,T;H^1(\Omega))$ as test function in (2.7), it follows that

$$\frac{d}{dt}\int_\Omega S_k(v_\varepsilon)dx = -a_\varepsilon(t,v_\varepsilon,T_k(v_\varepsilon))$$

$$= -a_\varepsilon(t,\chi_{_{v_\varepsilon<k}}v_\varepsilon,\chi_{_{v_\varepsilon<k}}v_\varepsilon) - \lambda k\int_\Omega v_\varepsilon\chi_{_{v_\varepsilon>k}}dx \le -\frac{d}{2}\parallel \chi_{_{v_\varepsilon<k}}v_\varepsilon \parallel^2_{H^1(\Omega)}. \tag{4.11}$$

Integration of (4.11) on $(0,t)$ yields

$$\int_\Omega S_k(v_\varepsilon)dx(t) + \frac{d}{2}\parallel \chi_{_{v_\varepsilon<k}}v_\varepsilon \parallel^2_{L^2(0,T;H^1(\Omega))} \le \int_\Omega S_k(u_{0,\varepsilon})dx. \tag{4.12}$$

Let $M>0$. We remark that $S_k(s) \le M^2 + ks\chi_{_{s>M}}$, for $s \ge 0$. Then, we deduce from (4.12) that

$$\frac{1}{k}\int_\Omega S_k(v_\varepsilon)dx(t) \le \frac{M^2}{k} + \int_{u_{0,\varepsilon}>M} u_{0,\varepsilon}dx. \tag{4.13}$$

Since $u_{0,\varepsilon}$ converges in $L^1(\Omega)$, M may be chosen so that

$$\sup_{\varepsilon>0}\int_{u_{0,\varepsilon}>M} u_{0,\varepsilon}dx$$

is arbitrarily small which allows us to claim that

$$\lim_{k\to\infty}\{\sup_{\varepsilon,t}\frac{1}{k}\int_\Omega S_k(v_\varepsilon)dx(t)\} = 0. \tag{4.14}$$

>From (4.12), we also note that

$$\lim_{k\to\infty}\{\sup_{\varepsilon}\frac{1}{k}\parallel \chi_{v_\varepsilon<k}v_\varepsilon\parallel^2_{L^2(0,T;H^1(\Omega))}\}=0. \tag{4.15}$$

Finally, combining (4.13) with $S_k(s)\geq k\chi_{|s|\geq k}\frac{|s|}{2}$ ends the proof of Lemma 2. □

Corollary 2 *There exists a subsequence such that for all $t\in(0,T)$, $u_\varepsilon(t,\cdot)$ converges weakly to $u(t,\cdot)$ in $L^1(\Omega)$. Moreover, let $\phi\in W^{1,q'}(\Omega)$, then the function of time*

$$t\longmapsto U(t)=\int_\Omega u(t,x)\phi(x)dx$$

is continuous on $(0,T)$.

Proof. Let $1\leq q<\frac{4}{3}$ and $\phi\in W^{1,q'}(\Omega)$. We set $U_\varepsilon(t)=\int_\Omega u_\varepsilon(t,x)\phi(x)dx$ Our proof starts with the observation of the following remarkable equicontinuity property

$$\sup_\varepsilon\mid U_\varepsilon(t)-U_\varepsilon(s)\mid\leq C\parallel\phi\parallel_{W^{1,q'}}\mid t-s\mid^\alpha \tag{4.16}$$

with $\alpha>0$ depending on q. Indeed, by the definition of u_ε, one has

$$\mid U_\varepsilon(t)-U_\varepsilon(s)\mid=\mid\int_s^t\int_\Omega(N(u_\varepsilon)u_\varepsilon+D(u_\varepsilon)\partial_x u_\varepsilon)\partial_x\phi dxd\tau\mid$$

$$\leq C_{u_0}\parallel\partial_x\phi\parallel_{L^{q'}}\left(\left(\int_s^t\int_\Omega u_\varepsilon^q dxd\tau\right)^{\frac{1}{q}}+\left(\int_s^t\int_\Omega\mid\partial_x u_\varepsilon\mid^q dxd\tau\right)^{\frac{1}{q}}\right).$$

Let $q<p<\frac{4}{3}$. By Proposition 3, u_ε is bounded in $L^p(0,T;W^{1,p}(\Omega))$. Hence, Holder's inequality yields

$$\mid U_\varepsilon(t)-U_\varepsilon(s)\mid\leq C_{u_0}K\mid t-s\mid^\alpha$$

with$\alpha=\frac{1}{q}-\frac{1}{p}>0$.

We turn to the proof of Corollary 2. Since $u_\varepsilon(t)$ is bounded in $L^1(\Omega)$ uniformly in t, ε, we observe by Lemma 2, that $u_\varepsilon(t)$ lies in a weakly compact set $\mathcal{K}$ of $L^1(\Omega)$ for all t, ε. Then, the diagonal process allows us to assume that $u_\varepsilon(\bar{t}) \rightharpoonup u(\bar{t})$ in $L^1(\Omega)$ for all rational time $\bar{t}$. We shall show that this convergence actually holds for all time $t \in (0, T)$. Indeed, assume that for a non rational t, $u_\varepsilon(t)$ does not weakly converge in $L^1(\Omega)$. Then, for some regular test function ϕ, we can extract two subsequences from $U_\varepsilon(t)$ with different limits, while

$$\mid U_\varepsilon(t) - U_\varepsilon(\bar{t}) \mid \leq C \mid t - \bar{t} \mid^\alpha$$

holds by (4.16) where $\bar{t}$ is a rational close to t. Thus, we are clearly led to a contradiction since $U_\varepsilon(\bar{t})$ converges as ε goes to 0. The continuity of $U(t)$ for regular ϕ is a immediate consequence of this weak L^1 compactness property and of (4.16). Observe also that the conservation law (1.4) follows from Corollary 2. □

Finally, we wish to pass to the limit $\varepsilon \to 0$ in the weak formulation satisfied by u_ε, namely

$$\begin{gathered}\int_0^t \int_\Omega u_\varepsilon \partial_t \phi dx ds + \int_\Omega u_{0,\varepsilon} \phi(0, \cdot) dx = \\ \int_\Omega u_\varepsilon \phi(t, \cdot) dx + \int_0^t \int_\Omega (N(u_\varepsilon) u_\varepsilon + D(u_\varepsilon) \partial_x u_\varepsilon) \partial_x \phi dx ds\end{gathered} \tag{4.17}$$

where $\phi \in C^0(0, T; W^{1,q'}(\Omega))$ with $\partial_t \phi \in C^0(0, T; L^{q'}(\Omega))$. There is no difficulty to deal with the left hand side term. In the right hand side, first, we use Corollary 2, and, next, we observe that $N(u_\varepsilon)$ and $D(u_\varepsilon)$ converge a.e. to $N(u)$ and $D(u)$ respectively and remain uniformly bounded in $L^\infty((0, T) \times \Omega)$. Thus, by the Lebesgue theorem, $D(u_\varepsilon)\partial_x \phi$ and $D(u_\varepsilon)\partial_x \phi$ converge stongly in $L^{q'}(Q)$ while u_ε and $\partial_x u_\varepsilon$ are weakly convergent in $L^q(Q)$. These arguments allow us to pass to the limit in the last integral of the right hand side. This proves that u is a weak solution of the Fokker-Planck equation (1.1- 1.3) in the sense that

$$\begin{gathered}\int_0^t \int_\Omega u\partial_t\phi dxds + \int_\Omega u_0\phi(0,\cdot)dx = \\ \int_\Omega u\phi(t,\cdot)dx + \int_0^t \int_\Omega (N(u)u + D(u)\partial_x u)\partial_x\phi dxds\end{gathered} \tag{4.18}$$

holds. □

Remark 5 The regularity of the obtained solution permits us to make more precise the sense in which the boundary condition (1.2) holds. We have $(M + \partial_x D)u + D\partial_x u = 0$ in $W^{-1,q}(0,T)$ at $x \in \{0,1\}$.

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