Homogenization of transport equations: A simple PDE approach to the Kubo formula

Thierry Goudon*, Frédéric Poupaud†

Abstract

We are interested in the behavior with respect to the small parameter ϵ of solutions ρ^ϵ of the conservative transport(-diffusion) equation $\partial_t \rho^\epsilon + \nabla_x(\rho^\epsilon u^\epsilon) = \eta \Delta_x \rho^\epsilon$, with $\eta \geq 0$, driven by a large random velocity field: $|u^\epsilon| = \mathcal{O}(1/\epsilon)$. Assuming that the velocity does not have long-time-range memory we justify the convergence of the expectation $\mathbb{E}\rho^\epsilon$ to the solution of a diffusion equation. Our proof relies only on PDE tools.

1 Introduction

We are interested in the evolution of a scalar quantity $\rho(t,x)$, advected by a "turbulent" velocity field $u(t,x)$. This quantity can also be submitted to diffusion effects, characterized by a non negative coefficient $\eta \geq 0$, possibly $\eta = 0$. Hence, ρ obeys the following equation

$$\partial_t \rho + \mathrm{div}_x(\rho u) = \eta \Delta_x \rho, \qquad t \geq 0,\ x \in \mathbb{R}^N. \tag{1}$$ fp

For the time being, "turbulent" remains a vague adjective roughly meaning that u has violent and fast variations and we aim at describing the average behavior of ρ over all these variations. To this end, let us denote by $\langle \cdot \rangle$ some averaging operation. We are led to

$$\partial_t \langle \rho \rangle = \mathrm{div}_x \langle \rho u \rangle + \eta \Delta_x \langle \rho \rangle. \tag{2}$$ fp2

*Laboratoire Paul Painlevé, UMR 8524, CNRS–Université des Sciences et Technologies de Lille, Cité Scientifique, F-59655 Villeneuve d'Ascq Cedex, France. E-mail: thierry.goudon@math.univ-lille1.fr

†Laboratoire J. A. Dieudonné, U.M.R. 6621, C.N.R.S.–Université Nice-Sophia Antipolis, Parc Valrose, F-06108 Nice Cedex 2, France. E-mail: poupaud@math.unice.fr

Therefore, while the problem (1) is linear, the question we address becomes non linear since we need to characterize $\langle \rho u \rangle$ and, as a common rule, usually the average of a product does not coincide with the product of the averages. This is the so called "turbulent moment closure problem" which consists in finding an expression of the correlation $\langle \rho u \rangle$ by means of $\langle \rho \rangle$. The problem arises in many different fields of application. Usual examples come from the prediction of temperature profiles in high Reynolds number turbulence [5] or the tracking of pollutants in the atmosphere [10]. The problem is also motivated by the study of the propagation of oscillations in fluid mechanics equations, as pointed out by Di Perna-Majda [11] and Mc Laughlin-Papanicolaou-Pironneau [28]. Closure of (2) has also been used recently to derive models for ionosperic plasmas by Besse *et al.* [6]. It leads to the so called "turbulent striation model" which is shown to have nice stability properties and gives a quite accurate model whose solution can be evaluated with a moderate computational cost.

There are two ways to model the fluctuations of the velocity field u: either we consider a deterministic field which oscillates fastly with respect to some periodic variables, or we consider a random field. The former situation (where $\langle \cdot \rangle$ stands for the average over the perodic variables) is analyzed in great generality in Goudon-Poupaud [15] where further references can be found; the latter (where $\langle \cdot \rangle$ stands for the expectation) is the object of this work. Let us point out immediatly that the role of the diffusivity can be completely different in both situations. When considering periodic oscillations, it is crucial to assume $\eta > 0$, the main difficulty being related to the well known fact that the Fredhom alternative does not apply to the transport operator $\mathrm{div}(u\cdot)$. When $\eta = 0$, rigorous results can be obtained only at the price of drastic restriction to very particular geometries:

- shear flows, where the oscillations of the velocity field hold in a direction transverse to the variable of derivation of the scalar field, see Tartar, [32, 33], Mascarenhas [29], Amirat-Hamdache-Ziani [1, 2], Hamdache [17, 18]...

- divergence free fields in 2 space dimension where we can use the fact that u is the curl of a potential, see Brenier [9], Hou-Xin [19], E [12]...

On the contrary, when dealing with random velocity fields, we can obtain a diffusive behavior for the average $\langle \rho \rangle$, even for $\eta = 0$, as we shall see below.

When u is a random field, instead of dealing directly with the PDE (1), a natural approach consists in looking at the evolution of the trajectories, solution of the differential equation

$$\mathrm{d}X(t) = u(t, X(t)) + \sqrt{2\eta}\,\mathrm{d}W(t), \tag{3}$$

W being the standard $N-$ dimensional Brownian motion. Actually, this differential equation is clearly related to the backward non conservative form of the PDE (1)

$$\partial_t \rho + u \cdot \nabla_x \rho + \eta \Delta_x \rho = 0, \qquad t < T,$$

completed by a final data $\rho(T, x) = \rho_T(x)$. The large time behavior of the trajectories is a question which goes back to TAYLOR [34]. Under appropriate assumptions on u, he conjectured that particles driven by (3) follow approximately a Brownian motion defined by the correlations matrix $\mathbb{E}(u \otimes u)$. More general and rigorous results are described in the seminal papers of KUBO [25], STRATONOVICH [31] and KHAS'MINSKII [24]. Within this viewpoint, the situation has been treated in full generality in the celebrated article of KESTEN-PAPANICOLAOU [20].

To be more precise, let us introduce a small parameter $\epsilon > 0$, associated to some physical quantities of the problem. Working with dimensionless quantities, the random velocity field becomes

$$\frac{1}{\epsilon}\, u(t/\epsilon^2, x/\epsilon^\gamma), \qquad 0 \leq \gamma \leq 1,$$

with u of order $\mathcal{O}(1)$. The crucial assumption relies on a strong separation of the time scales. This is the meaning of the so called mixing condition, which in some sense prohibits long-time-range memory of the velocity field. The case $\gamma < 1$ corresponds to a weak fluctuation scaling since, roughly speaking, space scales are not strongly separated. Then, under such a mixing condition on u, we can establish, according to KHAS'MINSKII [24], KESTEN-PAPANICOLAOU [20], the convergence of the measure on $C^0([0, \infty), \mathbb{R}^N)$ induced by X^ϵ, solution of $\frac{\mathrm{d}X^\epsilon}{\mathrm{d}t} = \frac{1}{\epsilon}\, u(t/\epsilon^2, X^\epsilon)$, $X^\epsilon(0) = x$, to a probability measure corresponding to a diffusion generator, identified by means of the correlation matrix of u. When $\gamma = 1$, the problem is much more difficult since space scales are strongly separated. Hence, we have to average over space fluctuations and the problem combines with homogenization effects. Note that this scaling is particularly relevant on a physical viewpoint since the change of variables $u(t, x) \to \frac{1}{\epsilon} u(t/\epsilon^2, x/\epsilon)$ leaves invariant the Navier-Stokes equation. This paper only deals with the weak fluctuation scaling $0 \leq \gamma < 1$ and relies on rapid temporal decorrelation.

Let us end this Introduction with some references, which are certainly far from exhaustive. A very complete presentation of the subject, both on mathematical, physical and numerical modelling, with nice and deep comments, is the review by KRAMER-MAJDA [23]. A shorter presentation can be found in FANNJIANG-RYZHIK-PAPANICOLAOU [14]. The problem is very often treated under the additional assumption that u is divergence free. It greatly simplifies the mathematical analysis. For instance, when $\eta > 0$, on the one hand the incompressibility condition leads to better functional estimates and on the other hand, by introducing a streaming skew-symetric matrix B verifying $\mathrm{Div}(B) = u$, we can reformulate the problem in the simpler framework of a purely parabolic equation. We refer on this aspect to the works by AVELLANEDA-MAJDA [4], FANNJIANG-PAPANICOLAOU [13], KOMOROWSKI-OLLA [21], LANDIM-OLLA-YAU [26]. Shear flows situations are analyzed by AVELLANEDA-MAJDA [3]. The strong fluctuation scaling ($\gamma = 1$) with $\eta = 0$ and an incompressible velocity field is treated by KOMOROWSKI-PAPANICOLAOU [22].

In this paper, we investigate the limit as ϵ goes to 0 of $\mathbb{E}\rho^\epsilon$, where ρ^ϵ satisfies

$$\left\{\begin{array}{l} \partial_t \rho^\epsilon + \mathrm{div}_x(u^\epsilon \rho^\epsilon) = \eta \Delta_x \rho^\epsilon, \\ \rho^\epsilon(0,x) = \rho^\epsilon_I(x), \\ u^\epsilon(t,x) = \dfrac{1}{\epsilon}\Big(\epsilon\, u^1(t,x) + u^{0,\epsilon}(t,x)\Big). \end{array}\right. \tag{4}$$

The scaling assumes large fluctuations of the random velocity field around a mean state u^1 which is assumed to be deterministic and which does not depend on ϵ. In the spirit of Kubo's approach, see [25], the crucial assumption will be that fluctuations decorrelates over large enough $\epsilon-$dependent time interval. Then, we justify the convergence to a diffusion process, in the sense that $\mathbb{E}\rho^\epsilon$ converges to ρ, in some suitable weak sense, solution of a diffusion equation. We point out once again that the result holds even when assuming $\eta = 0$, a situation completely different from the homogenization with periodic fields. Our method of proof uses the framework introduced in POUPAUD-VASSEUR [30] to study Schrödinger and Vlasov equation submitted to random potentials. This strategy has been successfully applied to general hyperbolic systems by BRASSART [8], to investigate turbulent regimes in tokamaks by LOEPER-VASSEUR [27] or to derive limit equations describing clouds of particles submitted to friction forces induced by turbulent surrounding fluid by GOUDON-POUPAUD [16]. The paper is organized as follows. In Section 2, we precise our assumptions on the velocity field u, and we state the convergence result, restricting to the simplest case $\gamma = 0$, $\eta = 0$. Section 3 is devoted to the proof and we point out that we use only a PDE approach to justify the diffusion approximation. We do not pretend to the full generallity nor to the optimality of our assumptions. Instead, we emphasize the simplicity of the proof and provide a result as neat and accessible as possible. Eventually, in Section 4 we sketch the modifications necessary to treat the situations $\eta > 0$ and of weak fluctuations scaling $0 < \gamma < 1/2$.

2 Random homogenization of advection equations

In (4), the singular term $\frac{1}{\epsilon}u^{0,\epsilon}(t,x)$ represents random fluctuations around the mean state u^1, assumed to be deterministic and independent on ϵ. More precisely, let us introduce the following notations.

Let $(\Omega, \mathcal{T}, \mu)$ be a probability space (with μ a $\sigma-$finite measure). A random variable X is a μ-measurable real function on Ω. When X is an integrable random variable, its expectation is given by

$$\text{For } X : \Omega \longrightarrow \mathbb{R}, \qquad \mathbb{E}(X) = \int_\Omega X(\omega)\,\mathrm{d}\mu(\omega).$$

Here, $u^{0,\epsilon}(t,x)$ is considered as an integrable random variable of $\mathbb{R}^N$. It is assumed to have a zero mean that is

$$\mathbb{E}(u^{0,\epsilon}(t,x)) = 0.$$

Accordingly, solution of (4) depends on the realization of the event ω. Then the solution ρ^ϵ is also a random function. Throughout the paper a quantity is said deterministic when it does not depend on the alea variable $\omega \in \Omega$. Since the mean field u^1 is of order ϵ compared to the random fluctuations, we are concerned with a weak mean field problem.

We recall the following definition from measure theory (see for instance [7]). Considering sequences $\left(\rho_n\right)_{n\in\mathbb{N}}$ of bounded measures on $\mathbb{R}^N$, we say that ρ_n converges vaguely to ρ if

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \varphi\,\rho_n\,\mathrm{d}x = \int_{\mathbb{R}^N} \varphi\,\rho\,\mathrm{d}x \tag{5}$$

for any continuous function with compact support $\varphi \in C_c(\mathbb{R}^N)$ (actually, the convergence (5) holds for any function φ which is continuous and vanishes at infinity). Considering bounded measures parametrized by $t \in [0,T]$, we say that $\rho_n(t)$ converges vaguely to $\rho(t)$ uniformly on $[0,T]$ when (5) holds uniformly on $[0,T]$.

We rewrite the transport equation (4) as follows

$$\partial_t \rho^\epsilon + \mathrm{div}_x(u^1 \rho^\epsilon) = \Theta^\epsilon[\rho^\epsilon], \qquad \text{in } \mathbb{R}_t \times \mathbb{R}^N_x, \tag{6}$$

where Θ^ϵ is the operator defined by

$$\Theta^\epsilon(t)[\rho] = -\frac{1}{\epsilon}\mathrm{div}_x(u^{0,\epsilon}\ \rho).$$

We wish to determine an effective limit equation for the expectation $\mathbb{E}(\rho^\epsilon)$ as $\epsilon \to 0$. Averaging (6) gives

$$\partial_t \mathbb{E}(\rho^\epsilon) + \mathrm{div}_x(u^1 \mathbb{E}(\rho^\epsilon)) = \mathbb{E}(\Theta^\epsilon[\rho^\epsilon]),$$

and we realize that we seek a description of the behavior of $\mathbb{E}(\Theta^\epsilon[\rho^\epsilon])$. We will follow the framework introduced in Poupaud-Vasseur [30] which crucially requires a time mixing hypothesis. Namely, the fluctuation field $u^{0,\epsilon}$ has random oscillations with respect to the time variable; we suppose it has the form

$$u^{0,\epsilon}(t,x) = v(t/\epsilon^2, x),$$

where the function v is required to satisfy

$$\begin{cases} \bullet\, v(t,x) \text{ is a } L^2 \text{ random variable with } \mathbb{E}(v(t,x)) = 0, \\ \bullet \text{ There exists a } (N\times N \text{ matrix valued) smooth function } R : [0,\infty)\times\mathbb{R}^{2N} \to \mathcal{M}_N, \\ \text{such that for every } x,\ y \in \mathbb{R}^N, \text{ for every } t,\ s\in[0,\infty) : \\ \mathbb{E}(v(t,x)\otimes v(s,y)) = R(s-t,x,y). \\ \bullet \text{ There exists a constant } \mathfrak{T} > 0 \text{ such that for every } x,\ y\in\mathbb{R}^N, \text{ for every } t,\ s \in [0,\infty), \\ \text{if } |t-s| \geq \mathfrak{T} \text{ then } v(t,x) \text{ and } v(s,y) \text{ are independent random variables,} \end{cases} \tag{7}$$

The velocity field v is assumed to be an L^2 random variable so that it admits finite covariance matrix. The first assumption means that v is centered. The second assumption means that v is stationary: the covariances of U evaluated at different times depends only on the differences between these times. The third assumption can be seen as a Markov like assumption on the trajectories. In [22], a random field v verifying such a property is said $\mathfrak{T}-$dependent. It means that events are decorrelated when occurring at times separated by $\mathfrak{T}$. This is reminiscent to the classical Kubo analysis, [25], see also [23]. This crucial assumption plays the role of time mixing hypothese which prohibits long-time memory effects. Note that the function R is compactly supported in $[-\mathfrak{T},\mathfrak{T}]\times\mathbb{R}^N\times\mathbb{R}^N$ since for $|t-s|\geq\mathfrak{T}$, we have $\mathbb{E}(v(t,x)\otimes v(s,y)) = R(s-t,x,y) = \mathbb{E}(v(t,x))\otimes\mathbb{E}(v(s,y)) = 0$. We also assume the following regularity assumptions:

$$\begin{cases} \bullet \text{ The mean velocity field } u^1 \in L^\infty(\mathbb{R}; W^{2,\infty}(\mathbb{R}^N)), \\ \bullet \text{ The random velocity field } v \in L^\infty(\mathbb{R}; W^{2,\infty}(\mathbb{R}^N)) \text{ satisfies for some } K>0, \\ \sup_{t,x} |\partial_x^\alpha v(t,x)| \leq K < \infty, \text{ almost surely for any } \alpha\in\mathbb{N}^N,\ |\alpha|\leq 2. \end{cases} \tag{8}$$

(With this assumption, note that $v(t,x)$ actually belongs to $L^\infty(\Omega) \subset L^2(\Omega)$.) Then, the homogenization process leads to a convection-diffusion equation, as described in the following statement.

theo:rand **Theorem 1** *Under the above assumptions* defvel hrand *(7) and (8), let*

$$u^\epsilon(t,x) = \frac{1}{\epsilon}\, v(\frac{t}{\epsilon^2}, x) + u^1(t,x), \qquad \forall (t,x) \in \mathbb{R}^{N+1}.$$

Let $\left(\rho_I^\epsilon\right)_{\epsilon>0}$ be a bounded sequence of deterministic non negative measures on $\mathbb{R}^N$. We suppose that $\left(\rho_I^\epsilon\right)_{\epsilon>0}$ converges vaguely to ρ_I. Let ρ^ϵ be the solution of the corresponding Cauchy problem

$$\partial_t \rho^\epsilon(t,x) + \mathrm{div}_x\Big(u^\epsilon\ \rho^\epsilon\Big)(t,x) = 0, \quad (t,x) \in \mathbb{R}^{N+1}, \tag{9}$$ eq:transp

$$\rho^\epsilon(0,x) = \rho_I^\epsilon(x), \quad x \in \mathbb{R}^N. \tag{10}$$ eq:cauchy

Then, up to the extraction of a subsequence, for any time $T > 0$, $\mathbb{E}(\rho^\epsilon)$ converges vaguely to $\rho(t) \geq 0$ uniformly on $[0,T]$ and the limit $\rho(t)$ is a solution of

$$\partial_t \rho + \mathrm{div}_x\Big((u^1 + c)\ \rho - D(x)\nabla\rho\Big) = 0.$$

in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} \varphi\, \rho(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^N} \Big((u^1 + c)\cdot\nabla_x\varphi + \mathrm{div}_x\cdot(D^T\nabla_x\varphi)\Big)\,\rho(t,x)\,\mathrm{d}x \tag{11}$$ limeqtn

holds in $\mathcal{D}'([0,\infty))$ for any $\varphi \in C_c^\infty(\mathbb{R}^N)$. The matrix D and the velocity field c are given by

$$\begin{cases} D_{i,j}(x) = \displaystyle\int_0^{\mathfrak{T}} R_{j,i}(\sigma,x,x)\,\mathrm{d}\sigma & i,j \in \{1,...,N\}, \\ c_i(x) = \displaystyle\sum_{j=1}^N \int_0^{\mathfrak{T}} \frac{\partial}{\partial x_j} R_{j,i}(\sigma,x,x)\,\mathrm{d}\sigma & i \in \{1,...,N\}. \end{cases}$$

These expressions of the limit coefficients are referred to as the Kubo formula. Note that the additional drift coefficient c vanishes when the velocity field is incompressible, as noted in KP [20]. Let us first detail some properties of the effective coefficients D and c. One important point is that the symmetric part of the diffusion matrix is non negative.

coefeff **Lemma 1** *For all indices $i,j \in \{1,...,N\}$, we have $R_{i,j} \in L^\infty(\mathbb{R}; W^{2,\infty}(\mathbb{R}^N \times \mathbb{R}^N))$, and $\mathrm{supp}(R_{i,j}) \subset [-\mathfrak{T}, +\mathfrak{T}] \times \mathbb{R}^N \times \mathbb{R}^N$. We also have the symmetry property $R_{i,j}(\sigma,x,y) = R_{j,i}(-\sigma,y,x)$. The matrix D has its coefficients in $W^{2,\infty}(\mathbb{R}^N)$ and it satisfies*

$$\sum_{i,j=1}^N D_{i,j}\xi_i\xi_j \geq 0$$

for any $\xi \in \mathbb{R}^N$. The velocity field c lies in $W^{1,\infty}(\mathbb{R}^N)$.

Proof. Regularity of the correlation function R is an immediate consequence of (8) while the support property follows from the second assumption in (7). Regularity of the coefficients D and c follows from those of R. Next, we remark that

$$R_{i,j}(\sigma,x,y)=\mathbb{E}(v_i(0,x)v_j(\sigma,y))=\mathbb{E}(v_j(\sigma,y)v_i(0,x))=R_{j,i}(-\sigma,y,x).$$

In particular, using this equality for $y=x$, we obtain

$$\sum_{i,j=1}^{N} D_{i,j}\xi_i\xi_j=\frac{1}{2}\sum_{i,j=1}^{N}\int_{-\mathfrak{T}}^{\mathfrak{T}} R_{i,j}(\sigma,x,x)\xi_i\xi_j\,\mathrm{d}\sigma=\frac{1}{2}\sum_{i,j=1}^{N}\int_{\mathbb{R}} R_{i,j}(\sigma,x,x)\xi_i\xi_j\,\mathrm{d}\sigma,$$

by using the support property. Then, according to the strategy of [30], we use the following property, satisfied by any integrable function F (see [30] for a proof),

$$\int_{\mathbb{R}} F(\sigma)\,\mathrm{d}\sigma=\lim_{R\to\infty}\frac{1}{2R}\int_{-R}^{R}\left(\int_{-R}^{R}F(s-t)\,ds\right)\,\mathrm{d}t. \tag{12}$$

We get

$$\begin{aligned}\sum_{i,j=1}^{N} D_{i,j}\xi_i\xi_j &= \lim_{R\to\infty}\frac{1}{4R}\sum_{i,j=1}^{N}\int_{-R}^{R}\left(\int_{-R}^{R}R_{i,j}(s-t,x,x)\xi_i\xi_j\,\mathrm{d}s\right)\,\mathrm{d}t\\ &= \lim_{R\to\infty}\frac{1}{4R}\int_{-R}^{R}\int_{-R}^{R}\mathbb{E}(v(t,x)\cdot\xi\ v(s,x)\cdot\xi)\,\mathrm{d}s\,\mathrm{d}t\\ &= \lim_{R\to\infty}\frac{1}{4R}\,\mathbb{E}\left(\Big[\int_{-R}^{R}v(t,x)\cdot\xi\,\mathrm{d}t\Big]^2\right)\geq 0,\end{aligned}$$

which proves the non negativeness of the symmetric part of D. ∎

It is worth illustrating this section with an example of random velocity field satisfying our requirements. Let $V\in C_c^\infty(\mathbb{R}^{N+1};\mathbb{R})$. Consider sequences $\big(\omega_n\big)_{n\in\mathbb{Z}}$, $\big(T_n\big)_{n\in\mathbb{Z}}$ of independent random variables in $\mathbb{R}$, with $\mathbb{E}(\omega_n)=0$, $\mathbb{E}(\omega_n\omega_p)=\delta_{np}$ and T_n equidistributed in $[-1/2,1/2]$. The random variables ω_n and T_n are independent. Then, we set

$$\mathcal{V}(t,x)=\sum_{n\in\mathbb{Z}}\omega_n\ V(t-n-T_n,x).$$

We define similarly $\mathcal{V}'(s,y)=\sum_{n\in\mathbb{Z}}\omega_n\ V'(s-n-T_n,y)$ from another function $V'\in C_c^\infty(\mathbb{R}^{N+1};\mathbb{R})$. Since V, V' have compact supports there is a positive constant $K>0$ such that $\mathcal{V}(t,x)$ depends only of ω_n, T_n with $n\in[t-K,t+K]$ and $\mathcal{V}'(s,y)$ depends only of ω_p, T_p with $p \in [s-K, s+K]$. Then as soon as $|t-s| > 2K$, $\mathcal{V}(t,x)$ and $\mathcal{V}'(s,y)$ are independent. We also have

$$\begin{aligned}\mathcal{V}(t,x)\mathcal{V}'(s,y) &= \sum_{n\in\mathbb{Z}} \omega_n^2 \; V(t-n-T_n,x)V'(s-n-T_n,y) \\ &\quad + \sum_{n,m\in\mathbb{Z},\, n\neq m} \omega_n\omega_m \; V(t-n-T_n)V(s-m-T_m,y).\end{aligned}$$

The independency of the random variables implies that the expectation of the last sum vanishes. Hence, we get (using that the T_n's are equidistributed)

$$\begin{aligned}\mathbb{E}(\mathcal{V}(t,x)\mathcal{V}'(s,y)) &= \sum_{n\in\mathbb{Z}} \mathbb{E}(\omega_n^2)\; \mathbb{E}\big(V(t-n-T_n,x)V'(s-n-T_n,y)\big) \\ &= \sum_{n\in\mathbb{Z}} \int_{-1/2}^{1/2} V(t-n-z,x)V'(s-n-z,y)\,\mathrm{d}z \\ &= \sum_{n\in\mathbb{Z}} \int_{n-1/2}^{n+1/2} V(t-z,x)V'(s-z,y)\,\mathrm{d}z \\ &= \int_{\mathbb{R}} V(t-z,x)V'(s-z,y)\,\mathrm{d}z = \int_{\mathbb{R}} V(t-s+z,x)V'(z,y)\,\mathrm{d}z.\end{aligned}$$

We denote this quantity by $R_{\mathcal{V},\mathcal{V}'}(t-s,x,y)$. Then $R_{\mathcal{V},\mathcal{V}'}$ is a compactly supported C^∞ function. Starting from functions $V_i = V_i(t,x) \in C_c^\infty(\mathbb{R}^{N+1};\mathbb{R})$, $i \in \{1,...,N\}$, we define the ith component of the velocity field as $v_i(t,x) = \sum_{n\in\mathbb{Z}} \omega_n V_i(t-n-T_n,x)$. The correlation functions are defined by $R_{i,j} = R_{\mathcal{V}_i,\mathcal{V}_j}$.

3 Proof of Theorem 1

The proof of Theorem 1 starts with the following bound and weak equicontinuity property. For a bounded measure μ on $\mathbb{R}^N$ we denote by $\|\mu\|_1$ its total mass, $\|\mu\|_1 = |\mu|(\mathbb{R}^N)$.

lemest **Lemma 2** *The quantities ρ^ϵ, $\mathbb{E}(\rho^\epsilon)$ are bounded uniformly with respect to ϵ. Indeed, let $M = \sup_{0<\epsilon\le 1} \|\rho_I^\epsilon\|_1$, then, we have for any $t \in \mathbb{R}$*

$$\|\rho^\epsilon(t)\|_1 = \|\rho_I^\epsilon\|_1 \le M, \qquad \|\mathbb{E}(\rho^\epsilon(t))\|_1 \le M.$$

Moreover, there exists some $C > 0$ depending only on the data u^1 and v, such that for any $\varphi \in C_c^\infty(\mathbb{R}^N)$, for any $t \in \mathbb{R}$, we have

$$\begin{aligned}\left| \int \rho^\epsilon(t+h,x)\varphi(x)\,\mathrm{d}x - \int \rho^\epsilon(t,x)\varphi(x)\,\mathrm{d}x \right| &\le C\; \|\varphi\|_{W^{1,\infty}}\; \frac{|h|}{\epsilon}, \\ \left| \int \mathbb{E}\rho^\epsilon(t+h,x)\varphi(x)\,\mathrm{d}z - \int \mathbb{E}\rho^\epsilon(t,x)\varphi(x)\,\mathrm{d}x \right| &\le C\; \|\varphi\|_{W^{1,\infty}}\; \frac{|h|}{\epsilon}.\end{aligned}$$

Proof. Since ρ^ϵ satisfies a transport equation in conservative form with bounded initial data, the first estimates follow. Next, the continuity property for the expectation is an immediate consequence of the continuity estimate for ρ^ϵ. To establish it, we simply remark that, in the sense of distributions,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi(x)\,\rho^\epsilon(t,x)\,\mathrm{d}x = \int \left(u^1 + \frac{1}{\epsilon}u^{0,\epsilon}\right)\cdot\nabla_x\varphi(x)\,\rho^\epsilon(t,x)\,\mathrm{d}x.$$

Since the velocity fields u^1 and v belong to $L^\infty(\mathbb{R}\times\mathbb{R}^N)$, the right hand side is readily estimated by

$$\|\rho^\epsilon\|_1\left(\|u^1\|_{L^\infty} + \frac{\|v\|_{L^\infty}}{\epsilon}\right)\|\nabla_x\varphi\|_{L^\infty},$$

which gives the two last estimates of Lemma lemest 2. ∎

The next crucial fact is the following independency property.

indep **Lemma 3** *Let $t \geq 0$, then $\rho^\epsilon(t)$ depends only on the realization of $u^{0,\epsilon}(s)$ on $[0,t]$; in particular $\rho^\epsilon(t)$ and $u^{0,\epsilon}(s)$ are independent provided $s \geq t + \epsilon^2\mathfrak{T}$.*

Proof. It is an immediate consequence of the fact that ρ_I^ϵ is deterministic. ∎

We expect that a part of the mean behavior is related to the free streaming operator

$$\partial_t \cdot + \mathrm{div}_x(u^1(t,x)\ \cdot). \tag{13}$$ freestream

Let us introduce the flow X defined from u^1 by the ODE

$$\begin{cases} \dfrac{\partial}{\partial t}X(t;s,x) = u^1(t,X(t;s,x)), \\ X(s;s,x) = x. \end{cases}$$

Since $u^1 \in L^\infty(\mathbb{R};C^1\cap W^{2,\infty}(\mathbb{R}^N))$, the characteristic X is a $W^{1,\infty}$ function of its arguments. For any bounded interval I and J we have

$$X \in L^\infty(I\times J; W^{2,\infty}(\mathbb{R}^N)), \tag{14}$$ eq:regX

and $x \longmapsto X(t;s,x)$ is a C^1 diffeomorphism on $\mathbb{R}^N$, with inverse $X(s;t,x)$. The differential operator (freestream 13) defines a family $\{S_{t,s},\ t,s\in\mathbb{R}\}$ of operators acting on measures μ as follows

$$\forall\phi\in C_0^0(\mathbb{R}^N),\quad \langle S_{t,s}[\mu],\phi\rangle = \langle\mu,\phi(X(t;s,x))\rangle.$$

The adjoint operator is defined by

$$S^\star_{t,s}[\phi](x) = \phi(X(t;s,x)).$$

Remark 1 *The family of measures $S_{t,s}[\mu]$ gives a solution $\nu(t) = S_{t,s}[\mu]$ in $\mathcal{D}'(\mathbb{R}\times\mathbb{R}^N)$ of the Cauchy problem $\partial_t \nu + \mathrm{div}_x(u^1(t,x)\nu) = 0$, $\nu_{|t=s} = \mu$. For a given s, $t \mapsto S_{t,s}[\mu]$ lies in $C^0(\mathbb{R};\mathcal{M}^1(\mathbb{R}^N - vaguely))$. Considering measure $\mu = f(x)\,\mathrm{d}x$ absolutely continuous with respect to the Lebesgue measure, we recover $S_{t,s}[f\,\mathrm{d}x] = f(X(s;t,x))\ J(s;t,x)\,\mathrm{d}x$ with $J(s;t,x) = |\det(\partial_x X(s;t,x))|$.*

Remark 2 *The operators $S_{t,s}$ satisfy the following group properties*

$$S_{s,s} = Id, \qquad S_{t,\tau}\circ S_{\tau,s} = S_{t,s}.$$

For the characteristics, this translates with the relations $X(t;\tau,X(\tau,s,x)) = X(t;s,x)$ and $X(s;s,x) = x$.

Then, starting from (6), we can express the solution ρ^ϵ as a fixed point of the Duhamel formula

$$\rho^\epsilon(t) = S_{t,t-s}[\rho^\epsilon(t-s)] + \int_{t-s}^{t} S_{t,\tau}\Theta^\epsilon(\tau)[\rho^\epsilon(\tau)]\,\mathrm{d}\tau. \tag{15}$$

eqduha

We interpret this equation and (6) as equalities between distributions in x. We denote by $\langle\ ,\ \rangle$ the distribution duality $\mathcal{D}', C_c^\infty$ on $\mathbb{R}^N$ and we remark that the (formal) adjoint operator of Θ^ϵ is given by

$$\big[\Theta^\epsilon(t)\big]^\star\varphi(x) = \frac{1}{\epsilon}u^{0,\epsilon}(t,x)\cdot\nabla_x\varphi(x).$$

It allows to obtain the following result.

lem:eqeps

Lemma 4 *For every $\varphi\in C_c^\infty(\mathbb{R}^N)$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\rho^\epsilon,\varphi\rangle \quad = \quad \langle\rho^\epsilon, u^1\cdot\nabla\varphi\rangle + H^\epsilon(\varphi) + I^\epsilon(\varphi) + J^\epsilon(\varphi)$$

with

$$\begin{aligned}
H^\epsilon(\varphi) &= \langle\Theta^\epsilon(t)[S_{t,t-\epsilon^2\mathfrak{T}}\rho^\epsilon(t-\epsilon^2\mathfrak{T})],\varphi\rangle,\\
I^\epsilon(\varphi) &= \Big\langle\int_{t-\epsilon^2\mathfrak{T}}^{t}\Theta^\epsilon(t)S_{t,\tau}\Theta^\epsilon(\tau)\mathbb{E}(\rho^\epsilon(t))\,\mathrm{d}\tau,\varphi\Big\rangle,\\
J^\epsilon(\varphi) &= \Big\langle\int_{t-\epsilon^2\mathfrak{T}}^{t}\Theta^\epsilon(t)S_{t,\tau}\Theta^\epsilon(\tau)\Big[\rho^\epsilon(\tau)-\mathbb{E}(\rho^\epsilon(t))\Big]\,\mathrm{d}\tau,\varphi\Big\rangle.
\end{aligned}$$

Proof. Starting from equation (6), we only have to compute $\Theta^\epsilon(t)\rho^\epsilon(t)$. We use formula (15) with $s = \epsilon^2\mathfrak{T}$, where $\mathfrak{T}$ is the decorrelating time in this equation. We obtain

$$\Theta^\epsilon(t)[\rho^\epsilon(t)] = \Theta^\epsilon(t)[S_{t,t-\epsilon^2\mathfrak{T}}\rho^\epsilon(t-\epsilon^2\mathfrak{T})] + \int_{t-\epsilon^2\mathfrak{T}}^{t}\Theta^\epsilon(t)S_{t,\tau}\Theta^\epsilon(\tau)[\rho^\epsilon(\tau)]\,\mathrm{d}\tau.$$

We rewrite the last integral by inserting $\mathbb{E}(\rho^\epsilon(t))$. By this way, we obtain the expressions of $H^\epsilon,\ I^\epsilon,\ J^\epsilon$. ∎

The following task is to compute the expectation of the quantities $H^\epsilon,\ I^\epsilon,\ J^\epsilon$.

lemH **Lemma 5** *For any $\varphi \in C_c^\infty(\mathbb{R}^N)$, we have $\mathbb{E}(H^\epsilon(\varphi)) = 0$.*

Proof. This comes from the fact that $u^{0,\epsilon}(t)$ and $\rho^\epsilon(s)$ are independent when $t \geq s + \epsilon^2\mathfrak{T}$, see Lemma indep 3. Indeed, we have for every $\varphi \in C_0^1(\mathbb{R}^N)$

$$\langle \Theta^\epsilon(t)[S_{t,t-\epsilon^2\mathfrak{T}}\rho^\epsilon(t-\epsilon^2\mathfrak{T})], \varphi\rangle = \langle \rho^\epsilon(t-\epsilon^2\mathfrak{T}), \frac{1}{\epsilon}(u^{0,\epsilon}(t)\cdot\nabla_x\varphi)(X(t;t-\epsilon^2\mathfrak{T},x))\rangle.$$

The independency result applies with $s = t - \epsilon^2\mathfrak{T}$. Then the expectation vanishes, by using $\mathbb{E}(u^{0,\epsilon}) = 0$. ∎

In the limit $\epsilon \to 0$, the expectation of J^ϵ vanishes.

lemJ **Lemma 6** *For any $\varphi \in C_c^\infty(\mathbb{R}^N)$, we have $\lim\limits_{\epsilon\to 0}\mathbb{E}(J^\epsilon(\varphi)) = 0$.*

Proof. The term J^ϵ reads

$$J^\epsilon(\varphi) = \int_{t-\epsilon^2\mathfrak{T}}^t \big\langle (\rho^\epsilon(\tau) - \mathbb{E}(\rho^\epsilon(t))),\ \Theta^{\epsilon,\star}(\tau)S^\star_{t,\tau}\Theta^{\epsilon,\star}(t)\varphi\big\rangle\,\mathrm{d}\tau.$$

We look at the expectation of this random quantity. The trick consists of inserting $\rho^\epsilon(\tau-\epsilon^2\mathfrak{T}) - \mathbb{E}(\rho^\epsilon(\tau-\epsilon^2\mathfrak{T}))$ because it does not change the expectation of J^ϵ. Indeed, we remark that $\rho^\epsilon(\tau-\epsilon^2\mathfrak{T})$ and $\Theta^{\epsilon,\star}(\tau)S^\star_{t,\tau}\Theta^{\epsilon,\star}(t)\varphi$ are independent since the last term depends only on $u^{0,\epsilon}(t)$ and $u^{0,\epsilon}(\tau)$ with $t,\tau \geq (\tau-\epsilon^2\mathfrak{T}) + \epsilon^2\mathfrak{T}$. It follows that

$$\begin{aligned}
\mathbb{E}\Big(\Big\langle \rho^\epsilon(\tau-\epsilon^2\mathfrak{T}),\ \Theta^{\epsilon,\star}(\tau)S^\star_{t,\tau}\Theta^{\epsilon,\star}(t)\varphi\Big\rangle\Big) &= \Big\langle \mathbb{E}\big(\rho^\epsilon(\tau-\epsilon^2\mathfrak{T})\big),\ \mathbb{E}\big(\Theta^{\epsilon,\star}(\tau)S^\star_{t,\tau}\Theta^{\epsilon,\star}(t)\varphi\big)\Big\rangle \\
&= \mathbb{E}\Big(\Big\langle \mathbb{E}\big(\rho^\epsilon(\tau-\epsilon^2\mathfrak{T})\big),\ \Theta^{\epsilon,\star}(\tau)S^\star_{t,\tau}\Theta^{\epsilon,\star}(t)\varphi\Big\rangle\Big).
\end{aligned}$$

Let us set

$$r^\epsilon(t,\tau) = \rho^\epsilon(\tau) - \rho^\epsilon(\tau-\epsilon^2\mathfrak{T}) + \mathbb{E}(\rho^\epsilon(\tau-\epsilon^2\mathfrak{T})) - \mathbb{E}(\rho^\epsilon(t)) = r_1^\epsilon(t,\tau) + r_2^\epsilon(t,\tau).$$

The previous remark yields

$$\mathbb{E}\big(J^\epsilon(\varphi)\big) = \mathbb{E}\left\{\int_{t-\epsilon^2\mathfrak{T}}^t \langle r^\epsilon(t,\tau),\ \Theta^{\epsilon,\star}(\tau)S^\star_{t,\tau}\Theta^{\epsilon,\star}(t)\varphi\rangle\,\mathrm{d}\tau\right\}.$$

Let $T > 0$ be an arbitrary positive time. We remark that for a constant $C > 0$ depending only on T, $\sup_{t\in\mathbb{R}} \|v(t)\|_{W^{p,\infty}}(\mathbb{R}^N)$ and on $\sup_{0\le t\le T} \|u^1(t)\|_{W^{p,\infty}}(\mathbb{R}^N)$ we have

$$\left\{ \begin{array}{l} \|\Theta^{\epsilon,\star}(t)[\varphi]\|_{W^{p,\infty}(\mathbb{R}^N)} \le \dfrac{C}{\epsilon} \; \|\varphi\|_{W^{p+1,\infty}(\mathbb{R}^N)}, \\ \|S^\star_{t,s}[\varphi]\|_{W^{p,\infty}(\mathbb{R}^N)} \le C \; \|\varphi\|_{W^{p,\infty}(\mathbb{R}^N)}, \end{array} \right. \tag{16}$$

sobo

for any $p \in \{0, 1, 2\}$. The estimate for $S^\star$ requires some regularity on the characteristics, which is given in (14). Then we have

$$\|\Theta^{\epsilon,\star}(\tau)S^\star_{t,\tau}\Theta^{\epsilon,\star}(t)\varphi\|_{W^{1,\infty}} \le \frac{C}{\epsilon^2} \; \|\varphi\|_{W^{3,\infty}(\mathbb{R}^N)}$$

Combining this estimate to Lemma 2, we deduce that

$$\begin{array}{rcl} \left|\langle r_i^\epsilon(t,\tau), \Theta^{\epsilon,\star}(\tau)S^\star_{t,\tau}\Theta^{\epsilon,\star}(t)\varphi\rangle\right| & \le & \dfrac{C}{\epsilon^2} \; \|\varphi\|_{W^{3,\infty}(\mathbb{R}^N)} \dfrac{\epsilon^2\mathfrak{T}}{\epsilon} \\ & \le & \dfrac{C}{\epsilon} \; \mathfrak{T} \; \|\varphi\|_{W^{3,\infty}(\mathbb{R}^N)}. \end{array}$$

Therefore, since we integrate over a $\tau-$interval with length $\epsilon^2\mathfrak{T}$, we finally obtain the desired result. ∎

The term I^ϵ in Lemma 4 gives the leading term; it reads

$$\mathbb{E}\big(I^\epsilon(\varphi)\big) = \mathbb{E}\left\{ \int_{t-\epsilon^2\mathfrak{T}}^t \big\langle \Theta^\epsilon(t)S_{t,\tau}\Theta^\epsilon(\tau) \; \mathbb{E}(\rho^\epsilon(t)), \; \varphi \big\rangle \, \mathrm{d}\tau \right\} = \langle \mathbb{E}(\rho^\epsilon(t)), \Phi^\epsilon(t)\rangle$$

where we have set

$$\Phi^\epsilon(t) = \int_{t-\epsilon^2\mathfrak{T}}^t \mathbb{E}\big\{\Theta^{\epsilon,\star}(\tau)S^\star_{t,\tau}\Theta^{\epsilon,\star}(t) \; \varphi\big\} \, \mathrm{d}\tau.$$

Thus we have to discuss the behavior of the quantity

$$\begin{array}{l} \Theta^{\epsilon,\star}(\tau)S^\star_{\tau,t}\Theta^{\epsilon,\star}(t) \; \varphi(x) = \dfrac{1}{\epsilon^2} \; v(\dfrac{\tau}{\epsilon^2}, x) \cdot \nabla_x \Big\{ v(\dfrac{t}{\epsilon^2}, X(t;\tau,x)) \cdot \nabla_x\varphi(X(t;\tau,x)) \Big\} \\ \quad = \dfrac{1}{\epsilon^2} \, \nabla_x \cdot \Big\{ v(\dfrac{\tau}{\epsilon^2}, x) \otimes v(\dfrac{t}{\epsilon^2}, X(t;\tau,x)) \; \nabla_x\varphi(X(t;\tau,x)) \Big\} \\ \qquad - \dfrac{1}{\epsilon^2} \; (\nabla_x \cdot v)(\dfrac{\tau}{\epsilon^2}, x) \; v(\dfrac{t}{\epsilon^2}, X(t;\tau,x)) \; \cdot \nabla_x\varphi(X(t;\tau,x)). \end{array}$$

Then, we take the expectation and we integrate over $\tau \in (t - \epsilon^2\mathfrak{T}, t)$ with the change of variable $\sigma = (t-\tau)/\epsilon^2$. We obtain

$$\begin{aligned}
\Phi^\epsilon(t,x) &= \nabla_x \cdot \left(\int_0^{\mathfrak{T}} \mathbb{E}\Big(v(\frac{t}{\epsilon^2} - \sigma, x) \otimes v(\frac{t}{\epsilon^2}, X(t; t-\epsilon^2\sigma, x)) \Big) \, \nabla_x \varphi(X(t; t-\epsilon^2\sigma, x)) \, \mathrm{d}\sigma \right) \\
&\quad - \int_0^{\mathfrak{T}} \mathbb{E}\Big((\nabla_x \cdot v)(\frac{t}{\epsilon^2} - \sigma, x) \; v(\frac{t}{\epsilon^2}, X(t; t-\epsilon^2\sigma, x)) \Big) \, \cdot \nabla_x \varphi(X(t; t-\epsilon^2\sigma, x)) \, \mathrm{d}\sigma \\
&= \sum_{i,j=1}^N \partial_{x_j} \left(\int_0^{\mathfrak{T}} R_{ji}(\sigma, x, X(t, t-\epsilon^2\sigma, x)) \; \partial_{x_i} \varphi(X(t; \epsilon^2\sigma, x)) \, \mathrm{d}\sigma \right) \\
&\quad - \sum_{i,j=1}^N \int_0^{\mathfrak{T}} (\partial_{x_j} R_{ji})(\sigma, x, X(t, t-\epsilon^2\sigma, x)) \; \partial_{x_i} \varphi(X(t; t-\epsilon^2\sigma, x)) \, \mathrm{d}\sigma.
\end{aligned}$$

As ϵ goes to 0, the convergences $X(t, t-\epsilon^2\sigma, x) \to X(t; t, x) = x$ holds uniformly; then, Φ^ϵ tends uniformly to

$$\begin{aligned}
&\sum_{i,j=1}^N \partial_{x_j} \left(\int_0^{\mathfrak{T}} R_{ji}(\sigma, x, x) \, \mathrm{d}\sigma \; \partial_{x_i} \varphi(x) \right) - \sum_{i,j=1}^N \int_0^{\mathfrak{T}} (\partial_{x_j} R_{ji})(\sigma, x, x) \; d\sigma \, \partial_{x_i} \varphi(x) \\
&\qquad = \nabla_x \cdot (D(x)^T \cdot \nabla_x \varphi) - c(x) \cdot \nabla_x \varphi(x),
\end{aligned}$$

where the matrix D and the velocity field c are defined by

$$D_{i,j}(x) = \int_0^{\mathfrak{T}} R_{j,i}(\sigma, x, x) \, \mathrm{d}\sigma, \qquad c_i(x) = \sum_{j=1}^N \int_0^{\mathfrak{T}} \frac{\partial}{\partial x_j} R_{j,i}(\sigma, x, x) \, \mathrm{d}\sigma$$

We have thus proved the following claim.

lemI **Lemma 7** *For any $\varphi \in C_c^\infty(\mathbb{R}^N)$, we have $\mathbb{E}\big(I^\epsilon(\varphi)\big) = \langle \mathbb{E}(\rho^\epsilon), \Phi^\epsilon(t) \rangle$ where the functions $\big(\Phi^\epsilon\big)_{\epsilon>0}$ converge uniformly to $\nabla_x \cdot (D^T \cdot \nabla_x \varphi) - c \cdot \nabla_x \varphi$.*

We are now ready to end the proof of Theorem 1. Using Lemma 4 and 5, the expectation of ρ^ϵ satisfies

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \langle \mathbb{E}(\rho^\epsilon), \varphi \rangle &= \langle \mathbb{E}(\rho^\epsilon), u^1 \cdot \nabla \varphi \rangle + \mathbb{E}(H^\epsilon(\varphi)) + \mathbb{E}(I^\epsilon(\varphi)) + \mathbb{E}(J^\epsilon(\varphi)) \\
&= \langle \mathbb{E}(\rho^\epsilon), u^1 \cdot \nabla \varphi \rangle + \mathbb{E}(I^\epsilon(\varphi)) + \mathbb{E}(J^\epsilon(\varphi)).
\end{aligned}$$

Therefore, by combining Lemma 6 and 7, we observe that the family $\Big(\langle \mathbb{E}(\rho^\epsilon(t)), \varphi \rangle \Big)_{\epsilon>0}$ is a family of Lipshitz functions, with a uniformly bounded Lipschitz constant. By the

Arzela-Ascoli theorem, we therefore obtain that for a subsequence, still denoted by ϵ, $\langle \mathbb{E}(\rho^\epsilon(t)), \varphi \rangle$ converges uniformly on any time interval $[0, T]$. By using the Cantor diagonal process, we can assume this convergence holds for any $\langle \mathbb{E}(\rho^\epsilon(t)), \varphi_n \rangle$ where $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of $C_c^\infty(\mathbb{R}^N)$ which is dense in $C_0^0(\mathbb{R}^N)$. By an $\epsilon/3$ argument we obtain that for any $\varphi \in C_0^0(\mathbb{R}^N)$, $\langle \mathbb{E}(\rho^\epsilon(t)), \varphi \rangle$ converges uniformly on any time interval $[0, T]$.

Let $\Phi = \nabla_x \cdot (D^T \cdot \nabla_x \varphi) - c \cdot \nabla_x \varphi$ we have

$$\Big| I^\epsilon(\varphi) - \langle \mathbb{E}(\rho^\epsilon(t)), \Phi \rangle \Big| = \Big| \langle \mathbb{E}(\rho^\epsilon(t)), \Phi^\epsilon - \Phi \rangle \Big| \leq M \|\Phi^\epsilon - \Phi\|_\infty.$$

With the convergence result on $\mathbb{E}(\rho^\epsilon(t))$ and Lemma 7, 6, letting $\epsilon \to 0$, we obtain the limit equation

$$\partial_t \rho + \mathrm{div}_x \Big((u^1 + c)\rho - D(x) \nabla_x \rho \Big) = 0,$$

in the sense of (11). ∎

4 Extensions

4.1 Convection-Diffusion equation

First, let us explain how the result extends easily to the case of a convection-diffusion equation.

Theorem 2 *Let the assumptions of Theorem 1 be fulfilled. Let ρ^ϵ be the solution of the corresponding Cauchy problem*

$$\partial_t \rho^\epsilon(t, x) + \mathrm{div}_x \Big(u^\epsilon \ \rho^\epsilon \Big)(t, x) = \eta \Delta_x \rho^\epsilon, \quad (t, x) \in \mathbb{R}^{N+1}, \tag{17}$$

$$\rho^\epsilon(0, x) = \rho_I^\epsilon(x), \quad x \in \mathbb{R}^N, \tag{18}$$

with $\eta > 0$. Then, up to the extraction of a subsequence, for any time $T > 0$, $\mathbb{E}(\rho^\epsilon)$ converges vaguely to $\rho(t) \geq 0$ uniformly on $[0, T]$ and the limit $\rho(t)$ is a solution of

$$\partial_t \rho + \mathrm{div}_x \Big((u^1 + c) \ \rho \Big) - \mathrm{div}_x ((\eta + D(x)) \nabla \rho) = 0,$$

with effective coefficients D and c defined as in Theorem 1.

Remark 3 *It is worth mentioning that the original diffusivity is enhanced by the homogenization limit: we always have $(\eta + D(x))\xi \cdot \xi \geq \eta \xi^2$. This is again a difference with the periodic case where diffusion coefficients can be depleted when passing to the limit, when the velocity field is not divergence free, see [15].*

Proof. We only mention the modifications in the proof detailed above. First, equation (6) reads

$$\partial_t \rho^\epsilon + \mathrm{div}_x(u^1 \rho^\epsilon) = \Theta^\epsilon[\rho^\epsilon] + \eta \Delta_x \rho^\epsilon,$$

which leads to

$$\partial_t \mathbb{E}\rho^\epsilon + \mathrm{div}_x(u^1 \mathbb{E}\rho^\epsilon) = \mathbb{E}\Theta^\epsilon[\rho^\epsilon] + \eta \Delta_x \mathbb{E}\rho^\epsilon.$$

Next, in the proof of Lemma 2, now we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi(x)\, \rho^\epsilon(t,x)\,\mathrm{d}x = \int \big(u^1 + \frac{1}{\epsilon} u^{0,\epsilon}\big) \cdot \nabla_x \varphi(x)\, \rho^\epsilon(t,x)\,\mathrm{d}x + \eta \int \Delta_x \varphi(x) \rho^\epsilon(t,x)\,\mathrm{d}x$$

so that the estimates in Lemma 2 holds with the $W^{1,\infty}$ norm of the test function replaced by $\|\varphi\|_{W^{2,\infty}}$.

Then, in the Duhamel formula (15), we add to the right hand side

$$\eta \int_{t-s}^t S_{t,\tau} \Delta_x \rho^\epsilon(\tau,x)\,\mathrm{d}\tau.$$

Accordingly, in Lemma 4, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \rho^\epsilon, \varphi\rangle = \langle \rho^\epsilon, u^1 \cdot \nabla\varphi\rangle + \eta\langle \rho^\epsilon, \Delta\varphi\rangle + H^\epsilon(\varphi) + I^\epsilon(\varphi) + J^\epsilon(\varphi) + K^\epsilon(\varphi),$$

with definition of H^ϵ, I^ϵ, J^ϵ left unchanged and

$$\begin{aligned} K^\epsilon(\varphi) &= \eta \int_{t-\epsilon^2\mathfrak{T}}^t \Big\langle \rho^\epsilon(\tau), \Delta_x\big(S^\star_{t,\tau}\Theta^{\epsilon,\star}(t)\varphi\big)\Big\rangle\,\mathrm{d}\tau \\ &= \eta \int_{t-\epsilon^2\mathfrak{T}}^t \Big\langle \mathbb{E}\rho^\epsilon(t), \Delta_x\big(S^\star_{t,\tau}\Theta^{\epsilon,\star}(t)\varphi\big)\Big\rangle\,\mathrm{d}\tau \\ &\quad + \eta \int_{t-\epsilon^2\mathfrak{T}}^t \Big\langle \big(\rho^\epsilon(\tau) - \mathbb{E}\rho^\epsilon(t)\big), \Delta_x\big(S^\star_{t,\tau}\Theta^{\epsilon,\star}(t)\varphi\big)\Big\rangle\,\mathrm{d}\tau. \end{aligned}$$

Taking the expectation, the first term vanishes since it reads

$$\frac{\eta}{\epsilon}\Big\langle \mathbb{E}\rho^\epsilon(t), \int_{t-\epsilon^2\mathfrak{T}}^t \Delta_x\big(\mathbb{E}v(t/\epsilon^2, X(t;\tau,x)) \cdot \nabla_x\varphi(X(t;\tau,x))\big)\,\mathrm{d}\tau\Big\rangle = 0$$

by using the condition $\mathbb{E}v(t,x) = 0$. Reasoning as in Lemma 6, the expectation of the second term rewrites

$$\eta \int_{t-\epsilon^2\mathfrak{T}}^t \mathbb{E}\Big\langle r^\epsilon(t,\tau), \Delta_x\big(S^\star_{t,\tau}\Theta^{\epsilon,\star}(t)\varphi\big)\Big\rangle\,\mathrm{d}\tau,$$

which is estimated using Lemma 2 and (16). We show it is dominated by

$$\frac{C\epsilon^2\mathfrak{T}}{\epsilon}\ \|S^\star_{t,\tau}\Theta^{\epsilon,\star}(t)\varphi\|_{W^{2,\infty}}\ \epsilon^2\mathfrak{T} \leq \frac{C\epsilon^2\mathfrak{T}}{\epsilon}\ \frac{1}{\epsilon}\ \|\varphi\|_{W^{3,\infty}}\ \epsilon^2\mathfrak{T} = C\ \epsilon^2\mathfrak{T}^2\|\varphi\|_{W^{3,\infty}}.$$

This ends the proof. ∎

4.2 Weak fluctuations

Our proof extends readily when the random velocity field $u^{0,\epsilon}$ oscillates also with respect to the space variable x with characteristic length ϵ^γ:

$$u^{0,\epsilon}(t,x) = v(t/\epsilon^2, x/\epsilon^\gamma),$$

and we assume that there is correlation function of the form $\mathbb{E}(v(t,x)\otimes v(s,y)) = R(s-t, y-x)$. Indeed, in such a case, the key estimate (16) is modified as follows

$$\|\Theta^{\epsilon,\star}(t)[\varphi]\|_{W^{p,\infty}(\mathbb{R}^N)} \leq \frac{C}{\epsilon}\Big(\frac{1}{\epsilon^{\gamma p}}\|\varphi\|_{W^{1,\infty}(\mathbb{R}^N)} + \frac{1}{\epsilon^{\gamma(p-1)}}\|\varphi\|_{W^{p+1,\infty}(\mathbb{R}^N)}\Big).$$

Therefore, the estimate on J^ϵ in Lemma 6 becomes

$$\frac{C}{\epsilon^2}\ \frac{1}{\epsilon^{2\gamma}}\ \|\varphi\|_{W^{3,\infty}}\ \frac{\epsilon^2\mathfrak{T}}{\epsilon}\ \epsilon^2\mathfrak{T} = C\ \mathfrak{T}^2\|\varphi\|_{W^{3,\infty}}\ \epsilon^{1-2\gamma}.$$

Hence, assuming $0<\gamma<1/2$, the space oscillations do not play any role and do not change the final result. With more technicalities, using in particular characteristics of the total velocity field u^ϵ it should be possible to treat the case $1/2 \leq \alpha < 1$. The problem changes completely of nature in the critical case $\alpha = 1$. Then there is an interaction between the time and the space fluctuations. It is a challenging open problem to determine the asymptotic behavior of $\mathbb{E}(\rho^\epsilon)$ in this case. Attempts for the special case of shear flows can be found in [3], and dealing with Gaussian incompressible flows in [22]. ∎

Acknowledgments. The authors are particularly grateful to Alain Pumir for fruitful discussions and nice comments about this problem. They also thank Grégoire Allaire and Sylvie Méléard for their help during the preparation of the manuscript.

References

AHZ1 [1] Amirat Y., Hamdache K., Ziani A., *Homogénéisation d'équations hyperboliques du premier ordre et application aux écoulements miscibles en milieux poreux*, Ann. IHP, An. Non linéaire, 6 (1989) 397–417.

AHZ2 [2] Amirat Y., Hamdache K., Ziani A., *Some results on homogenization of convection-diffusion equations*, Arch. Rat. Mech. Anal., 114 (1989) 155–178.

AM1 [3] Avellaneda M., Majda A., *Mathematical models with exact renormalization for turbulent transport*, Comm. Math. Phys., 131 (1990) 381–429.

AM2 [4] Avellaneda M., Majda A., *An integral representation and bounds on the effective diffusivity in passive advection by laminar and turbulent flows*, Comm. Math. Phys., 138 (1991) 339–391.

bat [5] Batchelor G. K., *The theory of homogeneous turbulence* (Cambridge University Press, 1982).

besse [6] Besse C., Claudel J., Degond P., Deluzet F., Gallice G., Tessieras T., *Ionospheric plasmas: model derivation, stability analysis and numerical simulations*, to appear in Numerical methods for hyperbolic and kinetic equations–CEMRACS 2003, Cordier S., Goudon T., Sonnendrucker E. Eds.

Bki [7] Bourbaki N., **Eléments de Mathématiques**, Fascicule XXXV, Livre VI, Chapitre IX: Intégration, Herrmann, Paris 1969.

Bra [8] Brassart M., *Limite semi-classique de transformées de Wigner dans des milieux périodiques ou aléatoires*, Thèse Université de Nice-Sophia Antipolis, Novembre 2002.

Br [9] Brenier Y., *Remarks on some linear hyperbolic equations with oscillatory coefficients*, in Third International Conference on Hyperbolic Problems: Theory, numerical methods and applications, Proceedings of the conference dedicated to Professor Heinz-Otto Kreiss on his 60th birthday held in Uppsala, June 11–15, 1990, Engquist B. and Gustafsson B. Eds., Vol. I, II (Studentlitteratur, 1991) pp. 119–130.

cs [10] Csanady G. T., *Turbulent diffusion in the environment*, Geophysics and Astrophysics Monographs (Reidel, 1973).

DiPM [11] Di Perna R., Majda A., *Oscillations and concentrations in weak solutions of the incompressible fluid equations*, Comm. Math. Phys., 108 (1987) 667–689.

E [12] E W., *Homogenization of linear and nonlinear transport equations*, Comm. Pure Appl. Math., 65 (1990) 301–326.

FaPa [13] Fannjiang A., Papanicolaou G., *Diffusion in turbulence*, Probab. Theory Related Fields, 105 (1996) 279–334.

FPR [14] Fannjiang A., Papanicolaou G., Ryzhik L., *Evolution of trajectory correlations in steady random flows*, in Recent advances in partial differential equations, Venice 1996, Proc. Sympos. Appl. Math., 54 (AMS, Providence, RI, 1998) pp. 105–130.

GP [15] Goudon T., Poupaud F., *Homogenization of transport equations; weak mean field approximation*, to appear in SIAM J. Math. Anal.

GP2 [16] Goudon T., Poupaud F., *On the modeling of the transport of particles in turbulent flows*, preprint.

H1 [17] Hamdache K., *Homogénéisation non locale d'équations hyperboliques*, in *Non linear pde's and their applications, Collège de France seminar*, vol. XII., Pitman Res. Notes in Math., 302 (Longman Sci. Tech., Harlow, 1994) pp. 97–112.

H2 [18] Hamdache K., *Equations de transport, homogénéisation*, Notes de cours, Université Bordeaux 1, 1994.

HX [19] Hou T., Xin X., *Homogenization of linear transport equations with oscillatory vector fields*, SIAM J. Appl. Math., 52 (1992) 34–45.

KP [20] Kesten H., Papanicolaou G., *A limit theorem for turbulent diffusion*, Comm. Math. Phys., 65 (1979) 97–128.

KO [21] Komorowski T., Olla S., *On homogenization of time dependent random flows*, Probab. Theory Relat. Fields, 121 (2001) 98–116.

KoPa [22] Komorowski T., Papanicolaou G., *Motion in a Gaussian incompressible flow*, Ann. Appl. Probab. 7 (1997) 229–264.

KM [23] Kramer P., Majda A., *Simplified models for turbulent diffusion: Theory, numerical modeling, and physical phenomena*, Physics Reports, 314 (1999) 237–574.

Kh [24] Khas'minskii R. Z., *A limit thorem for the solutions of differential equations with random right-hand sides*, Theory of Prob. Appl., 11 (1966) 390–406.

Ku [25] Kubo R., *Stochastic Liouville equations*, J. Math. Phys., 4 (1963) 174–183.

LOY [26] LANDIM-OLLA-YAU *Convection-diffusion equation with space-time ergodic random flow* Probab. Theory Relat. Fields, 112 (1998) 203–220.

lv [27] LOEPER G., VASSEUR A., *Electric turbulence in a plasma subject to a strong magnetic field*, preprint.

MPP [28] MC LAUGHLIN D., PAPANICOLAOU G., PIRONNEAU O., *Convection of microstructure and related problems*, SIAM J. Appl. Math., 45 (1985) 780–797.

M [29] MASCARENHAS L., *A linear homogenization problem with time dependent coefficient*, Trans. AMS, 281 (1984) 179–195.

PV [30] POUPAUD F., VASSEUR A., *Classical and quantum transport in random media*, to appear in J. Math. Pures et Appl.

Str [31] STRATONOVICH R. L., *Conditional Markov processes and their application to optimal control theory*, Doctoral dissertation, Moscow University, 1965.

T1 [32] TARTAR L., *Remarks on homogenization* in *Homogenization and effective moduli of material and media*, IMA Vol. in Math. and Appl., Springer, 1986, pp. 228–246.

T2 [33] TARTAR L., *Nonlocal effects induced by homogenization*, in *Partial differential equations and the calculus of variations, Essays of Mathematical analysis in honor of E. De Giorgi*, Vol. II, Progr. Nonlinear Differential Equations Appl., 2 (Birkhauser, 1989) pp. 925–938.

Tay [34] TAYLOR G. I., *Diffusion by continuous movements*, Proc. London Math. Soc., Ser. 2, 20 (1923) 196–211.