Parabolic equations involving 0^{th} and 1^{st} order terms with L^1 data

Thierry GOUDON[1,2] and Mazen SAAD[1]

[1] *Mathématiques Appliquées de Bordeaux*
CNRS-Université Bordeaux I
351, cours de la Libération, F-33405 Talence Cedex

[2] *Laboratoire J. A. Dieudonné UMR 6621*
Université Nice-Sophia Antipolis
Parc Valrose, 06108 Nice cedex 02

Abstract

This paper is devoted to general parabolic equations involving 0^{th} and 1^{st} order terms, in linear and nonlinear expressions, while the data only belong to L^1. Existence and entropic-uniqueness of solutions are proved.

1 Introduction

In this paper, we are concerned with the following general parabolic equation

$$(1.1) \qquad \begin{cases} \partial_t u - \nabla \cdot (A(t,x)\nabla u) + B(t,x,u,\nabla u) = f \;\; in \; (0,T) \times \Omega, \\ u_{|t=0} = u_0 \;\; in \; \Omega, \\ u = 0 \;\; on \; (0,T) \times \partial\Omega, \end{cases}$$

where Ω is a regular open bounded set in $\mathbb{R}^N$ and B involves the unknown u and its first derivatives. Precisely, B splits into terms which are linear with respect to u and ∇u and a nonlinear term as follows

$$(1.2) \qquad B(t,x,u,\nabla u) = b(t,x) \cdot \nabla u + d(t,x)u + g(t,x,u,\nabla u).$$

Here, A, b and d are given functions defined on $Q = (0,T) \times \Omega$ with values in $\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N$ and $\mathbb{R}$, respectively. Our basic requirement on A, b, d is

$$(1.3) \qquad A \in (L^\infty(Q))^{N\times N}, \;\; d \in L^\infty(Q),$$

$$(1.4) \qquad b \in (L^\infty(Q))^N, \;\; \nabla \cdot b \in L^\infty(Q).$$

As usual, we also assume that there exists $\underline{a} > 0$ such that the matrix A satisfies

$$(1.5) \qquad A(t,x)\xi \cdot \xi \geq \underline{a}|\xi|^2,$$

for a.e. $(t,x) \in Q$ and for all $\xi \in \mathbb{R}^N$. The function $g : Q \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is measurable on Q for all $\lambda \in \mathbb{R}, \xi \in \mathbb{R}^N$, continuous with respect to $\lambda \in \mathbb{R}, \xi \in \mathbb{R}^N$, a.e. in Q. Furthermore, g is required to satisfy both a sign condition and a growth condition with respect to the gradient variable since we suppose that

$$(1.6) \qquad \lambda g(t,x,\lambda,\xi) \geq 0,$$

$$(1.7) \qquad \exists 0 \le \sigma < 2; \quad |g(t,x,\lambda,\xi)| \le h(|\lambda|)(\gamma(t,x) + |\xi|^\sigma)$$

holds for all $\lambda \in \mathbb{R}, \xi \in \mathbb{R}^N$, and a.e. in Q, with $\gamma \in L^1(Q)$; h being a non decreasing function on $\mathbb{R}^+$. Main difficulties in this work arise from the fact that we consider data which only belong to L^1, namely

$$(1.8) \qquad u_0 \in L^1(\Omega), \; f \in L^1(Q).$$

Assuming $B = 0$, existence results for such parabolic problems with non regular data are established in [4] (see also [3], [10]) while uniqueness questions, in the sense of entropic or renormalized formulations, are considered in [15], [1]. Existence-uniqueness of renormalized solution for a linear parabolic equation involving a first order term with a free divergence coefficient is discussed in [14]. Taking into account the g term, the corresponding elliptic problem, with an integrable source term, is treated in [9] when $\sigma < 2$ and the critical case $\sigma = 2$ is dealt with in [5]. In [6], the g term appears in (1.1), still neglecting the linear terms involving b and d, with the restriction that g does not vanish for large value of u, which induces some regularizing effects in the equation. Note that in view of the quoted papers, our results extend to more general Leray-Lions operators ; however, to avoid technical complications and to emphasize the influence of the term B we restrict our attention on a simple operator satisfying (1.2). Let us now introduce some definitions and give the statement of our main results.

Definition 1 *By weak solution of (1.1), we shall mean any function $u \in L^q(0,T;W_0^{1,q}(\Omega)) \cap C^0(0,T;L^1(\Omega))$ such that $g(u,\nabla u)$ belongs to $L^1(Q)$ and satisfying*

$$(1.9) \qquad \begin{aligned} &\int_\Omega u\phi(T,x)dx - \int_\Omega u_0\phi(0,x)dx - \int_Q u\partial_t\phi(t,x)dxdt \\ &+ \int_Q A\nabla u\cdot\nabla\phi dxdt + \int_Q (g(u,\nabla u) + b\cdot\nabla u + du)\phi dxdt = \int_Q f\phi dxdt \end{aligned}$$

for all $T > 0$ and $\phi \in C^0(0,T;W^{1,q'}(\Omega)) \cap C^1(0,T;L^{q'}(\Omega))$ where $1 \le q < \frac{N+2}{N+1}$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

Theorem 1 *Assume that (1.3)-(1.8) hold. Then, there exists a weak solution of (1.1), in the sense of Definition 1.*

Let us recall the definition of the truncated function T_k. Let $k \in \mathbb{R}^+$. We set

$$(1.10) \qquad T_k(z) = \begin{cases} z & \text{if} \quad |z| \le k \\ k & \text{if} \quad z > k \\ -k & \text{if} \quad z < -k \end{cases}$$

and we denote $S_k(z) = \displaystyle\int_0^z T_k(\tau)d\tau$.

Definition 2 *Let $g = 0$. We say that u is a entropic solution of (1.1) if $u \in C^0(0,T;L^1(\Omega))$ satisfies $T_k(u) \in L^2(0,T;H_0^1(\Omega))$ for all $k > 0$, $\nabla u \in L^1(Q)$ and*

$$(1.11) \qquad \begin{aligned} &\int_\Omega S_k(u-\psi)(t)dx - \int_\Omega S_k(u_0 - \psi(0,\cdot))dx + \int_0^t \langle \partial_t\psi, T_k(u-\psi)\rangle_{H^{-1},H_0^1} dt \\ &+ \int_Q A\nabla u\cdot\nabla(T_k(u-\psi))dxdt + \int_Q (du + b\nabla u)T_k(u-\psi)dxdt \le \int_Q fT_k(u-\psi)dxdt \end{aligned}$$

for all $t \in (0,T), k > 0$ and $\psi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q) \cap C^0(0,T;L^1(\Omega))$ with $\partial_t\psi \in L^2(0,T;H^{-1}(\Omega))$.

Theorem 2 *Let $g = 0$. Assume that (1.3)-(1.8) hold. Then, there exists a unique entropic solution of (1.1).*

The strategy we adopt is rather close to those introduced in [4]. However, new difficulties arise essentially related to the influence of the linear 0^{th} and 1^{st} order terms. Then, this paper is organized as follows. First, Section 2 is devoted to an independent preliminary result which will be used to derive a bound in L^q on the gradient of the solutions, in despite of the perturbation induced by the additional terms of lower order. In Section 3, we deal with sequences u_ε of approximate solutions. We establish some a priori estimates on these solutions and we translate the obtained bounds in terms of compactness properties. Then, we explain how we can pass to the limit as $\varepsilon \to 0$ in the weak formulation satisfied by u_ε. In Section 4, we are concerned with the uniqueness of entropic solution. Finally, in Section 5, we slightly weaken the regularity assumption concerning the coefficient b.

2 A preliminary result

The main idea in the proof of Theorem 1 consists in deriving a $L^q(0,T;W_0^{1,q}(\Omega))$ estimate on the solutions depending only on the L^1 norm of the data f and u_0. Such an estimate will appear as a consequence of the following lemma.

Lemma 1 *Let $u \in L^2(0,T;H_0^1(\Omega))$ satisfy*

$$\sup_{t\in(0,T)} \int_\Omega |u|dx \leq \beta, \tag{2.1}$$

and

$$\int_{B_n} |\nabla u|^2 dxdt \leq C_0 + C_1 \int_{E_n} |\nabla u| dxdt, \tag{2.2}$$

where

$$B_n = \{(t,x) \in Q,\ n \leq |u(t,x)| \leq n+1\},$$

and

$$E_n = \{(t,x) \in Q,\ |u(t,x)| > n+1\}.$$

Let $1 \leq q < \frac{N+2}{N+1}$. Then, there exists $C > 0$, depending on β, C_0, C_1, $|\Omega|$, T, and q such that

$$\|u\|_{L^q(0,T;W_0^{1,q}(\Omega))} \leq C. \tag{2.3}$$

Proof. In [4], [10] inequality (2.2) appears with $C_1 = 0$ and is used to derive (2.3). Here, the additional term is related to the influence of the first order term $b \cdot \nabla u$ in the equation as we shall see in next section (see Proposition 1). However, exploiting carefully the fact that the integral in the right hand side is only taken over the large values of the unknow, we can obtain (2.3) as a consequence of (2.2).

Let $1 \leq q < 2$. From (2.2), we first notice that

$$\begin{aligned} \int_{B_n} |\nabla u|^2 dxdt &\leq C_0 + C_1 \left(\int_{E_n} |\nabla u|^q dx \right)^{\frac{1}{q}} |E_n|^{\frac{q-1}{q}} \\ &\leq C_0 + C_1 \|\nabla u\|_{L^q(Q)} |E_n|^{\frac{q-1}{q}} \end{aligned} \tag{2.4}$$

holds by using Holder's inequality. Thus, applying again Holder's inequality, we obtain

$$\begin{aligned} \int_{B_n} |\nabla u|^q dxdt &\leq |B_n|^{\frac{2-q}{2}} \left(\int_{B_n} |\nabla u|^2 dxdt \right)^{\frac{q}{2}} \\ &\leq |B_n|^{\frac{2-q}{2}} \left(C_0^{\frac{q}{2}} + C_1^{\frac{q}{2}} \|\nabla u\|_{L^q(Q)}^{\frac{q}{2}} |E_n|^{\frac{q-1}{2}} \right) \end{aligned} \tag{2.5}$$

by (2.4) and the elementary inequality $(a+b)^{\frac{q}{2}} \leq a^{\frac{q}{2}} + b^{\frac{q}{2}}$. Let $r \geq 0$ to be chosen later. Clearly, one has

$$\begin{cases} |B_n| \leq \dfrac{1}{n^r} \displaystyle\int_{B_n} |u|^r dxdt, \\ |E_n| \leq \dfrac{1}{n^r} \displaystyle\int_{E_n} |u|^r dxdt \leq \dfrac{1}{n^r} \|u\|_{L^r(Q)}^r. \end{cases} \tag{2.6}$$

Hence, (2.5) becomes

$$\begin{aligned} \int_{B_n} |\nabla u|^q dxdt &\leq C_0^{\frac{q}{2}} \left(\tfrac{1}{n}\right)^{r\frac{2-q}{2}} \left(\int_{B_n} |u|^r dxdt \right)^{\frac{2-q}{2}} \\ &+ C_1^{\frac{q}{2}} \|\nabla u\|_{L^q(Q)}^{\frac{q}{2}} \|u\|_{L^r(Q)}^{\frac{r(q-1)}{2}} \left(\tfrac{1}{n}\right)^{\frac{r}{2}} \left(\int_{B_n} |u|^r dxdt \right)^{\frac{2-q}{2}}. \end{aligned} \tag{2.7}$$

Let $K \in \mathbb{N}$ to be determined. We split $\|\nabla u\|_{L^q(Q)}^q$ as follows

$$\int_Q |\nabla u|^q dxdt = \sum_{n=0}^{K} \int_{B_n} |\nabla u|^q dxdt + \sum_{n=K+1}^{\infty} \int_{B_n} |\nabla u|^q dxdt. \tag{2.8}$$

Since $|B_n| \leq T|\Omega|$ and $|E_n| \leq T|\Omega|$, we simply evaluate the first term in the right hand side of (2.8) as follows

$$\sum_{n=0}^{K} \int_{B_n} |\nabla u|^q dxdt \leq KC_2 \left(1 + \|\nabla u\|_{L^q(Q)}^{\frac{q}{2}} \right), \tag{2.9}$$

by (2.5), where $C_2 = \max(C_0^{\frac{q}{2}} (T|\Omega|)^{\frac{2-q}{2}}, C_1^{\frac{q}{2}} (T|\Omega|)^{\frac{1}{2}})$. Thus, by using Young's inequality in (2.8)-(2.9), we get

$$\|\nabla u\|_{L^q(Q)}^q \leq C(K) + \sum_{n=K+1}^{\infty} \int_{B_n} |\nabla u|^q dxdt \tag{2.10}$$

where $C(K)$ tends to infinity as K becomes large. It remains to proceed to the study of the serie which appears in the right hand side.
Applying Holder's inequality on the series with exponents $\frac{2}{2-q}$ and $\frac{2}{q}$ and using (2.7), we

have

$$
\begin{aligned}
\sum_{n=K+1}^{\infty}\int_{B_n}|\nabla u|^q dxdt \;&\leq\; C_0^{\frac{q}{2}}\left(\sum_{n=K+1}^{\infty}\frac{1}{n^{r\frac{2-q}{q}}}\right)^{\frac{q}{2}}\left(\sum_{n=K+1}^{\infty}\int_{B_n}|u|^r dxdt\right)^{\frac{2-q}{2}}\\
&+\; C_1^{\frac{q}{2}}\|\nabla u\|_{L^q(Q)}^{\frac{q}{2}}\|u\|_{L^r(Q)}^{\frac{r(q-1)}{2}}\left(\sum_{n=K+1}^{\infty}\frac{1}{n^{\frac{r}{q}}}\right)^{\frac{q}{2}}\left(\sum_{n=K+1}^{\infty}\int_{B_n}|u|^r dxdt\right)^{\frac{2-q}{2}}\\
&\leq\; C_0^{\frac{q}{2}}\left(\sum_{n=K+1}^{\infty}\frac{1}{n^{r\frac{2-q}{q}}}\right)^{\frac{q}{2}}\|u\|_{L^r(Q)}^{\frac{r(2-q)}{2}}\\
&+\; C_1^{\frac{q}{2}}\|\nabla u\|_{L^q(Q)}^{\frac{q}{2}}\|u\|_{L^r(Q)}^{\frac{r}{2}}\left(\sum_{n=K+1}^{\infty}\frac{1}{n^{\frac{r}{q}}}\right)^{\frac{q}{2}}
\end{aligned}
\tag{2.11}
$$

Note that the conditions

$$r\frac{2-q}{q}>1,\ \text{and}\ \frac{r}{q}>1 \tag{2.12}$$

ensure the convergence of the series which appear in the right hand side of (2.11). Consequently, these terms become arbitrarily small when choosing K large enough as soon as (2.12) is fulfilled.

With the convention that $\delta(K)$ denotes quantities which tend to 0 as K goes to ∞, by combining (2.10) with (2.11), we get

$$\|\nabla u\|_{L^q(Q)}^q \leq C(K)+\delta(K)\left(\|u\|_{L^r(Q)}^{\frac{r(2-q)}{2}}+\|\nabla u\|_{L^q(Q)}^{\frac{q}{2}}\|u\|_{L^r(Q)}^{\frac{r}{2}}\right). \tag{2.13}$$

Therefore, by using Young's inequality on the last term in the right side, it follows that

$$\|\nabla u\|_{L^q(Q)}^q \leq C(K)+\delta(K)\left(\|u\|_{L^r(Q)}^{\frac{r(2-q)}{2}}+\|u\|_{L^r(Q)}^{r}\right) \tag{2.14}$$

where we keep the notation $C(K),\delta(K)$ while the value of these terms may have changed, still with the meaning that $C(K)\to\infty$, $\delta(K)\to 0$ when K becomes large.

We denote by $q_\star=\frac{Nq}{N-q}$ the Sobolev conjugate of q. The Sobolev imbedding theorem implies that

$$\int_0^T\left(\int_\Omega |u|^{q_\star}dx\right)^{\frac{q}{q_\star}}dt\leq C\int_Q|\nabla u|^q dxdt. \tag{2.15}$$

Assume now $1<r<q_\star$ and set $\frac{1}{r}=\theta+\frac{1-\theta}{q^\star}$ with $0<\theta<1$. For almost everywere $t\in(0,T)$, one has

$$\|u(t,.)\|_{L^r(\Omega)}^r\leq\|u(t,.)\|_{L^1(\Omega)}^{r\theta}\|u(t,.)\|_{L^{q_\star}(\Omega)}^{r(1-\theta)}. \tag{2.16}$$

Integrating (2.16) with respect to time and recalling the bound (2.1) in $L^\infty(0,T,L^1(\Omega))$ yield

$$\|u\|_{L^r(Q)}^r\leq\beta^{r\theta}\int_0^T\left(\int_\Omega |u|^{q_\star}dx\right)^{\frac{r(1-\theta)}{q_\star}}dt. \tag{2.17}$$

Choose now $r = \frac{q(N+1)}{N}$, noting that the convergence condition (2.12) is fulfilled as soon as $1 \le q < \frac{N+2}{N+1}$. Combining (2.14)-(2.17) with Young's inequality (since $\frac{2-q}{2} < 1$) leads to

$$
\begin{aligned}
\int_0^T \left(\int_\Omega |u|^{q_\star} dx\right)^{\frac{q}{q_\star}} dt \;&\le\; C(K) \\
&+\; \delta(K)\left(\left(\int_0^T \left(\int_\Omega |u|^{q_\star} dx\right)^{\frac{q}{q_\star}} dt\right)^{\frac{2-q}{2}} + \int_0^T \left(\int_\Omega |u|^{q_\star} dx\right)^{\frac{q}{q_\star}} dt\right) \qquad (2.18)\\
&\le\; C(K) + \delta(K)\int_0^T \left(\int_\Omega |u|^{q_\star} dx\right)^{\frac{q}{q_\star}} dt.
\end{aligned}
$$

We fix $K > 0$ so that, for instance, $1 - \delta(K) > \frac{1}{2}$. Hence, we deduce from (2.18) that

$$
\|u\|_{L^q(0,T,L^{q_\star}(\Omega))} \le C \qquad (2.19)
$$

holds, and the asserted estimate (2.3) follows easily from (2.17) and (2.14). $\square$

3 Proof of Theorem 1

The proof falls naturally into several steps and we detail each of them separetely.

3.1 Approximate solutions

We introduce the following smooth approximations of the data

$$
\begin{cases} u_{0,\varepsilon} \in C_0^\infty(\Omega), \; f_\varepsilon \in C_0^\infty(\Omega), \\ u_{0,\varepsilon} \to u_0 \; in \; L^1(\Omega), \; f_\varepsilon \to f \; in \; L^1(Q), \end{cases} \qquad (3.1)
$$

with

$$
\|u_{0,\varepsilon}\|_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)}, \quad \|f_\varepsilon\|_{L^1(Q)} \le \|f\|_{L^1(Q)}. \qquad (3.2)
$$

Moreover, we regularize the function g as follows

$$
g_\varepsilon(u, \nabla u) = \frac{g(u, \nabla u)}{1 + \varepsilon |g(u, \nabla u)|}. \qquad (3.3)
$$

Note that g_ε belongs in $L^\infty(Q)$ and satisfy the sign condition (1.6) and the growth condition in (1.7). Then, classical results, see e.g. [13], [11], [7], (or, in the linear case, use a Galerkin method), provide the existence of a sequence $u_\varepsilon \in C^0(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, with $\partial_t u_\varepsilon \in L^2(0,T;H^{-1}(\Omega))$, of solutions of (1.1) where u_0, f and g are replaced by $u_{0,\varepsilon}, f_\varepsilon$ and g_ε respectively. We have

$$
\begin{aligned}
&\langle \partial_t u_\varepsilon, \phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_\Omega A \nabla u_\varepsilon \cdot \nabla \phi dx \\
&+ \int_\Omega (g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) + b \cdot \nabla u_\varepsilon + d u_\varepsilon) \phi dx = \int_\Omega f_\varepsilon \phi dx
\end{aligned} \qquad (3.4)
$$

for all $T > 0$ and $\phi \in L^2(0,T;H_0^1(\Omega))$.

3.2 A priori estimates

In this section, we are concerned with a priori estimates satisfied by the sequence u_ε of solutions of (3.4) which lead to compactness properties essential to the proof.

Proposition 1 *Let A, b, d, g satisfy (1.3)-(1.7). Then, there exist $\beta > 0$, C_0 and C_1 depending only on $\|u_0\|_{L^1(\Omega)}$, $\|f\|_{L^1(Q)}$, $\|b\|_{L^\infty(Q)}$, $\|d\|_{L^\infty(Q)}$, $|\Omega|$ and T such that the sequence u_ε of solutions of (3.4) satisfies*

$$\sup_{\varepsilon>0,\, t\in(0,T)} \|u_\varepsilon(t)\|_{L^1(\Omega)} \leq \beta, \tag{3.5}$$

and

$$\int_{B_n} |\nabla u_\varepsilon|^2 dxdt \leq C_0 + C_1 \int_{E_n} |\nabla u_\varepsilon| dxdt. \tag{3.6}$$

In view of Lemma 1, we deduce immediately the following

Corollary 1 *Let A, b, d, g satisfy (1.3)-(1.7). Let $1 \leq q < \frac{N+2}{N+1}$. Then, there exists $C > 0$ depending only on the data, such that*

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{L^q(0,T;W_0^{1,q}(\Omega))} \leq C. \tag{3.7}$$

Proof of Proposition 1. Since T_k is a Lipschitz function and $u_\varepsilon \in L^2(0,T;H_0^1(\Omega))$, one has $T_k(u_\varepsilon) \in L^2(0,T;H_0^1(\Omega))$, see [17], [18], with, moreover,

$$\nabla T_k(u_\varepsilon) = \chi_{|u_\varepsilon|\leq k} \nabla u_\varepsilon,$$

where $\chi_{|u_\varepsilon|\leq k}$ denotes the characteristic function of the set $\{(t,x) \in Q,\ |u_\varepsilon(t,x)| \leq k\}$. Thus, we choose $\phi = T_k(u_\varepsilon)$ as test function in (3.4). Writing $b \cdot \nabla u_\varepsilon = \nabla \cdot (bu_\varepsilon) - (\nabla \cdot b)u_\varepsilon$, one gets

$$\begin{aligned} &\frac{d}{dt}\int_\Omega S_k(u_\varepsilon)dx + \int_\Omega \chi_{|u_\varepsilon|\leq k} A\nabla u_\varepsilon \cdot \nabla u_\varepsilon dx + \int_\Omega T_k(u_\varepsilon) g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)dx \\ &= \int_\Omega f_\varepsilon T_k(u_\varepsilon)dx + \int_\Omega \chi_{|u_\varepsilon|\leq k} u_\varepsilon b \cdot \nabla u_\varepsilon dx + \int_\Omega ((\nabla \cdot b) - d) u_\varepsilon T_k(u_\varepsilon)dx. \end{aligned} \tag{3.8}$$

By using Holder's and Young's inequalities, one obtains

$$\left|\int_\Omega \chi_{|u_\varepsilon|\leq k} u_\varepsilon b \cdot \nabla u_\varepsilon dx\right| \leq \frac{\underline{a}}{2}\int_\Omega \chi_{|u_\varepsilon|\leq k} |\nabla u_\varepsilon|^2 dx + \frac{1}{2\underline{a}}\|b\|^2_{L^\infty(Q)} \int_\Omega \chi_{|u_\varepsilon|\leq k} |u_\varepsilon|^2 dx. \tag{3.9}$$

Moreover, $u_\varepsilon T_k(u_\varepsilon)$ is non negative and we assume that d and $\nabla \cdot b$ belong to $L^\infty(Q)$. Hence, after integration of (3.8) with respect to t and using (1.5), we are led to

$$\begin{aligned} \int_\Omega S_k(u_\varepsilon)(t)dx \quad &+ \quad \int_0^t \int_\Omega T_k(u_\varepsilon) g_\varepsilon dxds + \frac{\underline{a}}{2}\int_0^t \int_\Omega \chi_{|u_\varepsilon|\leq k} |\nabla u|^2 dxds \\ &\leq \quad \int_0^t \int_\Omega |f_\varepsilon T_k(u_\varepsilon)| dxds + \int_\Omega S_k(u_{0,\varepsilon})dx \\ &+ \quad \tfrac{1}{2\underline{a}}\|b\|^2_{L^\infty(Q)} \int_0^t \int_\Omega \chi_{|u_\varepsilon|\leq k} |u_\varepsilon|^2 dxds \\ &+ \quad \left(\|d\|_{L^\infty(Q)} + \|\nabla \cdot b\|_{L^\infty(Q)}\right) \int_0^t \int_\Omega u_\varepsilon T_k(u_\varepsilon) dxds \end{aligned} \tag{3.10}$$

where, by the sign assumption (1.6) and the definition of S_k, all the terms in the left hand side of (3.10) are non negative. Next, we observe that

$$
\begin{aligned}
0 &\leq z^2\chi_{|z|\leq k} \leq zT_k(z) = z^2\chi_{|z|\leq k} + k|z|\chi_{|z|>k} \\
&\leq z^2\chi_{|z|\leq k} + (2k|z| - k^2)\chi_{|z|>k} = 2S_k(z)
\end{aligned}
\tag{3.11}
$$

which yields

$$
\int_\Omega S_k(u_\varepsilon)(t)dx \leq \int_0^t \int_\Omega |f_\varepsilon T_k(u_\varepsilon)|dxds + \int_\Omega S_k(u_{0,\varepsilon})dx + C(b,d) \int_0^t \int_\Omega S_k(u_\varepsilon)dxds \tag{3.12}
$$

where $C(b,d)$ stands for $2(\|d\|_{L^\infty(Q)} + \|\nabla \cdot b\|_{L^\infty(Q)}) + \frac{1}{\underline{a}}\|b\|_{L^\infty(Q)}$. We set $z(t) = \int_\Omega S_k(u_\varepsilon)(t)dx$. Thus, dropping non negative terms, we have

$$
0 \leq z(t) \leq z(0) + \int_0^t \int_\Omega |f_\varepsilon||T_k(u_\varepsilon)|dxds + C(b,d) \int_0^t z(s)ds
$$

and we apply Gronwall's lemma to deduce that

$$
z(t) \leq e^{C(b,d)T} \left(\int_\Omega S_k(u_{0,\varepsilon})dx + \int_Q |f_\varepsilon||T_k(u_\varepsilon)|dxdt \right) \tag{3.13}
$$

holds.

We set $k = 1$ in (3.13). Remarking that $|T_1(z)| \leq 1$ and $0 \leq S_1(z) \leq |z|$ leads to

$$
\int_\Omega S_1(u_\varepsilon)(t)dx \leq e^{C(b,d)T} \left(\|u_0\|_{L^1(\Omega)} + \|f\|_{L^1(Q)} \right)
$$

by (3.2). Therefore, we end the proof of (3.5) with the following observation

$$
\int_\Omega |u_\varepsilon|dx = \int_{|u_\varepsilon|\leq 1} |u_\varepsilon|dx + \int_{|u_\varepsilon|>1} |u_\varepsilon|dx
$$

$$
\leq \int_{|u_\varepsilon|\leq 1} dx + \int_{|u_\varepsilon|>1} (S_1(u_\varepsilon) + \frac{1}{2})dx \leq \frac{3}{2}|\Omega| + e^{C(b,d)T} \left(\|u_0\|_{L^1(\Omega)} + \|f\|_{L^1(Q)} \right) = \beta.
$$

To achieve the proof of Proposition 1, we are left with the task of showing that (3.6) holds. According to [4], we introduce the function

$$
\phi_n(z) = \begin{cases} 1 & \text{if} \quad z \geq n+1 \\ z-k & \text{if} \quad n \leq z < n+1 \\ 0 & \text{if} \quad -n < z < n \\ z+k & \text{if} \quad -n-1 < z \leq -n \\ -1 & \text{if} \quad z \leq -n-1 \end{cases} \tag{3.14}
$$

and we set $\Psi_n(z) = \int_0^z \phi_n(\tau)d\tau$. We note that ϕ_n is Lipschitz function. Thus, we have $\phi_n(u_\varepsilon) \in L^2(0,T;H_0^1(\Omega))$, see [17], [18], with

$$
\nabla\phi_n(u_\varepsilon) = \chi_{B_n}\nabla u_\varepsilon,
$$

χ_{B_n} denoting the characteristic function of the set $B_n = \{(t,x) \in Q,\ n \le |u_\varepsilon(t,x)| \le n+1\}$. Then, taking $\phi = \phi_n(u_\varepsilon) \in L^2(0,T; H_0^1(\Omega))$ as test function in (3.4) gives

$$\begin{aligned}\frac{d}{dt}\int_\Omega \Psi_n(u_\varepsilon)dx \quad &+ \quad \int_\Omega \chi_{B_n} A\nabla u_\varepsilon \cdot \nabla u_\varepsilon dx + \int_\Omega g(u_\varepsilon, \nabla u_\varepsilon)\phi_n(u_\varepsilon)dx \\ &= \quad \int_\Omega f_\varepsilon \phi_n(u_\varepsilon)dx - \int_\Omega du_\varepsilon \phi_n(u_\varepsilon)dx - \int_\Omega b\cdot\nabla u_\varepsilon \phi_n(u_\varepsilon)dx.\end{aligned}$$

Thus, integrating the above equation with respect to t, we have

$$\begin{aligned}\int_\Omega \Psi_n(u_\varepsilon)(t)dx \quad &+ \quad \int_0^t\int_\Omega \chi_{B_n} A\nabla u_\varepsilon \cdot \nabla u_\varepsilon dxdt + \int_0^t\int_\Omega g(u_\varepsilon,\nabla u_\varepsilon)\phi_n(u_\varepsilon)dxdt \\ &= \quad \int_\Omega \Psi_n(u_{0,\varepsilon})dx + \int_0^t\int_\Omega f_\varepsilon\phi_n(u_\varepsilon)dxdt - \int_0^t\int_\Omega du_\varepsilon\phi_n(u_\varepsilon)dxdt \\ &- \quad \int_0^t\int_\Omega b\cdot\nabla u_\varepsilon\phi_n(u_\varepsilon)dxdt.\end{aligned} \tag{3.15}$$

Since $|\phi_n(z)| \le 1$, and taking into account the estimate (3.5) we have

$$\left|\int_\Omega du_\varepsilon\phi_n(u_\varepsilon)dx\right| \le \beta\|d\|_{L^\infty(Q)}. \tag{3.16}$$

Furthermore, we remark that $u_\varepsilon\phi_n(u_\varepsilon) \ge 0$. Then, the third term in the left side is non negative. From the coercivity of A (see (1.5)), the positivity of $\Psi_n(.)$ and (3.15) we deduce that

$$\begin{aligned}\underline{a}\int_{B_n}|\nabla u_\varepsilon|^2dxdt \quad &\le \quad \int_Q|\phi_n(u_\varepsilon)f_\varepsilon|dxdt + \int_\Omega\Psi_n(u_{0,\varepsilon})dx \\ &+ \quad \beta\|d\|_{L^\infty(Q)} + \|b\|_{L^\infty(Q)}\int_Q|\nabla u_\varepsilon||\phi_n(u_\varepsilon)|dxdt \\ &\le \quad \|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)} + \beta\|d\|_{L^\infty(Q)} + \|b\|_{L^\infty(Q)}\int_Q|\nabla u_\varepsilon||\phi_n(u_\varepsilon)|dxdt.\end{aligned} \tag{3.17}$$

Let us split the last integral in (3.17) as follows

$$\begin{aligned}\int_Q|\nabla u_\varepsilon||\phi_n(u_\varepsilon)|dxdt \quad &= \quad \int_{B_n}|\nabla u_\varepsilon||\phi_n(u_\varepsilon)|dxdt + \int_{E_n}|\nabla u_\varepsilon|dxdt \\ &\le \quad \int_{B_n}|\nabla u_\varepsilon|dxdt + \int_{E_n}|\nabla u_\varepsilon|dxdt\end{aligned} \tag{3.18}$$

since $|\phi_n(u_\varepsilon)| = 1$ on $E_n = \{(t,x) \in Q,\ |u(t,x)| > n+1\}$ and $\phi_n(u_\varepsilon) = 0$ if $|u_\varepsilon(t,x)| < n$. Using the fact $0 \le \Psi_n(z) \le |z|$ and (3.18), we deduce from (3.17) that

$$\begin{aligned}\underline{a}\int_{B_n}|\nabla u_\varepsilon|^2dxdt \quad &\le \quad \|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)} + \beta\|d\|_{L^\infty(Q)} \\ &+ \quad \|b\|_{L^\infty(Q)}\left(\int_{B_n}|\nabla u_\varepsilon|dxdt + \int_{E_n}|\nabla u_\varepsilon|dxdt\right).\end{aligned}$$

By using Holder's and Young's inequalities, we have

$$\underline{a}\int_{B_n}|\nabla u_\varepsilon|^2dxdt \le C + \frac{\underline{a}}{2}\int_{B_n}|\nabla u_\varepsilon|^2dxdt + \frac{1}{2\underline{a}}\|b\|^2_{L^\infty(Q)}T|\Omega| + \int_{E_n}|\nabla u_\varepsilon|dxdt,$$

where $C = \|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)} + \beta\|d\|_{L^\infty(Q)}$. This finishes the proof of (3.6) with C_0 and C_1 depending on $\|f\|_{L^1(Q)}$, $\|u_0\|_{L^1(Q)}$, $\|b\|_{L^\infty(Q)}$, $\|d\|_{L^\infty(Q)}$, $\underline{a}$, $|\Omega|$, T and the bound β. $\square$

Now, we are interested in the nonlinear term g_ε. We have

Lemma 2 *Suppose A, b, d, g satisfy (1.3)-(1.7) and let u_ε be a sequence of solutions of (3.4). Then, there exists $C > 0$ depending only on the data such that the sequence $g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)$ satisfies*

$$\sup_{\varepsilon>0} \|g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\|_{L^1(Q)} \le C, \tag{3.19}$$

$$\lim_{k\to\infty} \sup_{\varepsilon>0} \int_{|u_\varepsilon|>k} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| dxdt = 0. \tag{3.20}$$

Proof. It is clear that

$$\begin{aligned} \int_{|u_\varepsilon|>n+1} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| dxdt &= \int_{|u_\varepsilon|>n+1} \phi_n(u_\varepsilon) g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) dxdt \\ &\le \int_Q \phi_n(u_\varepsilon) g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) dxdt \end{aligned} \tag{3.21}$$

since we recall that $|\phi_n(z)| = 1$ when $|z| > n+1$ and $\phi_n(u_\varepsilon) g_\varepsilon$ is non negative by the sign condition (1.6). In the sequel, we will often write $g_\varepsilon = g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)$ when no confusion can arise. >From the positivity of the first and the second terms in (3.15), we obtain

$$\begin{aligned} 0 \le \int_Q \phi_n(u_\varepsilon) g_\varepsilon dxdt &\le \left|\int_Q \phi_n(u_\varepsilon) f_\varepsilon dxdt\right| + \left|\int_\Omega \Psi_n(u_{0,\varepsilon}) dx\right| \\ &+ \left|\int_Q d\phi_n(u_\varepsilon) u_\varepsilon dxdt\right| + \left|\int_Q \phi_n(u_\varepsilon) b \cdot \nabla u_\varepsilon dxdt\right| \\ &\le \|f_\varepsilon\|_{L^1(Q)} + \|u_{0,\varepsilon}\|_{L^1(\Omega)} + \|d\|_{L^\infty(Q)} \int_Q |u_\varepsilon| dxdt \\ &+ \|b\|_{L^\infty(Q)} \int_Q |\nabla u_\varepsilon| dxdt \end{aligned} \tag{3.22}$$

since $|\phi_n(z)| \le 1$ and $0 \le \Psi_n(z) \le |z|$. By (3.2), (3.5) and the estimate (3.7) with $q = 1$, we deduce

$$\int_{|u_\varepsilon|>n+1} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| dxdt \le C. \tag{3.23}$$

It remains to evaluate the integral over $\{|u_\varepsilon| < n+1\}$. Assumption (1.6) yields

$$\int_{|u_\varepsilon|<n+1} |g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon)| dxdt \le h(n+1) \int_{|u_\varepsilon|<n+1} \left(|\nabla u_\varepsilon|^\sigma + \gamma(t,x)\right) dxdt \tag{3.24}$$

where we estimate as follows

$$\int_{|u_\varepsilon|<n+1} |\nabla u_\varepsilon|^\sigma dxdt = \sum_{j=0}^n \int_{B_j} |\nabla u_\varepsilon|^\sigma dxdt \le \sum_{j=0}^n |B_j|^{1-\frac{\sigma}{2}} \left(\int_{B_j} |\nabla u_\varepsilon|^2 dxdt\right)^{\frac{\sigma}{2}}. \tag{3.25}$$

By using (3.6), (3.7) and Holder's inequality, we get

$$\int_{|u_\varepsilon|<n+1} |\nabla u_\varepsilon|^\sigma dxdt \le (T|\Omega|)^{1-\frac{\sigma}{2}} \sum_{j=0}^n \left(C_0 + C_1 \int_Q |\nabla u_\varepsilon| dxdt\right)^{\frac{\sigma}{2}} \le C. \tag{3.26}$$

Combining (3.26) with (3.24) and (3.23), we conclude that g_ε is bounded in $L^1(Q)$ uniformly in ε.

We turn to the proof of (3.20). Obviously, one has

$$\int_{|u_\varepsilon|>k} |g_\varepsilon| dxdt \le \frac{1}{k} \int_Q T_k(u_\varepsilon) g_\varepsilon dxdt. \tag{3.27}$$

Similarly, replacing $\phi_n(u_\varepsilon)$ by $T_k(u_\varepsilon)$ in (3.15) we obtain, similarly to (3.22), that

$$\begin{aligned} 0 \le \int_Q T_k(u_\varepsilon) g_\varepsilon dxdt \;&\le\; \int_Q |f_\varepsilon T_k(u_\varepsilon)| dxdt + \int_\Omega S_k(u_{0,\varepsilon}) dx \\ &+\; \|d\|_{L^\infty(Q)} \int_Q T_k(u_\varepsilon) u_\varepsilon dxdt + \|b\|_{L^\infty(Q)} \int_Q |T_k(u_\varepsilon)| |\nabla u_\varepsilon| dxdt \end{aligned} \tag{3.28}$$

holds. Let $M > 0$. According to [14], we use the following trick

$$\begin{cases} 0 \le S_k(z) \le M^2 + k|z|\chi_{|z|>M}, \\ |T_k(z)| \le M + k\chi_{|z|>M} \end{cases} \tag{3.29}$$

which gives

$$\begin{aligned} \int_{|u_\varepsilon|>k} |g_\varepsilon| dxdt \;&\le\; \tfrac{M}{k} \|f_\varepsilon\|_{L^1(Q)} + \int_{|u_\varepsilon|>M} |f_\varepsilon| dxdt \\ &+\; \tfrac{M^2}{k} \|u_{0,\varepsilon}\|_{L^1(\Omega)} + \int_{|u_{0,\varepsilon}|>M} |u_{0,\varepsilon}| dx \\ &+\; \tfrac{M}{k} \|d\|_{L^\infty(Q)} \|u_\varepsilon\|_{L^1(Q)} + \int_{|u_\varepsilon|>M} |u_\varepsilon| dxdt \\ &+\; \tfrac{M}{k} \|b\|_{L^\infty(Q)} \|\nabla u_\varepsilon\|_{L^1(Q)} + \int_{|u_\varepsilon|>M} |\nabla u_\varepsilon| dxdt, \end{aligned} \tag{3.30}$$

by (3.28) and (3.29). Since, on the one hand, u_ε is bounded in $L^q(0, T, W_0^{1,q}(\Omega))$ for some $q > 1$ and $f_\varepsilon, u_{0,\varepsilon}$ are convergent sequences in $L^1(Q), L^1(\Omega)$ respectively, and, on the other hand,

$$\sup_{\varepsilon>0} meas\{(t,x) \in Q,\ |u_\varepsilon(t,x)| > M\} \le \frac{1}{M} \sup_{\varepsilon>0} \|u_\varepsilon\|_{L^1(Q)} \le \frac{\beta T}{M}$$

tends to 0 as M goes to ∞, we can choose M large enough so that the terms

$$\sup_{\varepsilon>0} \int_{|u_\varepsilon|>M} |f_\varepsilon| dxdt,\ \sup_{\varepsilon>0} \int_{|u_{0,\varepsilon}|>M} |u_{0,\varepsilon}| dx,\ \sup_{\varepsilon>0} \int_{|u_\varepsilon|>M} |u_\varepsilon| dxdt,\ \sup_{\varepsilon>0} \int_{|u_\varepsilon|>M} |\nabla u_\varepsilon| dxdt$$

are arbitrarily smalls, which, achieves the proof of (3.20). □

Let the assumptions of Proposition 1 be fulfilled. Then, u_ε is bounded in $L^q(0, T; W_0^{1,q}(\Omega))$, g_ε is bounded in $L^1(Q)$ which imply, in view of the equation satisfied by u_ε that $\partial_t u_\varepsilon$ is bounded in $L^1(0, T; W^{-1,q}(\Omega)) + L^1(Q)$. Therefore, possibly at the cost of extracting subsequences, see e.g. [16], [18], we can assume that

$$\begin{cases} u_\varepsilon \longrightarrow u & \textit{strongly in } L^q(Q) \textit{ and a.e.in } Q, \\ |u_\varepsilon(t,x)| \le \Gamma(t,x) & \textit{a.e.in } Q,\ \textit{with } \Gamma \in L^q(Q), \\ \nabla u_\varepsilon \rightharpoonup \nabla u & \textit{weakly in } L^q(Q). \end{cases} \tag{3.31}$$

3.3 Convergence almost everywhere of the gradients

The weak convergence of the gradients is clearly insufficient to pass to the limit when $\varepsilon \to 0$ in nonlinear terms. Then, we claim

Lemma 3 *Let the assumptions of Proposition 1 be fulfilled and let u_ε satisfy (3.31). Then, the sequence $(\nabla u_\varepsilon)_\varepsilon$ converges to ∇u almost everywhere as ε goes to zero.*

Proof. It suffices to show that $(\nabla u_\varepsilon)_\varepsilon$ is a Cauchy sequence in measure, see [8], i.e. for all $\mu > 0$

(3.32)
$$meas\{(t,x) \in Q,\ |\nabla u_{\varepsilon'} - \nabla u_\varepsilon| > \mu\} \longrightarrow 0$$

as $\varepsilon', \varepsilon \to 0$. Let us denote by $\mathcal{A}$ the subset of Q involved in (3.32). Let $k > 0$ and $\delta > 0$. Following [15], we remark that

$$\mathcal{A} \subset \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4 \tag{3.33}$$

where

$$\left\{\begin{array}{l} \mathcal{A}_1 = \{(t,x) \in Q,\ |\nabla u_\varepsilon| \geq k\}, \\ \mathcal{A}_2 = \{(t,x) \in Q,\ |\nabla u_{\varepsilon'}| \geq k\}, \\ \mathcal{A}_3 = \{(t,x) \in Q,\ |u_\varepsilon - u_{\varepsilon'}| \geq \delta\}, \\ \mathcal{A}_4 = \{(t,x) \in Q,\ |\nabla u_\varepsilon - \nabla u_{\varepsilon'}| \geq \mu,\ |\nabla u_\varepsilon| \leq k,\ |\nabla u_{\varepsilon'}| \leq k, |u_\varepsilon - u_{\varepsilon'}| \leq \delta\}. \end{array}\right. \tag{3.34}$$

By Corollary 1 and (3.31), we conclude easily for the three first sets. Indeed, one has

$$|\mathcal{A}_1| \leq \frac{1}{k}\|\nabla u_\varepsilon\|_{L^1(Q)} \leq \frac{C}{k}$$

and an analoguous estimate holds for $\mathcal{A}_2$. Hence, by choosing k large enough, $|\mathcal{A}_1| + |\mathcal{A}_2|$ is arbitrarily small. Similarly, one gets

$$|\mathcal{A}_3| \leq \frac{1}{\delta}\|u_\varepsilon - u_{\varepsilon'}\|_{L^1(Q)}$$

which, for $\delta > 0$ fixed, tends to 0 when $\varepsilon, \varepsilon' \to 0$ since, by (3.31), u_ε is a Cauchy sequence in $L^1(Q)$. Then, the proof is completed by choosing δ so that $|\mathcal{A}_4|$ is given arbitrarily small, uniformly with respect to $\varepsilon, \varepsilon'$. To this end, we shall use the equations satisfied by u_ε and $u_{\varepsilon'}$. Indeed, we observe that

$$\begin{array}{rcl} |\mathcal{A}_4| & \leq & \frac{1}{\mu^2}\displaystyle\int_{\mathcal{A}_4} |\nabla u_\varepsilon - \nabla u_{\varepsilon'}|^2 dxdt \\ & \leq & \frac{1}{\mu^2}\displaystyle\int_{|u_\varepsilon - u_{\varepsilon'}| \leq \delta} |\nabla u_\varepsilon - \nabla u_{\varepsilon'}|^2 dxdt = \frac{1}{\mu^2}\int_Q |\nabla(T_\delta(u_\varepsilon - u_{\varepsilon'}))|^2 dxdt. \end{array} \tag{3.35}$$

Substracting the relations obtained with $\phi = T_\delta(u_\varepsilon - u_{\varepsilon'})$ as test funtion in equation (3.4) satisfied successively by u_ε and $u_{\varepsilon'}$ leads to

$$\begin{array}{rcl} \displaystyle\frac{d}{dt}\int_\Omega S_\delta(u_\varepsilon - u_{\varepsilon'})dx & + & \displaystyle\int_\Omega A(\nabla u_\varepsilon - \nabla u_{\varepsilon'})\nabla T_\delta(u_\varepsilon - u_{\varepsilon'})dx = \int_\Omega (f_\varepsilon - f_{\varepsilon'})T_\delta(u_\varepsilon - u_{\varepsilon'})dx \\ & - & \displaystyle\int_\Omega \left(d(u_\varepsilon - u_{\varepsilon'}) - b\nabla(u_\varepsilon - u_{\varepsilon'})\right) T_\delta(u_\varepsilon - u_{\varepsilon'})dx \\ & - & \displaystyle\int_\Omega (g_\varepsilon - g_{\varepsilon'})T_\delta(u_\varepsilon - u_{\varepsilon'})dx. \end{array}$$

(3.36)

Since $|T_\delta(z)| \le \delta$ and $0 \le S_\delta(z) \le \delta|z|$, integrating (3.36) with respect to t and using the coercivity of A (see (1.5)) yield

$$\begin{aligned}\underline{a}\int_Q |\nabla(T_\delta(u_\varepsilon - u_{\varepsilon'}))|^2 dxdt &\le \delta\Big(\|f_\varepsilon - f_{\varepsilon'}\|_{L^1(Q)} + \|u_{0,\varepsilon} - u_{0,\varepsilon'}\|_{L^1(\Omega)} \\ &+ \|d\|_{L^\infty(Q)}\|u_\varepsilon - u_{\varepsilon'}\|_{L^1(Q)} + \|b\|_{L^\infty(Q)}\|\nabla(u_\varepsilon - u_{\varepsilon'})\|_{L^1(Q)} \\ &+ \|g_\varepsilon - g_{\varepsilon'}\|_{L^1(Q)}\Big). \end{aligned} \tag{3.37}$$

Therefore, by using (3.2) and the bounds (3.7), uniform in ε, on $\|u_\varepsilon\|_{L^1(Q)}$, $\|\nabla u_\varepsilon\|_{L^1(Q)}$ and on $\|g_\varepsilon\|_{L^1(Q)}$, we deduce from (3.37) that

$$\begin{aligned}\underline{a}\int_Q |\nabla(T_\delta(u_\varepsilon - u_{\varepsilon'}))|^2 dxdt &\le 2\delta\left(\|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\right) \\ &+ 2\delta C\left(1 + \|b\|_{L^\infty(Q)} + \|d\|_{L^\infty(Q)}\right), \end{aligned} \tag{3.38}$$

goes to zero as δ goes to zero, uniformly in $\varepsilon, \varepsilon'$. This completes the proof of Lemma 3. □

Having disposed of the proof of Lemma 3, let us consider the behaviour of g_ε as ε goes to 0, when it is assumed that $0 \le \sigma < 2$.

Corollary 2 *Let the assumptions of Proposition 1 be fulfilled and let u_ε satisfy (3.31). Then, (up to subsequences) the sequence $(g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon))_\varepsilon$ converges to $g(u, \nabla u)$ almost everywhere in Q and strongly in $L^1(Q)$.*

Proof. This result is similar to those obtained in [9] in the context of elliptic problems. For the sake of completeness, we sketch the proof. By combining Lemma 3 and (3.31), it is clear that

$$g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) \longrightarrow g(u, \nabla u)$$

a.e. in Q as ε tends to 0, since $g(t, x, \lambda, \xi)$ is a continuous function with respect to $\lambda \in \mathbb{R}, \xi \in \mathbb{R}^N$. Thus, by classical results, see e.g. [8], the sequence g_ε will be actually strongly convergent in $L^1(Q)$ if one shows that g_ε lies in weakly compact set in $L^1(Q)$. This property follows from (3.20) since $0 \le \sigma < 2$. Indeed, let $\mathcal{A}$ be a measurable set in Q. We split

$$\int_{\mathcal{A}} |g_\varepsilon| dxdt = \int_{\mathcal{A}\cap\{|u_\varepsilon|\le k\}} |g_\varepsilon| dxdt + \int_{\mathcal{A}\cap\{|u_\varepsilon|>k\}} |g_\varepsilon| dxdt \tag{3.39}$$

where it is clear that

$$\int_{\mathcal{A}\cap\{|u_\varepsilon|>k\}} |g_\varepsilon| dxdt \le \int_{|u_\varepsilon|>k} |g_\varepsilon| dxdt$$

tends to 0, uniformly in ε as $k \to \infty$, by (3.20). Moreover, the growth condition (1.6) and Holder's inequality yield

$$\begin{aligned}\int_{\mathcal{A}\cap\{|u_\varepsilon|\le k\}} |g_\varepsilon| dxdt &\le h(k)\left(\int_{\mathcal{A}\cap\{|u_\varepsilon|\le k\}} |\nabla u_\varepsilon|^\sigma dxdt + \int_{\mathcal{A}\cap\{|u_\varepsilon|\le k\}} \gamma(t,x) dxdt\right) \\ &\le h(k)\left(\int_{|u_\varepsilon|\le k} |\nabla u_\varepsilon|^2 dxdt\right)^{\frac{\sigma}{2}} |\mathcal{A}|^{\frac{2-\sigma}{2}} + h(k)\int_{\mathcal{A}} \gamma(t,x) dxdt \\ &\le h(k) C_k |\mathcal{A}|^{\frac{2-\sigma}{2}} + h(k)\int_{\mathcal{A}} \gamma(t,x) dxdt \end{aligned} \tag{3.40}$$

by using (3.6) and (3.7) as in (3.26). Since $\sigma < 2$ and $\gamma \in L^1(Q)$, the right hand side of this last inequality goes to 0 as $|\mathcal{A}| \to 0$. We conclude that

$$\lim_{|\mathcal{A}|\to 0} \sup_{\varepsilon>0} \int_{\mathcal{A}} |g_\varepsilon| dxdt = 0$$

which completes the proof of Corollary 2. □

3.4 Cauchy property in $C^0(0,T;L^1(\Omega))$ and passage to the limit

We end our review of the properties of the sequence u_ε with the following result.

Lemma 4 *Let the assumptions of Proposition 1 be fulfilled. We assume that the sequence $(u_\varepsilon)_\varepsilon$ satisfies (3.31). Then, $(u_\varepsilon)_\varepsilon$ is a Cauchy sequence in $C^0(0,T;L^1(\Omega))$.*

Proof. We set $w_{\varepsilon,\varepsilon'} = u_\varepsilon - u_{\varepsilon'}$, $F_{\varepsilon,\varepsilon'} = f_\varepsilon - f_{\varepsilon'}$ and $G_{\varepsilon,\varepsilon'} = g_\varepsilon - g_{\varepsilon'}$. We multiply the equations (3.4) satisfied respectively by u_ε and $u_{\varepsilon'}$ by $T_1(w_{\varepsilon,\varepsilon'})$. Substracting the obtained relations yields

$$(3.41)\quad \begin{aligned} \frac{d}{dt}\int_\Omega S_1(w_{\varepsilon,\varepsilon'})dx \quad &+ \quad \int_{|w_{\varepsilon,\varepsilon'}|\le 1} A\nabla w_{\varepsilon,\varepsilon'}\cdot\nabla w_{\varepsilon,\varepsilon'}dx = \int_\Omega F_{\varepsilon,\varepsilon'}T_1(w_{\varepsilon,\varepsilon'})dx \\ &- \quad \int_\Omega G_{\varepsilon,\varepsilon'}T_1(w_{\varepsilon,\varepsilon'})dx - \int_\Omega (b\cdot\nabla w_{\varepsilon,\varepsilon'} + dw_{\varepsilon,\varepsilon'})T_1(w_{\varepsilon,\varepsilon'})dx. \end{aligned}$$

Since $0 \le zT_1(z) = z^2\chi_{_{|z|\le 1}} + |z|\chi_{_{|z|>1}} \le z^2\chi_{_{|z|\le 1}} + (2|z|-1)\chi_{_{|z|>1}} = 2S_1(w_{\varepsilon,\varepsilon'})$, one gets

$$\left|\int_\Omega dw_{\varepsilon,\varepsilon'}T_1(w_{\varepsilon,\varepsilon'})dx\right| \le 2\|d\|_{L^\infty(Q)}\int_\Omega S_1(w_{\varepsilon,\varepsilon'})dx.$$

Moreover, one has $|T_1(w_{\varepsilon,\varepsilon'})| \le 1$. Then, integrating (3.41) between 0 and t and from the positivity of A, it follows

$$(3.42)\quad \begin{aligned} \int_\Omega S_1(w_{\varepsilon,\varepsilon'})(t)dx \quad &\le \quad \int_\Omega S_1(w^0_{\varepsilon,\varepsilon'})dx + \int_0^t\int_\Omega |F_{\varepsilon,\varepsilon'}|dxds + \int_0^t\int_\Omega |G_{\varepsilon,\varepsilon'}|dxds \\ &+ \quad \|b\|_{L^\infty(Q)}\int_0^t\int_\Omega |\nabla w_{\varepsilon,\varepsilon'}|dxds + 2\|d\|_{L^\infty(Q)}\int_0^t\int_\Omega S_1(w_{\varepsilon,\varepsilon'})dxds \end{aligned}$$

where $w^0_{\varepsilon,\varepsilon'} = u_{0,\varepsilon} - u_{0,\varepsilon'}$. Hence, Gronwall's lemma implies that

$$(3.43)\qquad \int_\Omega S_1(w_{\varepsilon,\varepsilon'})dx \le a_{\varepsilon,\varepsilon'}$$

where $a_{\varepsilon,\varepsilon'}$ stands for

$$\begin{aligned} a_{\varepsilon,\varepsilon'} \quad &= \quad e^{CT}\left(\int_\Omega S_1(w^0_{\varepsilon,\varepsilon'})dx + \int_Q |F_{\varepsilon,\varepsilon'}|dxdt + \int_Q |G_{\varepsilon,\varepsilon'}|dxdt + \int_Q |\nabla w_{\varepsilon,\varepsilon'}|dxdt\right) \\ &\le \quad e^{CT}\left(\|u_{0,\varepsilon} - u_{0,\varepsilon'}\|_{L^1(\Omega)} + \|f_\varepsilon - f_{\varepsilon'}\|_{L^1(Q)} + \|g_\varepsilon - g_{\varepsilon'}\|_{L^1(Q)} + \|\nabla u_\varepsilon - \nabla u_{\varepsilon'}\|_{L^1(Q)}\right), \end{aligned}$$

since $S_1(z) \le |z|$. By (3.1), $u_{0,\varepsilon}$ and f_ε are convergent sequences in $L^1(\Omega)$ and $L^1(Q)$, respectively and by Corollary 2, g_ε is a convergent sequence in $L^1(Q)$. Furthermore, by

Corollary 1 and Lemma 3, ∇u_ε is both bounded in $L^q(Q)$ and a.e. in Q convergent, which implies that ∇u_ε is actually strongly convergent in $L^p(Q)$ for $1 \le p < q$, and in particular in $L^1(Q)$. Hence, it is clear that $a_{\varepsilon,\varepsilon'}$ tends to 0 as $\varepsilon, \varepsilon' \to 0$. Finally, by Holder's inequality, we have

$$\begin{aligned}\int_\Omega |w_{\varepsilon,\varepsilon'}|dx &= \int_{|w_{\varepsilon,\varepsilon'}|\le 1} |w_{\varepsilon,\varepsilon'}|dx + \int_{|w_{\varepsilon,\varepsilon'}|>1} |w_{\varepsilon,\varepsilon'}|dx \\ &\le \left(\int_{|w_{\varepsilon,\varepsilon'}|\le 1} |w_{\varepsilon,\varepsilon'}|^2 dx\right)^{\frac{1}{2}} \left(\int_{|w_{\varepsilon,\varepsilon'}|\le 1} 1dx\right)^{\frac{1}{2}} + \int_{|w_{\varepsilon,\varepsilon'}|>1} |w_{\varepsilon,\varepsilon'}|dx \\ &\le \sqrt{|\Omega|}\left(\int_{|w_{\varepsilon,\varepsilon'}|\le 1} 2S_1(w_{\varepsilon,\varepsilon'})dx\right)^{\frac{1}{2}} + \int_{|w_{\varepsilon,\varepsilon'}|>1} 2S_1(w_{\varepsilon,\varepsilon'})dx,\end{aligned}$$

since $\frac{|z|}{2}\chi_{|z|>1} \le (\frac{|z|}{2} + \frac{|z|-1}{2})\chi_{|z|>1} = S_1(w_{\varepsilon,\varepsilon'})\chi_{|z|>1}$ and $\frac{|z|^2}{2}\chi_{|z|\le 1} = S_1(z)\chi_{|z|\le 1}$. By (3.43), we deduce that

$$\int_\Omega |u_\varepsilon - u_{\varepsilon'}|dx = \int_\Omega |w_{\varepsilon,\varepsilon'}|dx \le \sqrt{2|\Omega|}\sqrt{a_{\varepsilon,\varepsilon'}} + 2a_{\varepsilon,\varepsilon'}$$

tends to 0 as $\varepsilon, \varepsilon' \to 0$ which proves that u_ε is a Cauchy sequence in $C^0(0,T;L^1(Q))$. $\square$

Finally, we achieve the proof of Theorem 1 by passing easily to the limit $\varepsilon \to 0$ in the following weak formulation

$$\begin{aligned}\int_\Omega u_\varepsilon \phi(t)dx - \int_\Omega u_{0,\varepsilon}\phi(0,x)dx &- \int_0^t\int_\Omega u_\varepsilon \partial_t \phi dxdt + \int_0^t\int_\Omega A\nabla u_\varepsilon \cdot \nabla\phi dxdt \\ &+ \int_0^t\int_\Omega (b\cdot\nabla u_\varepsilon + du_\varepsilon + g_\varepsilon(u_\varepsilon, \nabla u_\varepsilon))\,\phi dxdt \\ &= \int_0^t\int_\Omega f_\varepsilon \phi dxdt\end{aligned} \tag{3.44}$$

with $\phi \in C^0(0,T,W_0^{1,q'}(\Omega)) \cap C^1(0,T,L^{q'}(\Omega))$, obtaining in this way that the limit u is a solution of (1.1) in the sense of (1.9).

Remark 1 *We point out the fact that the assumption on the derivatives of the coefficient b is useful uniquely to obtain the uniform bound (3.5) in $L^\infty(0,T;L^1(\Omega))$.*

Remark 2 *A similar existence result may be obtained if the strong convergences in (3.1) are replaced by weak L^1 convergences.*

4 Entropic solutions : End of proof of Theorem 2

In this Section, we assume $g = 0$. First, we prove that, besides the weak "natural" formulation (1.9), the limit u of the sequence of approximate solutions u_ε also satisfies the entropic relation (1.11). Having disposed of the existence of such a solution, we show that u is unique in the class of entropic solutions.

4.1 Existence of entropic solution

Let us recall the convergence properties obtained in Section 3 on the sequence u_ε, after suitable extraction of subsequences. First, u_ε converges to u strongly in $L^q(Q)$, with $1 \le q < \frac{N+2}{N+1}$, in $C^0(0,T;L^1(\Omega))$, a.e. in Q and is dominated. Moreover, ∇u_ε is bounded in $L^q(Q)$ and converges a.e. in Q to ∇u ; thus, the convergence actually holds strongly in $L^p(Q)$, for $1 \le p < q$ and in particular in $L^1(Q)$. We can also assume that ∇u_ε is dominated. Let $k > 0$. Since T_k is continuous and bounded by k, $T_k(u_\varepsilon)$ converges a.e. in Q and, by Lebesgue's theorem, strongly in $L^2(Q)$ to $T_k(u)$. Furthermore, from (3.10) it is easy to see that $\nabla T_k(u_\varepsilon)$ is bounded in $L^2(Q)$ (uniformly in ε, the bound depending on k). Therefore, we may suppose that $\nabla T_k(u_\varepsilon) \rightharpoonup \nabla T_k(u)$ weakly in $L^2(Q)$.

Fix $k > 0$ and let $\psi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$ with $\partial_t \psi \in L^2(0,T;H^{-1}(\Omega))$. We set $P = \|\psi\|_{L^\infty(Q)}$. It is clear that $|T_k(u_\varepsilon - \psi)| \le k$ and

$$|\nabla T_k(u_\varepsilon - \psi)| = \chi_{|u_\varepsilon - \psi| \le k} |\nabla (u_\varepsilon - \psi)| \le \chi_{|u_\varepsilon| \le k+P} |\nabla u_\varepsilon| + |\nabla \psi| \tag{4.1}$$

which implies that $T_k(u_\varepsilon - \psi)$ belongs to (a bounded set in) $L^2(0,T;H_0^1(\Omega))$. Then, plugging $\phi = T_k(u_\varepsilon - \psi)$ in (3.4) gives

$$\begin{aligned}\int_\Omega S_k(u_\varepsilon - \psi)(T)dx \quad &- \int_\Omega S_k(u_{0,\varepsilon} - \psi(0,\cdot))dx + \int_0^T \langle \partial_t \psi, T_k(u_\varepsilon - \psi)\rangle ds \\ &+ \int_Q A\nabla u_\varepsilon \nabla T_k(u_\varepsilon - \psi)dxds + \int_Q (b\cdot \nabla u_\varepsilon + du_\varepsilon)T_k(u_\varepsilon - \psi)dxds \\ &= \int_Q f_\varepsilon T_k(u_\varepsilon - \psi)dxds.\end{aligned} \tag{4.2}$$

We shall study the behaviour of (4.2) when we let ε go to 0. Since S_k is k-Lipschit, one has

$$\left| \int_\Omega S_k(u_\varepsilon - \psi) - S_k(u - \psi)dx \right| \le k \int_\Omega |u_\varepsilon - u| dx$$

for all $t \in [0,T]$ where the right hand side tends to 0 as $\varepsilon \to 0$. Next, since we have assumed that $\partial_t \psi$ lies in $L^2(0,T;H^{-1}(\Omega))$, we have to prove that

$$T_k(u_\varepsilon - \psi) \rightharpoonup T_k(u - \psi) \quad \textit{in } L^2(0,T;H_0^1(\Omega)). \tag{4.3}$$

Obviously, this convergence holds in $L^2(Q)$ since u_ε converges to u a.e. in Q and T_k is continuous and bounded by k. Derivating $T_k(u_\varepsilon - \psi)$ leads to

$$\nabla T_k(u_\varepsilon - \psi) = \nabla T_k(T_{k+P}(u_\varepsilon) - \psi) = \chi_{|T_{k+P}(u_\varepsilon) - \psi| \le k} (\nabla T_{k+P}(u_\varepsilon) - \nabla \psi) \tag{4.4}$$

where, by the above mentioned convergences, $\nabla T_{k+P}(u_\varepsilon)$ converges weakly in $L^2(Q)$ to $\nabla T_{k+P}(u)$ which proves (4.3). We also deal easily with the terms involving b, d and f_ε since it appears in these integrals a product of the sequence $T_k(u_\varepsilon - \psi)$ which converges a.e. in Q and is bounded in $L^\infty(Q)$ with a sequence which converges at least weakly in $L^1(Q)$. Finally, it remains to show that

$$\int_Q A\nabla u \cdot \nabla T_k(u - \psi)dxds \le \liminf_{\varepsilon \to 0} \int_Q A\nabla u_\varepsilon \cdot \nabla T_k(u_\varepsilon - \psi)dxds. \tag{4.5}$$

By using (4.4), we split the integral in the right hand side as follows

$$\begin{aligned}\int_Q A\nabla u_\varepsilon \cdot \nabla T_k(u_\varepsilon - \psi)dxds &= \int_Q \chi_{|T_{k+P}(u_\varepsilon)-\psi|\leq k} A\nabla u_\varepsilon \cdot \nabla T_{k+P}(u_\varepsilon)dxds \\ &- \int_Q \chi_{|T_{k+P}(u_\varepsilon)-\psi|\leq k} A\nabla u_\varepsilon \cdot \nabla\psi dxds = A_\varepsilon - B_\varepsilon,\end{aligned}$$

where, by the same argument as above, we have

$$\lim_{\varepsilon\to 0} B_\varepsilon = \int_Q \chi_{|T_{k+P}(u)-\psi|\leq k} A\nabla u \cdot \nabla\psi dxds.$$

Therefore (4.5) is a consequence of Fatou's lemma, applied by combining (4.3) with T_{k+P} and the positiveness property (1.5). Finally, letting $\varepsilon \to 0$ in (4.2), one gets (1.11). □

4.2 Uniqueness

Let v be an entropic solution. To obtain the uniqueness, we will show that $v = u$, u still being the solution obtained by approximation. To this end, it would be natural to choose $T_h(u_\varepsilon)$ as test function ψ in (1.11). However, as pointed out in [15], T_h is not regular enough which leads to difficulties in order to write the term involving the time derivative of the test function. Then, it is necessary to regularize the truncation. Let $\nu > 0$. We introduce $T_h^\nu \in C^2(\mathbb{R}, \mathbb{R})$ satisfying

$$\text{(4.6)} \qquad \begin{cases} (T_h^\nu)'(z) = 0 & if \quad |z| \geq h, \\ (T_h^\nu)'(z) = 1 & if \quad |z| \leq h-\nu, \\ 0 \leq (T_h^\nu)'(z) \leq (T_h)'(z) \leq 1. \end{cases}$$

Note that $|T_h^\nu(z)| \leq |T_h(z)|$, and $(T_h^\nu)"(z) = 0$ when $|z| \geq h$ or $|z| \leq h-\nu$.
In the sequel, let us denote

$$L(f,u) = f - b\cdot\nabla u - du.$$

We take $\psi = T_h^\nu(u_\varepsilon)$ as test function in the entropic formulation (1.11) satisfied by v, we have

$$\text{(4.7)} \qquad \begin{aligned}&\left[\int_\Omega S_k(v - T_h^\nu(u_\varepsilon))dx\right]_0^t + \int_0^t < \partial_t u_\varepsilon, (T_h^\nu)'(u_\varepsilon)T_k(v - T_h^\nu(u_\varepsilon)) > ds + \\ &\int_0^t\int_\Omega A\nabla v \nabla T_k(v - T_h^\nu(u_\varepsilon))dxds \leq \int_0^t\int_\Omega L(f,v)(T_h^\nu)'(u_\varepsilon)T_k(v - T_h^\nu(u_\varepsilon))dxds.\end{aligned}$$

By using (3.4), we write the term involving the time derivative of the test function as follows

$$\text{(4.8)} \qquad \int_0^t \langle \partial_t u_\varepsilon, \phi\rangle ds = \int_0^t\int_\Omega (L(f_\varepsilon, u_\varepsilon)\phi - A\nabla u_\varepsilon \nabla\phi)dxds$$

where $\phi = (T_h^\nu)'(u_\varepsilon)T_k(v - T_h^\nu(u_\varepsilon))$ and, consequently,

$$\nabla\phi = \nabla u_\varepsilon (T_h^\nu)"(u_\varepsilon)T_k(v - T_h^\nu(u_\varepsilon)) + (T_h^\nu)'(u_\varepsilon)\nabla(T_k(v - T_h^\nu(u_\varepsilon))).$$

By (4.8), the entropic formulation (4.7) is equivalent to

$$\text{(4.9)} \qquad \begin{aligned}\left[\int_\Omega S_k(v - T_h^\nu(u_\varepsilon))dx\right]_0^t &+ \int_0^t\int_\Omega A(\nabla v - (T_h^\nu)'(u_\varepsilon)\nabla u_\varepsilon)\nabla T_k(v - T_h^\nu(u_\varepsilon))dxds \\ &- \int_0^t\int_\Omega A\nabla u_\varepsilon \nabla u_\varepsilon (T_h^\nu)"(u_\varepsilon)T_k(v - T_h^\nu(u_\varepsilon))dxds \\ &\leq \int_0^t\int_\Omega (L(f,v) - L(f_\varepsilon, u_\varepsilon)(T_h^\nu)'(u_\varepsilon))T_k(v - T_h^\nu(u_\varepsilon))dxds.\end{aligned}$$

Now, according to [15], let successively $\nu \to 0$, $\varepsilon \to 0$ and $h \to \infty$. Difficulties only arise from the third integal in the left hand, denoted by I_ν which involves the second derivative of T_h^ν ; the remaining integrals being treated by using the Lebesgue theorem. Indeed, it is clear that $|S_k(v - T_h^\nu(u_\varepsilon))| \le k|v| + kh$, $|T_k(v - T_h^\nu(u_\varepsilon))| \le k$, $|(T_h^\nu)'(u_\varepsilon)T_k(v - T_h^\nu(u_\varepsilon))| \le k$, and $|\nabla T_k(v - T_h^\nu(u_\varepsilon))| \le (|\nabla T_{k+h}(v)| + |\nabla T_h(u_\varepsilon)|)$.

Next, we wish to obtain an estimate on I_ν. Following [15] (see also [2]), we define another C^2 function R_h^ν, satisfying for $z \ge 0$: $(R_h^\nu)'(z) = 1 - (T_h^\nu)'(z)$, $R_h^\nu(0) = 0$, $R_h^\nu(-z) = R_h^\nu(z)$. Take $(R_h^\nu)'(u_\varepsilon)$ as test function in (3.4). By using the positivity of R_h^ν and the fact that $(R_h^\nu)"(z) = |(T_h^\nu)"(z)|$, we obtain according to [15] the following estimate

$$|I_\nu| \le k \int_0^t \int_\Omega |L(f_\varepsilon, u_\varepsilon)| \chi_{|u_\varepsilon| > h-\nu} dxds + k \int_\Omega |u_{0,\varepsilon}| \chi_{|u_{0,\varepsilon}| > h-\nu} dx. \tag{4.10}$$

By Lebesgue's theorem, we can pass to the limit $\nu \to 0$ in the right hand side of (4.10), obtaining without difficulties

$$\limsup_{\nu \to 0} |I_\nu| \le k \int_0^t \int_\Omega |L(f_\varepsilon, u_\varepsilon)| \chi_{|u_\varepsilon| > h} dxds + k \int_\Omega |u_{0,\varepsilon}| \chi_{|u_{0,\varepsilon}| > h} dx.$$

Collecting these results, we get from (4.9) the following estimate

$$\begin{aligned} \left[\int_\Omega S_k(v - T_h(u_\varepsilon))dx\right]_0^t &+ \int_0^t \int_\Omega A\nabla(v - T_h(u_\varepsilon))\nabla T_k(v - T_h(u_\varepsilon))dxds \\ &\le \int_0^t \int_\Omega (L(f, v) - L(f_\varepsilon, u_\varepsilon)(T_h)'(u_\varepsilon))T_k(v - T_h(u_\varepsilon))dxds \\ &+ k \int_0^t \int_\Omega |L(f_\varepsilon, u_\varepsilon)| \chi_{|u_\varepsilon| > h} dxds + k \int_\Omega |u_{0,\varepsilon}| \chi_{|u_{0,\varepsilon}| > h} dx. \end{aligned} \tag{4.11}$$

The assumptions on the sequence of data and the properties of u_ε recalled above allow us to apply the Lebesgue theorem to pass to the limit as $\varepsilon \to 0$ the first term of the left hand side and as well as in the right hand side. In addition, the coersivity of A (see (1.5)) and the following almost everywhere convergence

$$\nabla T_k(v - T_h(u_\varepsilon)) = \chi_{|v - T_h(u_\varepsilon)| \le k}(\nabla v - \chi_{|u_\varepsilon| \le h} \nabla u_\varepsilon) \to \chi_{|v - T_h(u)| \le k}(\nabla v - \chi_{|u| \le h} \nabla u),$$

permit us to apply Fatou's lemma on the second term in the left hand side.

It remains to deal with $h \to \infty$ in the following relation

$$\begin{aligned} \left[\int_\Omega S_k(v - T_h(u))dx\right]_0^t &+ \int_0^t \int_\Omega A\nabla T_k(v - T_h(u))\nabla T_k(v - T_h(u))dxds \\ &\le \int_0^t \int_\Omega (L(f, v) - L(f, u)(T_h)'(u))T_k(v - T_h(u))dxds + k\mathcal{O}(h), \end{aligned} \tag{4.12}$$

where $\mathcal{O}(h)$ stands for

$$\int_0^t \int_\Omega \left(|f| + \|b\|_{L^\infty(Q)}|\nabla u| + \|d\|_{L^\infty(Q)}|u|\right) \chi_{|u| > h} dxds + \int_\Omega |u_0| \chi_{|u_0| > h} dx.$$

which goes to 0 as $h \longrightarrow \infty$ because $f, u, \nabla u$ belong to $L^1(Q)$.

We search for another expression of the integral in the right hand side of (4.12). We write, on the one hand,

$$
\begin{aligned}
L(f,v) &- L(f,u)(T_h)'(u) = f - b\cdot\nabla v - dv - (f - b\cdot\nabla u - du)(T_h)'(u) \\
&= L(f,u)(1-(T_h)'(u)) - b\cdot\nabla(v-u) - d(v-u)
\end{aligned} \tag{4.13}
$$

and, on the other hand

$$
\begin{aligned}
&\int_0^t\int_\Omega (b\cdot\nabla(v-u))T_k(v-T_h(u))dxds \\
= &-\int_0^t\int_\Omega (v-u)b\cdot\nabla T_k(v-T_h(u)) + (\nabla\cdot b)(v-u)T_k(v-T_h(u))dxds.
\end{aligned} \tag{4.14}
$$

By (4.13)-(4.14), inequality (4.12) becomes

$$
\begin{aligned}
\left[\int_\Omega S_k(v-T_h(u))dx\right]_0^t &+ \int_0^t\int_\Omega A\nabla T_k(v-T_h(u))\nabla T_k(v-T_h(u))dxds \\
&\leq \int_0^t\int_\Omega |(v-u)b\cdot\nabla T_k(v-T_h(u))|dxds \\
&+ \int_0^t\int_\Omega |(d-(\nabla\cdot b))(v-u)T_k(v-T_h(u))|dxds \\
&+ \int_0^t\int_\Omega |L(f,u)(1-(T_h)'(u))T_k(v-T_h(u))|dxds + k\mathcal{O}(h)
\end{aligned} \tag{4.15}
$$

We remark that $1-(T_h)'(u)$ tends to 0 as $h\to\infty$, and by Lebegue's theorem the third term of the right hand side can be included in the general expression $k\mathcal{O}(h)$ which tends to 0 as $h\longrightarrow\infty$.

Proceeding as in Section 4 leads to

$$
\begin{aligned}
\int_\Omega S_k(v-T_h(u))(t)dx &+ \frac{a}{2}\int_0^t\int_\Omega |\nabla T_k(v-T_h(u))|^2dxds \\
&\leq \int_\Omega S_k(v-T_h(u))(0)dx + \frac{1}{\underline{a}}\|b\|_{L^\infty(Q)}\int_0^t\int_\Omega \chi_{|v-T_h(u)|<k}|v-u|^2dxds \\
&+ (\|d\|_{L^\infty(Q)} + \|\nabla\cdot b\|_{L^\infty(Q)})\int_0^t\int_\Omega |(v-u)T_k(v-T_h(u))|dxds + k\mathcal{O}(h).
\end{aligned} \tag{4.16}
$$

In classical way, by Lebesgue's theorem and Fatou's lemma, letting h go to ∞, we are led to inequality (4.16) where $T_h(u)$ is replaced by u and the last term in the right hand side vanishes. By using $0\leq zT_k(z)\leq 2S_k(z)$, $0\leq z^2\chi_{|z|\leq k}\leq 2S_k(z)$, we deduce as in Section 2 that

$$
\begin{aligned}
\int_\Omega S_k(v-u)(t)dx &+ \tfrac{a}{2}\int_0^t\int_\Omega |\nabla T_k(v-u)|^2dxds \\
&\leq \int_\Omega S_k(v-u)(0)dx + C(b,d)\int_0^t\int_\Omega S_k(v-u)dxds
\end{aligned} \tag{4.17}
$$

where $C(b,d) = 2\|\nabla b\|_{L^\infty(Q)} + 2\|d\|_{L^\infty(Q)} + \frac{1}{\underline{a}}\|b\|_{L^\infty(Q)}$. Therefore, it suffices to apply Gronwall's lemma to deduce that

$$\int_\Omega S_k(v-u)(t)dx = 0$$

since $v_0 = u_0$, which gives $v = u$. $\square$

5 Lower regularity requirement on b

Our aim in this section is to weaken the regularity requirement on b, replacing the $L^\infty(Q)$ condition by $b \in L^s(Q)$ for $s > q'$; precisely one has

Theorem 3 *Let A, d, g satisfy (1.3)-(1.5) and let $b \in L^s(Q)$ with $s > q' = \frac{q}{q-1}$ (recall that $1 \le q < \frac{N+2}{N+1}$) and $\nabla \cdot b \in L^\infty(Q)$. Then, there exists a weak solution of (1.1) in the sense of Definition 1.*

Proof. The outline of the proof is the same of Theorem 1. Consider the approximate solution of (3.4). In the first step, we show, according to (3.5), that

$$u_\varepsilon \text{ is uniformly bounded in } L^\infty(0,T;L^1(\Omega)). \tag{5.1}$$

Reproducing the proof of Proposition 1, we take $\phi = T_k(u_\varepsilon)$ as test function in (3.4), and we find (3.8). All terms are treated as above except those involving $u_\varepsilon b \cdot \nabla T_k(u_\varepsilon)$ which becomes

$$\left| \int_Q u_\varepsilon b \cdot \nabla T_k(u_\varepsilon) dxdt \right| \le \frac{\underline{a}}{2} \int_Q |\nabla T_k(u_\varepsilon)|^2 dxdt + \frac{1}{2\underline{a}} \int_Q \chi_{|u_\varepsilon| \le k} |bu_\varepsilon|^2 dxdt,$$

by using Holder's and Young's inequalities. Since $s > 2$ the last integral is bounded by $\frac{1}{2\underline{a}} k^2 (T|\Omega|)^{\frac{s}{s-2}} \|b\|^2_{L^s(Q)}$. Then, from (3.12), we deduce that

$$\int_\Omega S_k(u_\varepsilon)(t)dx \le \alpha_0 + \alpha_1 \int_0^t \int_\Omega S_k(u_\varepsilon) dxds$$

holds where α_0 depends on $\|f\|_{L^1(Q)}, \|u_0\|_{L^1(\Omega)}$ and $\|b\|_{L^s(Q)}$ and α_1 depends on $\|d\|_{L^\infty(Q)}, \|\nabla b\|_{L^\infty(Q)}$. Gronwall's Lemma permits us to conclude as in Proposition 1 and leads to (5.1).

To establish an estimate on the solutions in $L^q(0,T,W_0^{1,q}(\Omega))$, we follow step by step the proofs of estimate (3.6) and of Lemma 1 which need to be adapted. For that, take $\phi = \phi_n(u_\varepsilon)$ in (3.4). We deduce from (3.15)

$$\begin{aligned} \underline{a} \int_{B_n} |\nabla u_\varepsilon|^2 dxdt &\le \int_Q |\phi_n(u_\varepsilon) f_\varepsilon| dxdt + \int_\Omega \Psi_n(u_{0,\varepsilon}) dx \\ &+ \beta \|d\|_{L^\infty(Q)} + \int_{E_n} |b \cdot \nabla u_\varepsilon| |\phi_n(u_\varepsilon)| dxdt \\ &\le \|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)} + \beta \|d\|_{L^\infty(Q)} + \left(\int_{E_n} |b|^{q'} dxdt \right)^{\frac{1}{q'}} \left(\int_Q |\nabla u_\varepsilon|^q dxdt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $s > q'$, using Holder's inequality, with exponents $\frac{s}{q'}$ and $\frac{s}{s-q'}$, yields the following substitue to (3.6)

$$\int_{B_n} |\nabla u_\varepsilon|^2 dxdt \le C_0 + C_1 \|\nabla u_\varepsilon\|_{L^q(Q)} |E_n|^{\frac{s-q'}{sq'}}, \tag{5.2}$$

where C_0 stands for $\frac{1}{\underline{a}}(\|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)} + \beta \|d\|_{L^\infty(Q)})$ and $C_1 = \|b\|_{L^s(Q)}$.
Recall that $q < 2$. Therefore, Holder's inequality yields

$$\begin{aligned} \int_{B_n} |\nabla u_\varepsilon|^q dxdt &\le |B_n|^{\frac{2-q}{2}} \left(\int_{B_n} |\nabla u_\varepsilon|^2 dxdt \right)^{\frac{q}{2}} \\ &\le |B_n|^{\frac{2-q}{2}} \left(C_0^{\frac{q}{2}} + C_1^{\frac{q}{2}} \|\nabla u_\varepsilon\|^{\frac{q}{2}}_{L^q(Q)} |E_n|^{\frac{s-q'}{sq'} \frac{q}{2}} \right). \end{aligned} \tag{5.3}$$

Let r and K as in Lemma 1. By using (2.6), one gets

$$\begin{aligned}\int_{B_n}|\nabla u_\varepsilon|^q dxdt &\leq C_0^{\frac{q}{2}}\frac{1}{n^{r\frac{2-q}{2}}}\left(\int_{B_n}|u_\varepsilon|^r dxdt\right)^{\frac{2-q}{2}}\\ &+ C_1^{\frac{q}{2}}\|\nabla u_\varepsilon\|_{L^q(Q)}^{\frac{q}{2}}\|u_\varepsilon\|_{L^r(Q)}^{r\frac{s-q'}{sq'}\frac{q}{2}}\frac{1}{n^{r\frac{s-q'}{sq'}\frac{q}{2}+r\frac{2-q}{2}}}\left(\int_{B_n}|u_\varepsilon|^r dxdt\right)^{\frac{2-q}{2}}.\end{aligned}$$

Repeated use of Holder's inequality, as in (2.11), implies

$$\begin{aligned}\sum_{n=K+1}^{\infty}\int_{B_n}|\nabla u_\varepsilon|^q dxdt &\leq C_0^{\frac{q}{2}}\left(\sum_{n=K+1}^{\infty}\frac{1}{n^{r\frac{2-q}{q}}}\right)^{\frac{q}{2}}\|u_\varepsilon\|_{L^r(Q)}^{\frac{r(2-q)}{2}}\\ &+ C_1^{\frac{q}{2}}\|\nabla u_\varepsilon\|_{L^q(Q)}^{\frac{q}{2}}\|u_\varepsilon\|_{L^r(Q)}^{r(\frac{s-q'}{sq'}\frac{q}{2}+\frac{2-q}{2})}\left(\sum_{n=K+1}^{\infty}\frac{1}{n^{r(\frac{s-q'}{sq'}+\frac{2-q}{q})}}\right)^{\frac{q}{2}}.\end{aligned} \tag{5.4}$$

The conditions

$$r\frac{2-q}{q}>1, \text{ and } r(\frac{s-q'}{sq'}+\frac{2-q}{q})>1 \tag{5.5}$$

ensure the convergence of the series. As in Section 2, we deduce from (2.8), that

$$\|\nabla u_\varepsilon\|_{L^q(Q)}^q \leq C(K)+\delta(K)\left(\|u_\varepsilon\|_{L^r(Q)}^{\frac{r(2-q)}{2}}+\|\nabla u_\varepsilon\|_{L^q(Q)}^{\frac{q}{2}}\|u_\varepsilon\|_{L^r(Q)}^{r(\frac{s-q'}{sq'}\frac{q}{2}+\frac{2-q}{2})}\right),$$

holds where $\delta(K)$ tends to zero as K goes to infinity. Therefore, Young's inequality yields

$$\|\nabla u_\varepsilon\|_{L^q(Q)}^q \leq C(K)+\delta(K)\left(\|u_\varepsilon\|_{L^r(Q)}^{\frac{r(2-q)}{2}}+\|u_\varepsilon\|_{L^r(Q)}^{r(\frac{s-q'}{sq'}q+2-q)}\right). \tag{5.6}$$

If we choose $r=q\frac{N+1}{N}$, estimate (2.17) becomes

$$\|u_\varepsilon\|_{L^r(Q)}^r \leq C\|u_\varepsilon\|_{L^q(0,T;L^{q^\star}(\Omega))}^q.$$

Using Sobolev's theorem, as in Section 2, we derive the following estimate on u_ε in $L^q(0,T;L^{q^\star}(\Omega))$

$$\|u_\varepsilon\|_{L^q(0,T;L^{q^\star}(\Omega))}^q \leq C(K)+\delta(K)\left(\|u_\varepsilon\|_{L^q(0,T;L^{q^\star}(\Omega))}^{q\frac{2-q}{2}}+\|u_\varepsilon\|_{L^q(0,T;L^{q^\star}(\Omega))}^{q(\frac{s-q'}{sq'}q+2-q)}\right).$$

Since $\frac{2-q}{2}<1$ and $\frac{s-q'}{sq'}q+2-q<1$, we can use again Young's inequality which, choosing K large enough, leads to a bound on u_ε in $L^q(0,T;L^{q^\star}(\Omega))$ and, thus, in $L^q(0,T;W_0^{1,q}(\Omega))$. Finally, let us verify the compatibility of conditions (5.5). For $r=q\frac{N+1}{N}$, the first condition is equivalent to $1\leq q<\frac{N+2}{N+1}$ and the second condition means that $s>(N+1)\frac{q'}{q'-1}$ which is clearly satisfied since it is yet required $s>q'$.
Finally, one can easily verify that Lemma 2, Lemma 3, Corollary 2 and Lemma 4 are valid in the context of Theorem 3 and the proof follows. □

Appendix A : Weak L^1 convergence

Lemma 5 *The sequence u_ε solution of (3.4) satisfies*

$$\lim_{k\to\infty} \sup_{\varepsilon>0,\, t\in(0,T)} \int_{|u_\varepsilon|>k} |u_\varepsilon| dx = 0. \tag{A.1}$$

Proof.
We look at (3.13) without precising the value of $k>0$. Let $M>0$. Inequalities (3.29) leads to

$$\begin{aligned} \frac{1}{k}\int_\Omega S_k(u_\varepsilon)(t)dx &\le e^{C(b,d)T}\Bigg(\frac{M}{k}\|f_\varepsilon\|_{L^1(Q)} + \int_{|u_\varepsilon|>M}|f_\varepsilon|dxdt \\ &+ \frac{M^2}{k}|\Omega| + \int_{|u_{0,\varepsilon}|>M}|u_{0,\varepsilon}|dx\Bigg). \end{aligned} \tag{A.2}$$

Since f_ε and $u_{0,\varepsilon}$ are convergent sequences in $L^1(Q)$ and $L^1(\Omega)$ respectively and remarking, by (3.5), that

$$\sup_{\varepsilon>0}(meas\{(t,x)\in Q,\ |u_\varepsilon(t,x)|>M\}) \le \frac{1}{M}T\sup_{\varepsilon>0}\|u_\varepsilon\|_{L^\infty(0,T,L^1(\Omega))} \le \frac{T\beta}{M}$$

tends to 0 as $M\to\infty$, the quantities

$$\sup_{\varepsilon>0}\left(\int_{|u_{0,\varepsilon}|>M}|u_{0,\varepsilon}|dx\right),\ \sup_{\varepsilon>0}\left(\int_{|u_\varepsilon|>M}|f_\varepsilon|dxdt\right)$$

may be given arbitrarily small by choosing M large enough. Obviously $\frac{|z|}{2}\chi_{|z|>k} \le \frac{1}{k}S_k(z)$, hence, it follows from (A.2) that

$$\sup_{\varepsilon>0,t\in(0,T)}\int_{|u_\varepsilon|>k}|u_\varepsilon|dx \longrightarrow 0$$

as k goes to ∞. □

We end this appendix with the remark that an existence result similar to Theorem 1 may be obtained if the strong convergences in (3.1) are replaced by weak L^1 convergences. Indeed, by the Dunford-Pettis theorem, see [8], everything we did in Section 3 still holds, except Lemma 2 (since we loss that $\|f_\varepsilon - f_{\varepsilon'}\|_{L^1(Q)}$ and $\|u_{0,\varepsilon}-u_{0,\varepsilon'}\|_{L^1(\Omega)}$ go to 0). However, by Corollary 1, Lemma 3 and Corollary 2, the convergences (3.44) hold and, moreover, $\nabla u_\varepsilon \to \nabla u$ a.e. Q, $g_\varepsilon \to g(u,\nabla u)$ a.e. Q and strongly in $L^1(Q)$. Furthermore, we can assume that, for all $t\in(0,T)$,

$$u_\varepsilon(t) \rightharpoonup u(t) \quad in\ L^1(\Omega). \tag{A.3}$$

Indeed, by Lemma 3, $u_\varepsilon(t)$ belongs to a weakly compact set in $L^1(\Omega)$, for all $\varepsilon>0$ and $t\in(0,T)$. Thus, by using the Cantor process, we can suppose that (A.3) holds for all rational time $\overline{t} \in (0,T) \cap I\!\!Q$. Now, let $\phi \in L^\infty(\Omega) \cap W^{1,q'}(\Omega)$. We set $U_\varepsilon(t) = \int_\Omega u_\varepsilon(t)\phi dx$. Then, one has

(A.4) $$\sup_{\varepsilon>0} |U_\varepsilon(t) - U_\varepsilon(s)| \to 0.$$

as $|t-s| \to 0$. Hence, if (A.3) did not hold for all $t \in (0,T)$, there would exist some $\phi \in L^\infty(\Omega) \cap W^{1,q'}(\Omega)$ and $t \in (0,T)$ and two subsequences extracted from $U_\varepsilon(t)$ with different limits. This clearly contredicts (A.2) where we consider a rational $s = \overline{t}$ close to t, since it is known that$U_\varepsilon(\overline{t})$ converges. It remains to justify (A.4). Integrating (3.4) between s and t gives

$$\begin{aligned} |U_\varepsilon(t) - U_\varepsilon(s)| &\leq \left| \int_s^t \int_\Omega (A\nabla u_\varepsilon \cdot \nabla\phi + (b\cdot\nabla u_\varepsilon + du_\varepsilon + g_\varepsilon + f_\varepsilon)\phi)\, dxdt \right| \\ &\leq \|\phi\|_{L^\infty(\Omega)} \left(\|b\|_{L^\infty(Q)} \|\nabla u_\varepsilon\|_{L^q(Q)} + \|d\|_{L^\infty(Q)} \|u_\varepsilon\|_{L^q(q)} \right) |\Omega|^{\frac{1}{q'}} |t-s|^{\frac{1}{q'}} \\ &+ \|A\|_{L^\infty(Q)} \|\nabla\phi\|_{L^{q'}(\Omega)} \|\nabla u_\varepsilon\|_{L^q(Q)} |t-s|^{\frac{1}{q'}} \\ &+ \|\phi\|_{L^\infty(\Omega)} \left(\left| \int_s^t \int_\Omega |f_\varepsilon| dxdt \right| + \left| \int_s^t \int_\Omega |g_\varepsilon| dxdt \right| \right) \end{aligned}$$

by using Holder's inequality. Since f_ε and g_ε are weakly convergent sequences in $L^1(Q)$, we get

$$\sup_{\varepsilon>0} \left(\left| \int_s^t \int_\Omega |f_\varepsilon| dxdt \right| + \left| \int_s^t \int_\Omega |g_\varepsilon| dxdt \right| \right) \leq \omega(|t-s|)$$

with $\lim_{h\to 0} \omega(h) = 0$. The asserted convergences allow us to pass to the limit in each terms of (3.44) and, then, we are conclude that u is a weak solution in the sense of (1.9), dropping in Definition 1 the requirement that u lies in $C^0(0,T;L^1(\Omega))$.

Annex B : Details of Proof of uniqueness of entropic solutions

Let v be an entropic solution. To obtain the uniqueness, we will show that $v = u$, u still being the solution obtained by approximation. To this end, it would be natural to choose $T_h(u_\varepsilon)$ as test function ψ in (1.11). However, as pointed out in [15], T_h is not regular enough which leads to difficulties in order to write the term involving the time derivative of the test function. Then, it is necessary to regularize the truncation. Let $\nu > 0$. We introduce T_h^ν satisfying

$$\begin{cases} (T_h^\nu)'(z) = 0 & if \quad |z| \geq h, \\ 0 \leq (T_h^\nu)'(z) \leq (T_h)'(z) \leq 1, & \\ (T_h^\nu)'(z) = 1 & if \quad |z| \leq h - \nu. \end{cases} \tag{B.1}$$

In particular, T_h^ν is twice differentiable, with $(T_h^\nu)"(z) = 0$ when $|z| \geq h$ or $|z| \leq h - \nu$, and approximate T_h as ν goes to 0. Notice also that $|T_h^\nu(z)| \leq |T_h(z)|$. >From now on, we require that, besides the strong convergences (3.1), $u_{0,\varepsilon}$ and f_ε are dominated and converge a.e. in Ω and Q, respectively.

We take $\psi = T_h^\nu(u_\varepsilon)$ as test function in relation (1.11) satisfied by v. By using (3.4), we write the term involving the time derivative of the test function as follows

$$\int_0^t \langle \partial_t u_\varepsilon, \phi \rangle ds = \int_0^t \int_\Omega ((f_\varepsilon - b \cdot \nabla u_\varepsilon - d u_\varepsilon)\phi - A \nabla u_\varepsilon \nabla \phi) dx ds \tag{B.2}$$

where $\phi = (T_h^\nu)'(u_\varepsilon) T_k(v - T_h^\nu(u_\varepsilon))$ and, consequently,

$$\nabla \phi = \nabla u_\varepsilon (T_h^\nu)"(u_\varepsilon) T_k(v - T_h^\nu(u_\varepsilon)) + (T_h^\nu)'(u_\varepsilon) \nabla (T_k(v - T_h^\nu(u_\varepsilon))).$$

By (B.2), the entropic formulation (1.11) satisfied by v with $\psi = T_h^\nu(u_\varepsilon)$ becomes

$$\begin{aligned} &\left[\int_\Omega S_k(v - T_h^\nu(u_\varepsilon)) dx\right]_0^t + \int_0^t \int_\Omega ((f_\varepsilon - b \cdot \nabla u_\varepsilon - d u_\varepsilon)(T_h^\nu)'(u_\varepsilon) T_k(v - T_h^\nu(u_\varepsilon)) dx ds \\ &+ \int_0^t \int_\Omega (b \cdot \nabla v + dv) T_k(v - T_h^\nu(u_\varepsilon)) dx ds + \int_0^t \int_\Omega A \nabla v \nabla T_k(v - T_h^\nu(u_\varepsilon)) dx ds \\ &- \int_0^t \int_\Omega A \nabla u_\varepsilon \nabla (T_k(v - T_h^\nu(u_\varepsilon)))(T_h^\nu)'(u_\varepsilon) dx ds \\ &- \int_0^t \int_\Omega A \nabla u_\varepsilon \nabla u_\varepsilon (T_h^\nu)"(u_\varepsilon) T_k(v - T_h^\nu(u_\varepsilon)) dx ds \leq \int_0^t \int_\Omega f T_k(v - T_h^\nu(u_\varepsilon)) dx ds \end{aligned} \tag{B.3}$$

We denote by $I_1, ..., I_7$ the integrals which appear in (B.3). Now, according to [15], let successively $\nu \to 0$, $\varepsilon \to 0$ and $h \to \infty$.

Difficulties only arise from I_6, involving the second derivative of T_h^ν, the remaining integrals being treated by using the Lebesgue theorem. Indeed, it is clear that, on the one hand,

$$\begin{cases} S_k(v - T_h^\nu(u_\varepsilon)) & \to & S_k(v - T_h(u_\varepsilon)), \\ T_k(v - T_h^\nu(u_\varepsilon)) & \to & T_k(v - T_h(u_\varepsilon)), \\ (T_h^\nu)'(u_\varepsilon) T_k(v - T_h^\nu(u_\varepsilon)) & \to & (T_h)'(u_\varepsilon) T_k(v - T_h(u_\varepsilon)), \end{cases}$$

a.e. Q, as $\nu \to 0$, and, on the other hand,

$$\begin{cases} |S_k(v - T_h^\nu(u_\varepsilon))| & \leq & k|v| + kh, \\ |T_k(v - T_h^\nu(u_\varepsilon))| & \leq & k, \\ |(T_h^\nu)'(u_\varepsilon) T_k(v - T_h^\nu(u_\varepsilon))| & \leq & k. \end{cases}$$

Furthermore, we write the gradient terms as follows

$$\nabla T_k(v - T_h^\nu(u_\varepsilon)) = \chi_{|v-T_h^\nu(u_\varepsilon)|\leq k}(\nabla v - (T_h^\nu)'(u_\varepsilon)\nabla u_\varepsilon)$$

which tends a.e. Q to

$$\chi_{|v-T_h(u_\varepsilon)|\leq k}(\nabla v - (T_h)'(u_\varepsilon)\nabla u_\varepsilon) = \nabla T_k(v - T_h(u_\varepsilon))$$

and which is dominated by the following $L^2(Q)$ function

$$\chi_{|v|\leq k+h}(|\nabla v| + |(T_h)'(u_\varepsilon)\nabla u_\varepsilon)|) \leq \chi_{|v|\leq k+h}(|\nabla T_{k+h}(v)| + |\nabla T_h(u_\varepsilon)|)$$

since $|T_h^\nu(z)| \leq |T_h(z)|$ and $0 \leq (T_h^\nu)'(z) \leq (T_h)'(z)$. Next, we wish to obtain an estimate on I_6. Following [15] (see also [2]), we define another C^2 function R_h^ν, satisfying for $z \geq 0$

$$\text{(B.4)} \qquad \begin{cases} (R_h^\nu)'(z) = 1 - (T_h^\nu)'(z), \\ R_h^\nu(0) = 0, \\ R_h^\nu(-z) = R_h^\nu(z). \end{cases}$$

Take $(R_h^\nu)'(u_\varepsilon)$ as test function in (3.4) ; it follows that

$$\begin{aligned} \left[\int_\Omega R_h^\nu(u_\varepsilon)dx\right]_0^t &+ \int_0^t\int_\Omega (b\cdot\nabla u_\varepsilon + du_\varepsilon)(R_h^\nu)'(u_\varepsilon)dxds \\ &+ \int_0^t\int_\Omega A\nabla u_\varepsilon\nabla u_\varepsilon(R_h^\nu)"(u_\varepsilon)dxds = \int_0^t\int_\Omega f_\varepsilon(R_h^\nu)'(u_\varepsilon).dxds \end{aligned}$$

By constuction, one has $(R_h^\nu)"(z) = |(T_h^\nu)"(z)|$. Hence, by using (1.5), we deduce that

$$\text{(B.5)} \qquad \begin{aligned} |I_6| &= \left|\int_0^t\int_\Omega A\nabla u_\varepsilon\nabla u_\varepsilon(T_h^\nu)"(u_\varepsilon)T_k(v - T_h^\nu(u_\varepsilon))dxds\right| \\ &\leq k\int_0^t\int_\Omega A\nabla u_\varepsilon\nabla u_\varepsilon(R_h^\nu)"(u_\varepsilon)dxds \\ &\leq k\int_0^t\int_\Omega (f_\varepsilon - b\nabla u_\varepsilon - du_\varepsilon)(R_h^\nu)'(u_\varepsilon)dxds + k\int_\Omega R_h^\nu(u_{0,\varepsilon})dx, \end{aligned}$$

since $|T_k(z)| \leq k$ and $R_h^\nu(z) \geq 0$. The definition of R_h^ν implies that $0 \leq R_h^\nu(z) = (|z| - |T_h^\nu(z)|) \leq |z|\chi_{|z|>h-\nu}$ and $0 \leq (R_h^\nu)'(z) \leq \chi_{|z|>h-\nu}$, so that we are led to

$$\text{(B.6)} \qquad \begin{aligned} |I_6| &\leq k\int_0^t\int_\Omega \left(|f_\varepsilon| + \|b\|_{L^\infty(Q)}|\nabla u_\varepsilon| + \|d\|_{L^\infty(Q)}|u_\varepsilon|\right)\chi_{|u_\varepsilon|>h-\nu}dxds \\ &+ k\int_\Omega |u_{0,\varepsilon}|\chi_{|u_{0,\varepsilon}|>h-\nu}dx \end{aligned}$$

by (B.5). By Lebesgue's theorem, we can pass to the limit $\nu \to 0$ in the right hand side of (B.6), obtaining without difficulties

$$\limsup_{\nu\to 0}|I_6| \leq k\int_0^t\int_\Omega \left(|f_\varepsilon| + \|b\|_{L^\infty(Q)}|\nabla u_\varepsilon| + \|d\|_{L^\infty(Q)}|u_\varepsilon|)\right)\chi_{|u_\varepsilon|>h}dxds + k\int_\Omega |u_{0,\varepsilon}|\chi_{|u_{0,\varepsilon}|>h}dx.$$

Collecting these results, we get from (B.3) the following estimate

$$
\text{(B.7)}\qquad
\begin{aligned}
\left[\int_\Omega S_k(v-T_h(u_\varepsilon))dx\right]_0^t &+ \int_0^t\int_\Omega ((f_\varepsilon - b\cdot\nabla u_\varepsilon - du_\varepsilon)(T_h)'(u_\varepsilon)T_k(v-T_h(u_\varepsilon))dxds \\
&+ \int_0^t\int_\Omega (b\cdot\nabla v + dv)T_k(v-T_h(u_\varepsilon))dxds \\
&+ \int_0^t\int_\Omega A\nabla v\nabla T_k(v-T_h(u_\varepsilon))dxds \\
&- \int_0^t\int_\Omega A\nabla u_\varepsilon\nabla T_k(v-T_h(u_\varepsilon))(T_h)'(u_\varepsilon)dxds \\
&\le \int_0^t\int_\Omega fT_k(v-T_h(u_\varepsilon))dxds \\
&+ k\int_0^t\int_\Omega \left(|f_\varepsilon| + \|b\|_{L^\infty(Q)}|\nabla u_\varepsilon| + \|d\|_{L^\infty(Q)}|u_\varepsilon|\right)\chi_{|u_\varepsilon|>h}dxds \\
&+ k\int_\Omega |u_{0,\varepsilon}|\chi_{|u_{0,\varepsilon}|>h}dx.
\end{aligned}
$$

Let $J_1, ..., J_5$ stand for the integrals in the left hand side and let us denote by J_6 the right hand side side. The assumptions on the sequence of data and the properties of u_ε recalled above allow us to apply the Lebesgue theorem to pass to the limit as $\varepsilon \to 0$ in J_1, J_2, J_3 and J_6. Next, since $(T_h)'(u_\varepsilon)\nabla u_\varepsilon = \nabla T_h(u_\varepsilon)$, we write

$$
J_4 - J_5 = \int_0^t\int_\Omega A\nabla(v-T_h(u_\varepsilon))\nabla T_k(v-T_h(u_\varepsilon))dxds,
$$

where, by (1.5) and the following a.e. convergence

$$
\nabla T_k(v-T_h(u_\varepsilon)) = \chi_{|v-T_h(u_\varepsilon)|\le k}(\nabla v - \chi_{|u_\varepsilon|\le h}\nabla u_\varepsilon) \to \chi_{|v-T_h(u)|\le k}(\nabla v - \chi_{|u|\le h}\nabla u),
$$

we can use Fatou's lemma.

It remains to deal with $h\to\infty$ in the following relation

$$
\text{(B.8)}\qquad
\begin{aligned}
\left[\int_\Omega S_k(v-T_h(u))dx\right]_0^t &+ \int_0^t\int_\Omega ((f - b\cdot\nabla u - du)(T_h)'(u)T_k(v-T_h(u))dxds \\
&+ \int_0^t\int_\Omega (b\cdot\nabla v + dv)T_k(v-T_h(u))dxds \\
&+ \int_0^t\int_\Omega A\nabla T_k(v-T_h(u))\nabla T_k(v-T_h(u))dxds \\
&\le \int_0^t\int_\Omega fT_k(v-T_h(u))dxds + kR(h),
\end{aligned}
$$

where $R(h)$ stands for

$$
\int_0^t\int_\Omega \left(|f| + \|b\|_{L^\infty(Q)}|\nabla u| + \|d\|_{L^\infty(Q)}|u|\right)\chi_{|u|>h}dxds + \int_\Omega |u_0|\chi_{|u_0|>h}dx.
$$

Since $meas\{(t,x)\in Q,\ |u|>h\} \le \frac{1}{h}T\|u\|_{L^\infty(0,T;L^1(\Omega))}$ and $f, u, \nabla u$ belong to $L^1(Q)$, it is clear that

$$
\lim_{h\to\infty} R(h) = 0.
$$

Let us combine the second and third integrals in (B.8) by writing, on the one hand,

$$
\text{(B.9)}\qquad
\begin{aligned}
&b\cdot\nabla v + dv - (b\cdot\nabla u + du)(T_h)'(u) \\
=\ &b\cdot\nabla(v-u) + d(v-u) + (b\cdot\nabla u + du)(1-T_h)'(u)
\end{aligned}
$$

and, on the other hand

$$
\begin{aligned}
&\int_0^t\int_\Omega (b\cdot\nabla(v-u))T_k(v-T_h(u))dxds \\
\text{(B.10)}\qquad = &-\int_0^t\int_\Omega (v-u)b\cdot\nabla T_k(v-T_h(u))+(\nabla\cdot b)(v-u)T_k(v-T_h(u))dxds.
\end{aligned}
$$

By (B.9-B.10), (B.8) becomes

$$
\begin{aligned}
\left[\int_\Omega S_k(v-T_h(u))dx\right]_0^t &+ \int_0^t\int_\Omega A\nabla T_k(v-T_h(u))\nabla T_k(v-T_h(u))dxds \\
&\le \int_0^t\int_\Omega f(1-(T_h)'(u))T_k(v-T_h(u))dxds + kR(h) \\
\text{(B.11)}\qquad &+ \left|\int_0^t\int_\Omega (v-u)b\cdot\nabla T_k(v-T_h(u))dxds\right| \\
&+ \left|\int_0^t\int_\Omega (d-(\nabla\cdot b))(v-u)T_k(v-T_h(u))dxds\right| \\
&+ \left|\int_0^t\int_\Omega (b\cdot\nabla u+du)(1-(T_h)'(u))T_k(v-T_h(u))dxds\right|.
\end{aligned}
$$

Proceeding as in Section 4 leads to

$$
\begin{aligned}
\int_\Omega S_k(v-T_h(u))(t)dx &+ \tfrac{a}{2}\int_0^t\int_\Omega |\nabla T_k(v-T_h(u))|^2dxds \\
&\le \int_\Omega S_k(v-T_h(u))(0)dx + k\int_0^t\int_\Omega |f||1-(T_h)'(u)|dxds + kR(h) \\
&+ C(b,d)\int_0^t\int_\Omega S_k(v-T_h(u))dxds \\
&+ k\int_0^t\int_\Omega (\|b\|_{L^\infty(Q)}|\nabla u|+\|d\|_{L^\infty(Q)}|u|)|1-(T_h)'(u)|dxds,
\end{aligned}
$$

(B.12)
by using $0\le zT_k\le 2S_k(z)$, $0\le z^2\chi_{|z|\le k}\le 2S_k(z)$, the bound on $\nabla\cdot b$ and (1.3-1.5) and with $C(b,d)=2\|\nabla b\|_{L^\infty(Q)}+2\|d\|_{L^\infty(Q)}+\frac{1}{a}\|b\|_{L^\infty(Q)}$. Then, we remark that $1-(T_h)'(u)$ tends to 0 as $h\to\infty$ while $|1-(T_h)'(u)|\le 2$ and $f,u,\nabla u$ belong to $L^1(Q)$. We have also $0\le S_k(v-T_h(u))\le k(|v|+|u|)$ and its limit is $S_k(v-u)$ a.e. Q and for all t, a.e. Ω. Moreover, $\nabla T_k(v-T_h(u))=\chi_{|v-T_h(u)|\le k}(\nabla v-\chi_{|u|\le h}\nabla u)$ converges a.e. Q to $\chi_{|v-u|\le k}(\nabla v-\nabla u)=T_k(v-u)$. Hence, by Lebesgue's theorem and Fatou's lemma, letting h go to ∞ in (B.12) yields

$$
\begin{aligned}
\text{(B.13)}\qquad \int_\Omega S_k(v-u)(t)dx &+ \tfrac{a}{2}\int_0^t\int_\Omega |\nabla T_k(v-u)|^2dxds \\
&\le \int_\Omega S_k(v-u)(0)dx + C(b,d)\int_0^t\int_\Omega S_k(v-u)dxds.
\end{aligned}
$$

Therefore, it suffices to apply Gronwall's lemma to deduce that

$$\int_\Omega S_k(v-u)(t)dx = 0$$

since $v_0=u_0$, which gives $v=u$. $\square$

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