DDFV method for Navier-Stokes problem with outflow boundary conditions

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Abstract

We propose a Discrete Duality Finite Volume scheme (DDFV for short) for the unsteady incompressible Navier-Stokes problem with outflow boundary conditions. As in the continuous case, those conditions are derived from a weak formulation of the equations and they provide an energy estimate of the solution. We prove wellposedness of the scheme and a discrete energy estimate. Finally we perform some numerical tests simulating the flow behind a cylinder inside a long channel to show the robustness of such conditions in the DDFV framework.

1 Introduction

The problem we are interested in is the computation of a flow whose velocity is prescribed at one part of the boundary and it flows freely on the other one. In this framework, we are often required to truncate the physical domain to obtain a reduced computational domain, either because we want to save computational ressources or because the physical domain is unbounded.

The aim of this paper is to design and analyze a finite volume approximation of the 2D unsteady incompressible Navier-Stokes problem:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mathrm{div}(\sigma(\mathbf{u},\mathrm{p})) = 0; & \text{in} \quad \Omega_T = \Omega \times [0,T] \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in} \quad \Omega_T, \\ \mathbf{u} = \mathbf{g}_1 & \text{on} \quad \Gamma_1 \times (0,T), \\ \sigma(\mathbf{u},\mathrm{p})\cdot\vec{\mathbf{n}} + \dfrac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})^-(\mathbf{u}-\mathbf{u}_{ref}) = \sigma_{ref}\cdot\vec{\mathbf{n}} & \text{on} \quad \Gamma_2 \times (0,T), \\ \mathbf{u}(0) = \mathbf{u}_{init} & \text{in} \quad \Omega \end{cases} \tag{1}$$

with $0 < T < \infty$, Ω an open bounded polygonal domain of $\mathbb{R}^2$, whose boundary is $\partial\Omega = \Gamma_1 \cup \Gamma_2$ and whose outern normal is $\vec{\mathbf{n}}$, $\mathbf{u}_{init} \in (L^\infty(\Omega))^2$, $\mathbf{g}_1 \in (H^{\frac{1}{2}}(\partial\Omega))^2$ and where $\mathbf{u} : \Omega_T \to \mathbb{R}^2$ is the velocity, $\mathrm{p} : \Omega_T \to \mathbb{R}$ is the pressure and $\sigma(\mathbf{u},\mathrm{p}) = \dfrac{2}{\mathrm{Re}}\mathrm{D}\mathbf{u} - \mathrm{pId}$ is the stress tensor, with $\mathrm{Re} > 0$. In particular, the strain rate tensor is defined by the symmetric part of the velocity gradient $\mathrm{D}\mathbf{u} = \dfrac{1}{2}(\nabla\mathbf{u} + {}^t\nabla\mathbf{u})$.
On the physical part of the boundary Γ_1 we impose Dirichlet boundary conditions. On the "non-physical" part, Γ_2, we impose the artificial boundary condition

$$\sigma(\mathbf{u},\mathrm{p})\cdot\vec{\mathbf{n}} + \frac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})^-(\mathbf{u}-\mathbf{u}_{ref}) = \sigma_{ref}\cdot\vec{\mathbf{n}} \tag{2}$$

that was first introduced in [10] and then further studied in [9] and [4]. We use the notation $(a)^- = -\min(a,0)$. In order to build it, we need to choose some *reference flow* $\mathbf{u}_{ref}$, which is any $\mathbf{u}_{ref} \in (H^1(\Omega))^2$ such that $\mathbf{u}_{ref} = \mathbf{g}_1$ on Γ_1, chosen so as to be a reasonable approximation of the expected flow near Γ_2, and a *reference stress tensor* σ_{ref} such that $\sigma_{ref}\cdot\vec{\mathbf{n}} \in (H^{-\frac{1}{2}}(\Omega))^2$. This nonlinear condition is physically meaningful: if the flow is outward, we impose the constraint coming from the selected reference flow; if it is inward, we need to control the increase of energy, so we add a term that is quadratic with respect to velocity. Other techniques to model artificial boundaries have been studied during the years. For instance, in [16] an artificial boundary condition is designed for the Navier-Stokes equations under the hypothesis of small viscosity. The method consists into the approximation of the transparent boundary conditions, since they are non-local. The technique was then generalized to parabolic perturbations of hyperbolic systems in [15] and to compressible flows in [21]. We choose to work with the condition (2) of [10] since it is defined locally and it does not add hypothesis on the viscosity. It has been derived by a particular weak formulation of Navier-Stokes equation that ensures an energy estimate: we would like to reproduce the same property at a discrete level with the DDFV formalism.

The DDFV method has been developed to approximate anisotropic diffusion problems on general meshes. More precisely, it has been first introduced and studied in [1, 12] to approximate the Laplace equation with Dirichlet boundary conditions or homogeneous Neumann boundary conditions on a large class of 2D meshes including non-conformal and distorted meshes. Such schemes require unknowns on both vertices and centers of primal control volumes and allow us to build two-dimensional discrete gradient and divergence operators being in duality in a discrete sense. The DDFV scheme is extended in [1] to the case of the approximation of solutions to general linear and nonlinear elliptic problems with non homogeneous Dirichlet boundary conditions, including the case of anisotropic elliptic problems.

The analysis of problem (1) is done in [9] and [3] from the continuous point of view and simulations are performed in [10] by the use of Finite Differences schemes in the case of cartesian meshes. Thanks to DDFV method, we are able to reproduce those simulations by extending to the case of general meshes and to make a complete analysis of the discrete problem, perspective that was never addressed in the literature.

To approximate problem (1), we start from the theory developed in [17] and [19] for the Navier-Stokes equations in the case of Dirichlet boundary conditions. We modify the convection term presented in [19] on the boundary, since we want to preserve an energy estimate at the discrete level. For this reason, we also have to prove a Korn inequality, in order to control the norm of the gradient with the norm of the strain rate tensor, and a trace theorem, useful to estimate the boundary terms.
Our wellposedness result relies on a uniform discrete inf-sup condition, see [5] and Section 4.1. In the case of Stokes problem [14, 18] and of Navier-Stokes with Dirichlet boundary conditions [19], this difficulty was overcome by adding a stabilization term in the equation of conservation of mass. This stabilization term is inspired by the Brezzi-Pitkäranta method [7] in the finite element framework. We could have used the same technique in order to generalize the result of wellposedness to general meshes, but since our proof for Korn's inequality requires the hypothesis of inf-sup stability, we decided not to stabilize the equation.
We finally validate our theoretical results by numerical simulations, by first showing convergence results and then by reproducing the test cases proposed in [10] and [22].

Outline. This paper is organized as follows. In Section 2, we recall the DDFV framework and we show how we approximate the nonlinear convection term. In Section 3, we introduce the DDFV scheme for the Navier-Stokes problem (1) and we prove its well-posedness in Section 4 (see Theorem (4.2)). In Section 5 we show an estimate of the convection term. In Section 6, we prove a discrete Korn inequality, useful for the discrete energy estimate that we prove in Section 7. Finally, in Section 8, theoretical results are illustrated with numerical simulations. Conclusions are given in Section 9.

2 DDFV framework

Here and below, we adopt the main definitions and notation introduced in [1] and [17].

2.1 Meshes

A DDFV mesh $\mathfrak{T}$ is constituted by a primal mesh $\mathfrak{M}$ and a dual mesh $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$, see Figure 1. We consider a primal mesh $\mathfrak{M}$ consisting of disjoints polygons κ called primal cells, whose union covers Ω. We denote $\partial\mathfrak{M}$ the set of edges of the primal mesh included in $\partial\Omega$, that are considered as degenerated primal cells. We associate to each κ a point x_κ, called center. For the volumes of the boundary, the point x_κ is situated at the mid point of the edge. When κ and $\textsc{l}$ are neighboring volumes, we suppose that $\partial\kappa \cap \partial\textsc{l}$ is a segment that we denote $\sigma = \kappa|\textsc{l}$, edge of the primal mesh $\mathfrak{M}$. We denote with $\mathcal{E}$ the set of all edges and with $\mathcal{E}_{int} = \mathcal{E} \setminus \{\sigma \in \mathcal{E} \text{ such that } \sigma \subset \partial\Omega\}$. The DDFV framework is free of further "admissibility constraint", in particular we do not need to assume the orthogonality of the segment $x_\kappa, x_\textsc{l}$ with $\sigma = \kappa|\textsc{l}$. Here we suppose:

Hp 2.1 *All control volumes* κ *are star-shaped with respect to* x_κ.

From this primal mesh, we build the associated dual mesh. A dual cell κ^* is associated to a vertex x_{κ^*} of the primal mesh. The dual cells are obtained by joining the centers of the primal control volumes that have x_{κ^*} as vertex. Then, the point x_{κ^*} is called center of κ^*. We will distinguish interior dual mesh, for which x_{κ^*} does not belong to $\partial\Omega$, denoted by $\mathfrak{M}^*$ and the boundary dual mesh, for which x_{κ^*} belongs to

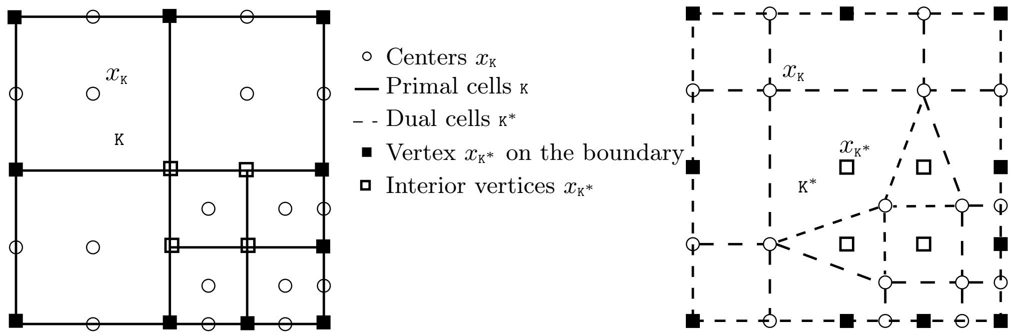


Figure 1: DDFV meshes.

$\partial\Omega$, denoted by $\partial\mathfrak{M}^*$. We denote with $\sigma^* = \text{K}^*|\text{L}^*$ the edges of the dual mesh $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$ and $\mathcal{E}^*$ the set of those edges. In what follows, we assume:

Hp 2.2 *All control volumes* K^* *are star-shaped with respect to* x_{K^*}*.*

The diamond mesh is made of quadrilaterals with disjoint interiors (thanks to Hp 2.1), such that their principal diagonals are a primal edge $\sigma = \text{K}|\text{L} = [x_{\text{K}^*}, x_{\text{L}^*}]$ and the dual edge $\sigma^* = [x_{\text{K}}, x_{\text{L}}]$. Those quadrilaterals are called diamonds and they are denoted with D or $\text{D}_{\sigma,\sigma^*}$. Thus a diamond is a quadrilateral with vertices $x_{\text{K}}, x_{\text{L}}, x_{\text{K}^*}$ and x_{L^*}.
We remark that diamonds are the union of two disjoints triangles $(x_{\text{K}}, x_{\text{K}^*}, x_{\text{L}^*})$ and $(x_{\text{L}}, x_{\text{K}^*}, x_{\text{L}^*})$ and that diamonds are not necessarily convexes.
Moreover, if $\sigma \in \mathcal{E} \cap \partial\Omega$, the quadrilateral $\text{D}_{\sigma,\sigma^*}$ degenerates into a triangle.
The set of all diamonds is denoted with $\mathfrak{D}$ and we have $\Omega = \bigcup_{\text{D}\in\mathfrak{D}} \mathfrak{D}$. We distinguish the diamonds on the interior and of the boundary:

$$\mathfrak{D}_{ext} = \{\text{D}_{\sigma,\sigma^*} \in \mathfrak{D}, \text{ such that } \sigma \subset \partial\Omega\}$$
$$\mathfrak{D}_{int} = \mathfrak{D} \backslash \mathfrak{D}_{ext}.$$

Remark 2.3 *We have a bijection between the diamonds* $D \in \mathfrak{D}$ *and the edges* $\mathcal{E}$ *of the primal mesh; also between the diamonds* $D \in \mathfrak{D}$ *and the edges* $\mathcal{E}^*$ *of the dual mesh.*

2.2 Notations

The following notation will be used throughout the paper. The reader familiar with DDFV may want to skip this section.

For a volume $\text{V} \in \mathfrak{M} \cup \partial\mathfrak{M} \cup \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ we define:

- m_{V} the measure of the cell V,
- $\mathfrak{D}_{\text{V}} = \{\text{D}_{\sigma,\sigma^*} \in \mathfrak{D}, \sigma \in \mathcal{E}_{\text{V}}\}$,
- d_{V} the diameter of V

For a diamond $\text{D}_{\sigma,\sigma^*}$ whose vertices are $(x_{\text{K}}, x_{\text{K}^*}, x_{\text{L}}, x_{\text{L}^*})$, we denote:

- x_{D} the center of the diamond D
- m_σ the length of the edge σ
- m_{σ^*} the length of σ^*
- m_{D} the measure of the diamond $\text{D}_{\sigma,\sigma^*}$
- d_{D} the diameter of the diamond $\text{D}_{\sigma,\sigma^*}$
- α_{D} the angle between σ and σ^*

We introduce for every diamond two orthonormal basis $(\vec{\boldsymbol{\tau}}_{\text{K}^*,\text{L}^*}, \vec{\mathbf{n}}_{\sigma\text{K}})$ and $(\vec{\mathbf{n}}_{\sigma^*\text{K}^*}, \vec{\boldsymbol{\tau}}_{\text{K},\text{L}})$, where:

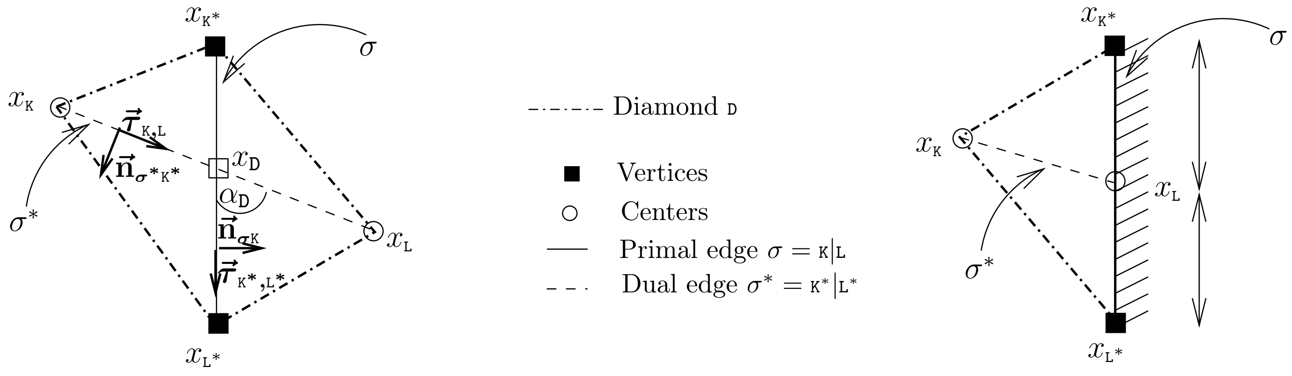


Figure 2: A diamond $\text{D} = \text{D}_{\sigma,\sigma^*}$, on the interior (left) and on the boundary (right).

- $\vec{\mathbf{n}}_{\sigma\text{K}}$ the unit normal to σ going out from K
- $\vec{\tau}_{\text{K},\text{L}}$ the unit tangent vector to σ oriented from K^* to L^*
- $\vec{\mathbf{n}}_{\sigma^*\text{K}^*}$ the unit normal vector to σ^* going out from K^*
- $\vec{\tau}_{\text{K}^*,\text{L}^*}$ the unit tangent vector to σ^* oriented from K to L.

We denote for each diamond:

- his sides $\mathfrak{s}$ (for example $\mathfrak{s} = [x_{\text{K}}, x_{\text{K}^*}]$)
- $\mathcal{E}_{\text{D}} = \{\mathfrak{s}, \mathfrak{s} \subset \partial\text{D} \text{ and } \mathfrak{s} \not\subseteq \partial\Omega\}$ the set of all interior sides of the diamond
- $m_{\mathfrak{s}}$ the length of $\mathfrak{s}$
- $\vec{\mathbf{n}}_{\mathfrak{s}\text{D}}$ the unit normal to $\mathfrak{s}$ going out from D

Remark 2.4 *Every diamond is star-shaped with respect to x_D.*

2.3 Regularity of the mesh

Let size$(\mathfrak{T})$ be the maximum of the diameters of the diamonds cells in $\mathfrak{D}$. To measure the flattening of the triangles we denote with $\alpha_{\mathfrak{T}}$ the only real in $]0, \frac{\pi}{2}]$ such that $\sin(\alpha_{\mathfrak{T}}) := \min_{\text{D}\in\mathfrak{D}} |\sin(\alpha_{\text{D}})|$.
We introduce a positive number reg$(\mathfrak{T})$ that measures the regularity of the mesh. It is defined as:

$$\begin{aligned} \text{reg}(\mathfrak{T}) = \max\Big(& \frac{1}{\sin(\alpha_T)}, \mathcal{N}, \mathcal{N}^*, \max_{\text{D}\in\mathfrak{D}} \max_{\mathfrak{s}\in\mathcal{E}_{\text{D}}} \frac{d_{\text{D}}}{m_{\mathfrak{s}}}, \max_{\text{K}\in\mathfrak{M}} \frac{d_{\text{K}}}{\sqrt{m_{\text{K}}}}, \max_{\text{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} \left(\frac{d_{\text{K}^*}}{\sqrt{m_{\text{K}^*}}}\right), \\ & \max_{\text{K}\in\mathfrak{M}} \max_{\text{D}\in\mathfrak{D}_K} \left(\frac{d_{\text{K}}}{d_{\text{D}}}\right), \max_{\text{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} \max_{\text{D}\in\mathfrak{D}_{\text{K}^*}} \left(\frac{d_{\text{K}^*}}{d_{\text{D}}}\right)\Big). \end{aligned} \tag{3}$$

where $\mathcal{N}$ and $\mathcal{N}^*$ are the maximum of edges of each primal cell and the maximum of edges incident to any vertex. The number reg$(\mathfrak{T})$ should be uniformly bounded when size$(\mathfrak{T}) \to 0$ for the convergence to hold.
From the definition of reg$(\mathfrak{T})$, the following geometrical result holds: there exist two constants C_1 and C_2 depending on reg$(\mathfrak{T})$ such that $\forall \text{K} \in \mathfrak{M}, \forall \text{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ and $\forall \text{D} \in \mathfrak{D}$ such that $\text{D} \cap \text{K} \neq 0$ and $\text{D} \cap \text{K}^* \neq 0$ we have:

$$C_1 m_{\text{K}} \leq m_{\text{D}} \leq C_2 m_{\text{K}}, \qquad C_1 m_{\text{K}^*} \leq m_{\text{D}} \leq C_2 m_{\text{K}^*}$$

and

$$C_1\, d_{\text{K}} \leq d_{\text{D}} \leq C_2\, d_{\text{K}}, \qquad C_1\, d_{\text{K}^*} \leq d_{\text{D}} \leq C_2\, d_{\text{K}^*}.$$

2.4 Unknown and meshes

The DDFV method for Navier-Stokes problem uses staggered unknowns. We associate to each primal volume $\text{K} \in \mathfrak{M} \cup \partial\mathfrak{M}$ an unknown $\text{u}_{\text{K}} \in \mathbb{R}^2$ for the velocity, to every dual volume $\text{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ an unknown $\text{u}_{\text{K}^*} \in \mathbb{R}^2$ for the velocity and to each diamond $\text{D} \in \mathfrak{D}$ an unknown $\text{p}^{\mathfrak{D}} \in \mathbb{R}$ for the pressure. Those unknowns are collected in the families:

$$\mathbf{u}^{\mathfrak{T}} = \left((\text{u}_{\text{K}})_{\text{K}\in(\mathfrak{M}\cup\partial\mathfrak{M})}, (\text{u}_{\text{K}^*})_{\text{K}^*\in(\mathfrak{M}^*\cup\partial\mathfrak{M}^*)}\right) \in (\mathbb{R}^2)^{\mathfrak{T}} \quad \text{and} \quad \text{p}^{\mathfrak{D}} = \left((\text{p}^{\mathfrak{D}})_{\text{D}\in\mathfrak{D}}\right) \in \mathbb{R}^{\mathfrak{D}}.$$

We define two subspaces of the boundary mesh, where to impose Dirichlet and "outflow" boundary conditions:

$$\begin{aligned}
\partial\mathfrak{M}_D &= \{\mathtt{K} \in \partial\mathfrak{M} : x_{\mathtt{K}} \in \Gamma_1\};\\
\partial\mathfrak{M}_O &= \{\mathtt{K} \in \partial\mathfrak{M} : x_{\mathtt{K}} \in \Gamma_2 \setminus \Gamma_1\};\\
\partial\mathfrak{M}^*_D &= \{\mathtt{K}^* \in \partial\mathfrak{M}^* : x_{\mathtt{K}^*} \in \Gamma_1\};\\
\partial\mathfrak{M}^*_O &= \{\mathtt{K}^* \in \partial\mathfrak{M}^* : x_{\mathtt{K}^*} \in \Gamma_2 \setminus \Gamma_1\};
\end{aligned}$$

We also define a discrete subspace of $(\mathbb{R}^2)^{\mathfrak{T}}$ useful to take in account Dirichlet boundary conditions:

$$\mathbb{E}^D_g = \{\mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}, \text{s. t. } \forall \mathtt{K} \in \partial\mathfrak{M}_D,\ \mathbf{u}_{\mathtt{K}} = g(x_{\mathtt{K}}) \text{ and } \forall \mathtt{K}^* \in \partial\mathfrak{M}^*_D,\ \mathbf{u}_{\mathtt{K}^*} = g(x_{\mathtt{K}^*})\}.$$

2.5 Discrete operators

In this section we define discrete operators that are necessary to write and to analyse the DDFV scheme.

Definition 2.5 *We define the discrete gradient of a vector field of $(\mathbb{R}^2)^{\mathfrak{T}}$ as the operator*

$$\nabla^{\mathfrak{D}} : \mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}} \mapsto (\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}})_{\mathtt{D}\in\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}},$$

such that for $\mathtt{D} \in \mathfrak{D}$:

$$\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} = \frac{1}{\sin(\alpha_{\mathtt{D}})}\left[\frac{\mathbf{u}_{\mathtt{L}} - \mathbf{u}_{\mathtt{K}}}{m_{\sigma^*}} \otimes \vec{\mathbf{n}}_{\sigma\mathtt{K}} + \frac{\mathbf{u}_{\mathtt{L}^*} - \mathbf{u}_{\mathtt{K}^*}}{m_{\sigma}} \otimes \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\right], \tag{4}$$

where $\otimes$ represents the tensor product. It can also be written in the following way:

$$\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} = \frac{1}{2m_{\mathtt{D}}}\left[m_\sigma(\mathbf{u}_{\mathtt{L}} - \mathbf{u}_{\mathtt{K}}) \otimes \vec{\mathbf{n}}_{\sigma\mathtt{K}} + m_{\sigma^*}(\mathbf{u}_{\mathtt{L}} - \mathbf{u}_{\mathtt{K}^*}) \otimes \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\right].$$

Definition 2.6 *We define the discrete divergence of a discrete tensor field of $(\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$ as the operator*

$$\boldsymbol{div}^{\mathfrak{T}} : \xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}} \mapsto \boldsymbol{div}^{\mathfrak{T}}\xi^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{T}}.$$

Let $\xi^{\mathfrak{D}} = (\xi^D)_{D\in\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$, we set:

$$\boldsymbol{div}^{\mathfrak{T}}\xi^{\mathfrak{D}} = (\boldsymbol{div}^{\mathfrak{M}}\xi^{\mathfrak{D}}, \boldsymbol{div}^{\partial\mathfrak{M}}\xi^{\mathfrak{D}}, \boldsymbol{div}^{\mathfrak{M}^*}\xi^{\mathfrak{D}}, \boldsymbol{div}^{\partial\mathfrak{M}^*}\xi^{\mathfrak{D}}),$$

where we define $\boldsymbol{div}^{\mathfrak{M}}\xi^{\mathfrak{D}} = (\mathbf{div}^{\mathtt{K}}\xi^{\mathfrak{D}})_{\mathtt{K}\in\mathfrak{M}}$, $\boldsymbol{div}^{\partial\mathfrak{M}}\xi^{\mathfrak{D}} = 0$, $\boldsymbol{div}^{\mathfrak{M}^*}\xi^{\mathfrak{D}} = (\mathbf{div}^{\mathtt{K}^*}\xi^{\mathfrak{D}})_{\mathtt{K}\in\mathfrak{M}^*}$ and $\boldsymbol{div}^{\partial\mathfrak{M}^*}\xi^{\mathfrak{D}} = (\mathbf{div}^{\mathtt{K}^*}\xi^{\mathfrak{D}})_{\mathtt{K}^*\in\partial\mathfrak{M}^*}$ with:

$$\begin{aligned}
\mathbf{div}^{\mathtt{K}}\xi^{\mathfrak{D}} &= \frac{1}{m_{\mathtt{K}}}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma \xi^{\mathtt{D}}\vec{\mathbf{n}}_{\sigma\mathtt{K}}, \quad \forall \mathtt{K} \in \mathfrak{M}\\
\mathbf{div}^{\mathtt{K}^*}\xi^{\mathfrak{D}} &= \frac{1}{m_{\mathtt{K}^*}}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\xi^{\mathtt{D}}\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}, \quad \forall \mathtt{K}^* \in \mathfrak{M}^*\\
\mathbf{div}^{\mathtt{K}^*}\xi^{\mathfrak{D}} &= \frac{1}{m_{\mathtt{K}^*}}\left(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}} m_{\sigma^*}\xi^{\mathtt{D}}\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}\cap\mathfrak{D}_{ext}} \frac{m_\sigma}{2}\xi^{\mathtt{D}}\vec{\mathbf{n}}_{\sigma\mathtt{K}}\right) \quad \forall \mathtt{K}^* \in \partial\mathfrak{M}^*.
\end{aligned}$$

Definition 2.7 *We define the discrete divergence of a vector field of $(\mathbb{R}^2)^{\mathfrak{T}}$ as the operator*

$$div^{\mathfrak{D}} : \mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}} \mapsto (div^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}})_{\mathtt{D}\in\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$$

with

$$div^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} = Tr(\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}}), \quad \forall \mathtt{D} \in \mathfrak{D}.$$

Definition 2.8 *We define the discrete strain rate tensor of a vector field in $(\mathbb{R}^2)^{\mathfrak{T}}$ as the operator*

$$\mathrm{D}^{\mathfrak{D}} : \mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}} \mapsto (\mathrm{D}^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}})_{\mathtt{D}\in\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$$

such that for $\mathtt{D} \in \mathfrak{D}$:

$$\mathrm{D}^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} = \frac{\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} + {}^t(\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}})}{2}. \tag{5}$$

Definition 2.9 *We define the discrete curl of a vector field of* $(\mathbb{R}^2)^{\mathfrak{T}}$ *as the operator*

$$curl^{\mathfrak{D}} : \mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}} \mapsto curl^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}},$$

such that for $\mathtt{D} \in \mathfrak{D}$*:*

$$curl^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} = \frac{1}{2m_{\mathtt{D}}}\left[m_\sigma(\mathbf{u}_{\mathtt{L}} - \mathbf{u}_{\mathtt{K}}) \otimes \vec{\boldsymbol{\tau}}_{\boldsymbol{\sigma^*},\mathtt{K}^*} - m_{\sigma^*}(\mathbf{u}_{\mathtt{L}^*} - \mathbf{u}_{\mathtt{K}^*}) \otimes \vec{\boldsymbol{\tau}}_{\boldsymbol{\sigma},\mathtt{K}}\right].$$

Definition 2.10 *We define the discrete rotational of a vector field of* $(\mathbb{R}^2)^{\mathfrak{T}}$ *as the operator*

$$rot^{\mathfrak{D}} : \mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}} \mapsto rot^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} \in \mathbb{R}^{\mathfrak{D}}$$

with

$$rot^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} = -Tr(curl^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}}), \quad \forall \mathtt{D} \in \mathfrak{D}.$$

2.6 Scalar products and norms

We define the *trace operators* on $(\mathbb{R}^2)^{\mathfrak{T}}$ and $\mathbb{R}^{\mathfrak{D}}$. Let $\gamma^{\mathfrak{T}} : \mathbf{u}^{\mathfrak{T}} \mapsto \gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}) = (\gamma_\sigma(\mathbf{u}^{\mathfrak{T}}))_{\sigma\in\partial\mathfrak{M}} \in (\mathbb{R}^2)^{\mathfrak{T}}$, such that:

$$\gamma_\sigma(\mathbf{u}^{\mathfrak{T}}) = \frac{\mathbf{u}_{\mathtt{K}^*} + 2\mathbf{u}_{\mathtt{L}} + \mathbf{u}_{\mathtt{L}^*}}{4} \quad \forall\sigma = [x_{\mathtt{K}^*}, x_{\mathtt{L}^*}] \in \partial\mathfrak{M}.$$

We can also define $\widetilde{\gamma}^{\mathfrak{T}} : \mathbf{u}^{\mathfrak{T}} \mapsto \widetilde{\gamma}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}) = (\widetilde{\gamma}_\sigma(\mathbf{u}^{\mathfrak{T}}))_{\sigma\in\partial\mathfrak{M}} \in (\mathbb{R}^2)^{\mathfrak{T}}$, such that:

$$\widetilde{\gamma}_\sigma(\mathbf{u}^{\mathfrak{T}}) = \frac{\mathbf{u}_{\mathtt{K}^*} + 2\mathbf{u}_{\mathtt{K}} + \mathbf{u}_{\mathtt{L}^*}}{4} \quad \forall\sigma = [x_{\mathtt{K}^*}, x_{\mathtt{L}^*}] \in \partial\mathfrak{M},$$

that will appear in the equations of the scheme and later in the proof of the property of the nonlinear convection term.
On the diamond mesh, we define $\gamma^{\mathfrak{D}} : \Phi^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}} \mapsto (\Phi^D)_{\mathtt{D}\in\mathfrak{D}_{ext}} \in (\mathbb{R}^2)^{\mathfrak{D}_{ext}}$, the operator of restriction on $\mathfrak{D}_{ext}$.
Now we define the scalar products on the approximation spaces:

$$[[\mathbf{v}^{\mathfrak{T}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} = \frac{1}{2}\Bigg(\sum_{\mathtt{K}\in\mathfrak{M}} m_{\mathtt{K}}\,\mathbf{u}_{\mathtt{K}} \cdot \mathbf{v}_{\mathtt{K}} + \sum_{\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} m_{\mathtt{K}^*}\,\mathbf{u}_{\mathtt{K}^*} \cdot \mathbf{v}_{\mathtt{K}^*}\Bigg) \quad \forall \mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$$

$$(\Phi^{\mathfrak{D}}, \mathbf{v}^{\mathfrak{T}})_{\partial\Omega} = \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma \Phi^D \cdot \mathbf{v}_\sigma \quad \forall\Phi^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}_{ext}}, \mathbf{v}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\partial\mathfrak{M}}$$

$$(\xi^{\mathfrak{D}} : \Phi^{\mathfrak{D}})_{\mathfrak{D}} = \sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}}\big(\xi^D : \Phi^D\big) \quad \forall \xi^{\mathfrak{D}}, \Phi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$$

$$(\mathrm{p}^{\mathfrak{D}}, q^{\mathfrak{D}})_{\mathfrak{D}} = \sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}} p^{\mathtt{D}} q^{\mathtt{D}} \quad \forall \mathrm{p}^{\mathfrak{D}}, q^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}},$$

and the corresponding norms:

$$\|\mathbf{u}^{\mathfrak{T}}\|_2 = [[\mathbf{u}^{\mathfrak{T}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}}^{1/2}, \quad \forall \mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}},$$
$$\|\xi^{\mathfrak{D}}\|_2 = (\xi^{\mathfrak{D}} : \xi^{\mathfrak{D}})_{\mathfrak{D}}^{1/2}, \quad \forall \xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}},$$
$$\|\mathrm{p}^{\mathfrak{D}}\|_2 = (\mathrm{p}^{\mathfrak{D}}, \mathrm{p}^{\mathfrak{D}})_{\mathfrak{D}}^{1/2}, \quad \forall \mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}.$$

We can generalize $\forall p \geq 1$ and we can define for any $\mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ and $\xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$:

$$\|\mathbf{u}^{\mathfrak{T}}\|_p = \frac{1}{2}\Bigg(\sum_{\mathtt{K}\in\mathfrak{M}} m_{\mathtt{K}}|\mathbf{u}_{\mathtt{K}}|^p + \sum_{\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} m_{\mathtt{K}^*}|\mathbf{u}_{\mathtt{K}^*}|^p\Bigg)^{1/p} = \frac{1}{2}\Bigg(\|\mathbf{u}^{\mathfrak{M}}\|_p^p + \|\mathbf{u}^{\mathfrak{M}^*\cup\partial\mathfrak{M}^*}\|_p^p\Bigg)^{1/p},$$

$$\|\xi^{\mathfrak{D}}\|_p = \Bigg(\sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}}|\xi^{\mathfrak{D}}|^p\Bigg)^{1/p},$$

$$\|\mathbf{u}^{\mathfrak{T}}\|_{1,p} = \|\mathbf{u}^{\mathfrak{T}}\|_p + \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_p,$$

$$\|\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{p,\partial\Omega} = \Bigg(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma|\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})|^p\Bigg)^{1/p} \text{ and } \|\widetilde{\gamma}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{p,\partial\Omega} = \Bigg(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma|\widetilde{\gamma}^\sigma(\mathbf{u}^{\mathfrak{T}})|^p\Bigg)^{1/p}.$$

2.7 Green's formula

In [1], [12] the discrete gradient and discrete divergence for a scalar-valued function are linked by a discrete Stokes formula. This is precisely the duality property that gives its name to the method.

Theorem 2.11 ***Discrete Green's formula: (Thm. IV.9 in [17])***
For all $\xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$*, we have:*

$$[[\mathbf{div}^{\mathfrak{T}}\xi^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} = -(\xi^{\mathfrak{D}} : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})_{\mathfrak{D}} + (\gamma^{\mathfrak{D}}(\xi^{\mathfrak{D}}) \cdot \overrightarrow{\boldsymbol{n}}, \gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}))_{\partial\Omega},$$

where $\overrightarrow{\boldsymbol{n}}$ *is the unitary outer normal.*

2.8 Approximation of the nonlinear convection term

We will consider the weak formulation of (1), thus we will need to discretize the nonlinear convection term $\displaystyle\int_{\Omega}(\mathbf{u}\cdot\nabla)\mathbf{u}$.
As in [17, 19], we construct a bilinear form $\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}})$ as an approximation of $\displaystyle\int_{\Omega}(\mathbf{u}\cdot\nabla)\mathbf{v}$.
The form introduced in [17, 19] is built in order to take into account homogeneous Dirichlet boundary conditions, so we need to modify it in order to handle boundary terms.

To obtain the approximation of the convection term, we need to integrate the equation over the primal and dual mesh; we approximate $\displaystyle\int_{\kappa}(\mathbf{u}\cdot\nabla)\mathbf{v}$ when $\kappa \in \mathfrak{M}$ with $m_{\kappa}\,\mathbf{b}^{\kappa}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}})$.
We remark that for $\mathbf{u}$ and $\mathbf{v}$ smooth functions:

$$\int_{\kappa}(\mathbf{u}\cdot\nabla)\mathbf{v} = \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\kappa}} \int_{\sigma}(\mathbf{u}\cdot\vec{\mathbf{n}}_{\sigma\kappa})\mathbf{v}, \quad \forall \kappa \in \mathfrak{M}.$$

Such as for the Dirichlet case [19], we look for an approximation of the fluxes: $\displaystyle\int_{\sigma}(\mathbf{u}\cdot\vec{\mathbf{n}}_{\sigma\kappa}) \rightsquigarrow F_{\sigma,\kappa}(\mathbf{u}^{\mathfrak{T}})$.
We obtain them by calculating the fluxes on the sides $\mathfrak{s}$ of diamonds for the interior edges (see Fig. 3). For what concerns the boundary edges, the definition depends on the trace $\gamma^{\sigma}(\mathbf{u}^{\mathfrak{T}})$. So we impose:

$$F_{\sigma,\kappa}(\mathbf{u}^{\mathfrak{T}}) = \begin{cases} -\displaystyle\sum_{\mathfrak{s}\in\mathfrak{S}_{\kappa}\cap\mathcal{E}_{\mathtt{D}}} G_{\mathfrak{s},\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}) & \text{if } \sigma \in \mathcal{E}_{int} \\ m_{\sigma}\gamma^{\sigma}(\mathbf{u}^{\mathfrak{T}})\cdot\vec{\mathbf{n}}_{\sigma\kappa} & \text{if } \sigma \in \partial\Omega \end{cases} \tag{6}$$

and with an equivalent argument, we define for the dual edges:

$$F_{\sigma^*,\kappa^*}(\mathbf{u}^{\mathfrak{T}}) = \begin{cases} -\displaystyle\sum_{\mathfrak{s}\in\mathfrak{S}_{\kappa^*}\cap\mathcal{E}_{\mathtt{D}}} G_{\mathfrak{s},\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}) & \text{if } \kappa^* \in \mathfrak{M}^*\cup\partial\mathfrak{M}^*,\ \sigma^*\cap\partial\Omega = \emptyset \\ -G_{\mathfrak{s},\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}) - \dfrac{1}{2}F_{\sigma,\kappa}(\mathbf{u}^{\mathfrak{T}}) & \text{if } \kappa^* \in \partial\mathfrak{M}^*,\ \sigma^*\cap\partial\Omega \neq \emptyset \end{cases} \tag{7}$$

where

$$G_{\mathfrak{s},\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}) = m_{\mathfrak{s}}\frac{\mathbf{u}_{\kappa}+\mathbf{u}_{\kappa^*}}{2}\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}}, \qquad \forall\mathfrak{s} = [x_{\kappa}, x_{\kappa^*}].$$

Remark that we have conservativity of the fluxes $F_{\sigma,\kappa}$ and F_{σ^*,κ^*}:

$$F_{\sigma,\kappa}(\mathbf{u}^{\mathfrak{T}}) = -F_{\sigma,\mathtt{L}}(\mathbf{u}^{\mathfrak{T}}), \quad \forall\sigma = \kappa|\mathtt{L} \quad \text{and} \quad F_{\sigma^*,\kappa^*}(\mathbf{u}^{\mathfrak{T}}) = -F_{\sigma^*,\mathtt{L}^*}(\mathbf{u}^{\mathfrak{T}}), \quad \forall\sigma^* = \kappa^*|\mathtt{L}^*. \tag{8}$$

Unlike in [17, 19], we do not stabilize the solenoidal constraint thus we do not need to add a stabilization term in the flux $G_{\mathfrak{s},\mathtt{D}}$.
Consequently, we denote:

$$\mathfrak{D}_{\kappa}^{int} = \{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\kappa}\cap\mathfrak{D}_{int}\}, \qquad \mathfrak{D}_{\kappa}^{ext} = \{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\kappa}\cap\mathfrak{D}_{ext}\},$$
$$\mathfrak{D}_{\kappa^*}^{int} = \{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\kappa^*}\cap\mathfrak{D}_{int}\}, \quad \mathfrak{D}_{\kappa^*}^{ext} = \{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\kappa^*}\cap\mathfrak{D}_{ext}\}.$$

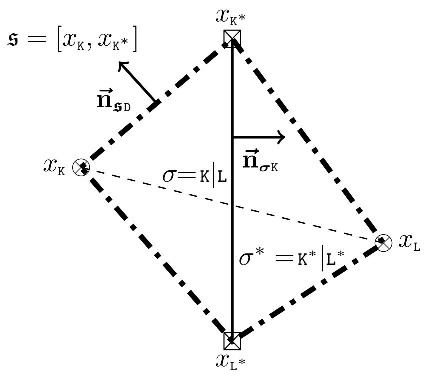


Figure 3: A diamond $D = D_{\sigma,\sigma^*}$ with $\sigma \subset \mathcal{E}_{int}$

We define our bilinear form on the primal mesh as:

$$m_K \mathbf{b}^K(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}}) = \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_K^{int}} F_{\sigma,K}(\mathbf{u}^{\mathfrak{T}}) \mathbf{v}_\sigma^+ + \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_K^{ext}} F_{\sigma,K}(\mathbf{u}^{\mathfrak{T}}) \gamma^\sigma(\mathbf{v}^{\mathfrak{T}}) \qquad \forall K \in \mathfrak{M}$$

where

$$\mathbf{v}_\sigma^+ = \begin{cases} \mathbf{v}_K & \text{if } F_{\sigma,K} \geq 0 \\ \mathbf{v}_L & \text{otherwise} \end{cases} \qquad \forall \sigma \in \mathcal{E}_{int},$$

and on the dual mesh as:

$$m_{K^*} \mathbf{b}^{K^*}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}}) = \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_{K^*}} F_{\sigma^*,K^*}(\mathbf{u}^{\mathfrak{T}}) \mathbf{v}_{\sigma_*}^+ \qquad \forall K^* \in \mathfrak{M}^*$$

$$m_{K^*} \mathbf{b}^{K^*}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}}) = \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_{K^*}} F_{\sigma^*,K^*}(\mathbf{u}^{\mathfrak{T}}) \mathbf{v}_{\sigma_*}^+ + \frac{1}{2} \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_{K^*}^{ext}} F_{\sigma,K}(\mathbf{u}^{\mathfrak{T}}) \gamma^\sigma(\mathbf{v}^{\mathfrak{T}}) \quad \forall K^* \in \partial\mathfrak{M}^*$$

where

$$\mathbf{v}_{\sigma_*}^+ = \begin{cases} \mathbf{v}_{K^*} & \text{if } F_{\sigma^*,K^*} \geq 0 \\ \mathbf{v}_{L^*} & \text{otherwise} \end{cases} \qquad \forall \sigma^* \in \mathcal{E}^*.$$

We choose to do upwinding on the interior diamonds because we started from the analysis done in [17] and [19]. In the case of Dirichlet boundary conditions in [17, 19], it is necessary to upwind in order to get wellposedness of the scheme and an energy estimate, since it is the key to prove an inequality of the type $[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}}), \mathbf{w}^{\mathfrak{T}}]]_{\mathfrak{T}} \geq 0$. In our case, in the weak formulation the convection term is symmetrized, so the upwinding is not necessary. But since it does not introduce additional difficulties, we prefer to keep the same definitions to show the continuity of the work. Passing to a centered scheme would not impact the analysis; we would expect from it a better accuracy.

3 DDFV scheme

Let $N \in \mathbb{N}^*$. We note $\delta t = \frac{T}{N}$ and $t_n = n\delta t$ for $n \in \{0, \dots N\}$. To obtain our DDFV scheme, we choose to use an implicit Euler time discretization, except for the nonlinear term, which is linearized by using a semi-implicit approximation.
We look for $\mathbf{u}^{\mathfrak{T},[0,T]} = (\mathbf{u}^n)_{n \in \{0,\dots N\}} \in \left(\mathbb{E}_{\mathbf{g}_1}^D\right)^{N+1}$ and $\mathrm{p}^{\mathfrak{D},[0,T]} = (\mathrm{p}^n)_{n \in \{0,\dots N\}} \in (\mathbb{R}^{\mathfrak{D}})^{N+1}$, that we initialize with:

$$\mathbf{u}^0 = \mathbb{P}_c^{\mathfrak{T}} \mathbf{u}_0 \in \mathbb{E}_{\mathbf{g}_1}^D \tag{9}$$

where we define the centered projection on the mesh $\mathfrak{T}$ as:

$$\mathbb{P}_c^{\mathfrak{T}} \mathbf{v} = ((\mathbf{v}(x_K))_{K \in (\mathfrak{M} \cup \partial\mathfrak{M})}, (\mathbf{v}(x_{K^*}))_{K^* \in (\mathfrak{M}^* \cup \partial\mathfrak{M}^*)}), \quad \forall \mathbf{v} \in (H^2(\Omega))^2.$$

We would like to write the system (1) in our setting.

For what concerns the **equation of the momentum**, we start by finding the discrete equivalent of the variational formulation of the problem. For the continuous problem, as presented in [4], the velocity $\mathbf{u}$ satisfies:

$$\int_\Omega \partial_t \mathbf{u}\cdot\Psi + \frac{2}{\mathrm{Re}}\int_\Omega \mathrm{D}(\mathbf{u}):\mathrm{D}(\Psi) + \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Psi - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\Psi\cdot\mathbf{u}$$
$$= -\frac{1}{2}\int_{\Gamma_2}(\mathbf{u}\cdot\vec{\mathbf{n}})^+(\mathbf{u}\cdot\Psi) + \frac{1}{2}\int_{\Gamma_2}(\mathbf{u}\cdot\vec{\mathbf{n}})^-(\mathbf{u}_{ref}\cdot\Psi) + \int_{\Gamma_2}\sigma_{ref}\cdot\vec{\mathbf{n}}\cdot\Psi \quad (10)$$

where Ψ is a test function in the space

$$V = \{\Psi \in (H^1(\Omega))^2,\ \Psi|_{\Gamma_1} = 0,\ \mathrm{div}(\Psi) = 0\}.$$

This weak formulation (10) can be rewritten in the DDFV framework (with the operators introduced in section 2) as:

$$[[\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\delta t},\Psi^{\mathfrak{T}}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1},\mathrm{D}^{\mathfrak{D}}\Psi^{\mathfrak{T}})_{\mathfrak{D}} + \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^n,\mathbf{u}^{n+1}),\Psi^{\mathfrak{T}}]]_{\mathfrak{T}} - \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^n,\Psi^{\mathfrak{T}}),\mathbf{u}^{n+1}]]_{\mathfrak{T}}$$
$$= -\frac{1}{2}\sum_{\mathtt{D}\in\mathtt{D}_{ext}\cap\Gamma_2}(F_{\sigma,\mathtt{K}}(\mathbf{u}^n))^+\gamma^\sigma(\mathbf{u}^{n+1})\cdot\gamma^\sigma(\Psi^{\mathfrak{T}}) + \frac{1}{2}\sum_{\mathtt{D}\in\mathtt{D}_{ext}\cap\Gamma_2}(F_{\sigma,\mathtt{K}}(\mathbf{u}^n))^-\gamma^\sigma(\mathbf{u}_{ref})\cdot\gamma^\sigma(\Psi^{\mathfrak{T}})$$
$$+\sum_{\mathtt{D}\in\mathtt{D}_{ext}\cap\Gamma_2} m_\sigma(\sigma^{\mathfrak{D}}_{\mathrm{ref}}\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}})\cdot\gamma^\sigma(\Psi^{\mathfrak{T}}), \quad (11)$$

where $\Psi^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ is a test function in the discrete space that satisfies similar properties compared to the continuous test function Ψ:

$$\begin{cases}\Psi^{\mathfrak{T}} \in \mathbb{E}_0^D,\\ \mathrm{div}^{\mathfrak{D}}(\Psi^{\mathfrak{T}}) = 0.\end{cases} \quad (12)$$

To simplify the computations, as in the continuous case (see [4]), the reference flow $(\mathbf{u}^{\mathfrak{T}}_{ref},\mathrm{p}^{\mathfrak{D}}_{ref}) \in \mathbb{E}^D_{\mathbf{g}_1}\times\mathbb{R}^{\mathfrak{D}}$ is supposed to be a solution of the under-determined steady Stokes problem:

$$\begin{cases}-\mathbf{div}^{\mathfrak{T}}\left(\frac{2}{\mathrm{Re}}\mathrm{D}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}_{ref}) - \mathrm{p}^{\mathfrak{D}}\mathrm{Id}\right) = 0,\\ \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}_{ref}) = 0.\end{cases}$$

From this formulation, we design our DDFV scheme as follows:

- For all $\mathtt{K}\in\mathfrak{M}$:

$$m_{\mathtt{K}}\frac{\mathbf{u}^{n+1}_{\mathtt{K}}-\mathbf{u}^n_{\mathtt{K}}}{\delta t} - m_{\mathtt{K}}\mathbf{div}^{\mathtt{K}}(\sigma^{\mathfrak{D}}(\mathbf{u}^{n+1},\mathrm{p}^{n+1})) + \frac{1}{2}m_{\mathtt{K}}\mathbf{b}^{\mathtt{K}}(\mathbf{u}^n,\mathbf{u}^{n+1})$$
$$-\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}^{int}_{\mathtt{K}}}\left(F^+_{\sigma,\mathtt{K}}(\mathbf{u}^n)\mathbf{u}^{n+1}_{\mathtt{K}} - F^-_{\sigma,\mathtt{L}}(\mathbf{u}^n)\mathbf{u}^{n+1}_{\mathtt{L}}\right) = 0; \quad (13)$$

- For all $\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*_O$:

$$m_{\mathtt{K}^*}\frac{\mathbf{u}^{n+1}_{\mathtt{K}^*}-\mathbf{u}^n_{\mathtt{K}^*}}{\delta t} - m_{\mathtt{K}^*}\mathbf{div}^{\mathtt{K}^*}(\sigma^{\mathfrak{D}}(\mathbf{u}^{n+1},\mathrm{p}^{n+1})) + \frac{1}{2}m_{\mathtt{K}^*}\mathbf{b}^{\mathtt{K}^*}(\mathbf{u}^n,\mathbf{u}^{n+1})$$
$$-\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}^*}}\left(F^+_{\sigma^*,\mathtt{K}^*}(\mathbf{u}^n)\mathbf{u}^{n+1}_{\mathtt{K}^*} - F^-_{\sigma^*,\mathtt{L}^*}(\mathbf{u}^n)\mathbf{u}^{n+1}_{\mathtt{L}^*}\right) = 0; \quad (14)$$

- For all $\sigma\in\partial\mathfrak{M}_O$:

$$m_\sigma\sigma^{\mathtt{D}}(\mathbf{u}^{n+1},\mathrm{p}^{n+1}))\vec{\mathbf{n}}_{\sigma\mathtt{L}} - \frac{1}{4}F_{\sigma,\mathtt{L}}(\mathbf{u}^n)\left(\mathbf{u}^{n+1}_{\mathtt{K}}-\mathbf{u}^{n+1}_{\mathtt{L}}\right)$$
$$= -\frac{1}{2}(F_{\sigma,\mathtt{L}}(\mathbf{u}^n))^-\left(\gamma^\sigma(\mathbf{u}^{n+1}) - \gamma^\sigma(\mathbf{u}_{ref})\right) + m_\sigma(\sigma^{\mathtt{D}}_{\mathrm{ref}}\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}}); \quad (15)$$

- For all $\mathtt{D}\in\mathfrak{D}$:

$$\mathrm{div}^{\mathtt{D}}(\mathbf{u}^{n+1}) = 0. \quad (16)$$

4 Well-posedness

We now prove the existence and uniqueness of the solution of our DDFV scheme.
The well-posedness result relies on a uniform discrete *inf-sup condition*. We could have add a stabilization term to the equation of conservation of mass to generalize the result to general meshes, as done in [14, 18] for Stokes and in [19] for Navier-Stokes, but since our proof for Korn's inequality (that is crucial to prove the energy estimate) requires the hypothesis of inf-sup stability, we decided not to stabilize the equation. This hypothesis is not restrictive, and in the following section we recall the definition and the related properties.

4.1 Inf-sup stability

Inf-sup stability for DDFV method was studied in [5].
It has been proven to hold unconditionally for conforming acute triangle meshes, non-conforming triangle meshes and chechkerboard meshes. For some conforming or non-conforming Cartesian meshes, it holds up to a single unstable pressure mode. Moreover, it has been proven numerically for many other families of meshes and it has still not been found a mesh that does not satisfy it.

Theorem 4.1 *A given DDFV mesh $\mathfrak{T}$ is said to satisfy the Inf-Sup stability if the following condition holds:*

$$\beta_{\mathfrak{T}} := \inf_{\mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}} \left(\sup_{\mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0} \frac{a^{\mathfrak{T}}(\mathbf{v}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})}{\|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathfrak{T}}\|_2 \|\mathrm{p}^{\mathfrak{D}} - m(\mathrm{p}^{\mathfrak{D}})\|_2} \right) > 0,$$

where $a^{\mathfrak{T}}(\mathbf{v}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}) = (\mathrm{div}^{\mathfrak{D}} \mathbf{v}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})_{\mathfrak{D}}$ *and* $m(\mathrm{p}^{\mathfrak{D}}) = \sum_{\mathtt{D} \in \mathfrak{D}} m_{\mathtt{D}} \mathrm{p}^{\mathtt{D}}$.

From this inequality, two important properties follow:

- There exists $C > 0$, depending on $\beta_{\mathfrak{T}}$, such that $\forall \mathrm{p}^{\mathfrak{D}} \in \mathfrak{D}$, $\forall \mathbf{v}^{\mathfrak{T}} \in \mathbb{E}_0$:

$$\|\mathrm{p}^{\mathfrak{D}} - m(\mathrm{p}^{\mathfrak{D}})\|_2 \leq C \|\nabla^{\mathfrak{D}} \mathbf{v}^{\mathfrak{T}}\|_2, \tag{17}$$

- For every $\mathrm{p}^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ such that $m(\mathrm{p}^{\mathfrak{D}}) = 0$, there exists $\mathbf{w}^{\mathfrak{T}} \in \mathbb{E}_0$ such that:

$$\begin{aligned} \mathrm{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}) &= \mathrm{p}^{\mathfrak{D}} \\ \|\nabla^{\mathfrak{D}} \mathbf{w}^{\mathfrak{T}}\|_2 &\leq C \|\mathrm{p}^{\mathfrak{D}}\|_2, \end{aligned} \tag{18}$$

  where $\mathbb{E}_0 = \{\mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}, \text{s t. } \forall \mathtt{K} \in \partial\mathfrak{M},\ \mathbf{u}_{\mathtt{K}} = 0 \text{ and } \forall \mathtt{K}^* \in \partial\mathfrak{M}^*,\ \mathrm{u}_{\mathtt{K}^*} = 0\}$.

4.2 Existence and uniqueness

Theorem 4.2 *(Well-posedness) Let $\mathfrak{T}$ be a DDFV mesh associated to Ω that satisfies the inf-sup stability condition. The scheme* (9), (13)-(16) *has a unique solution* $(\boldsymbol{u}^{\mathfrak{T},[0,T]}, p^{\mathfrak{D},[0,T]}) \in (\mathbb{E}^D_{\boldsymbol{g}_1})^{N+1} \times (\mathbb{R}^{\mathfrak{D}})^{N+1}$.

Proof The scheme issued from the equations (13)-(16) is a linear system $Av = b$ with A square matrix at each time step. We want to show that A is injective, thus we study the kernel of the matrix. Let $v = (\mathbf{u}^{n+1}, \mathrm{p}^{n+1}) \in \mathbb{E}^D_{\mathbf{g}_1} \times \mathbb{R}^{\mathfrak{D}}$ be in $ker(A)$: we then obtain the system $Av = 0$. If we multiply this relation by a test function $\Psi^{\mathfrak{T}}$ that satisfies (12), this is equivalent to consider the discrete variational formulation (11) in the form:

$$\begin{aligned} \frac{1}{\delta t}[[\mathbf{u}^{n+1}, \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1}, \mathrm{D}^{\mathfrak{D}}\Psi^{\mathfrak{T}})_{\mathfrak{D}} + \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^n, \mathbf{u}^{n+1}), \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} - \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^n, \Psi^{\mathfrak{T}}), \mathbf{u}^{n+1}]]_{\mathfrak{T}} \\ = -\frac{1}{2} \sum_{\mathtt{D} \in \mathtt{D}_{ext} \cap \Gamma_2} (F_{\sigma,\mathtt{K}}(\mathbf{u}^n))^+ \gamma^{\sigma}(\mathbf{u}^{n+1}) \cdot \gamma^{\sigma}(\Psi^{\mathfrak{T}}). \end{aligned}$$

The choice $\Psi^{\mathfrak{T}} = \mathbf{u}^{n+1}$ leads to:

$$\frac{1}{\delta t}||\mathbf{u}^{n+1}||_2^2 + \frac{2}{\mathrm{Re}}||\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1}||_2^2 + \underbrace{\frac{1}{2}\sum_{\mathtt{D} \in \mathtt{D}_{ext}} (F_{\sigma,\mathtt{K}}(\mathbf{u}^n))^+ \left|\gamma^{\sigma}(\mathbf{u}^{n+1})\right|^2}_{\geq 0} = 0,$$

that implies

$$\frac{1}{\delta t}||\mathbf{u}^{n+1}||_2^2 + \frac{2}{\mathrm{Re}}||\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1}||_2^2 \leq 0$$

from which we deduce that $\mathbf{u}^{n+1} = 0$.
To conclude the proof, we need to show that p^{n+1} is equal to zero too. Thanks to the inequality (17) ensured by inf-sup stability and to the fact that $\mathbf{u}^{n+1} = 0$, we can deduce that p^{n+1} is constant. Then, the condition (15) on $\partial\mathfrak{M}_O$ implies that $\mathrm{p}^{n+1} = 0$ on the boundary, since we recall that $m_\sigma \sigma^{\mathrm{D}}(\mathbf{u}^{n+1}, \mathrm{p}^{n+1}))\vec{\mathbf{n}}_{\sigma\mathrm{L}} = m_\sigma \left(\frac{2}{\mathrm{Re}}\mathrm{D}^{\mathfrak{D}}(\mathbf{u}^{n+1}) - \mathrm{p}^{\mathfrak{D}}\mathrm{Id}\right)\vec{\mathbf{n}}_{\sigma\mathrm{L}}$, that $\mathbf{u}^{\mathfrak{T}}_{ref} = \sigma^{\mathfrak{D}}_{ref} = 0$ because we are studying $ker(A)$ and that $\mathbf{u}^{n+1} = 0$.
Thus, by putting together the fact that p^{n+1} is constant and it is zero on the boundary, we have $\mathrm{p}^{n+1} = 0$ in all the domain.

■

Remark 4.3 *Supposing inf-sup condition is not that restrictive; just in the case of Cartesian meshes the stability is proved up to a checkerboard mode for the pressure, but thanks to the boundary conditions that we impose even in this case we can deduce that $p^{n+1} = 0$. Moreover, lots of numerical tests have been done and it still has not been found another mesh that does not satisfy the condition, see [5].*

5 Property of the convection term

We need to prove the following estimate in order to establish a discrete energy estimate:

Proposition 5.1 *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. For all $(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}}, \mathbf{w}^{\mathfrak{T}}) \in \mathbb{E}^D_{\boldsymbol{g}_1} \times \mathbb{E}^D_{\boldsymbol{g}_1} \times \mathbb{E}^D_{\boldsymbol{g}_1}$, there exists a constant $C > 0$ that depens only on Ω and reg($\mathfrak{T}$) such that:*

$$\begin{aligned}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}}), \mathbf{w}^{\mathfrak{T}}]]_{\mathfrak{T}} \leq & C\left(\|\mathbf{u}^{\mathfrak{T}}\|_3 + \|\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{3,\partial\Omega}\right)\|\mathbf{v}^{\mathfrak{T}}\|_6\|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2 \\ & + C\,\|\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{\frac{8}{3},\partial\Omega}\|\gamma^{\mathfrak{T}}(\mathbf{v}^{\mathfrak{T}})\|_{\frac{8}{3},\partial\Omega}\|\widetilde{\gamma}^{\mathfrak{T}}(\mathbf{w}^{\mathfrak{T}})\|_{4,\partial\Omega}\end{aligned}$$

Before starting the proof, we recall the following Hölder's inequality:

Theorem 5.2 *(Hölder's inequality) Let $p, q, r \in (1, +\infty)$ with $1/p+1/q+1/r = 1$. For every $(x_1, \ldots x_n)$, $(y_1, \ldots y_n), (z_1, \ldots z_n) \in \mathbb{R}^n$ it holds*

$$\sum_{i=1}^n |x_i\, y_i\, z_i| \leq \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}\left(\sum_{i=1}^n |y_i|^q\right)^{1/q}\left(\sum_{i=1}^n |z_i|^r\right)^{1/r}$$

Proof (of Proposition 5.1) By the definition of the scalar product $[[\cdot,\cdot]]_{\mathfrak{T}}$ and of the convection term:

$$\begin{aligned}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}}), \mathbf{w}^{\mathfrak{T}}]]_{\mathfrak{T}} &= \frac{1}{2}\sum_{\mathtt{K}\in\mathfrak{M}} m_{\mathtt{K}}\mathbf{w}_{\mathtt{K}}\cdot\mathbf{b}^{\mathtt{K}}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}}) + \frac{1}{2}\sum_{\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} m_{\mathtt{K}^*}\mathbf{w}_{\mathtt{K}^*}\cdot\mathbf{b}^{\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}}) \\ &= \frac{1}{2}\sum_{\mathtt{K}\in\mathfrak{M}}\mathbf{w}_{\mathtt{K}}\cdot\Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}^{int}_{\mathtt{K}}} F_{\sigma,\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})\mathbf{v}^+_\sigma + \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}^{ext}_{\mathtt{K}}} F_{\sigma,\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})\gamma^\sigma(\mathbf{v}^{\mathfrak{T}})\Big) \\ &+ \frac{1}{2}\sum_{\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*}\mathbf{w}_{\mathtt{K}^*}\cdot\Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{\mathtt{K}^*}} F_{\sigma^*,\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}})\mathbf{v}^+_{\sigma_*} + \frac{1}{2}\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}^{ext}_{\mathtt{K}^*}} F_{\sigma,\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})\gamma^\sigma(\mathbf{v}^{\mathfrak{T}})\Big).\end{aligned}$$

If we reorganize the sum over diamonds, since the fluxes are conservatives (see (8)) , we get:

$$\begin{aligned}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}, \mathbf{v}^{\mathfrak{T}}), \mathbf{w}^{\mathfrak{T}}]]_{\mathfrak{T}} &= \frac{1}{2}\Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{int}} F_{\sigma,\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})\mathbf{v}^+_\sigma\cdot(\mathbf{w}_{\mathtt{K}} - \mathbf{w}_{\mathtt{L}}) + 2\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} F_{\sigma,\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})\gamma^\sigma(\mathbf{v}^{\mathfrak{T}})\cdot\widetilde{\gamma}^\sigma(\mathbf{w}^{\mathfrak{T}}) \\ &+ \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}} F_{\sigma^*,\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}})\mathbf{v}^+_{\sigma_*}\cdot(\mathbf{w}_{\mathtt{K}^*} - \mathbf{w}_{\mathtt{L}^*})\Big) \\ &:= \frac{1}{2}(\mathrm{T}_1 + 2\mathrm{T}_2 + \mathrm{T}_3).\end{aligned}$$

Estimate of T_1:

By the definition of $\mathbf{v}^+_\sigma$, we have:

$$\begin{aligned}|\mathrm{T}_1| = \Big|\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{int}} F_{\sigma,\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})\mathbf{v}^+_\sigma\cdot(\mathbf{w}_{\mathtt{K}} - \mathbf{w}_{\mathtt{L}})\Big| &= \Big|\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{int}}\big(F^+_{\sigma,\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})\,\mathbf{v}_{\mathtt{K}} - F^-_{\sigma,\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})\,\mathbf{v}_{\mathtt{L}}\big)\cdot(\mathbf{w}_{\mathtt{K}} - \mathbf{w}_{\mathtt{L}})\Big| \\ &\leq \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{int}}\big|F_{\sigma,\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})\big||\mathbf{v}_{\mathtt{K}} + \mathbf{v}_{\mathtt{L}}||\mathbf{w}_{\mathtt{K}} - \mathbf{w}_{\mathtt{L}}|.\end{aligned}$$

If we look at the flux $F_{\sigma,\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})$, we remark that $\forall \mathtt{D} \in \mathfrak{D}_{\mathtt{K}}^{int}$:

$$|F_{\sigma,\mathtt{K}}(\mathbf{u}^{\mathfrak{T}})| = \Big| - \sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}} m_{\mathfrak{s}} \frac{\mathbf{u}_{\mathtt{K}} + \mathbf{u}_{\mathtt{K}^*}}{2} \cdot \vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}} \Big| \leq C m_\sigma \sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}} \Big| \frac{\mathbf{u}_{\mathtt{K}} + \mathbf{u}_{\mathtt{K}^*}}{2} \Big|,$$

where C depends on $\mathrm{reg}(\mathfrak{T})$ (see (3)). We use this result in the estimate of T_1 to obtain:

$$|\mathrm{T}_1| \leq C \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{int}} m_\sigma m_{\sigma^*} |\mathbf{v}_{\mathtt{K}} + \mathbf{v}_{\mathtt{L}}| \Big| \frac{\mathbf{w}_{\mathtt{K}} - \mathbf{w}_{\mathtt{L}}}{m_{\sigma^*}} \Big| \sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}} \Big| \frac{\mathbf{u}_{\mathtt{K}} + \mathbf{u}_{\mathtt{K}^*}}{2} \Big|.$$

We apply Hölder's inequality with $p = 6$, $q = 2$, $r = 3$:

$$|\mathrm{T}_1| \leq C\Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{int}} m_\sigma m_{\sigma^*} |\mathbf{v}_{\mathtt{K}} + \mathbf{v}_{\mathtt{L}}|^6 \Big)^{1/6} \Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{int}} m_\sigma m_{\sigma^*} \Big| \frac{\mathbf{w}_{\mathtt{K}} - \mathbf{w}_{\mathtt{L}}}{m_{\sigma^*}} \Big|^2 \Big)^{1/2}$$
$$\Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{int}} m_\sigma m_{\sigma^*} \sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}} \Big| \frac{\mathbf{u}_{\mathtt{K}} + \mathbf{u}_{\mathtt{K}^*}}{2} \Big|^3 \Big)^{1/3}$$

and thanks to the definition (4) of the gradient operator and (3), we can write:

$$|\mathrm{T}_1| \leq C\Big(\sum_{\mathtt{K}\in\mathfrak{M}} m_{\mathtt{K}} |\mathbf{v}_{\mathtt{K}}|^6 \Big)^{1/6} \Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_{\mathtt{D}} |\nabla^{\mathtt{D}} \mathbf{w}^{\mathfrak{T}}|^2 \Big)^{1/2} \Big(\frac{1}{2} \sum_{\mathtt{K}\in\mathfrak{M}} m_{\mathtt{K}} |\mathbf{u}_{\mathtt{K}}|^3 + \frac{1}{2} \sum_{\mathtt{K}^*\in\mathfrak{M}^*} m_{\mathtt{K}^*} |\mathbf{u}_{\mathtt{K}^*}|^3 \Big)^{1/3}$$
$$\leq C \|\mathbf{v}^{\mathfrak{T}}\|_6 \; \|\nabla^{\mathfrak{D}} \mathbf{w}^{\mathfrak{T}}\|_2 \; \|\mathbf{u}^{\mathfrak{T}}\|_3.$$

<u>Estimate of T_2:</u>

For what concerns boundary terms, the definition of fluxes changes (see (6)), so we can estimate by:

$$|\mathrm{T}_2| = \Big| \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} F_{\sigma,\mathtt{K}}(\mathbf{u}^{\mathfrak{T}}) \gamma^\sigma(\mathbf{v}^{\mathfrak{T}}) \cdot \widetilde{\gamma}^\sigma(\mathbf{w}^{\mathfrak{T}}) \Big| \leq \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma |\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})| |\gamma^\sigma(\mathbf{v}^{\mathfrak{T}})| |\widetilde{\gamma}^\sigma(\mathbf{w}^{\mathfrak{T}})|$$

By applying Hölder's inequality with $p = \frac{8}{3}$, $q = \frac{8}{3}$, $r = 4$ we get

$$|\mathrm{T}_2| \leq \Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma |\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})|^{8/3} \Big)^{3/8} \Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma |\gamma^\sigma(\mathbf{v}^{\mathfrak{T}})|^{8/3} \Big)^{3/8} \Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma |\widetilde{\gamma}^\sigma(\mathbf{w}^{\mathfrak{T}})|^4 \Big)^{1/4}$$
$$\leq \|\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{\frac{8}{3},\partial\Omega} \; \|\gamma^{\mathfrak{T}}(\mathbf{v}^{\mathfrak{T}})\|_{\frac{8}{3},\partial\Omega} \; \|\widetilde{\gamma}^{\mathfrak{T}}(\mathbf{w}^{\mathfrak{T}})\|_{4,\partial\Omega}.$$

<u>Estimate of T_3:</u>

As we did for T_1, by the definition of $\mathbf{v}_{\sigma^*}^+$, we have:

$$|\mathrm{T}_3| \leq \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}} \big| F_{\sigma^*,\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}}) \big| |\mathbf{v}_{\mathtt{K}^*} + \mathbf{v}_{\mathtt{L}^*}| |\mathbf{w}_{\mathtt{K}^*} - \mathbf{w}_{\mathtt{L}^*}|.$$

By the definition of the flux (see (7)), this term can be split into two contributions:

$$|\mathrm{T}_3| \leq \sum_{\substack{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D} \\ \sigma^*\cap\partial\Omega=\emptyset}} \big| F_{\sigma^*,\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}}) \big| |\mathbf{v}_{\mathtt{K}^*} + \mathbf{v}_{\mathtt{L}^*}| |\mathbf{w}_{\mathtt{K}^*} - \mathbf{w}_{\mathtt{L}^*}| + \sum_{\substack{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D} \\ \sigma^*\cap\partial\Omega\neq\emptyset}} \big| F_{\sigma^*,\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}}) \big| |\mathbf{v}_{\mathtt{K}^*} + \mathbf{v}_{\mathtt{L}^*}| |\mathbf{w}_{\mathtt{K}^*} - \mathbf{w}_{\mathtt{L}^*}| = \mathrm{T}_3^1 + \mathrm{T}_3^2.$$

For what concerns the estimate of T_3^1, the definition of the flux $F_{\sigma^*,\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}})$ is the same as the one of $F_{\sigma,\mathtt{K}}$ when $\sigma \in \mathcal{E}_{int}$.
Thus we can proceed as for the estimate of T_1 and we get:

$$\mathrm{T}_3^1 \leq C \, \|\mathbf{v}^{\mathfrak{T}}\|_6 \; \|\nabla^{\mathfrak{D}} \mathbf{w}^{\mathfrak{T}}\|_2 \; \|\mathbf{u}^{\mathfrak{T}}\|_3.$$

For the term T_3^2, the definition of the flux changes and we can estimate it by:

$$|F_{\sigma^*,\mathtt{K}^*}(\mathbf{u}^{\mathfrak{T}})| = \Big| - \sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}} m_{\mathfrak{s}} \frac{\mathbf{u}_{\mathtt{K}} + \mathbf{u}_{\mathtt{K}^*}}{2} \cdot \vec{\mathbf{n}}_{\mathfrak{s}\mathtt{D}} - \frac{1}{2} m_\sigma \gamma^\sigma(\mathbf{u}^{\mathfrak{T}}) \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}} \Big| \leq C m_\sigma \Big(\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}} \Big| \frac{\mathbf{u}_{\mathtt{K}} + \mathbf{u}_{\mathtt{K}^*}}{2} \Big| + |\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})| \Big).$$

In this case, we can write:

$$|\mathrm{T}_3^2| \le \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma^2 |\mathbf{v}_{\mathtt{K}^*}+\mathbf{v}_{\mathtt{L}^*}| \left|\frac{\mathbf{w}_{\mathtt{K}^*}-\mathbf{w}_{\mathtt{L}^*}}{m_\sigma}\right| \Big(\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}} \left|\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{K}^*}}{2}\right| + |\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})| \Big)$$

We split the right hand side into two terms. The first one is estimated exactly as T_1:

$$\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma^2 |\mathbf{v}_{\mathtt{K}^*}+\mathbf{v}_{\mathtt{L}^*}| \left|\frac{\mathbf{w}_{\mathtt{K}^*}-\mathbf{w}_{\mathtt{L}^*}}{m_\sigma}\right| \sum_{\mathfrak{s}\in\mathfrak{S}_{\mathtt{K}}\cap\mathcal{E}_{\mathtt{D}}} \left|\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{K}^*}}{2}\right| \le C\,\|\mathbf{v}^{\mathfrak{T}}\|_6\,\|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2\,\|\mathbf{u}^{\mathfrak{T}}\|_3.$$

For the second one, we apply Hölder's inequality with $p=6$, $q=2$, $r=3$ and we obtain:

$$\begin{aligned}
&\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma^2 |\mathbf{v}_{\mathtt{K}^*}+\mathbf{v}_{\mathtt{L}^*}| \left|\frac{\mathbf{w}_{\mathtt{K}^*}-\mathbf{w}_{\mathtt{L}^*}}{m_\sigma}\right| |\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})| \\
&\quad\le C \Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma^2 |\mathbf{v}_{\mathtt{K}^*}+\mathbf{v}_{\mathtt{L}^*}|^6 \Big)^{\frac16} \Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma^2 \left|\frac{\mathbf{w}_{\mathtt{K}^*}-\mathbf{w}_{\mathtt{L}^*}}{m_\sigma}\right|^2 \Big)^{\frac12} \Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma |\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})|^3 \Big)^{\frac13} \\
&\quad\le C \Big(\sum_{\mathtt{K}\in\mathfrak{M}^*} m_{\mathtt{K}^*} |\mathbf{v}_{\mathtt{K}^*}|^6 \Big)^{\frac16} \Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}} m_{\mathtt{D}} |\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}|^2 \Big)^{\frac12} \Big(\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma |\gamma^\sigma(\mathbf{u}^{\mathfrak{T}})|^3 \Big)^{\frac13} \\
&\quad\le C \|\mathbf{v}^{\mathfrak{T}}\|_6 \|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2 \|\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{3,\partial\Omega}.
\end{aligned}$$

By collecting the estimates we find the announced result:

$$\mathrm{T}_3 = \mathrm{T}_3^1 + \mathrm{T}_3^2 \le C\,\|\mathbf{v}^{\mathfrak{T}}\|_6\,\|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2\,(\|\mathbf{u}^{\mathfrak{T}}\|_3 + \|\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}})\|_{3,\partial\Omega}).$$

■

6 Korn inequality

The proof of the discrete Korn's inequality is inspired by the continuous version in [6]. In DDFV setting in the case of homogeneous Dirichlet boundary conditions, i.e. if $\mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0$, the theorem was proved in [17]. In this case the proof relies on the definition of the operators and the constant of the estimate can be measured. By adding a part of the boundary with non-zero data, we introduce some difficulties and we are able to prove the result only by contradiction, just as in the continuous setting.

Theorem 6.1 *(Korn's inequality) Let $\mathfrak{T}$ be a mesh that satisfies inf-sup stability condition. Then there exists $C>0$ such that :*

$$\boxed{\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \le C\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2} \qquad \forall \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^D$$

In order to prove this result, it is necessary to first consider the case in which $\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}$ has zero mean.

Lemma 6.2 *Let $\mathfrak{T}$ be a mesh that satisfies inf-sup stability condition. Then there exists $C>0$ such that $\forall \mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ that satisfies $m(rot^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}) = \sum_{\mathtt{D}\in\mathfrak{D}} m_{\mathtt{D}} rot^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} = 0$ it holds:*

$$\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \le C\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2$$

Proof (of Lemma 6.2) Let $\mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ such that $m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}) = 0$. If we consider the function

$$\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} = \sum_{\mathtt{D}\in\mathfrak{D}} \mathrm{rot}^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}}\mathbf{1}_{\mathtt{D}},$$

this is an L^2 funcion with zero mean, by hypothesis. This means that, by infsup stability condition (18), $\exists \mathbf{w}^{\mathfrak{T}} \in \mathbb{E}_0$ such that:

$$\begin{aligned}
\mathrm{div}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}) &= \mathrm{rot}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) \\
\|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2 &\le C\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2.
\end{aligned} \tag{19}$$

Moreover, if we define the matrix $\chi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we have the following property:

$$\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} = \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \frac12 \mathrm{rot}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}})\chi \quad \forall \mathbf{u}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{D}}.$$

If we compute:

$$
\begin{aligned}
(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{curl}^{\mathfrak{D}}(\mathbf{w}^{\mathfrak{T}}))_{\mathfrak{D}} =& (\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \frac{1}{2}\mathrm{rot}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}})\chi : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{curl}^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}})_{\mathfrak{D}} \\
=& \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 - (\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} : \mathrm{curl}^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}})_{\mathfrak{D}} + \frac{1}{2}(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\chi : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{curl}^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}})_{\mathfrak{D}} \\
=& \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 + 0 + \frac{1}{2}(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}, \underbrace{-\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{div}^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}}_{=0 \text{ by infsup}})_{\mathfrak{D}} \\
=& \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2
\end{aligned}
\tag{20}
$$

where we use the fact that

$$
(\chi : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})_{\mathfrak{D}} = -\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} \quad , \quad (\chi : \mathrm{curl}^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}})_{\mathfrak{D}} = -\mathrm{div}^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}
$$

and that

$$
(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} : \mathrm{curl}^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}})_{\mathfrak{D}} = 0.
$$

This means that, if we apply the Cauchy-Schwarz inequality and triangle inequality to (20), we deduce:

$$
\begin{aligned}
\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 \leq& \|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} - \mathrm{curl}^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2 \\
\leq& \|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2(\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 + \|\mathrm{curl}^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2).
\end{aligned}
$$

By applying the definition of $\mathrm{curl}^{\mathfrak{D}}$ and (19) we get:

$$
\begin{aligned}
\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2^2 \leq& \|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2(\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 + \|\nabla^{\mathfrak{D}}\mathbf{w}^{\mathfrak{T}}\|_2) \\
\leq& C\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2
\end{aligned}
$$

We conclude that:

$$
\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \leq C\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2.
$$

■

Thanks to this result, we give the proof of Korn's inequality in the general case.

Proof (of Theorem 6.1) Let $\mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^D$. We define $\mathbf{z}^{\mathfrak{T}} \in \left(\mathbb{R}^2\right)^{\mathfrak{T}}$ as:

$$
\mathbf{z}^{\mathfrak{T}} = \mathbf{u}^{\mathfrak{T}} + \frac{1}{2}m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})\mathbf{x}^{\mathfrak{T}},
$$

where $\mathbf{x}^{\mathfrak{T}} = \mathbb{P}_c^{\mathfrak{T}}\begin{pmatrix} y \\ -x \end{pmatrix}$ is a vector that satisfies for all $\mathtt{D} \in \mathfrak{D}$: $\nabla^{\mathtt{D}}\mathbf{x}^{\mathfrak{T}} = \chi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathrm{D}^{\mathtt{D}}\mathbf{x}^{\mathfrak{T}} = 0$ and $\mathrm{rot}^{\mathtt{D}}\mathbf{x}^{\mathfrak{T}} = -2$.
As a consequence, we have that $m(\mathrm{rot}^{\mathfrak{D}}\mathbf{z}^{\mathfrak{T}}) = 0$ and $\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} = \mathrm{D}^{\mathfrak{D}}\mathbf{z}^{\mathfrak{T}}$. By Theorem 6.2 to $\mathbf{z}^{\mathfrak{T}}$, there exists a constant $C > 0$ such that:

$$
\|\nabla^{\mathfrak{D}}\mathbf{z}^{\mathfrak{T}}\|_2 \leq C\|\mathrm{D}^{\mathfrak{D}}\mathbf{z}^{\mathfrak{T}}\|_2. \tag{21}
$$

If we compute $\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}$, we obtain:

$$
\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} = \nabla^{\mathfrak{D}}\mathbf{z}^{\mathfrak{T}} + \frac{1}{2}m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})\chi,
$$

from which we deduce

$$
\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 = C(\|\nabla^{\mathfrak{D}}\mathbf{z}^{\mathfrak{T}}\|_2 + |m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})|).
$$

By (21)

$$
\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \leq C(\|\mathrm{D}^{\mathfrak{D}}\mathbf{z}^{\mathfrak{T}}\|_2 + |m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})|)
$$

that by the definition of $\mathbf{z}^{\mathfrak{T}}$ becomes:

$$
\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \leq C(\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 + |m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})|). \tag{22}
$$

It remains to prove that $\exists \widetilde{C} > 0$ such that:

$$
|m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})| \leq \widetilde{C}\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2 \quad \forall \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^D. \tag{23}
$$

Let $(h_n)_{n\in\mathbb{N}}$ be a sequence such that $h_n \to 0$ as $n \to +\infty$, and let $(\mathfrak{T}_n)_n$ be a sequence of meshes such that $\mathrm{size}(\mathfrak{T}_n) = h_n$ while $\mathrm{reg}(\mathfrak{T}_n)$ is bounded. For every n, there exists a constant C_n such that:

$$|m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_n})| \le C_n \|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_n}\|_2 \quad \forall \mathbf{u}^{\mathfrak{T}_n} \in \mathbb{E}_0^D, \tag{24}$$

with $C_n := \sup_{\mathbf{u}^{\mathfrak{T}_n}\in\mathbb{E}_0^D} \dfrac{|m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_n})|}{\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_n}\|_2}$. Inequality (24) is true because of Theorem 11.1, that ensures that $\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_n}\|_2$ is actually a norm.

Proving (23), it is equivalent to show that the bound (24) is a uniform bound. Thus we argue by contradiction, and we suppose that:

$$\forall k \in \mathbb{N},\ \exists n_k \text{ with } n_k \ge k \text{ such that } C_{n_k} \ge k,$$

that is

$$\forall k \in \mathbb{N},\ \exists \widetilde{\mathbf{u}}^{\mathfrak{T}_{n_k}} \text{ such that } |m(\mathrm{rot}^{\mathfrak{D}}\widetilde{\mathbf{u}}^{\mathfrak{T}_{n_k}})| \ge k\,\|\mathrm{D}^{\mathfrak{D}}\widetilde{\mathbf{u}}^{\mathfrak{T}_{n_k}}\|_2 \quad \forall \widetilde{\mathbf{u}}^{\mathfrak{T}_n} \in \mathbb{E}_0^D.$$

Let $\mathbf{u}^{\mathfrak{T}_{n_k}} = \dfrac{\widetilde{\mathbf{u}}^{\mathfrak{T}_{n_k}}}{m(\mathrm{rot}^{\mathfrak{D}}\widetilde{\mathbf{u}}^{\mathfrak{T}_{n_k}})}$, so that:

$$m(\mathrm{rot}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_{n_k}}) = 1, \quad \|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_{n_k}}\|_2 \le \frac{1}{k}. \tag{25}$$

From (22), we can deduce that $\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_{n_k}}$ is bounded as $k \to +\infty$, since:

$$\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_{n_k}}\|_2 \le C\left(\frac{1}{k} + 1\right).$$

We can thus apply the compactness result of [1, Lemma 3.6], which implies the existence of $\mathbf{u} \in H_D^1(\Omega)$ such that, up to a subsequence:

$$\mathbf{u}^{\mathfrak{T}_{n_k}} \to \mathbf{u} \quad \text{in } L^2(\Omega)$$
$$\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_{n_k}} \rightharpoonup \nabla\mathbf{u} \quad \text{in } L^2(\Omega).$$

From (25) and from the weak convergence of $\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}_{n_k}}$, it follows that $m(\mathrm{rot}\,\mathbf{u}) = 1$ and $\mathrm{D}\mathbf{u} = 0$, i.e. $\mathbf{u}$ is a rigid motion. The only rigid motion that satisfies $\mathbf{u}|_{\Gamma_D} = 0$ is $\mathbf{u} = 0$ since $\mathrm{meas}(\Gamma_D) > 0$. We have therefore a contradiction.

■

7 Discrete energy estimate

The open boundary condition (2) that we study is derived from a weak formulation of the Navier-Stokes equation that ensures an energy estimate, presented in [9]. In this section we prove a discrete version of the energy estimate.

In order to do so, we will need to consider the variational formulation (11) and select the solution as a test function. Since the solution $\mathbf{u}^{\mathfrak{T},[0,T]}$ is not zero on the Dirichlet boundary Γ_1, it does not satisfy the hypothesis (12). We decompose it as $\mathbf{u}^{\mathfrak{T},[0,T]} = \mathbf{v}^{\mathfrak{T},[0,T]} + \mathbf{u}^{\mathfrak{T}}_{ref}$ and, thanks to the definition of $\mathbf{u}^{\mathfrak{T}}_{ref}$ (see (3)) , $\mathbf{v}^{\mathfrak{T},[0,T]}$ will be a good candidate to be our test function.

Theorem 7.1 *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω that satisfies inf-sup stability condition. Let $(\boldsymbol{u}^{\mathfrak{T},[0,T]}, p^{\mathfrak{D},[0,T]}) \in \left(\mathbb{E}^D_{\boldsymbol{g}_1}\right)^{N+1} \times (\mathbb{R}^{\mathfrak{D}})^{N+1}$ be the solution of the DDFV scheme (9), (13)-(16) , where $\boldsymbol{u}^{\mathfrak{T},[0,T]} = \boldsymbol{v}^{\mathfrak{T},[0,T]} + \boldsymbol{u}^{\mathfrak{T}}_{ref}$.*
For $N > 1$, there exists a constant $C > 0$, depending on $\Omega, \boldsymbol{u}^{\mathfrak{T}}_{ref}, \boldsymbol{u}_0, Re$ and T such that:

$$\sum_{j=0}^{N-1} \|\boldsymbol{v}^{j+1} - \boldsymbol{v}^j\|_2^2 \le C, \quad \|\boldsymbol{v}^N\|_2^2 \le C,$$
$$\sum_{j=0}^{N-1} \delta t \frac{1}{Re}\|\mathrm{D}^{\mathfrak{D}}\boldsymbol{v}^{j+1}\|_2^2 \le C, \quad \delta t \frac{1}{Re}\|\mathrm{D}^{\mathfrak{D}}\boldsymbol{v}^N\|_2^2 \le C,$$
$$\sum_{j=0}^{N-1} \delta t \sum_{\mathtt{D}\in\mathfrak{D}_{ext}} (F_{\sigma,\mathtt{K}}(\boldsymbol{v}^j + \boldsymbol{u}^{\mathfrak{T}}_{ref}))^+(\gamma^\sigma(\boldsymbol{v}^{j+1}))^2 \le C.$$

Proof The first step to obtain the energy inequality consists in rewriting the variational formulation (11) for the unknown $\mathbf{v}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}^{\mathfrak{T}}_{ref}$. Going back to the definition of $\mathbf{u}^{\mathfrak{T}}_{ref}$, it becomes:

$$
\begin{aligned}
&[[\frac{\mathbf{v}^{n+1}-\mathbf{v}^n}{\delta t},\Psi^{\mathfrak{T}}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1},\mathrm{D}^{\mathfrak{D}}\Psi^{\mathfrak{T}})_{\mathfrak{D}} \\
&\quad + \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^n+\mathbf{u}^{\mathfrak{T}}_{ref},\mathbf{v}^{n+1}+\mathbf{u}^{\mathfrak{T}}_{ref}),\Psi^{\mathfrak{T}}]]_{\mathfrak{T}} - \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^n+\mathbf{u}^{\mathfrak{T}}_{ref},\Psi^{\mathfrak{T}}),\mathbf{v}^{n+1}+\mathbf{u}^{\mathfrak{T}}_{ref}]]_{\mathfrak{T}} \\
&\qquad + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{ext}}(F_{\sigma,\mathtt{K}}(\mathbf{v}^n+\mathbf{u}^{\mathfrak{T}}_{ref}))^+\gamma^\sigma(\mathbf{v}^{n+1}+\mathbf{u}^{\mathfrak{T}}_{ref})\cdot\gamma^\sigma(\Psi^{\mathfrak{T}}) \\
&\qquad\qquad = -\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{ext}}F_{\sigma,\mathtt{K}}(\mathbf{v}^n+\mathbf{u}^{\mathfrak{T}}_{ref})^-\gamma^\sigma(\mathbf{u}^{\mathfrak{T}}_{ref})\cdot\gamma^\sigma(\Psi^{\mathfrak{T}}).
\end{aligned}
$$

The second step consists in selecting $\Psi^{\mathfrak{T}} = (\mathbf{v}^{n+1}+\mathbf{u}^{\mathfrak{T}}_{ref}) - \mathbf{u}^{\mathfrak{T}}_{ref}$ as a test function. It follows that:

$$
\begin{aligned}
E :=& [[\frac{\mathbf{v}^{n+1}-\mathbf{v}^n}{\delta t},\mathbf{v}^{n+1}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2^2 + \frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{ext}}(F_{\sigma,\mathtt{K}}(\mathbf{v}^n+\mathbf{u}^{\mathfrak{T}}_{ref}))^+(\gamma^\sigma(\mathbf{v}^{n+1}))^2 \\
\leq& \left|\frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^n+\mathbf{u}^{\mathfrak{T}}_{ref},\mathbf{v}^{n+1}),\mathbf{u}^{\mathfrak{T}}_{ref}]]_{\mathfrak{T}} - \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^n+\mathbf{u}^{\mathfrak{T}}_{ref},\mathbf{u}^{\mathfrak{T}}_{ref}),\mathbf{v}^{n+1}]]_{\mathfrak{T}}\right| \\
&+ \left|\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{ext}}F_{\sigma,\mathtt{K}}(\mathbf{v}^n+\mathbf{u}^{\mathfrak{T}}_{ref})^-\gamma^\sigma(\mathbf{u}^{\mathfrak{T}}_{ref})\cdot\gamma^\sigma(\mathbf{v}^{n+1})\right|
\end{aligned}
$$

We apply property of Thm. 5.1 to the convection terms $[[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^n+\mathbf{u}^{\mathfrak{T}}_{ref},\mathbf{v}^{n+1}),\mathbf{u}^{\mathfrak{T}}_{ref}]]_{\mathfrak{T}}$ and $[[\mathbf{b}^{\mathfrak{T}}(\mathbf{v}^n+\mathbf{u}^{\mathfrak{T}}_{ref},\mathbf{u}^{\mathfrak{T}}_{ref}),\mathbf{v}^{n+1}]]_{\mathfrak{T}}$; for what concerns the boundary term, thanks to the definition of $F_{\sigma,\mathtt{K}}$ for $\sigma\in\partial\Omega$, we have:

$$
\left|\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}}F_{\sigma,\mathtt{K}}\gamma^\sigma(\mathbf{u}^{\mathfrak{T}}_{ref})\cdot\gamma^\sigma(\mathbf{v}^{n+1})\right| \leq \sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma|\gamma^\sigma(\mathbf{v}^n+\mathbf{u}^{\mathfrak{T}}_{ref})||\gamma^\sigma(\mathbf{u}^{\mathfrak{T}}_{ref})||\gamma^\sigma(\mathbf{v}^{n+1})|
$$

and by applying Hölder's inequality with $p=\frac{8}{3}$, $q=\frac{8}{3}$, $r=4$ we get:

$$
\left|\sum_{\mathtt{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}}F_{\sigma,\mathtt{K}}\gamma^\sigma(\mathbf{u}^{\mathfrak{T}}_{ref})\cdot\gamma^\sigma(\mathbf{v}^{n+1})\right| \leq C\|\gamma^{\mathfrak{T}}(\mathbf{v}^n+\mathbf{u}^{\mathfrak{T}}_{ref})\|_{\frac{8}{3},\partial\Omega}\|\gamma^{\mathfrak{T}}(\mathbf{v}^{n+1})\|_{\frac{8}{3},\partial\Omega}\|\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}_{ref})\|_{4,\partial\Omega}.
$$

Thus we are led to:

$$
\begin{aligned}
E \leq& C\Big(\|\mathbf{v}^n+\mathbf{u}^{\mathfrak{T}}_{ref}\|_3 + \|\gamma^{\mathfrak{T}}(\mathbf{v}^n+\mathbf{u}^{\mathfrak{T}}_{ref})\|_{3,\partial\Omega}\Big)\Big(\|\mathbf{v}^{n+1}\|_6\;\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}_{ref}\|_2 + \|\mathbf{u}^{\mathfrak{T}}_{ref}\|_6\;\|\nabla^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2\Big) \\
&+ C\|\gamma^{\mathfrak{T}}(\mathbf{v}^n+\mathbf{u}^{\mathfrak{T}}_{ref})\|_{\frac{8}{3},\partial\Omega}\|\gamma^{\mathfrak{T}}(\mathbf{v}^{n+1})\|_{\frac{8}{3},\partial\Omega}\|\gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}_{ref})\|_{4,\partial\Omega}.
\end{aligned}
$$

By Sobolev inequalities of [2, Thm. 9], we bound $\|\mathbf{v}^n\|_3$ from above by $C\|\nabla^{\mathfrak{D}}\mathbf{v}^n\|_2^{\frac{1}{3}}\|\mathbf{v}^n\|_2^{\frac{2}{3}}$ and $\|\mathbf{v}^{n+1}\|_6$ by $C\|\nabla^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2^{\frac{2}{3}}\|\mathbf{v}^{n+1}\|_2^{\frac{1}{3}}$. Moreover, thanks to the trace Theorem 10.4 and to [2, Thm. 9], we bound $\|\gamma^{\mathfrak{T}}(\mathbf{v}^{n+1})\|_{\frac{8}{3},\partial\Omega}$ and $\|\gamma^{\mathfrak{T}}(\mathbf{v}^n)\|_{3,\partial\Omega}$ by $C\|\nabla^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2^{\frac{5}{8}}\|\mathbf{v}^{n+1}\|_2^{\frac{3}{8}}$ and $C\|\nabla^{\mathfrak{D}}\mathbf{v}^n\|_2^{\frac{2}{3}}\|\mathbf{v}^n\|_2^{\frac{1}{3}}$.

We then apply the discrete Poincaré inequality, Thm. 10.3, to get rid of the norms of $\mathbf{v}^{n+1}$.
Finally we recall that $\mathbf{u}^{\mathfrak{T}}_{ref}$ is a fixed reference steady flow. Hence there exists a constant $C>0$ that depends only on Ω, $\mathrm{reg}(\mathfrak{T})$ and $\mathbf{u}^{\mathfrak{T}}_{ref}$ such that:

$$
\begin{aligned}
E \leq C\Big(& 2\|\nabla^{\mathfrak{D}}\mathbf{v}^n\|_2^{\frac{1}{3}}\|\mathbf{v}^n\|_2^{\frac{2}{3}}\|\nabla^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2 + 2\|\nabla^{\mathfrak{D}}\mathbf{v}^n\|_2^{\frac{2}{3}}\|\mathbf{v}^n\|_2^{\frac{1}{3}}\|\nabla^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2 \\
&+ 5\|\nabla^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2 + \|\nabla^{\mathfrak{D}}\mathbf{v}^n\|_2^{\frac{5}{8}}\|\mathbf{v}^n\|_2^{\frac{3}{8}}\|\nabla^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2\Big).
\end{aligned}
$$

We control the norm of the gradients with the norms of $\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}$ and $\mathrm{D}^{\mathfrak{D}}\mathbf{v}^n$ thanks to Korn's inequality, Thm. 6.1:

$$
\begin{aligned}
E \leq C\Big(& 2\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^n\|_2^{\frac{1}{3}}\|\mathbf{v}^n\|_2^{\frac{2}{3}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2 + 2\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^n\|_2^{\frac{2}{3}}\|\mathbf{v}^n\|_2^{\frac{1}{3}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2 \\
&+ 5\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2 + \|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^n\|_2^{\frac{5}{8}}\|\mathbf{v}^n\|_2^{\frac{3}{8}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2\Big).
\end{aligned}
$$

Hence, by suitable use of Young's inequality (Lemma 10.1) we end up with:

$$[[\frac{\mathbf{v}^{n+1}-\mathbf{v}^n}{\delta t},\mathbf{v}^{n+1}]]_{\mathfrak{T}}+\frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2^2+\frac{1}{2}\sum_{\mathtt{D}\in\mathfrak{D}_{ext}}(F_{\sigma,\mathtt{K}}(\mathbf{v}^n+\mathbf{u}_{ref}^{\mathfrak{T}}))^+(\gamma^\sigma(\mathbf{v}^{n+1}))^2$$
$$\le 16\mathrm{Re}^2C^3\|\mathbf{v}^n\|_2^2+\frac{1}{2\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^n\|_2^2+\frac{1}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2^2+\frac{25}{2}\mathrm{Re}C^2$$

We combine $\frac{1}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2^2$ with the LHS, we multiply this relation by δt and we apply

$$2[[\mathbf{v}^{n+1}-\mathbf{v}^n,\mathbf{v}^{n+1}]]_{\mathfrak{T}}=\|\mathbf{v}^{n+1}-\mathbf{v}^n\|_2^2+\|\mathbf{v}^{n+1}\|_2^2-\|\mathbf{v}^n\|_2^2.$$

We obtain:

$$\|\mathbf{v}^{n+1}-\mathbf{v}^n\|_2^2+\|\mathbf{v}^{n+1}\|_2^2+\delta t\frac{2}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2^2+\delta t\sum_{\mathtt{D}\in\mathfrak{D}_{ext}}(F_{\sigma,\mathtt{K}}(\mathbf{v}^n+\mathbf{u}_{ref}^{\mathfrak{T}}))^+(\gamma^\sigma(\mathbf{v}^{n+1}))^2$$
$$\le\|\mathbf{v}^n\|_2^2+32\mathrm{Re}^2C^3\delta t\|\mathbf{v}^n\|_2^2+\frac{1}{\mathrm{Re}}\delta t\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^n\|_2^2+25\mathrm{Re}C^2\delta t.$$

We sum over $n=0\dots m-1$ with $m\in\{1\dots N\}$ to obtain:

$$\sum_{n=0}^{m-1}\|\mathbf{v}^{n+1}-\mathbf{v}^n\|_2^2+\|\mathbf{v}^m\|_2^2+\sum_{n=0}^{m-1}\delta t\frac{1}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{n+1}\|_2^2$$
$$+\delta t\frac{1}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^m\|_2^2+\sum_{n=0}^{m-1}\delta t\sum_{\mathtt{D}\in\mathfrak{D}_{ext}}(F_{\sigma,\mathtt{K}}(\mathbf{v}^n+\mathbf{u}_{ref}^{\mathfrak{T}}))^+\left|\gamma^\sigma(\mathbf{v}^{n+1})\right|^2$$
$$\le\|\mathbf{v}^0\|_2^2+\frac{1}{\mathrm{Re}}\delta t\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^0\|_2^2+25\mathrm{Re}C^2T+32\mathrm{Re}^2C^3\delta t\sum_{n=0}^{m-1}\|\mathbf{v}^n\|_2^2. \quad (26)$$

We can now apply Grönwall's lemma (Lemma 10.2), with:

$$a_0:=\|\mathbf{v}^0\|_2^2+\frac{1}{\mathrm{Re}}\delta t\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^0\|_2^2$$

$$a_m:=\sum_{j=0}^{m-1}\|\mathbf{v}^{j+1}-\mathbf{v}^j\|_2^2+\|\mathbf{v}^m\|_2^2+\sum_{j=0}^{m-1}\delta t\frac{1}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{j+1}\|_2^2$$
$$+\delta t\frac{1}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^m\|_2^2+\sum_{j=0}^{m-1}\delta t\sum_{\mathtt{D}\in\mathfrak{D}_{ext}}(F_{\sigma,\mathtt{K}}(\mathbf{v}^j+\mathbf{u}_{ref}^{\mathfrak{T}}))^+\left|\gamma^\sigma(\mathbf{v}^{j+1})\right|^2$$

for $m=\{1\dots N\}$. In fact, we deduce from (26) that

$$a_m\le\underbrace{a_0+25\mathrm{Re}C^2T}_{:=A}+\underbrace{32\mathrm{Re}^2C^3}_{:=B}\delta t\sum_{i=0}^{m-1}a_i,$$

that implies:

$$\max_{m=1\dots N}a_m\le Ae^{BT}.$$

This proves our initial statement, since we can choose $m=N$ and write:

$$\sum_{j=0}^{N-1}\|\mathbf{v}^{j+1}-\mathbf{v}^j\|_2^2+\|\mathbf{v}^N\|_2^2+\sum_{j=0}^{N-1}\delta t\frac{1}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^{j+1}\|_2^2$$
$$+\delta t\frac{1}{\mathrm{Re}}\|\mathrm{D}^{\mathfrak{D}}\mathbf{v}^N\|_2^2+\sum_{j=0}^{N-1}\delta t\sum_{\mathtt{D}\in\mathfrak{D}_{ext}}(F_{\sigma,\mathtt{K}}(\mathbf{v}^j+\mathbf{u}_{ref}^{\mathfrak{T}})^+\left|\gamma^\sigma(\mathbf{v}^{j+1})\right|^2\le C(T).$$

■

8 Numerical results

We validate the scheme through a series of numerical experiments. First, we study numerically the consistency properties of the scheme. Second, we reproduce the simulations of a flow in a channel presented in [10] and [22].

8.1 Convergence results

Test case 1. The computational domain is $\Omega = [0,1]^2$, whose boundary is divided into $\partial\Omega = \Gamma_1 \cup \Gamma_2$. We impose Dirichlet boundary conditions on Γ_1, composed by the two orizontal boundaries and the left vertical one. The open boundary condition (2) is imposed on Γ_2, the right vertical boundary. We set the viscosity to 1.
For the tests we give the expression of the exact solution ($\mathbf{u}$,p), from which we deduce a source term $\mathbf{f}$ for the momentum equation and the Dirichlet boundary condition g_1. As a reference flow, we consider the exact solution. We will compare the L^2-norm of the error (difference between a centered projection of the exact solution and the approximated solution obtained with DDFV scheme) for the velocity (denoted Ervel), the velocity gradient (Ergradvel) and the pressure (Erpre). In particular we denote:

$$\text{Ergradvel} = \frac{\left(\sum_{n=0}^{N} \delta t \|\nabla^{\mathfrak{D}}(\mathbb{P}_c^{\mathfrak{T}}\mathbf{u})^n - \nabla^{\mathfrak{D}}\mathbf{u}^n\|_2^2\right)^{1/2}}{\left(\sum_{n=0}^{N} \delta t \|\nabla^{\mathfrak{D}}\left(\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}\right)^n\|_2^2\right)^{1/2}}, \qquad \text{Erpre} = \frac{\left(\sum_{n=0}^{N} \delta t \|(\mathbb{P}_c^{\mathfrak{D}}\mathrm{p})^n - \mathrm{p}^n\|_2^2\right)^{1/2}}{\left(\sum_{n=0}^{N} \delta t \|\left(\mathbb{P}_c^{\mathfrak{D}}\mathrm{p}\right)^n\|_2^2\right)^{1/2}},$$

$$\text{Ervel} = \frac{\max\limits_{n=0\ldots N} \|\left(\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}\right)^n - \mathbf{u}^n\|_2}{\max\limits_{n=1\ldots N} \|\left(\mathbb{P}_c^{\mathfrak{T}}\mathbf{u}\right)^n\|_2},$$

where $(\mathbb{P}_c^{\mathfrak{T}}\mathbf{u})^n$ and $(\mathbb{P}_c^{\mathfrak{D}}\mathrm{p})^n$ are the centered projections of $\mathbf{u}$ and p at the time step $t_n = n\delta t$.
On Table 1 we give the number of primal cells (NbCell) and the convergence rates (Ratio). We remark that, to discuss the error estimates, a family of meshes (Fig. 4) is obtained by refining successively and uniformly the original mesh. The exact solutions is:

$$\mathbf{u}(x,y) = \begin{pmatrix} -2\pi\cos(\pi x)\sin(2\pi y)\exp(-5\eta t\pi^2), \\ \pi\sin(\pi x)\cos(2\pi y)\exp(-5\eta t\pi^2) \end{pmatrix}$$

$$\mathrm{p}(x,y) = -\frac{\pi^2}{4}(4\cos(2\pi x) + \cos(4\pi y))\exp(-10t\eta\pi^2)$$

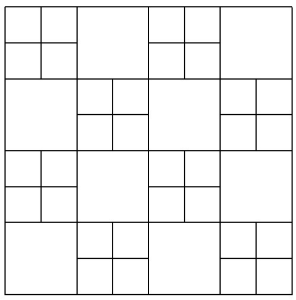


Figure 4: Non conformal square mesh.

The final time is $\mathrm{T} = 0.03s$ and we set $\delta t = 3 \times 10^{-5}s$. As we can see in Table 1, we observe super convergence in L^2 norm of the velocity, that is a classical result for Finite Volume methods. For what concerns the gradient of the velocity and the pressure, we remark that the non-conformity of the mesh does not influence the good convergence of the method. We get a first order accuracy on the velocity gradient, and an order of 1.5 for the pressure, that is better than what we expected. We tested many

NbCell	**Ervel**	**Ratio**	**Ergradvel**	**Ratio**	**Erpre**	**Ratio**
64	1.424E-01	-	1.612E-01	-	6.127E+00	-
208	4.095E-02	1.80	7.316E-02	1.14	1.725E+00	1.83
736	1.019E-02	2.00	3.489E-02	1.07	5.836E-01	1.56
2752	2.559E-03	1.99	1.710E-02	1.03	1.947E-01	1.58
10624	6.493E-04	**1.98**	8.474E-03	**1.01**	6.189E-02	**1.65**

Table 1: Test case 1 on the non conformal square mesh Fig. 4.

other meshes and the results do not change. The geometry of the mesh does not influence the accuracy of the approximation.

8.2 Simulations of a flow in a pipe

Figure 5 describes the situation we are dealing with: we consider Ω, a bounded polygonal domain of $\mathbb{R}^2$, whose boundary $\partial\Omega$ is split into Γ_0, Γ_1 and Γ_2 and whose outer normal is denoted by $\vec{\mathbf{n}}$. We add a cylindrical obstacle inside Ω. The Dirichlet part of the boundary is composed by Γ_0 and Γ_1: on the physical boundary Γ_0 we impose no slip boundary conditions and on the inflow boundary Γ_1 the velocity is prescribed. On the *artificial* boundary Γ_2, that we wish to set as close as possible to the obstacle, we impose the nonlinear boundary condition (2). We reproduce two different test cases, proposed in [10] and

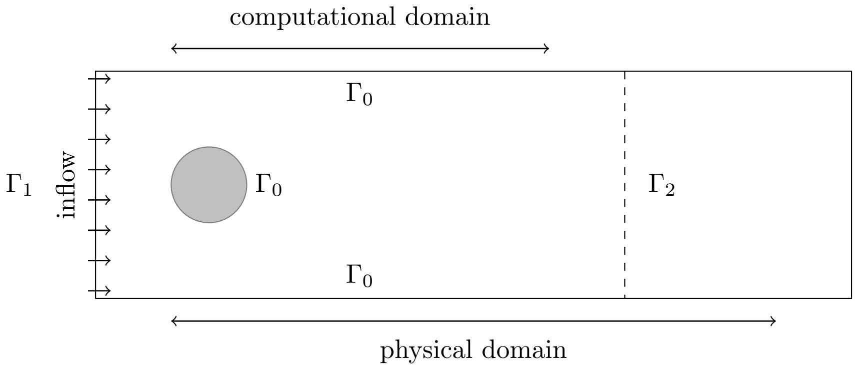


Figure 5: Domain and notations.

in [22]. In both cases, the simulations are performed on a triangular mesh, generated by GMSH, that is locally refined around the cylinder.

Test case 2. We show that by adding an artificial boundary, thanks to condition (2), we do not introduce any perturbation to the flow. For this purpose, we consider an original domain that we cut into two smaller domains and we draw the streamlines of the respective solution.
We consider the symmetric domain $\Omega = [0,5] \times [0,1]$ with a cylindrical obstacle of diameter $L = 0.4$m. The smaller domains are obtained by cutting at the horizontal axis first in $x = 3$, then in $x = 1.5$. The mesh for Ω is composed by 12118 cells, and we pass to $\Omega' = [0,3] \times [0,1]$ with 8636 cells and to $\Omega'' = [0,1.5] \times [0,1]$ with 6534 cells. The time step is $\delta t = 0.035s$. The inflow on Γ_1 is:

$$\mathbf{g}_1 = (6y(1-y), 0).$$

Since our simulations are performed with $\text{Re} = 100$, it makes sense to set as reference flow a Poiseuille flow. Therefore we choose $\mathbf{u}_{ref} = \mathbf{g}_1$ and $\sigma_{ref}(\mathbf{u}, \text{p}) \cdot \vec{\mathbf{n}} = (0, 6\eta(1-2y))$. As initial condition, we impose $\mathbf{u}_{init} = \mathbf{g}_1$ and the final time is $T = 3.5s$. The Reynold's number is $\text{Re} = \dfrac{\mathbf{u}_{moy}\, L}{\eta} = \dfrac{0.4}{\eta}$, with average velocity $\mathbf{u}_{moy} = 1$. If $\mathbf{u} = (\text{u}_1, \text{u}_2)$, the stream function Ψ is defined in the continuous setting as the solution of

$$\frac{\partial \Psi}{\partial y} = \text{u}_1, \quad \frac{\partial \Psi}{\partial x} = -\text{u}_2,$$

in particular it has to satisfy the following system:

$$\begin{cases} \Delta \Psi = \text{rot}(\mathbf{u}) & \text{in } \Omega \\ \nabla \Psi \cdot \vec{\mathbf{n}} = \mathbf{u} \cdot \vec{\tau} & \text{on } \partial\Omega \end{cases}$$

where $\text{rot}(\mathbf{u}) = -\dfrac{\partial \text{u}_1}{\partial x} + \dfrac{\partial \text{u}_1}{\partial y}$, $\vec{\mathbf{n}}$ is the outer normal to the domain and $\vec{\tau}$ is the unitary tangent to the boundary. In the DDFV setting, we look for $\Psi^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ solution of:

$$\begin{cases} \text{div}^{\mathfrak{D}} \boldsymbol{\nabla}^{\mathfrak{T}} \Psi^{\mathfrak{D}} = \text{rot}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) \\ \nabla^{\text{K}} \Psi^{\mathfrak{D}} \cdot \vec{\mathbf{n}}_{\sigma\text{K}} = \gamma^{\sigma}(\mathbf{u}^{\mathfrak{T}}) \cdot \vec{\boldsymbol{\tau}}_{\text{K}^*,\text{L}^*} \quad \forall \sigma \in \partial\Omega \end{cases}$$

where $\boldsymbol{\nabla}^{\mathfrak{T}} \Psi^{\mathfrak{D}} = \mathbf{div}^{\mathfrak{T}}(\Psi^{\mathfrak{D}} \text{Id})$.

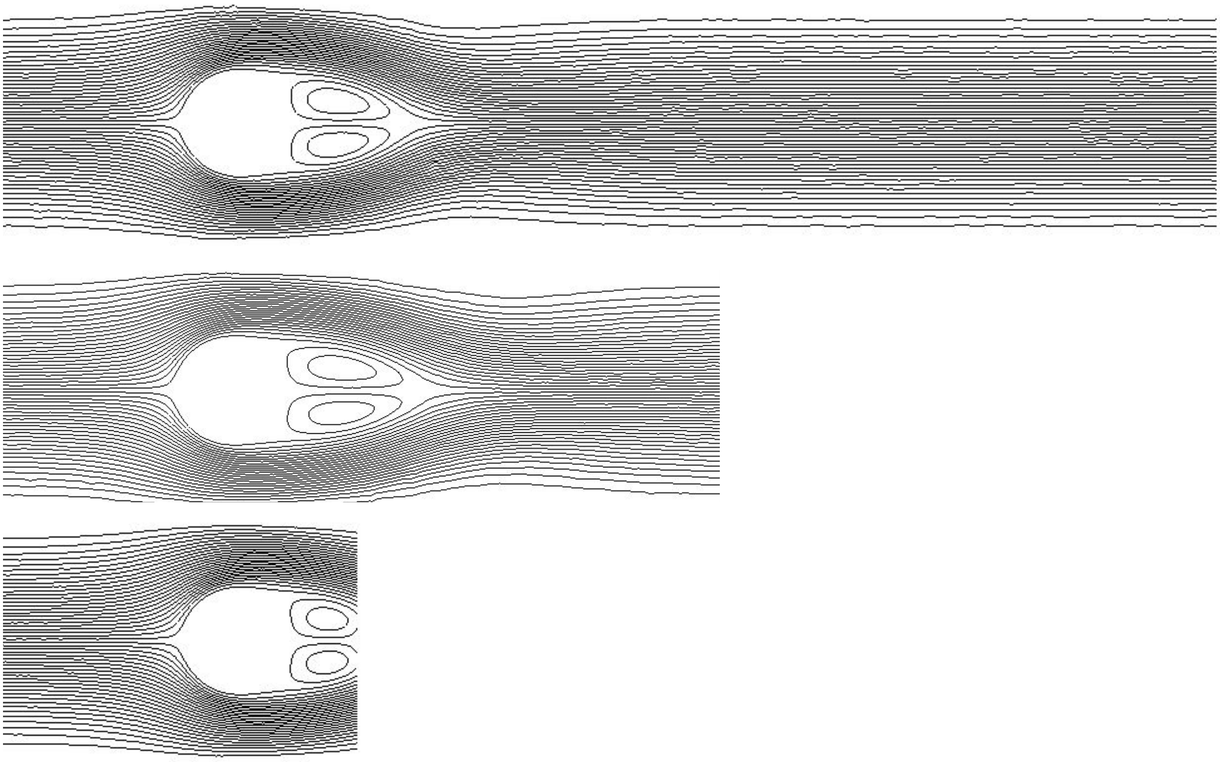


Figure 6: Streamline of Test case 2 at $T = 1.5s$, Re $= 100$, $\eta = 4 \times 10^{-3}$. On the top: $\Omega = [0, 5] \times [0, 1]$, NbCell= 12118. In the middle: $\Omega' = [0, 3] \times [0, 1]$, NbCell=8636. On the bottom: $\Omega'' = [0, 1.5] \times [0, 1]$, NbCell=6534.

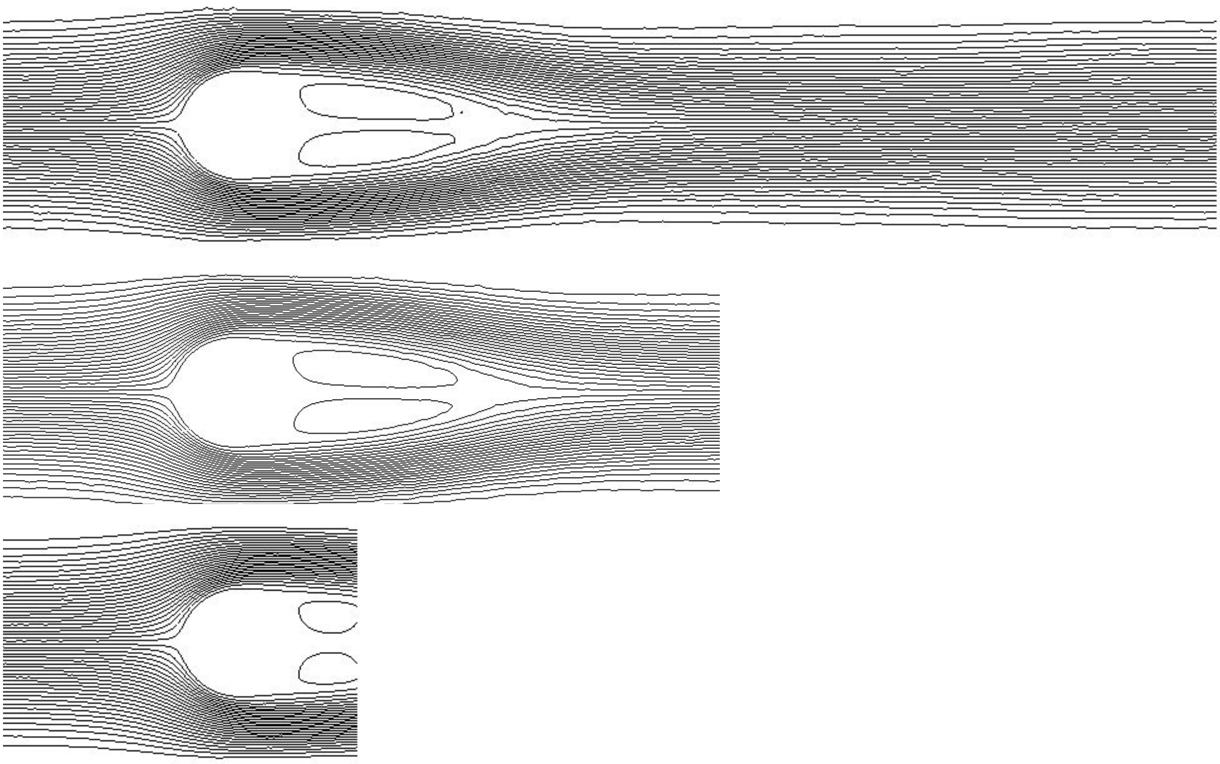


Figure 7: Streamline of Test case 2 at $T = 3.5s$, Re $= 100$, $\eta = 4 \times 10^{-3}$. On the top: $\Omega = [0, 5] \times [0, 1]$, NbCell= 12118. In the middle: $\Omega' = [0, 3] \times [0, 1]$, NbCell=8636. On the bottom: $\Omega'' = [0, 1.5] \times [0, 1]$, NbCell=6534.

We observe by the streamlines in Fig. 6 and Fig. 7 that we can cut close to the obstacle without adding any perturbation to the whole flow. The recirculations are well located and there is no presence of parasite vortices. Clearly, the closer we cut, the more we loose in precision in the cells right before the artificial boundary. This is due to the artificiality of the conditions and to the choice of the reference flow. But in any case, the boundary can cut the recirculation right in the middle without affecting the whole flow.
The choice of the *reference flow* is crucial. In [10], it is proposed to use a Poiseuille flow as we reproduce in our numerical tests. In [8], since to write down the variational formulation (10) the reference flow is assumed to be the solution of a steady Stokes problem with $\mathbf{u} = \mathbf{g}_1$ on Γ_1, the author chooses the flow at infinity: $\mathbf{u}_{ref} = \mathbf{u}_\infty$, $\sigma_{ref} = \sigma_\infty$. Nevertheless, when the flow is chaotic or turbulent such a reference flow does not give a good equivalent of the traction. Thus for higher Reynold's numbers, other techniques can be envisaged for the choice of the reference flow; for example, it looks reasonable to choose a reference flow that changes at each time step. We might think that a good approximation of the solution at the boundary Γ_2 is the solution computed at the previous time step (or even just before the boundary at the same time step), but actually we numerically observed that this techniques lead to strong instabilities. Remark that by replacing $\mathbf{u}_{ref}^{\mathfrak{T}}$ with $\mathbf{u}^n$, the energy stalibity is no longer guaranteed. A way to overcome this difficulty could be to compute the flow on a strictly larger domain (with respect to the smaller one) with a less refined mesh, and then take as reference flow the trace of the solution on Γ_2.

Test case 3. To measure the quality of the solution we obtain on the shorter domain, as a second experiment we computed the drag and the lift coefficient, whose reference values can be found in [22]. We consider a long channel $\Omega = [0, 2.2] \times [0, 0.41]$, that we cut at $x = 0.6$m, with a cylindrical obstacle S whose center is in $(0.15, 0.15)$.
In [22], the coefficients are computed in the long domain with a Dirichlet type condition on the boundary $x = 2.2$m. We perform the same computations by considering the smaller domain $\Omega' = [0, 0.6] \times [0, 0.41]$, with the outflow boundary condition (2) on Γ_2 (at $x = 0.6$m). The triangular mesh that we considered on Ω', obtained with GMSH, has 8020 cells and it is locally refined around the cylinder.

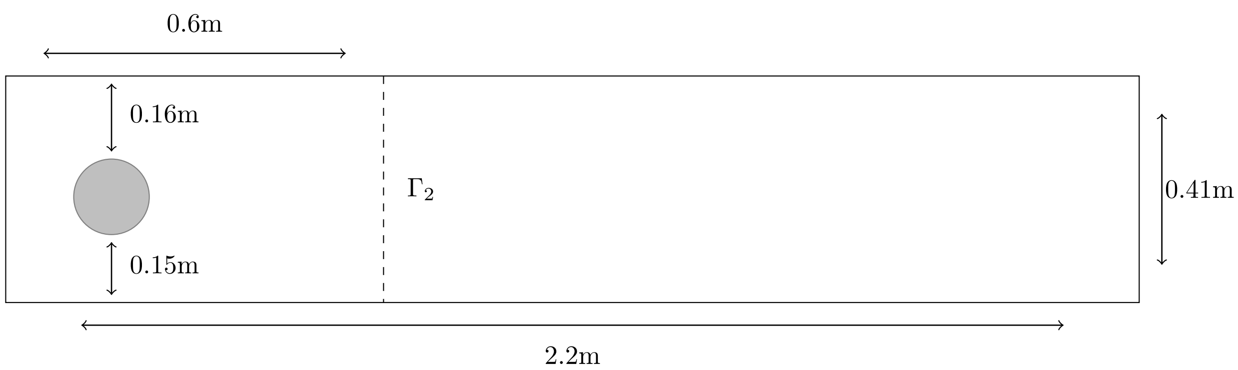


Figure 8: Domains $\Omega = [0, 2.2] \times [0, 0.41]$ and $\Omega' = [0, 0.6] \times [0, 0.41]$.

The viscosity of the fluid is set to $\eta = 10^{-3}\text{m}^2\text{s}^{-1}$ and the final time is $T = 8$s. The time-dependent inflow on Γ_1 is:

$$\mathbf{g}_1 = 0.41^{-2} \sin(\pi t/8)(6y(0.41 - y), 0),$$

and as a reference flow on Γ_2 we choose $\mathbf{u}_{ref} = \mathbf{g}_1$ and $\sigma_{ref} \cdot \vec{\mathbf{n}} = \sigma(\mathbf{u}_{ref}, 0) \cdot \vec{\mathbf{n}}$, where $\vec{\mathbf{n}}$ is the outer normal to Ω. The initial condition is $\mathbf{u}_{init} = (0, 0)$. The density of the fluid is given by $\rho = 1\text{kgm}^{-3}$, and the maximum velocity is $U_{max} = 1\text{ms}^{-1}$. The diameter of the cylinder is $L = 0.1$m, so that the Reynold's number is $0 \leq \text{Re}(t) \leq 100$. We define the drag coefficient $c_d(t)$ and the lift coefficient $c_l(t)$ as:

$$c_d(t) = \frac{2}{\rho L U_{max}^2} \int_S \left(\rho\eta \frac{\partial \mathbf{u}_{t_S}(t)}{\partial n} n_y - p(t) n_x \right),$$
$$c_l(t) = -\frac{2}{\rho L U_{max}^2} \int_S \left(\rho\eta \frac{\partial \mathbf{u}_{t_S}(t)}{\partial n} n_x + p(t) n_y \right),$$

where here $\vec{\mathbf{n}}_S = (n_x, n_y)$ is the normal vector on S directing into Ω , $\mathbf{t}_S = (n_y, -n_x)$ the tangential vector and $\mathbf{u}_{t_S}$ the tangential velocity.

To corresponding formula in the DDFV setting is:

$$c_d^n = \frac{2}{\rho L U_{max}^2} \sum_{\mathtt{D}\in\mathfrak{D}_{ext}\cap S} m_\sigma \left(\rho\eta\nabla^{\mathtt{D}}(\mathbf{u}^n \cdot \vec{\boldsymbol{\tau}}_{\mathtt{K}^*,\mathtt{L}^*}) \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}}\, n_y - \mathrm{p}^n n_x\right),$$

$$c_l^n = -\frac{2}{\rho L U_{max}^2} \sum_{\mathtt{D}\in\mathfrak{D}_{ext}\cap S} m_\sigma \left(\rho\eta\nabla^{\mathtt{D}}(\mathbf{u}^n \cdot \vec{\boldsymbol{\tau}}_{\mathtt{K}^*,\mathtt{L}^*}) \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}}\, n_x + \mathrm{p}^n n_y\right),$$

where

$$\nabla^{\mathtt{D}}(\mathbf{u}^n \cdot \vec{\boldsymbol{\tau}}_{\mathtt{K}^*,\mathtt{L}^*}) \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}} = \frac{m_\sigma}{2m_{\mathtt{D}}}(\mathbf{u}_{\mathtt{L}}^n - \mathbf{u}_{\mathtt{K}}^n) \cdot \vec{\boldsymbol{\tau}}_{\mathtt{K}^*,\mathtt{L}^*} + \frac{m_{\sigma^*}}{2m_{\mathtt{D}}}(\mathbf{u}_{\mathtt{L}^*}^n - \mathbf{u}_{\mathtt{K}^*}^n) \cdot \vec{\boldsymbol{\tau}}_{\mathtt{K}^*,\mathtt{L}^*}\; \vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*} \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}}.$$

We study the evolution of the coefficients in Fig. 9 and their maximum value in Table 2, defined as:

$$c_{d,max} = \max_{n\in\{0\ldots N\}} c_d^n, \qquad c_{l,max} = \max_{n\in\{0\ldots N\}} c_l^n.$$

The results shown in Table 2 and in Fig. 9 prove that the boundary conditions are robust and the solution we find is quantitatively correct. The small difference in the coefficients, with respect to the reference values, is due to the level of refinement of the mesh and to the different kind of condition on the boundary.
In Figure 9 we can also observe how the time step and the choice of the scheme influences the result for the lift coefficient: for the reference values, [22] considers a time step $\delta t = 0.0025s$ with a second order scheme in time. We thus implement a second order backward difference formula in time to see if the approximation improves. The first iteration of the scheme remains unchanged, and for $n \in \{1, \ldots N\}$ the variational formulation (11) becomes:

$$\begin{aligned}
&[[\frac{1}{\delta t}\left(\frac{3}{2}\mathbf{u}^{n+1} - 2\mathbf{u}^n + \frac{1}{2}\mathbf{u}^{n-1}\right), \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} + \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{n+1}, \mathrm{D}^{\mathfrak{D}}\Psi^{\mathfrak{T}})_{\mathfrak{D}} + \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(2\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{u}^{n+1}), \Psi^{\mathfrak{T}}]]_{\mathfrak{T}} \\
&\quad - \frac{1}{2}[[\mathbf{b}^{\mathfrak{T}}(2\mathbf{u}^n - \mathbf{u}^{n-1}, \Psi^{\mathfrak{T}}), \mathbf{u}^{n+1}]]_{\mathfrak{T}} = -\frac{1}{2}\sum_{\mathtt{D}\in\mathtt{D}_{ext}\cap\Gamma_2} (F_{\sigma,\mathtt{K}}(2\mathbf{u}^n - \mathbf{u}^{n-1}))^+ \gamma^\sigma(\mathbf{u}^{n+1}) \cdot \gamma^\sigma(\Psi^{\mathfrak{T}}) \\
&\qquad + \frac{1}{2}\sum_{\mathtt{D}\in\mathtt{D}_{ext}\cap\Gamma_2} (F_{\sigma,\mathtt{K}}(2\mathbf{u}^n - \mathbf{u}^{n-1}))^- \gamma^\sigma(\mathbf{u}_{ref}) \cdot \gamma^\sigma(\Psi^{\mathfrak{T}}) + \sum_{\mathtt{D}\in\mathtt{D}_{ext}\cap\Gamma_2} m_\sigma(\sigma_{\mathrm{ref}}^{\mathfrak{D}} \cdot \vec{\mathbf{n}}_{\sigma\mathtt{K}}) \cdot \gamma^\sigma(\Psi^{\mathfrak{T}}).
\end{aligned}$$

We observe in Fig. 9 that this technique actuallly improves the quality of the approximation of the lift coefficient.

	DDFV	Reference
$\mathbf{c_{d,max}}$	**2.9**754	**2.9**509
$\mathbf{c_{l,max}}$	**0.4**4902	**0.4**7795

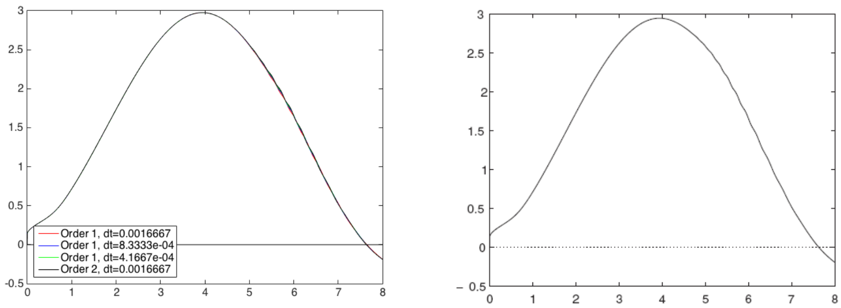
Table 2: Comparison between the values of $c_{d,max}, c_{l,max}$ obtained with DDFV scheme (left) and the reference values of [22] (right).

9 Conclusions

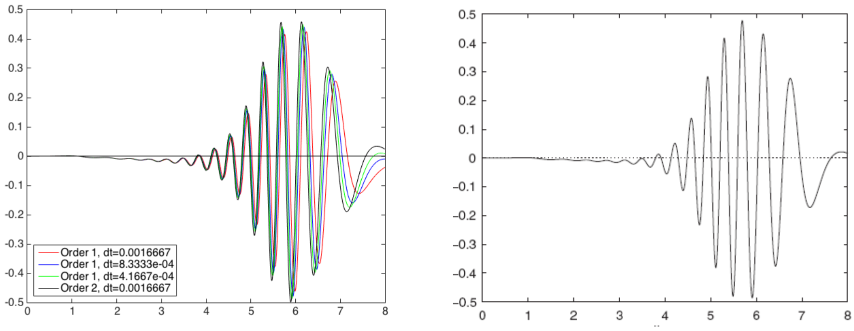
In this article, we propose DDFV schemes for the Navier-Stokes problem with outflow boundary conditions. The DDFV scheme is proved to be well-posed and it satisfies a discrete energy estimate. We numerically proved the good convergence of the scheme and we performed numerical tests that show the accuracy of this condition. These results are proved in the case of a constant viscosity, but they could be extended to the case of variable viscosity, by starting from the works of [3, 19].

Acknowledgments

We would like to show our gratitude to Franck Boyer for his useful help during the preparation of this work.



(a) Evolution of the drag coefficient c_d^n



(b) Evolution of the lift coefficient c_l^n

Figure 9: Comparison between the evolution of c_d^n, c_l^n on the time interval $[0, 8]$ obtained with DDFV scheme (left) and the reference values of [22] (right). We plot the results for the scheme of order 1 in time, with respect to different time steps, and for the scheme of order 2.

10 Appendix A

Lemma 10.1 *(Young's inequality)*
Let a, b, c be three non negative numbers. Let p_1, p_2 and p_3 be positive real numbers such that $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$. Then, we have:

$$abc \le \frac{C_1}{p_1}\, a^{p_1} + \frac{C_2}{p_2}\, b^{p_2} + \frac{1}{p_3\, C_1 C_2}\, c^{p_3}$$

for some positive constants C_1, C_2,

We adapted the proof of Grönwall's lemma, Lemma 16.I.6 in [20], to obtain the following:

Lemma 10.2 *(Discrete Grönwall's lemma)*
If a sequence $(a_n)_n$, $n = 0 \dots N$, satisfies

$$a_0 \le A, \qquad a_n \le A + B\delta t \sum_{i=0}^{n-1} a_i \quad \forall n \in 1, \dots N, \delta t = \frac{T}{N}$$

where A and B are two positive constants independent of δt, then

$$\max_{n=1\dots N} a_n \le A e^{BT}.$$

10.1 Trace theorem

10.1.1 Definitions

Given a vector $\mathbf{u}^{\mathfrak{T}} = ((\mathbf{u}_{\mathrm{K}})_{\mathrm{K}\in\mathfrak{M}\cup\partial\mathfrak{M}}, (\mathbf{u}_{\mathrm{K}^*})_{\mathrm{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*})$ defined on a DDFV mesh $\mathfrak{T}$, we associate the approximate solution on the boundary in two different ways:

$$\widetilde{\gamma}(\mathbf{u}^{\mathfrak{T}}) = \frac{1}{2}\sum_{\mathrm{K}\in\mathfrak{M}} \mathbf{u}_{\mathrm{K}} \mathbf{1}_{\bar{\mathrm{K}}\cap\partial\Omega} + \frac{1}{2}\sum_{\mathrm{K}^*\in\partial\mathfrak{M}^*} \mathbf{u}_{\mathrm{K}^*} \mathbf{1}_{\bar{\mathrm{K}}^*\cap\partial\Omega}.$$

$$\gamma(\mathbf{u}^{\mathfrak{T}}) = \frac{1}{2}\sum_{\mathrm{L}\in\partial\mathfrak{M}} \mathbf{u}_{\mathrm{L}} \mathbf{1}_{\mathrm{L}} + \frac{1}{2}\sum_{\mathrm{K}^*\in\partial\mathfrak{M}^*} \mathbf{u}_{\mathrm{K}^*} \mathbf{1}_{\bar{\mathrm{K}}^*\cap\partial\Omega}.$$

With this definition, we use simultaneously the values on the primal mesh and the values on the dual mesh. We can also consider two different reconstructions based either on the primal values or the dual values:

$$\widetilde{\gamma}^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}}) = \sum_{\mathtt{K}\in\mathfrak{M}} \mathbf{u}_{\mathtt{K}} \mathbf{1}_{\bar{\mathtt{K}}\cap\partial\Omega} \quad \text{or} \quad \gamma^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}}) = \sum_{\mathtt{L}\in\partial\mathfrak{M}} \mathbf{u}_{\mathtt{L}} \mathbf{1}_{\mathtt{L}}$$

$$\widetilde{\gamma}^{\partial\mathfrak{M}^*}(\mathbf{u}^{\mathfrak{T}}) = \gamma^{\partial\mathfrak{M}^*}(\mathbf{u}^{\mathfrak{T}}) = \sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*} \mathbf{u}_{\mathtt{K}^*} \mathbf{1}_{\bar{\mathtt{K}}^*\cap\partial\Omega}(\mathrm{x}).$$

We point out that, if we consider the object we want to estimate, we have for both cases (by Minkowski's inequality):

$$\|\widetilde{\gamma}(\mathbf{u}^{\mathfrak{T}})\|_{q,\partial\Omega} \le \|\widetilde{\gamma}^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}})\|_{q,\partial\Omega} + \|\widetilde{\gamma}^{\partial\mathfrak{M}^*}(\mathbf{u}^{\mathfrak{T}})\|_{q,\partial\Omega},$$
$$\|\gamma(\mathbf{u}^{\mathfrak{T}})\|_{q,\partial\Omega} \le \|\gamma^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}})\|_{q,\partial\Omega} + \|\gamma^{\partial\mathfrak{M}^*}(\mathbf{u}^{\mathfrak{T}})\|_{q,\partial\Omega}.$$

Before proving the trace theorem, we introduce a discrete Poincaré inequality.

Theorem 10.3 *(Discrete Poincaré inequality)[2, Thm. 11] Let Ω be an open connected bounded polygonal domain of $\mathbb{R}^2$ and Γ_0 be a part of the boundary such that $m(\Gamma_0) > 0$. Let $\mathfrak{T}$ be a DDFV mesh associated to Ω.*

- *If $1 \le p < 2$, let $1 \le q \le p^*$*
- *If $p \ge 2$, let $1 \le q < \infty$.*

There exists a constant $C > 0$, depending only on p,q, Γ_0 and Ω such that $\forall \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_0}$:

$$\|\mathbf{u}^{\mathfrak{T}}\|_q \le \frac{C}{\sin(\alpha_{\mathfrak{T}})^{\frac{1}{p}} \; reg(\mathfrak{T})^{\frac{p-1}{p}}} \|\nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}\|_p.$$

Theorem 10.4 *(Trace inequality) Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. For all $p > 1$ there exists a constant $C > 0$, depending only on p, $\sin(\alpha_{\mathfrak{T}})$, reg($\mathfrak{T}$) and Ω such that $\forall \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_0}$ and $\forall s \ge 1$:*

$$\boxed{\boxed{\|\widetilde{\gamma}(\mathbf{u}^{\mathfrak{T}})\|_{s,\partial\Omega}^s \le C \|\mathbf{u}^{\mathfrak{T}}\|_{1,p} \|\mathbf{u}^{\mathfrak{T}}\|_{\frac{p(s-1)}{p-1}}^{s-1}.}} \tag{27}$$

The computations of the proof are similar to those present in [13] and [11]. In [13], the proof is given for finite volume methods; in [11], the proof is given for DDFV method but in the case of L^1 norm. Our proof has been adapted to the vectorial case and to general L^s, L^p norms.

Proof
Boundary properties: By compactness of $\partial\Omega$, there exists a finite number of open hyper-rectangles $\{R_i, i = 1 \dots N\}$, and normalized vectors of $\mathbb{R}^2$, $\{\eta_i, i = 1, \dots, N\}$, such that:

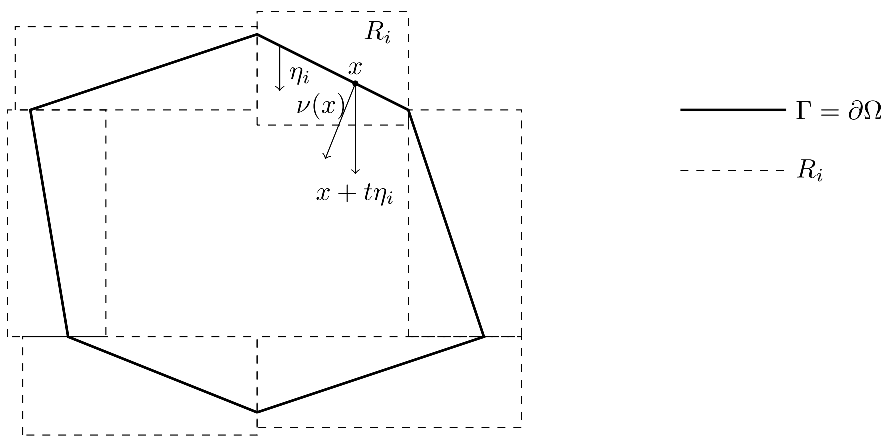


Figure 10: Properties of the boundary $\partial\Omega$.

$$\begin{cases} \partial\Omega \subset \bigcup_{i=1}^{N} R_i \\ (\eta_i, \overrightarrow{\nu}(x)) \geq \lambda > 0 \quad \forall x \in R_i \cap \partial\Omega, i \in \{1 \ldots N\} \\ \{x + t\eta_i, x \in R_i \cap \partial\Omega, t \in \mathbb{R}^+\} \cap R_i \subset \Omega, \end{cases}$$

where λ is a strictly positive number and $\overrightarrow{\nu}(x)$ is the normal vector to $\partial\Omega$ at x, inward to Ω (see Figure 1). Let $\{\lambda_i, i = 1 \ldots N\}$ be a family of functions such that $\sum_{i=1}^{N} \lambda_i(x) = 1$, for all $x \in \partial\Omega$, $\lambda_i \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^+)$ and $\lambda_i = 0$ outside of R_i, for all $i = 1 \ldots N$. Let $\partial\Omega_i = R_i \cap \partial\Omega$; we shall prove that there exists $C_i > 0$ depending only on $\lambda, \mathrm{reg}(\mathfrak{T})$ and λ_i such that

$$\int_{\partial\Omega_i} \lambda_i(x) |\widetilde{\gamma}^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}})(x)|^s \mathrm{d}x + \int_{\partial\Omega_i} \lambda_i(x) |\widetilde{\gamma}^{\partial\mathfrak{M}^*}(\mathbf{u}^{\mathfrak{T}})(x)|^s \mathrm{d}x \leq C_i \|\mathbf{u}^{\mathfrak{T}}\|_{1,p} \|\mathbf{u}^{\mathfrak{T}}\|^{s-1}_{\frac{p(s-1)}{p-1}}.$$

Then it will be sufficient to define $C := \sum_{i=1}^{N} C_i$ to get (27).
We study separately the two terms.

On the primal mesh: We introduce the functions to determine the successive neighbours of a cell $\mathbf{u}_\kappa$. Consider $x, y \in \Omega$, then:

$$\text{for } \sigma \in \mathcal{E} \quad \Psi_\sigma(x,y) := \begin{cases} 1 & \text{if } [x,y] \cap \sigma \neq \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{for } \kappa \in \mathfrak{M} \quad \Psi_{\mathtt{K}}(x,y) := \begin{cases} 1 & \text{if } [x,y] \cap \kappa \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

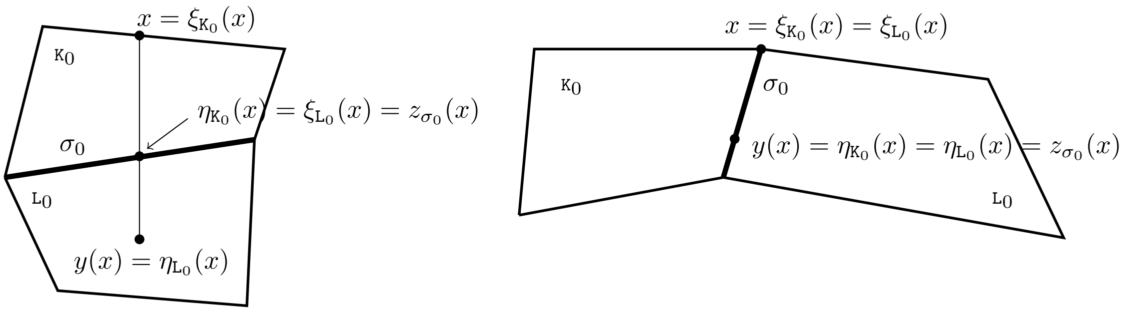


Figure 11: (Left) $[x, y(x)] \cap \sigma_0$ is reduced to a point $z_{\sigma_0}(x)$. (Right) $[x, y(x)] \cap \sigma_0$ is the segment $[x, y(x)]$.

Now, we fix $i \in \{1 \ldots N\}$ and $x \in \partial\Omega_i$.
Then there exists a unique $t > 0$ such that $x + t\eta_i = y(x) \in \partial R_i$. Then, for $\sigma \in \mathcal{E}$, if $[x, y(x)] \cap \sigma \neq \emptyset$, then it is:

- either a point: $z_\sigma(x) := [x, y(x)] \cap \sigma$
- either a segment: $[a(x), b(x)] := [x, y(x)] \cap \sigma$ and let $z_\sigma(x) := b(x)$.

For $K \in \mathfrak{M}$, if $[x, y(x)] \cap \kappa \neq \emptyset$ we have:

$$[\xi_{\mathtt{K}}(x), \eta_{\mathtt{K}}(x)] := [x, y(x)] \cap K.$$

Let us fix $x \in \kappa_0$, with $\kappa_0 \in \mathfrak{M}$ such that $y(x) \in \mathtt{L}_0$, $\sigma_0 = \kappa_0|\mathtt{L}_0$. We distinguish the following two cases:

1. For the left case (see Fig. 11):

$$\begin{aligned} \lambda_i(x)|\mathbf{u}_{\mathtt{K}_0}|^s = &\big(\lambda_i(\xi_{\mathtt{K}_0}(x)) - \lambda_i(\eta_{\mathtt{K}_0}(x))\big)|\mathbf{u}_{\mathtt{K}_0}|^s \\ &+ \big(\lambda_i(\xi_{\mathtt{L}_0}(x)) - \lambda_i(\eta_{\mathtt{L}_0}(x))\big)|\mathbf{u}_{\mathtt{L}_0}|^s \\ &+ \lambda_i(z_{\sigma_0}(x))\big(|\mathbf{u}_{\mathtt{K}_0}|^s - |\mathbf{u}_{\mathtt{L}_0}|^s\big), \end{aligned}$$

2. for the right case (see Fig. 11):

$$\lambda_i(x)|\mathbf{u}_{\mathtt{K}_0}|^s = \big(\lambda_i(\xi_{\mathtt{K}_0}(x)) - \lambda_i(\eta_{\mathtt{K}_0}(x))\big)|\mathbf{u}_{\mathtt{K}_0}|^s.$$

In both cases:

$$\begin{aligned}\lambda_i(x)|\mathbf{u}_{\mathtt{K}_0}|^s \le & \sum_{\mathtt{D}\in\mathfrak{D}} \Psi_\sigma(x,y(x))\lambda_i(z_\sigma(x))\big||\mathbf{u}_{\mathtt{K}}|^s - |\mathbf{u}_{\mathtt{L}}|^s\big| \\ & + \sum_{\mathtt{K}\in\mathfrak{M}} \Psi_{\mathtt{K}}(x,y(x))|\lambda_i(\xi_{\mathtt{K}}(x)) - \lambda_i(\eta_{\mathtt{K}}(x))||\mathbf{u}_{\mathtt{K}}|^s,\end{aligned}$$

that we can write as

$$\lambda_i(x)|\mathbf{u}_{\mathtt{K}_0}|^s \le A(x) + B(x)$$

by defining

$$\begin{aligned}A(x) &:= \sum_{\mathtt{D}\in\mathfrak{D}} \Psi_\sigma(x,y(x))\lambda_i(z_\sigma(x))||\mathbf{u}_{\mathtt{K}}|^s - |\mathbf{u}_{\mathtt{L}}|^s| \\ B(x) &:= \sum_{\mathtt{K}\in\mathfrak{M}} \Psi_{\mathtt{K}}(x,y(x))|\lambda_i(\xi_{\mathtt{K}}(x)) - \lambda_i(\eta_{\mathtt{K}}(x))||\mathbf{u}_{\mathtt{K}}|^s.\end{aligned}$$

We proceed by estimating separately the two terms.

<u>Estimate of A:</u>
Since λ_i is bounded, we get:

$$A(x) \le \|\lambda_i\|_\infty \sum_{\mathtt{D}\in\mathfrak{D}} \Psi_\sigma(x,y(x))\Big||\mathbf{u}_{\mathtt{K}}|^s - |\mathbf{u}_{\mathtt{L}}|^s\Big|;$$

We now use the following estimate (with $c_\sigma = |(\eta_i, \overrightarrow{\nu}_\sigma(x))|$)

$$\int_{\partial\Omega_i} \Psi_\sigma(x,y(x))\mathrm{d}x \le \frac{c_\sigma}{\lambda} m_\sigma,$$

that is proved in [13], to conclude:

$$\begin{aligned}A = \int_{\partial\Omega_i} A(x)\mathrm{d}x &\le \|\lambda_i\|_\infty \sum_{\mathtt{D}\in\mathfrak{D}} \Big(\int_{\partial\Omega_i} \Psi_\sigma(x,y(x))\mathrm{d}x\Big)\Big||\mathbf{u}_{\mathtt{K}}|^s - |\mathbf{u}_{\mathtt{L}}|^s\Big| \\ &\le C_i \sum_{\mathtt{D}\in\mathfrak{D}} m_\sigma \Big||\mathbf{u}_{\mathtt{K}}|^s - |\mathbf{u}_{\mathtt{L}}|^s\Big|\end{aligned}$$

where in the 3rd inequality we used [17][Lemma I.19].
Now, as in [2], we use the inequality:

$$\Big||\mathbf{u}_{\mathtt{K}}|^s - |\mathbf{u}_{\mathtt{L}}|^s\Big| \le s(|\mathbf{u}_{\mathtt{K}}|^{s-1} + |\mathbf{u}_{\mathtt{L}}|^{s-1})|\mathbf{u}_{\mathtt{K}} - \mathbf{u}_{\mathtt{L}}|$$

that leads to:

$$\begin{aligned}\sum_{\mathtt{D}\in\mathfrak{D}} m_\sigma \Big||\mathbf{u}_{\mathtt{K}}|^s - |\mathbf{u}_{\mathtt{L}}|^s\Big| &\le s \sum_{\mathtt{D}\in\mathfrak{D}} m_\sigma(|\mathbf{u}_{\mathtt{K}}|^{s-1} + |\mathbf{u}_{\mathtt{L}}|^{s-1})|\mathbf{u}_{\mathtt{K}} - \mathbf{u}_{\mathtt{L}}| \\ &\le C \sum_{\mathtt{D}\in\mathfrak{D}} m_\sigma m_{\sigma^*}(|\mathbf{u}_{\mathtt{K}}|^{s-1} + |\mathbf{u}_{\mathtt{L}}|^{s-1})\Big|\frac{\mathbf{u}_{\mathtt{K}} - \mathbf{u}_{\mathtt{L}}}{m_{\sigma^*}}\Big| \qquad \text{(Int. by parts and Hölder)} \\ &\le C\Big(\sum_{\mathtt{K}\in\mathfrak{M}}\sum_{\mathtt{D}\in\mathfrak{D}_{\mathtt{K}}} m_\sigma m_{\sigma^*}|\mathbf{u}_{\mathtt{K}}|^{\frac{(s-1)p}{p-1}}\Big)^{\frac{p-1}{p}} \Big(\sum_{\mathtt{D}\in\mathfrak{D}} m_\sigma m_{\sigma^*}\Big|\frac{\mathbf{u}_{\mathtt{K}} - \mathbf{u}_{\mathtt{L}}}{m_{\sigma^*}}\Big|^p\Big)^{\frac{1}{p}}.\end{aligned}$$

By regularity hypothesis on the mesh and the definition of the discrete gradient we can write:

$$A \le \frac{C}{\sin(\alpha_{\mathfrak{T}})^{\frac{1}{p}}\ \mathrm{reg}(\mathfrak{T})^{\frac{p-1}{p}}}\ \|\mathbf{u}^{\mathfrak{T}}\|^{s-1}_{\frac{(s-1)p}{p-1}}\ \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_p.$$

<u>Estimate of B:</u>
Since λ_i is C^∞, we have, by Talylor's formula:

$$B(x) \le \|\nabla\lambda_i\|_\infty \sum_{K\in\mathfrak{M}} \Psi_{\mathtt{K}}(x,y(x))|\xi_{\mathtt{K}}(x) - \eta_{\mathtt{K}}(x)||\mathbf{u}_{\mathtt{K}}|^s$$

and thanks to the inequality that can be found in [13, Lemma 3.10]

$$\int_{\partial\Omega_i} \Psi_{\mathtt{K}}(x, y(x))|\xi_{\mathtt{K}}(x) - \eta_{\mathtt{K}}(x)|\mathrm{d}x \leq \frac{m_{\mathtt{K}}}{\lambda},$$

we can conclude:

$$\begin{aligned} B = \int_{\partial\Omega_i} B(x) &\leq \|\nabla\lambda_i\|_\infty \sum_{\mathtt{K}\in\mathfrak{M}} \Big(\int_{\partial\Omega_i} \Psi_{\mathtt{K}}(x, y(x))|\xi_{\mathtt{K}}(x) - \eta_{\mathtt{K}}(x)|\Big)|\mathbf{u}_{\mathtt{K}}|^s \\ &\leq C_i \sum_{\mathtt{K}\in\mathfrak{M}} m_{\mathtt{K}}|\mathbf{u}_{\mathtt{K}}|^s. \end{aligned}$$

Thus

$$B \leq C\|\mathbf{u}^{\mathfrak{T}}\|_s^s.$$

Putting together the terms, we find:

$$\int_{\partial\Omega_i} \lambda_i(x)|\widetilde{\gamma}^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}})|^s \leq C_i \left(\|\mathbf{u}^{\mathfrak{T}}\|^{s-1}_{\frac{(s-1)p}{p-1}} \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_p + \|\mathbf{u}^{\mathfrak{T}}\|_s^s \right).$$

By proceeding as in the proof of [2, Lemma 1], we use interpolation between L^p spaces and we write:

$$\|\mathbf{u}^{\mathfrak{T}}\|_s^s \leq \|\mathbf{u}^{\mathfrak{T}}\|^{s-1}_{\frac{(s-1)p}{p-1}} \|\mathbf{u}^{\mathfrak{T}}\|_p$$

that leads to

$$\int_{\partial\Omega_i} \lambda_i(x)|\widetilde{\gamma}^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}})|^s \leq C_i\|\mathbf{u}^{\mathfrak{T}}\|^{s-1}_{\frac{(s-1)p}{p-1}} \|\mathbf{u}^{\mathfrak{T}}\|_{1,p}$$

that proves our theorem.

On the dual mesh: the computations are exactly the same, exchanging $\mathtt{K}$ with $\mathtt{K}^*$ and σ in σ^* .

■

Corollary 10.5 *Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C > 0$, depending only on p,q, $\sin(\alpha_{\mathfrak{T}})$, reg($\mathfrak{T}$) and Ω such that $\forall \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_0}$ and for all $s \geq 1$, $p > 1$:*

$$\|\gamma(\mathbf{u}^{\mathfrak{T}})\|^s_{s,\partial\Omega} \leq C\|\mathbf{u}^{\mathfrak{T}}\|_{1,p}\|\mathbf{u}^{\mathfrak{T}}\|^{s-1}_{\frac{p(s-1)}{p-1}}.$$

Proof The proof is almost the same as Theorem 10.4.
What changes is just that we now fix $x \in \mathtt{L}$, $\mathtt{L} \in \partial\mathfrak{M}$ and $\mathtt{K}_0 \in \mathfrak{M}$ such that $\mathtt{L} \subset \mathtt{K}_0$, $y(x) \in \mathtt{K}_0$, $\sigma_0 = \mathtt{K}_0|\mathtt{L}$.
The term that we want to study now is $\lambda_i(x)|\mathbf{u}_{\mathtt{L}}|^s$, since we are focusing on the boundary. It can be written as:

$$\lambda_i(x)|\mathbf{u}_{\mathtt{L}}|^s = \lambda_i(x)(|\mathbf{u}_{\mathtt{L}}|^s - |\mathbf{u}_{\mathtt{K}_0}|^s) + \lambda_i(x)|\mathbf{u}_{\mathtt{K}_0}|^s \tag{28}$$

that can be estimated by:

$$\begin{aligned} \lambda_i(x)|\mathbf{u}_{\mathtt{L}}|^s &\leq \lambda_i(x)\Big||\mathbf{u}_{\mathtt{L}}|^s - |\mathbf{u}_{\mathtt{K}_0}|^s\Big|\mathbf{1}_{\mathtt{L}}(x) + \lambda_i(x)|\mathbf{u}_{\mathtt{K}_0}|^s \\ &:= A_b(x) + \lambda_i(x)|\mathbf{u}_{\mathtt{K}_0}|^s \end{aligned}$$

Estimate of A_b:
Since λ is bounded, we have:

$$A_b = \int_{\partial\Omega_i} A_b(x) \leq \|\lambda_i\|_\infty \sum_{\mathtt{D}\in\mathfrak{D}} m_\sigma \Big||\mathbf{u}_{\mathtt{L}}|^s - |\mathbf{u}_{\mathtt{K}}|^s\Big|.$$

We can proceed exactly as in the proof of Thm 10.4 for A, so we get:

$$A_b \leq \frac{C}{\sin(\alpha_{\mathfrak{T}})^{\frac{1}{p}} \operatorname{reg}(\mathfrak{T})^{\frac{p-1}{p}}} \|\mathbf{u}^{\mathfrak{T}}\|^{s-1}_{\frac{(s-1)p}{p-1}} \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_p.$$

Putting together all the terms, we find:

$$\left(\int_{\partial\Omega_i} \lambda_i(x)|\gamma^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}})|^s\right) \leq A_b + \left(\int_{\partial\Omega_i} \lambda_i(x)|\widetilde{\gamma}^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}})|^s\right).$$

Thanks to the previous theorem, we conclude:

$$\int_{\partial\Omega_i} \lambda_i(x)|\gamma^{\partial\mathfrak{M}}(\mathbf{u}^{\mathfrak{T}})|^s \leq C_i \|\mathbf{u}^{\mathfrak{T}}\|^{s-1}_{\frac{(s-1)p}{p-1}} \ \|\mathbf{u}^{\mathfrak{T}}\|_{1,p}$$

that proofs our statement.

On the dual mesh: the computations are the same as the previous theorem.

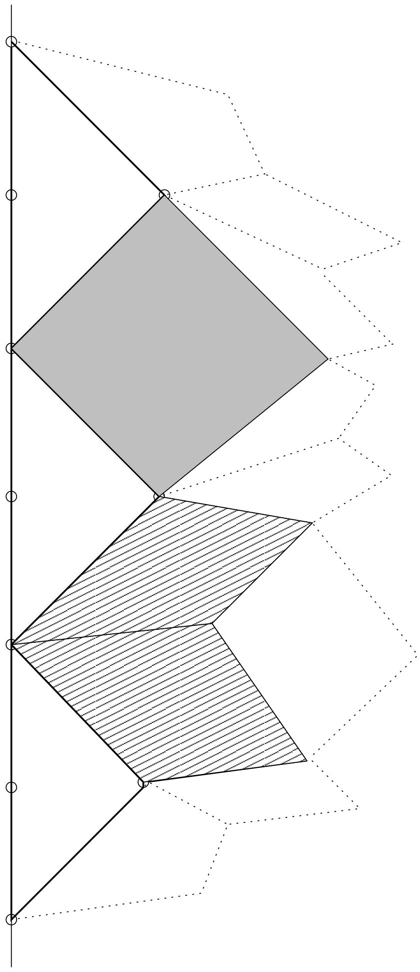
■

11 Appendix B: Study of the kernel of $\mathrm{D}^{\mathfrak{D}}$

Theorem 11.1 *Let Ω be an open connected bounded polygonal domain of $\mathbb{R}^2$ and Γ_0 be a part of the boundary such that $m(\Gamma_0) > 0$.*
Let $\mathfrak{T}$ be a DDFV mesh associated to Ω that satisfies inf-sup stability condition. Then $\forall \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_0^{\Gamma_0}$ such that $\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} = 0$ we have $\mathbf{u}^{\mathfrak{T}} = 0$ in Ω.

Proof Since we are not able to give a general proof of this theorem for all meshes, we focus on particular families, namely Cartesian meshes and all the ones that are unconditionally *inf-sup* stable (see [5]), since to prove Thm. 6.2 we need this last hypothesis.

Figure 12: Structures.



When studying those meshes, we observe a propagation phenomenon of the zero boundary data on Γ_0 to the entire mesh.
In fact, it is important to remark that in DDFV meshes all boundary diamonds are triangles (see Fig. 2). If we focus on one of those diamonds, the condition on Γ_0 implies that the velocity is zero on the three vertices L, K^* and L^*:

$$\mathbf{u}_{\mathrm{L}} = \begin{pmatrix} \mathrm{u}_{\mathrm{L}}^x \\ \mathrm{u}_{\mathrm{L}}^y \end{pmatrix} = 0, \qquad \mathbf{u}_{\mathrm{K}^*} = \begin{pmatrix} \mathrm{u}_{\mathrm{K}^*}^x \\ \mathrm{u}_{\mathrm{K}^*}^y \end{pmatrix} = 0, \qquad \mathbf{u}_{\mathrm{L}^*} = \begin{pmatrix} \mathrm{u}_{\mathrm{L}^*}^x \\ \mathrm{u}_{\mathrm{L}^*}^y \end{pmatrix} = 0.$$

Since we are supposing $\mathrm{D}^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} = 0$ for all $\mathrm{D} \in \mathfrak{D}$, this is true in particular for the boundary diamonds (the white ones in Fig. 12). By the definition of the discrete strain rate tensor (5) we are led to the following system:

$$\begin{cases} m_\sigma\, \mathrm{u}_{\mathrm{K}}^x\, \mathrm{n}_{\sigma,\mathrm{K}}^x = 0 \\ m_\sigma\, \mathrm{u}_{\mathrm{K}}^y\, \mathrm{n}_{\sigma,\mathrm{K}}^y = 0 \\ m_\sigma\, (\mathrm{u}_{\mathrm{K}}^x\, \mathrm{n}_{\sigma,\mathrm{K}}^y + \mathrm{u}_{\mathrm{K}}^y\, \mathrm{n}_{\sigma,\mathrm{K}}^x) = 0, \end{cases} \tag{29}$$

that implies $\mathbf{u}_{\mathrm{K}} = \begin{pmatrix} \mathrm{u}_{\mathrm{K}}^x \\ \mathrm{u}_{\mathrm{K}}^y \end{pmatrix} = 0$, since the outer normal $\vec{\mathbf{n}}_{\sigma\mathrm{K}} = \begin{pmatrix} \mathrm{n}_{\sigma,\mathrm{K}}^x \\ \mathrm{n}_{\sigma,\mathrm{K}}^y \end{pmatrix}$ cannot be zero.
This means that for all diamonds in $\mathfrak{D}_{ext} \cap \Gamma_0$ the four components of the velocity, $\mathbf{u}_{\mathrm{K}}, \mathbf{u}_{\mathrm{L}}, \mathbf{u}_{\mathrm{K}^*}, \mathbf{u}_{\mathrm{L}^*}$, are zero.

We now look at the diamonds that are adjacent to ones on the boundary: for the meshes under consideration, we can distinguish two possible situations that we illustrate in Fig. 12.

The first one is the case of the shaded diamond, for which the situation is equivalent to the one of boundary diamonds. In fact, we know that the velocity is zero on three of its vertices. So we can conclude, by solving a system similar to (29) deduced by $\mathrm{D}^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} = 0$, that even the last component of the velocity is zero on that diamond.

The second structure is described by the hatched diamonds. This is the case of two neighbors, that we will denote with $\mathrm{D}^1,\mathrm{D}^2$ which share a common vertex. Remark that on that vertex the velocity is zero and both diamonds have one more vertex with zero velocity. Thus we are considering a structure composed by 6 vertices, where the values of the velocity are zero on 3 among them.
In this case, we denote the normal vectors of $\mathrm{D}^1, \mathrm{D}^2$ with

$$\vec{\mathbf{n}}_{\sigma\mathrm{K}}^i = \begin{pmatrix} \mathrm{n}_\sigma^{x,i} \\ \mathrm{n}_\sigma^{y,i} \end{pmatrix}, \quad \vec{\mathbf{n}}_{\sigma^*\mathrm{K}^*}^i = \begin{pmatrix} \mathrm{n}_{\sigma^*}^{x,i} \\ \mathrm{n}_{\sigma^*}^{y,i} \end{pmatrix} \qquad \text{for } i = 1, 2$$

and we write the system of equations equivalent to the conditions $\mathrm{D}^{\mathtt{D}^i}\mathbf{u}^{\mathfrak{T}} = 0$ for $i = 1, 2$.
The 6×6 matrix of that system has determinant

$$det = (\mathrm{n}^{x,2}_{\sigma^*}\,\mathrm{n}^{y,2}_{\sigma} - \mathrm{n}^{x,2}_{\sigma}\,\mathrm{n}^{y,2}_{\sigma^*})(\mathrm{n}^{x,1}_{\sigma^*}\,\mathrm{n}^{y,1}_{\sigma} - \mathrm{n}^{x,1}_{\sigma}\,\mathrm{n}^{y,1}_{\sigma^*})(\mathrm{n}^{x,1}_{\sigma}\,\mathrm{n}^{y,2}_{\sigma} - \mathrm{n}^{x,2}_{\sigma}\,\mathrm{n}^{y,1}_{\sigma}) \neq 0$$

that is always different from zero, except in a degenerate case that we treat in the following section where the normals of the two diamonds are parallel. Thus the matrix is invertible, that implies that all the six components of the velocity on those two diamonds are zero: $\mathbf{u}^i_{\mathtt{K}} = \mathbf{u}^i_{\mathtt{L}} = \mathbf{u}^i_{\mathtt{K}^*} = \mathbf{u}^i_{\mathtt{L}^*} = 0$ for $i = 1, 2$.

Degenerate case: checkerboard mesh

This is a particular case of the second structure, in which the normal vectors of the two hatched diamonds are parallel. In order to have an invertible system to solve, it is necessary to consider a third diamond.
In particular, if we call $\mathtt{D}_1, \mathtt{D}_2$ the hatched diamonds and $\mathtt{D}_3$ the white one, we have for instance: $\vec{\mathbf{n}}^i_{\sigma\mathtt{K}} = \begin{pmatrix}0\\1\end{pmatrix}$ for $i = 1, 2$ and $\vec{\mathbf{n}}^3_{\sigma\mathtt{K}} = \begin{pmatrix}1\\0\end{pmatrix}$.
If, as we did in the previous cases, we write the system of equations equivalent to $\mathrm{D}^{\mathtt{D}^i}\mathbf{u}^{\mathfrak{T}} = 0$, but this time for $i = 1, 2, 3$, we get again an invertible system, this time of size 8×8. As before, we find that all the components of the velocity are zero on the three diamonds.

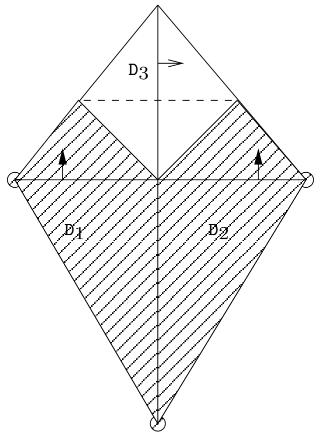


Figure 13: Degenerate case

By proceeding step by step, we can prove that the velocity $\mathbf{u}^{\mathfrak{T}}$ is zero on the entire domain Ω.

■

Remark 11.2 *We suppose that the mesh satisfies inf-sup stability condition because this hypothesis is necessary to prove Theorem 6.2. Since the inf-sup constant it is not involved in the proof of Theorem 11.1, we could extend the technique of the proof to all geometries, considering one mesh at a time.*

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