SIAM J. APPL. MATH.
Vol. , No. , pp.

A FLUID–KINETIC MODEL FOR PARTICULATE FLOWS WITH COAGULATION AND BREAKUP: STATIONARY SOLUTIONS, STABILITY AND HYDRODYNAMIC REGIMES*

THIERRY GOUDON†, MAMADOU SY‡, AND LÉON M. TINÉ §

Abstract. We consider coupled models for particulate flows, where the disperse phase is made of particles subject to size variations. We are thus led to kinetic equations with coagulation and breakup operators, coupled to fluid mechanics equations. We discuss the existence and stability of stationary solutions. We also derive macroscopic models through asymptotic hydrodynamic regimes, once relevant scaling parameters have been identified.

Key words. coagulation and breakup, particulate flows, coupled fluid-kinetic models, hydrodynamic regimes

AMS subject classifications. 35Q35 82C21 45K05

DOI.

1. Introduction. We are interested in models describing a large set of particles interacting with a fluid. The study of such two–phase flows (where particles represent the disperse phase evolving in a dense fluid) is motivated by applications like the dispersion of dusts, smokes or pollutants [25, 21, 48, 49], the modeling of biomedical sprays [4, 40], the optimization of combustion processes [23], the formation of powder-snow avalanches [11], etc. On the mathematical viewpoint the modeling leads to non standard systems of PDEs. The disperse phase is described through a distribution function in phase space, $f(t, x, v, r)$ where $t \geq 0$ stands for the time variable, $x \in \mathbb{R}^3$ the space variable, $v \in \mathbb{R}^3$ the velocity of the particles and $r \geq 0$ is related to the particles size. The dense phase is described, as usual in fluid mechanics, by macroscopic quantities (mass density, velocity and temperature, say) depending only on the time and space variables. Hence the unknowns do not depend on the same set of variables, which makes part of the difficulty for mathematical analysis, together with the fact that we consider systems coupled through nonlinear terms. Furthermore, in view of numerical experiments, the kinetic framework leads to high computational cost, both in terms of size and time. This remark motivates to seek reduced models, which are of purely hydrodynamic type, by means of asymptotic arguments that take advantage of some relaxation processes embodied into the model. Anticipating on the detailed presentation of the model, we are concerned with PDEs where the evolution of the particle distribution function is driven by the combination of the following phenomena:

- the drag force exerted by the surrounding fluid on the particles,
- the influence of an external potential $x \mapsto \Phi(x)$ (gravity, electrostatic force, centrifugal force, etc),
- the Brownian motion of the particles,

*Received by the editors ; accepted for publication (in revised form) ; published electronically .
http://www.siam.org/journals/siap/.html

†Team COFFEE, INRIA Sophia Antipolis Méditerrannée & Labo. J. A. Dieudonné, UMR 7351 CNRS–Université Nice Sophia Antipolis, Parc Valrose, F-06108 Nice, France

‡Laboratoire d'Analyse Numérique et d'Informatique (LANI) Université Gaston Berger, B.P. 234 Saint-Louis, Sénégal

§Laboratoire d'Analyse Numérique et d'Informatique (LANI) Université Gaston Berger, B.P. 234 Saint-Louis, Sénégal & Laboratoire MAP5, UMR 8145 CNRS–Université Paris Descartes, 45 rue des Saints Pères, F-75270 Paris cedex 06

- coagulation and breakup which modify the size of the particles.

It leads to a Vlasov–Fokker–Planck equation, which furthermore involves a non linear "collision" operator describing the size variations. The fluid quantities obey Euler or Navier-Stokes equations, depending on the physical context. We shall deal with "Two-Way Coupled" systems where the drag force exerted by the particles on the dense phase is accounted for in the momentum balance. We refer to [50] for introduction to such coupled models in combustion theory and to [44] for further details on the modeling of such multiphase flows; recent developments can be found in [3, 37]. Investigating existence, uniqueness and regularity issues depends on the nature of the coupling and the complexity of the equations used for describing the fluid. We refer for instance to [5] (strong solutions locally in time), [33, 8, 38] (weak solution for viscous flows), [31] (solutions close to equilibrium). Asymptotic issues are introduced and analyzed in [26, 27, 39, 16]. The rationale consists in adapting the reasoning used to derive gas dynamic equations from kinetic models, like e. g. the Boltzmann equation [46]. Identifying physically relevant scaling parameters, we can bring out relaxation effects in the coupled system that lead asymptotically to purely macroscopic models for describing the mixture flow. Numerical aspects on such fluid-kinetic models are devised in [1, 17, 43, 42, 36, 28, 30, 29]. Most of these references do not address the question of the influence of the size variations. This is the aim of the present work: we discuss several aspects of the role of a Smoluchowski operator in the stability and dissipation properties of the system, and we study hydrodynamic regimes.

In the next Section, we describe precisely the PDEs system we are interested in. In Section 3, we write the equations in dimensionless form, which permits to identify the scaling parameters by means of the physical characteristics of the flows (respective densities of the two phases, particles' radius, etc). In Section 4, we discuss how detailed balance principles of the coagulation/breakup dynamics and transport phenomena combine to define stationary solutions of the problem. The fact that the equilibrium functions depend on the phase space variable introduces new conditions for the equilibria to be physically admissible (that is with finite mass and energy). Then, we adapt the discussion in [16] to the poly-disperse framework: we exhibit a relevant functional of the unknowns which encompasses the relaxation effects of the system. It allows to establish the stability of the stationary states. Finally, in Section 5 we discuss hydrodynamic regimes. It leads to new macroscopic systems for describing the complex flow. These models still consider particles of different sizes, and the details of the coupling with the dense phase relies on the underlying microscopic description. Our discussion remains mostly at the formal level: complete and rigorous proofs of existence-uniqueness and asymptotic convergence are beyond the scope of the present work. Nevertheless the dissipation estimates we establish can be seen as the necessary preliminary step and a convincing hint towards such statements.

2. A fluid-particle model with coagulation and breakup. Here and below, we adopt a discrete modeling of the size variable. Let $i \in \mathbb{N} \setminus \{0\}$. We refer to "a particle of size i" as to be a assembly of i monomers. Therefore, denoting by $a > 0$ the radius of a monomer and ρ_P its mass density, the volume of a $i-$mer is $\frac{4}{3}\pi a^3 i$, the radius is $r_i = ai^{1/3}$, the mass is $m_i = \frac{4}{3}\pi a^3 i \rho_P$. Let $f_i(t, x, v)$ stand for the density of i-mers in phase space: $f_i(t, x, v)\ \mathrm{d}v\ \mathrm{d}x$ represents the number of particles with size i having at time $t \geq 0$ their position and velocity in the infinitesimal domain centered at (x, v) with volume $\mathrm{d}v\ \mathrm{d}x$. Particles are subject to a drag force, which is proportional to the relative velocity with the fluid. The Stokes law defines the proportionality factor as $6\pi\mu r_i = 6\pi\mu a i^{1/3}$, with μ the dynamic viscosity of the fluid. For the sake of simplicity, we assume throughout the paper that μ is a positive given constant. Brownian motion produces velocity fluctuation, described by a diffusion operator with coefficient (Einstein formula)

$$\frac{k\theta 6\pi\mu a i^{1/3}}{(\frac{4}{3}\pi a^3 i\rho_P)^2} = \frac{k\theta}{m_i}\frac{9\mu}{2\rho_P r_i^2},$$

where k is the Boltzmann constant and $\theta > 0$ the temperature of the fluid. Therefore, f_i satisfies the following equation

$$\partial_t f_i + v\cdot\nabla_x f_i - \nabla_x\Phi\cdot\nabla_v f_i = \frac{9\mu}{2\rho_P r_i^2}\nabla_v\cdot\Big((v-u)f_i + \frac{k\theta}{m_i}\nabla_v f_i\Big) + \frac{1}{\tau_c}Q_i(f). \tag{2.1}$$

Note that

$$\tau_i = \frac{m_i}{6\pi\mu r_i} = \frac{2\rho_P r_i^2}{9\mu} = i^{2/3}\ \tau_1, \qquad \text{with } \tau_1 = \frac{2\rho_P a^2}{9\mu}$$

has the dimension of time: this is the Stokes settling time, typical of the effect of the drag force on the $i-$particle. In the right hand side of (2.1) the so–called (discrete) Smoluchowski operator Q describes binary coagulation and breakup and τ_c is the characteristic time scale of the coagulation and breakup phenomena. The operator is defined by

$$Q_i(f) = \frac{1}{2}\sum_{j=1}^{i-1}\kappa_{j,i-j}f_j f_{i-j} - \sum_{j=1}^{\infty}\kappa_{i,j}f_i f_j + \sum_{j=i+1}^{\infty}\sigma_{j-i,i}f_j - \frac{1}{2}\sum_{j=1}^{i-1}\sigma_{j,i-j}f_i$$

according to the formation of $(i+j)-$mers from i-mers and $j-$mers ($X_i + X_j \to X_{i+j}$) which occurs with a rate $\kappa_{i,j}$ and the breakup of $i-$mers into smaller pieces with, assuming $j < i$, size j and $i-j$ ($X_i \to X_{i-j} + X_j$). As a specific case, the Becker-Döring operator restricts to the case where, for $i \geq 2$, the coefficients vanish but for $j = 1$: only monomers can be added to or removed from $i-$mers. Monomers are involved in all reactions so that the collision term Q_1 has a different expression. More precisely, the Becker-Döring operator [6] casts as follows

$$\begin{aligned} &J_i(f) = \kappa_i f_i f_1 - \sigma_{i+1}f_{i+1}, \\ &Q_i(f) = J_{i-1} - J_i \text{ for } i \geq 2 \text{ and } Q_1(f) = -2J_1 - \sum_{i=2}^{\infty} J_i. \end{aligned}$$

In this expression, κ_i is the rate of the coagulation reaction $X_i + X_1 \to X_{i+1}$ and σ_{i+1} is the rate of the breakup $X_{i+1} \to X_i + X_1$. For a thorough presentation of the Smoluchowski and Becker-Döring operators, we refer to [20, 41]. The effect of coagulation and breakup is to reduce the total number of particles but to maintain the total mass: at least formally, we have

$$\sum_{i=1}^{\infty} Q_i(f) \leq 0, \qquad \sum_{i=1}^{\infty} iQ_i(f) = 0. \tag{2.2}$$

Taking (2.2) for granted, we obtain the following local mass conservation law

$$\partial_t\Big(\sum_{i=1}^{\infty} if_i\Big) + v\cdot\nabla_x\Big(\sum_{i=1}^{\infty} if_i\Big) - \nabla_x\Phi\cdot\nabla_v\Big(\sum_{i=1}^{\infty} if_i\Big) = \sum_{i=1}^{\infty}\frac{9\mu}{2\rho_P r_i^2}\nabla_v\cdot\Big((v-u)if_i + \frac{k\theta}{m_i}\nabla_v if_i\Big).$$

The possibility of making these manipulations rigorous depends on the precise definition of the kinetic coefficients κ and σ. Indeed, many coagulation/breakup models are known to exhibit gelation phenomena, where the solution loses mass in finite time: for the homogeneous problem $\partial_t f_i = Q_i(f)$ there are situations where a solution can be shown to exist for a given initial data $\big(f_{\mathrm{init},i}\big)_{i\in\mathbb{N}\setminus\{0\}}$, but for some $t_{\mathrm{gel}} < \infty$, we have $\sum_i i f_i(t_{\mathrm{gel}}) < \sum_i i f_{\mathrm{init},i}$. It might appear in contradiction to (2.2), but this phenomena is due to the fact that the sums in (2.2) can be recast under these circumstances as the difference of infinite quantities. To decide whether of not gelation occurs is the object of fine analysis; we refer to [2] for discussion on the Becker-Döring case, to [24] for an overview on this question and to [12] for further results in non homogeneous cases, with particles subject to space-diffusion. Roughly speaking gelation can be avoided under restrictions on the growth of the coagulation coefficients, or assuming that fragmentation is strong enough compared to coagulation, see [24] for precise statements. To our knowledge the analysis of non homogeneous coagulation-fragmentation equations and the occurrence of gelation in the framework of kinetic models is open. Throughout this work, we assume that gelation does not occur. Finally, let us mention that the size variation modeling we adopt is quite crude. In particular, the operators we consider only affect the size variable, but do not modify the energy of the interacting partners. A more intricate modeling is described e. g. in [3, Chapter 2].

The fluid is described by its density $n(t,x)$ and velocity $u(t,x)$ which obey the following mass conservation and momentum balance relations

$$\begin{aligned}&\partial_t n + \nabla_x\cdot(n\,u) = 0\\ &\rho_F\partial_t(n\,u) + \rho_F \mathrm{Div}_x(n\,u\otimes u) + \rho_F\alpha n\nabla_x\Phi + \nabla_x p - \mu\Delta_x u = 6\pi\mu\sum_{i=1}^{\infty}\int_{\mathbb{R}^3}(v-u)f_i r_i\,\mathrm{d}v,\end{aligned} \tag{2.3}$$

where $\rho_F > 0$ is a typical mass density for the fluid. It means that, given a subdomain $\mathscr{O}$, the integral $\int_{\mathscr{O}}\rho_F n(t,x)\,\mathrm{d}x$ gives the total mass of fluid enclosed in $\mathscr{O}$ at time t. The definition relies here implicitly on the assumption that the fluid mass density does not vary too much over the observation scales. Note that the model belongs to the class of "Two-Way coupling" because both phase exerts an influence on the other through the drag force: the momentum equation takes into account the drag force exerted on the fluid, at a given position x, by the particles. By the way, the following observation is also worthwhile: integrating by parts, the right hand side of (2.3) can be recast as

$$\begin{aligned}&-\sum_{i=1}^{\infty} m_i\times\frac{9\mu}{2\rho_P r_i^2}\int_{\mathbb{R}^3} v\nabla_v\cdot\Big((v-u)f_i + \frac{k\theta}{m_i}\nabla_v f_i\Big)\,\mathrm{d}v\\ &\qquad = -\sum_{i=1}^{\infty} m_i\times\left(\frac{9\mu}{2\rho_P r_i^2}\int_{\mathbb{R}^3} v\nabla_v\cdot\Big((v-u)f_i + \frac{k\theta}{m_i}\nabla_v f_i\Big)\,\mathrm{d}v + \int_{\mathbb{R}^3} vQ_i(f)\,\mathrm{d}v\right)\end{aligned} \tag{2.4}$$

as far as (2.2) holds. In (2.3), the coefficient $\alpha\in\mathbb{R}$ takes into account the fact that the external force can act differently on the two phases, both in amplitude and direction. For the pressure, various options can be considered:

- *incompressible model* in which case p is the Lagrange multiplier associated to the divergence-free constraint $\nabla_x\cdot u = 0$,
- *compressible model* which needs an equation of state. We can restrict our discussion to situations where the temperature is assumed to remain constant, at least as seen from the

particles. We disregard the energy equation and assume in such a case a simple relation $p = p(n)$; the isothermal case corresponds to $p(n) = n$. More complete models incorporate energy exchanges, as discussed in [7, 28], possibly including "turbulent" effects with k/ϵ models [37] (and for further discussions on "turbulent" effects, we refer to [32, 29]): we set $p = R\rho\theta$, with R the perfect gas constant, and $E = \frac{u^2}{2} + \frac{R\theta}{\gamma-1}$, the total energy, with $\gamma > 1$ the adiabatic constant. Then, we have

(2.5)

$$\rho_P\big(\partial_t(nE)+\nabla_x\cdot((nE+p)u)+\alpha nu\cdot\nabla_x\Phi\big) = 6\pi\mu\sum_{i=1}^{\infty}\int_{\mathbb{R}^3} r_i\frac{v^2}{2}\nabla_v\cdot\big((v-u)f_i+\frac{k\theta}{m_i}\nabla_v f_i\big)\,\mathrm{d}v.$$

A remark similar to (2.4) then applies to the energy equation (2.5).

3. Dimensionless equations. Following [16], we write the equations in dimensionless form. To this end, we need time and length scales L and T respectively, which defines the velocity unit $U = L/T$. Velocity fluctuations are measured by means of the thermal velocity

$$V_{th} = \sqrt{\frac{k\bar\theta}{m_1}} \quad \text{with } m_1 = \frac{4}{3}\pi a^3\rho_P, \text{ and } \bar\theta > 0 \text{ a reference temperature.}$$

Denoting $\cdot'$ dimensionless quantities, we set

- $t = Tt'$, $x = Lx'$, $v = V_{th}v'$,
- $n(Tt', Lx') = n'(t', x')$, $u(Tt', Lx') = Uu'(t', x')$,
- $p(Tt', Lx') = \mathcal{P}p'(t', x')$, $f_i'(t', x', v') = \frac{4}{3}\pi a^3\ V_{th}^3\ f_i(Tt', Lx', V_{th}v')$,

where $\mathcal{P}$ stands for a suitable pressure unit. Since $\mathrm{d}v = V_{th}^3\,\mathrm{d}v'$, for any given function φ, we have

$$\int_{\mathbb{R}^3}\varphi(v)f_i(t,x,v)\,\mathrm{d}v = \frac{1}{\frac{4}{3}\pi a^3}\int_{\mathbb{R}^3}\varphi(V_{th}v')f_i'(t',x',v')\,\mathrm{d}v'.$$

If the temperature is not assumed constant, we set similarly $\theta(Tt', Lx') = \bar\theta\theta'(t', x')$. For the external potential, we set $\Phi(Lx') = \frac{\vartheta_s L}{\tau_1}\Phi'(x')$ where ϑ_s has the dimension of velocity (for gravity driven flows, it is nothing but the Stokes settling velocity). We arrive at

$$\begin{aligned}\frac{1}{T}\partial_{t'}(f_i') &+ \frac{V_{th}}{L}v'\cdot\nabla_{x'}(f_i') - \frac{\vartheta_s}{\tau_1 V_{th}}\nabla_{x'}\Phi'\cdot\nabla_{v'}(f_i)\\ &= \frac{1}{\tau_i}\nabla_{v'}\cdot\left((v'-\frac{U}{V_{th}}u')f_i' + \frac{k\bar\theta}{m_iV_{th}^2}\theta'\nabla_{v'}f_i'\right) + \frac{1}{\tau_c}Q_i'(f'),\end{aligned}$$

with $Q_i'(f') = Q_i\big(\frac{f}{\frac{4}{3}\pi a^3 V_{th}^3}\big)$. This is coupled to

$$\begin{aligned}&\frac{1}{T}\partial_{t'}n' + \frac{U}{L}\nabla_{x'}\cdot(n'u') = 0,\\ &\frac{U}{T}\partial_{t'}(n'u') + \frac{U^2}{L}\mathrm{Div}_{x'}(n'u'\otimes u') + \alpha\frac{\vartheta_s}{\tau_1}n'\nabla_{x'}\Phi' + \frac{\mathcal{P}}{L\rho_F}\nabla_{x'}p'\\ &\qquad = \frac{\rho_P}{\tau_1\rho_F}\sum_{i=1}^{\infty}\int_{\mathbb{R}^3}(V_{th}v'-Uu')f_i'i^{1/3}\,\mathrm{d}v' + \frac{\mu U}{\rho_F L^2}\Delta_{x'}u'.\end{aligned}$$

(A similar work can be done considering the energy equation if necessary.) The system is governed by the following set of dimensionless parameters

$$\beta = \frac{T}{L}V_{th} = \frac{V_{th}}{U}, \qquad \frac{1}{\varepsilon} = \frac{T}{\tau_1}, \qquad \eta = \frac{\vartheta_s T}{V_{th}\tau_1}, \qquad \chi = \frac{\mathcal{P}T}{\rho_F LU} = \frac{\mathcal{P}}{\rho_F U^2},$$

together with the density ratio ρ_P/ρ_F. Finally, we obtain (dropping the prime marks)

$$\begin{cases} \partial_t f_i + \beta v\cdot\nabla_x f_i - \eta\nabla_x\Phi\cdot\nabla_v f_i = \dfrac{1}{\varepsilon}\dfrac{1}{i^{2/3}}\nabla_v\cdot\left((v-\dfrac{1}{\beta}u)f_i + \dfrac{\theta}{i}\nabla_v f_i\right) + \dfrac{1}{\tau_c}Q_i(f), \\ \partial_t n + \nabla_x\cdot(n\,u) = 0, \\ \partial_t(n\,u) + \mathrm{Div}_x(n\,u\otimes u) + \alpha\beta\eta\, n\nabla_x\Phi + \chi\nabla_x p = \dfrac{\rho_P}{\varepsilon\rho_F}\displaystyle\sum_{i=1}^{\infty}\int_{\mathbb{R}^3}(\beta v - u)f_i i^{1/3}\,\mathrm{d}v + \mu\Delta_x u. \end{cases} \tag{3.1}$$

Here, μ and τ_c stand for the rescaled and dimensionless version of the fluid viscosity and coagulation relaxation (that are, with the notation in physical units, $\frac{\mu U}{\rho_F L^2}$ and $\frac{T}{\tau_c}$, respectively). Note that in many applications it is relevant to get rid of the diffusion term in the momentum equation (3.1) because the rescaled viscosity μ is very small, hence dealing with the Euler equations instead of the Navier-Stokes system. In what follows the problem is considered on a domain Ω where

- either $\Omega = \mathbb{R}^3$, in which case the analysis will rely on suitable confining assumption on the potential Φ,
- or Ω is a smooth bounded subset in $\mathbb{R}^3$. In such a case the problem is completed by boundary conditions, for instance specular reflection for the particles and the no-slip condition $u_{|\partial\Omega} = 0$ for the fluid (or $u\cdot\nu = 0$ on $\partial\Omega$, with $\nu(x)$ the unit outward normal at $x\in\partial\Omega$ when working with the Euler equations). For more intricate boundary conditions, see [16].

The initial condition are denoted as follows

$$f_i(t=0,x,v) = f_{\mathrm{init},i}(x,v), \qquad n(t=0,x) = n_{\mathrm{init}}(x), \qquad u(t=0,x) = u_{\mathrm{init}}(x).$$

For further purposes, let us set

$$M_P = \sum_{i=1}^{\infty}\int_\Omega\int_{\mathbb{R}^3} i f_i\,\mathrm{d}v\,\mathrm{d}x, \qquad M_F = \int_\Omega n\,\mathrm{d}x$$

which are thus conserved quantities (as far as we assume that gelation does not occur).

4. Equilibria, dissipation and relative entropies. In this section we wish to exhibit some conservation and dissipation properties satisfied by the model. These dissipation properties will induce the existence of equilibrium solutions and their stability; they also provide the basic estimates needed for the analysis of the system.

4.1. Detailed balance equilibria; dissipation properties of the coagulation/breakup operator. First of all, we are interested in sequences of size distributions $\{\mathscr{M}_1, \mathscr{M}_2, ...\}$ which make the coagulation/breakup operator vanish. We thus recall the material on the homogeneous problem, with unknowns that do not depend on the phase space variables x, v. We shall study later the new conditions introduced by taking into account the transport operator. The condition $Q_i(\mathscr{M}) = 0$ is realized when we impose

$$\kappa_{i,j}\mathscr{M}_i\mathscr{M}_j = \sigma_{i,j}\mathscr{M}_{i+j} \qquad \text{for any } i,j\geq 1. \tag{4.1}$$

A solution of (4.1), if it exists, is referred to as a "detailed balance equilibrium". Given a detailed balance equilibrium, we set

$$\mathscr{L}(f) = \sum_{i=1}^{\infty} f_i \Big(\ln \Big(\frac{f_i}{\mathscr{M}_i} \Big) - 1 \Big). \tag{4.2}$$

When dealing with the free coagulation-breakup problem (e. g. without coupling inducing space dependence of the solution) $\mathscr{L}$ plays the role of a Lyapounov functional for the underlying infinite system of ODEs. We refer to [2, 13, 14, 20, 34, 35, 47] for thorough details on the role of the detailed balance assumption and the functional (4.2) in the analysis of coagulation-fragmentation phenomena and of the large time behavior of the solutions.

Using (4.1) with $j = 1$ leads to a recursion formula for defining the equilibria. In turn, detailed balance equilibria can be parametrized by the monomers concentration $\mathscr{M}_1$ as follows:

$$\mathscr{M}_i = \mathscr{Q}_i \ (\mathscr{M}_1)^i \text{ where } \mathscr{Q}_1 = 1 \ , \ \mathscr{Q}_i = \prod_{j=1}^{i-1} \frac{\kappa_{j,1}}{\sigma_{j,1}} \text{ for } i > 1.$$

In the sequel, we assume the existence of detailed balance equilibria. In order to define equilibria with finite mass, it is natural to further request that

$$\text{the radius of convergence } Z_\star \text{ of the series } z \mapsto \sum_{i=1}^{\infty} i \mathscr{Q}_i z^i \text{ is } > 0.$$

Note that $Z_\star$ may be finite or not, which strongly influences the large time asymptotics, see [2, 13, 14, 20, 34, 35, 47]. The saturation density is defined by

$$\varrho_\star = \sum_{i=1}^{\infty} i \mathscr{Q}_i Z_\star^i \in (0, \infty].$$

Observe that, given $0 < \rho < \varrho_\star$, there exists a unique detailed balance equilibrium, characterized by $0 < \mathscr{M}_1 < Z_\star$ and the relation $\sum_{i=1}^{\infty} i \mathscr{Q}_i \mathscr{M}_1^i = \rho$. Finally, the key observation which makes (4.2) a relevant functional for studying the coagulation/breakup dynamics, is simply that

$$\sum_{i=1}^{\infty} Q_i(f) \ln \Big(\frac{f_i}{\mathscr{M}_i} \Big) \leq 0,$$

and this entropy dissipation term vanishes iff $f_i = \mathscr{M}_i$.

4.2. Detailed balance and stationary solutions. Let us now adapt the discussion to particles' distributions depending on the phase space variable (x, v). We search for stationary solutions $n_S(x)$, $u_S = 0$, $\tilde{\mathscr{M}}_i(x, v)$ of (3.1). Making the coagulation/breakup term vanish still leads to (4.1), while we have additionally

$$(\beta v \cdot \nabla_x - \eta \nabla_x \Phi \cdot \nabla_v) \tilde{\mathscr{M}}_i = 0 \ \text{ and } \ \nabla_v \cdot \Big(v \tilde{\mathscr{M}}_i + \frac{\theta}{i} \nabla_v \tilde{\mathscr{M}}_i \Big) = \frac{\theta}{i} \nabla_v \cdot \Big(e^{-i \frac{v^2}{2\theta}} \nabla_v \Big(\frac{\tilde{\mathscr{M}}_i}{e^{-i \frac{v^2}{2\theta}}} \Big) \Big) = 0.$$

These two last relations determine how $\tilde{\mathscr{M}_i}$ depends on the variables x and v. We arrive at

$$\tilde{\mathscr{M}_i}(x,v) = \mathscr{Q}_i \left(\tilde{\mathscr{M}_1}(x,v)\right)^i, \qquad \tilde{\mathscr{M}_1}(x,v) = \omega \exp\Big(-\frac{v^2}{2\theta} - \frac{\eta}{\beta}\frac{\Phi(x)}{\theta}\Big). \tag{4.3}$$

It can be convenient to rewrite

$$\tilde{\mathscr{M}_i}(x,v) = \mathscr{M}_i \ \exp\Big(-i\frac{v^2}{2\theta} - i\frac{\eta}{\beta}\frac{\Phi(x)}{\theta}\Big)$$

where $\mathscr{M}_i = \mathscr{Q}_i\omega^i$ is an equilibrium for the homogeneous equation. The dependence of the equilibrium with respect to x and v introduces new conditions to define certain physical quantities.

In order to guaranty the finiteness of the mass and energy of the equilibrium, we need to assume that the potential Φ fulfills the following confinement conditions

(HC0) Φ is bounded from below: there exists $C \in \mathbb{R}$ such that $\Phi(x) \geq C$ holds a.e. Ω.

(HC1) $x \mapsto e^{-\eta\Phi(x)/(\beta\theta)} \in L^1(\Omega)$.

(HC2) $x \mapsto \Phi(x)e^{-\eta\Phi(x)/(\beta\theta)} \in L^1(\Omega)$.

The total mass of such an equilibrium is now defined as

$$\sum_{i=1}^{\infty} \int_\Omega \int_{\mathbb{R}^3} i\tilde{\mathscr{M}_i}(x,v)\,\mathrm{d}v\,\mathrm{d}x = (2\pi\theta)^{3/2} \ \sum_{i=1}^{\infty} \frac{\mathscr{Q}_i\Gamma_i}{\sqrt{i}} \ \omega^i$$

where we denote from now on $\Gamma_i = \int_\Omega e^{-i\eta\Phi(x)/(\beta\theta)}\,\mathrm{d}x$. By **(HC0)** and **(HC1)**, the Γ_i's are finite. Therefore, we introduce $\omega_\star > 0$ as to be the radius of convergence of the series $\sum_{i=1}^{\infty} \frac{\mathscr{Q}_i\Gamma_i}{\sqrt{i}} \ z^i$ and we set

$$M_{P\star} = (2\pi\theta)^{3/2} \ \sum_{i=1}^{\infty} \frac{\mathscr{Q}_i\Gamma_i}{\sqrt{i}} \ \omega_\star^i \in (0,\infty].$$

Clearly, for any $0 \leq M_P < M_{P\star}$, there exists a unique ω such that the equilibrium parametrized by ω has total mass M_P. Observe that, for a given set of kinetic coefficients, $\omega_\star \neq \rho_\star$ (for instance when the Γ_i's are bounded, we have $\omega_\star \geq \rho_\star$). Similarly, the total energy of an equilibrium is defined as the sum of the kinetic and potential energies, that is

$$\sum_{i=1}^{\infty} \int_\Omega \int_{\mathbb{R}^3} i \ \Big(\beta\frac{v^2}{2} + \eta\Phi(x)\Big)\tilde{\mathscr{M}_i}\,\mathrm{d}v\,\mathrm{d}x = (2\pi\theta)^{3/2} \sum_{i=1}^{\infty} \mathscr{Q}_i\omega^i \Big(\beta\frac{3\theta}{2}\frac{\Gamma_i}{i^{3/2}} + \eta\frac{\tilde{\Gamma}_i}{\sqrt{i}}\Big)$$

with $\tilde{\Gamma}_i = \int_\Omega \Phi(x)e^{-i\eta\Phi(x)/(\beta\theta)}\,\mathrm{d}x$.

We turn briefly to the stationary solution n_S, recalling the material from [16]. Note that the bulk velocity of the equilibrium functions $\mathscr{M}_i$ vanishes: $\int v\mathscr{M}_i\,\mathrm{d}v = 0$. In turn, at equilibrium with $u_S = 0$, the exchange term in (3.1) vanishes. For the sake of concreteness, we detail the computations for a compressible model, assuming that the pressure is defined by a simple law $p : n \mapsto p(n)$. The function p is required to satisfy:

(HP1) $p : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, of class C^2 on $]0,\infty[$, it is increasing and verifies $p(0) = 0$.

(HP2) We set $h(n) := \int_1^n \frac{p'(s)}{s}\,\mathrm{d}s$ (enthalpy function) for $n \in]0,\infty[$ and we assume $h \in L^1_{\mathrm{loc}}([0,\infty[)$.

Owing to **(HP1)-(HP2)**, we can introduce

$$\Pi : n \in [0,\infty[\longmapsto \int_0^n h(s)\,\mathrm{d}s \in \mathbb{R}$$

which can be interpreted as an internal energy. We have $\Pi'(n) = h(n)$ and $n\Pi''(n) = p'(n)$ for any $n \in \mathbb{R}_+$, while $\Pi(0) = \Pi'(1) = 0$. For instance, in the isothermal case, $p(n) = n$ and $\Pi(n) = n\ln(n) - n$, and in the isentropic case $p(n) = n^\gamma$ for some $\gamma > 1$, and $\Pi(n) = \frac{n^\gamma - \gamma n}{\gamma - 1}$.

At equilibrium, the fluid equation degenerates to

$$\Pi'(n_S(x)) = h(n_S(x)) = Z - \frac{\alpha\beta\eta}{\chi}\Phi(x)$$

where Z is a normalizing constant. Let us introduce the generalized inverse of h

$$\Upsilon : \mathbb{R} \to [0,\infty], \quad \Upsilon(s) = \begin{cases} 0 & \text{for} \quad s \le h(0+), \\ h^{-1}(s) & \text{for} \quad h(0+) < s < h(\infty), \\ \infty & \text{for} \quad h(\infty) \le s, \end{cases}$$

and set $n_S(x) = \Upsilon\big(Z - \frac{\alpha\beta\eta}{\chi}\Phi(x)\big)$. The constant Z is defined by the mass condition $M_F = \int_\Omega n_{\mathrm{init}}\,\mathrm{d}x = \int_\Omega n_S\,\mathrm{d}x$. Again, the definition needs that some requirements on the potential and the pressure law are fulfilled:

(HC3) $\Phi \in \mathrm{W}^{1,1}(\Omega)$ if Ω is bounded, or $\Phi \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$ otherwise.

(HC4) $\alpha\Phi$ is bounded from below on Ω: there exists $C \in \mathbb{R}$ such that $\alpha\Phi(x) \ge C$ a.e. $x \in \Omega$.

(HC5) $\alpha\Phi$ is coercive on Ω: for any $A \in \mathbb{R}$ the set $\{x \in \Omega | \alpha\Phi(x) \le A\}$ is bounded.

(HC6) Consider the family of functions $n_Z(x) = \Upsilon\left(Z - \frac{\alpha\beta\eta}{\chi}\Phi(x)\right)$, parametrized by $Z \in \mathbb{R}$. We suppose the existence of some $Z \in \mathbb{R}$ such that $n_Z \in L^1_+(\Omega)$. Hence, we define $\zeta_\star = \sup\{Z \in \mathbb{R} | n_Z \in L^1(\Omega)\}$.

(HC7) For $Z \in (-\infty, \zeta_\star)$, we denote $TM(Z) = \int_\Omega n_Z(x)\,\mathrm{d}x$. Clearly $Z \mapsto TM(Z)$ is strictly increasing. We denote $M_{F\star} = \lim_{Z\to\zeta_\star} TM(Z)$. Hence, for any $M_F \in (0, M_{F\star})$, there exists a unique normalization constant Z_{M_F} such that the associated equilibrium n_S is well defined and has mass M_{F}.

(HC8) To a non negative integrable function n we associate the quantity, that belongs to $\mathbb{R} \cup \{\infty\}$,

$$E_F(n) = \begin{cases} \displaystyle\int_\Omega \left(\frac{\alpha\beta\eta}{\chi} n\Phi + \Pi^+(n)\right)\mathrm{d}x - \int_\Omega \Pi^-(n)\,\mathrm{d}x & \text{if } \Pi^-(n) \in L^1(\Omega) \\ \infty & \text{otherwise.} \end{cases} \tag{4.4}$$

The equilibrium n_S is required to have finite free energy. Thus, we further assume: $E_F(n_Z) < \infty$ and $\Pi^-(n_Z) \in L^1(\Omega)$ for any $Z \in (-\infty, \zeta_\star)$.

Remark 1. *In order to clarify this set of assumptions, let us discuss relevant examples:*

- *If $h(0^+) > -\infty$ and $h(\infty) = \infty$, then hypotheses* **(HC5)-(HC7)** *are trivially satisfied with $M_{F\star} = \infty$. This is the case assuming the polytropic gas law $p(n) = n^\gamma$, for some $\gamma > 1$, which yields*

$$\Upsilon(s) = \left(\left[\frac{\gamma-1}{\gamma}s + 1\right]^+\right)^{1/(\gamma-1)}.$$

- *If* $h(0^+) = -\infty$ *and* $h(\infty) = \infty$, *then* $M_{F\star} = \infty$. *This is the case for an isothermal fluid* $p(n) = n$*:* **(HC5)**-**(HC7)** *are equivalent to* **(HC1)**-**(HC2)** *and* $\Upsilon(s) = e^s$.
- *When* Ω *is bounded and* Φ *is bounded the conditions* **(HC3)**-**(HC7)** *are trivially satisfied.*

The equilibrium n_S can be interpreted has a minimizer of the functional E_F, under the constraint of prescribed mass: the following result is proven in [18, Proposition 5, Lemma 6].

PROPOSITION 4.1. *Assuming the conditions* **(HP1)**-**(HP2)** *on the pressure and the conditions* **(HC0)**-**(HC8)** *on the potential, then the functional* $E_F(n)$ *has a unique minimizer in the set of non negative integrable functions with total fluid mass* M_F *given by*

$$n_S(x) = \Upsilon\left(Z_{M_F} - \frac{\alpha\beta\eta}{\chi}\Phi(x)\right). \tag{4.5}$$

Moreover:

$$E_F(n) - E_F(n_S) \geq \int_\Omega \left[\Pi(n) - \Pi(n_S) - \Pi'(n_S)\,(n - n_S)\right](x)\,\mathrm{d}x \tag{4.6}$$

holds, with equality if and only if

$$\frac{\alpha\beta\eta}{\chi}\Phi(x) + h(n(x)) = Z_{M_F}, \quad \textit{for almost all } x \in \Omega \ (\textit{that is } n = n_S).$$

4.3. Dissipation and stability properties of the fluid–particles system. We are now in position to derive the crucial dissipation estimate satisfied by the system (3.1). We restrict to the case where (3.1) is closed by the pressure law $p = p(n)$, the temperature $\theta > 0$ in the Fokker–Planck term being fixed; we assume without loss of generality that the units are such that $\theta = 1$.

THEOREM 4.2. *We assume that the conditions* **(HP1)**-**(HP2)** *and* **(HC0)**-**(HC8)** *are fulfilled. We suppose that*

$$\frac{\rho_P}{\rho_F} = \frac{1}{\beta^2}, \quad \eta = \beta, \quad \theta = 1 \tag{4.7}$$

holds. Let $\mathscr{M}_i = \mathscr{Q}_i(\mathscr{M}_1)^i$, *with* $\mathscr{M}_1 > 0$, *be a detailed balance equilibrium of the coagulation/breakup operator. We define the following free energy functionals, associated respectively to the particles and to the fluid*

$$\mathcal{F}_P(f) = \sum_{i=1}^\infty \int_\Omega \int_{\mathbb{R}^3} \left(f_i \Big(\ln\Big(\frac{f_i}{\mathscr{M}_i}\Big) - 1\Big) + i\frac{v^2}{2} f_i + i\Phi\, f_i \right) \mathrm{d}v\,\mathrm{d}x,$$
$$\mathcal{F}_F(n,u) = \int_\Omega \left(n\frac{|u|^2}{2} + \chi\Pi(n) + \alpha\beta^2\, n\Phi \right) \mathrm{d}x.$$

Then, the total free energy $\mathcal{F}(f,n,u) = \mathcal{F}_P(f) + \mathcal{F}_F(n,u)$ *is dissipated: solutions of* (3.1) *satisfy*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(f,n,u) + \mu \int_\Omega |\nabla_x u|^2\,\mathrm{d}x + \frac{1}{\varepsilon}\sum_{i=1}^\infty \int_\Omega \int_{\mathbb{R}^3} \left| \Big(v - \frac{1}{\beta}u\Big)\sqrt{i^{1/3} f_i} + \frac{1}{i^{5/6}}\frac{\nabla_v f_i}{\sqrt{f_i}} \right|^2 \mathrm{d}v\,\mathrm{d}x \\ &\qquad = \frac{1}{\tau_c}\sum_{i=1}^\infty \int_\Omega \int_{\mathbb{R}^3} Q_i(f) \ln\Big(\frac{f_i}{\mathscr{M}_i}\Big)\,\mathrm{d}v\,\mathrm{d}x \leq 0. \end{aligned} \tag{4.8}$$

Remark that $\mathcal{F}_F$ can be rewritten by means of the functional E_F

$$\mathcal{F}_F(n(t),u(t)) = \int_\Omega n\frac{|u|^2}{2}\,\mathrm{d}x + \chi\left(E_F(n) - E_F(n_S)\right) + \chi E_F(n_S).$$

However, we notice that the right-hand side of (4.6) is positive and equal to zero iff $n = n_S$, so that we can set, with $u_S = 0$,

$$RE_F((n,u)|(n_S,u_S)) = \int_\Omega n\frac{|u|^2}{2}\,\mathrm{d}x + \chi\left(E_F(n) - E_F(n_S)\right)$$

and consider it as a relative entropy, that is a functional that controls the distance from the pair $(n(t),u(t))$ to the equilibrium solution $(n_S, u_S = 0)$. Similarly, when the total mass M_P is subcritical the free energy $\mathcal{F}_P$ can be interpreted as a relative entropy with respect to the equilibrium having the same mass. In turn, we can establish a non linear stability statement.

Corollary 4.3. *We assume that* $0 < M_P \le M_{P\star}$ *and we denote*

$$\tilde{\mathscr{M}}_i(x,v) = \omega^i \mathscr{Q}_i \exp\Big(- i\frac{v^2}{2} - i\Phi(x)\Big)$$

the equilibrium with total mass M_P. *Therefore, we set*

$$RE_P(f|\tilde{\mathscr{M}}) = \sum_{i=1}^{\infty}\int_\Omega\int_{\mathbb{R}^3}\Big(f_i\ln\Big(\frac{f_i}{\tilde{\mathscr{M}}_i}\Big) - f_i + \tilde{\mathscr{M}}_i\Big)\,\mathrm{d}v\,\mathrm{d}x = \mathcal{F}_P(f) + (2\pi)^{3/2}\sum_{i=1}^{\infty}\frac{\Gamma_i\mathscr{Q}_i}{i^{3/2}}\omega^i.$$

Let n_S *be the equilibrium state defined by* (4.5) *with mass* M_F. *Then, for any* $\delta > 0$, *there exits* $\kappa > 0$ *such that if initially* $RE_P(f_{\mathrm{init}}|\tilde{\mathscr{M}}) + RE_F((n_{\mathrm{init}}, u_{\mathrm{init}})|(n_S, 0)) \le \kappa$ *holds, then, for any* $t \ge 0$, *the solution satisfies*

$$\sum_{i=1}^{\infty}\int_\Omega\int_{\mathbb{R}^3}|f_i(t,x,v) - \tilde{\mathscr{M}}_i(x,v)|\,\mathrm{d}v\,\mathrm{d}x \le \delta, \qquad \int_\Omega |n(t,x) - n_S(x)|\,\mathrm{d}x \le \delta, \qquad \int_\Omega n\frac{|u|^2}{2}\,\mathrm{d}x \le \delta.$$

Corollary 4.3 follows exactly as in [16], once the entropy dissipation has been established. These arguments go back to [10, 22, 45] to which we refer for further details. The crucial step is the identification of the relative entropies RE_P and RE_F, which are the appropriate tools to evaluate how far the solution is from the equilibrium. The remarkable fact we bring out here is that structure properties are preserved when dealing with polydisperse solutions and the entropy dissipation still holds.

Remark 2. *Similar conclusions hold for incompressible models or models including the energy equation. We restrict the discussion to the free space problem, but it applies to problems set on a domain with suitable boundary conditions, say no-slip boundary condition for the velocity* u, *and specular reflection of the particles; for more intricate reflection operator we refer to [16].*

Remark 3. *The condition* (4.7) *might look a bit arbitrary and artificial; it is adopted for notational convenience only in order not to keep track of complicated coefficients in the energy balance. The value of* θ *and of the ratio* η/β *and* $\beta^2\rho_P/\rho_F$ *can be fixed arbitrarily, just adapting accordingly the definition of the free energy. Similarly, when discussing the hydrodynamic regimes,* *we can keep ε as the scaling parameter which characterizes the asymptotics and assume that η/β and $\beta^2\rho_P/\rho_F$ are any positive constants (or even that they tend to positive constants as $\varepsilon \to 0$).*

Proof of Theorem 4.2. Let us start by computing

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{\infty}\int_\Omega\int_{\mathbb{R}^3}\Big(f_i\Big(\ln\Big(\frac{f_i}{\mathscr{M}_i}\Big)-1\Big)+i\frac{v^2}{2}f_i\Big)\,\mathrm{d}v\,\mathrm{d}x$$
$$=-\frac{1}{\varepsilon}\sum_{i=1}^{\infty}\int_\Omega\int_{\mathbb{R}^3}\Big((v-\frac{1}{\beta}u)f_i+\frac{1}{i}\nabla_v f_i\Big)\cdot\Big(\frac{1}{i^{2/3}}\frac{\nabla_v f_i}{f_i}+i^{1/3}v\Big)\,\mathrm{d}v\,\mathrm{d}x$$
$$+\frac{1}{\tau_c}\int_\Omega\int_{\mathbb{R}^3}\sum_{i=1}^{\infty}\ln\Big(\frac{f_i}{\mathscr{M}_i}\Big)Q(f_i)\,\mathrm{d}v\,\mathrm{d}x-\eta\sum_{i=1}^{\infty}\int_\Omega\int_{\mathbb{R}^3}i\nabla_x\Phi\cdot vf_i\,\mathrm{d}v\,\mathrm{d}x.$$

Next the potential energy of the particles satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{\infty}\int_\Omega\int_{\mathbb{R}^3}i\Phi f_i\,\mathrm{d}v\,\mathrm{d}x=\beta\sum_{i=1}^{\infty}\int_\Omega\int_{\mathbb{R}^3}i\nabla_x\Phi\cdot\, vf_i\,\mathrm{d}v\,\mathrm{d}x.$$

We turn to the kinetic energy of the fluid and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}n\frac{u^2}{2}\,\mathrm{d}x=\frac{1}{\varepsilon}\frac{\rho_P}{\rho_F}\sum_{i=1}^{\infty}\int_\Omega\int_{\mathbb{R}^3}(\beta v-u)f_i i^{1/3}\cdot u\,\mathrm{d}v\,\mathrm{d}x-\chi\int_{\mathbb{R}^3}u\cdot p'(n)\nabla_x n\,\mathrm{d}x$$
$$-\alpha\,\beta^2\int_{\mathbb{R}^3}n\,u\cdot\nabla_x\Phi\,\mathrm{d}x-\mu\int_\Omega|\nabla_x u|^2\,\mathrm{d}x.$$

The evolution of entropy associated to the fluid is driven by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\chi\Pi(n)\,\mathrm{d}x=-\chi\int_{\mathbb{R}^3}\nabla_x\cdot(n\,u)\Pi'(n)\,\mathrm{d}x=\chi\int_{\mathbb{R}^3}\Pi''(n)\nabla_x n\cdot(n\,u)\,\mathrm{d}x=\chi\int_\Omega p'(n)\nabla_x n\cdot u\,\mathrm{d}x.$$

Finally, the potential energy for the fluid satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}n\Phi\,\mathrm{d}x=\int_{\mathbb{R}^3}n\,u\cdot\nabla_x\Phi\,\mathrm{d}x.$$

We sum all these contributions, using (4.7) and the fact that $\int_{\mathbb{R}^3}u\cdot\nabla_v f_i\,\mathrm{d}v=0$ holds. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(f,n,u)+\mu\int_\Omega|\nabla_x u|^2\,\mathrm{d}x$$
$$=-\frac{1}{\varepsilon}\sum_{i=1}^{\infty}\int_\Omega\int_{\mathbb{R}^3}\big(v-\frac{1}{\beta}u\big)^2 f_i i^{1/3}\,\mathrm{d}v\,\mathrm{d}x-\frac{2}{\varepsilon}\sum_{i=1}^{\infty}\int_\Omega\int_{\mathbb{R}^3}(v-\frac{1}{\beta}u)\frac{\nabla_v f_i}{i^{2/3}}\,\mathrm{d}v\,\mathrm{d}x$$
$$-\frac{1}{\varepsilon}\sum_{i=1}^{\infty}\int_\Omega\int_{\mathbb{R}^3}\frac{1}{i^{5/3}}\frac{|\nabla_v f_i|^2}{f_i}\,\mathrm{d}v\,\mathrm{d}x+\frac{1}{\tau_c}\sum_{i=1}^{\infty}\int_\Omega\int_{\mathbb{R}^3}Q(f)\ln\Big(\frac{f_i}{\mathscr{M}_i}\Big)\,\mathrm{d}v\,\mathrm{d}x.$$

In the first three terms in the right hand side we recognize $\big|(v-u/\beta)\sqrt{i^{1/3}f_i}+\frac{\nabla_v f_i}{\sqrt{i^{5/3}f_i}}\big|^2$. This observation ends the proof. □

As said in the Introduction, the construction of solutions to the system (3.1) is beyond the scope of this work. We perform formally several manipulations which lead to remarkable properties of the system. In turn, Theorem 4.2 provides information on the natural functional properties that solutions can be expected to satisfy. In view of the a priori estimates we have derived so far, a possible strategy would be to construct an approximate model, which can be proved to be well-posed. The approximation should be constructed in a suitable way, so that the a priori estimates apply to the approximated solutions. We warn the reader that this part of the program can already be quite difficult and technical; [9, 38] provides examples of such intricate construction for closely related problems. The next step relies on the application of compactness techniques to remove the approximation. Again, this step is likely to be highly technical due to the nonlinearities. Another viewpoint, which would also use as a starting point the results of the present paper, consists in studying by energy methods solutions close to equilibrium, see [15, 19, 31] for such techniques applied to fluid/particles flows. Of course, as mentioned above, another keypoint in such analysis is to determine whether or not gelation can occur, depending on growth assumptions on the coagulation and breakup coefficients.

5. Hydrodynamic regimes. We are interested in regimes where $0 < \varepsilon \ll 1$. As we shall see this regime leads to relaxation effects, which tend to prescribe how the particles distribution functions depend on the velocity variable v. In turn, we obtain models of purely hydrodynamic nature, where the unknowns depend only on the time and space variables. Indeed, as ε goes to 0, the f_i's tend to make the Fokker–Planck operator vanish. Looking at the dissipation term

$$\frac{1}{\varepsilon}\sum_{i=1}^{\infty}\int_\Omega\int_{\mathbb{R}^3} i^{1/3}\Big|\big(v-\frac{1}{\beta}u\big)\sqrt{f_i}+\frac{2}{i}\nabla_v\sqrt{f_i}\Big|^2\,\mathrm{d}v\,\mathrm{d}x$$

in (4.8), we guess that

$$f_i(t,x,v)\simeq\frac{\rho_i(t,x)}{(2\pi/i)^{3/2}}\ \exp\Big(-i\frac{|v-u(t,x)/\beta|^2}{2}\Big)\qquad\text{as }\varepsilon\to0. \tag{5.1}$$

The limit distribution function makes the Fokker–Planck operator, that is the leading term in (3.1), vanish. Then the behavior of the particulate flows can be described through the macroscopic particle density $\rho_i(t,x)$, the fluid velocity $u(t,x)$ and the fluid density $n(t,x)$. Identifying the limit system, that belongs to the class of complex models for mixture flows [44], satisfied by these quantities is the object of the present section. Of course, the asymptotic analysis depends on the prescribed behavior with respect to ε of the other scaling parameters. According to [16], we identify two regimes of interest: the so–called flowing and bubbling regimes.

Before detailing the asymptotic analysis, let us set up a few notation and discuss remarkable estimates, which can be seen as a preliminary step towards a complete justification. We associate to $f_i(t,x,v)$ the following macroscopic quantities

$$\rho_i(t,x)=\int_{\mathbb{R}^3}f_i(t,x,v)\,\mathrm{d}v,\quad J_i(t,x)=\beta i\int_{\mathbb{R}^3}v\ f_i(t,x,v)\,\mathrm{d}v,\quad \mathbb{P}_i(t,x)=i\int_{\mathbb{R}^3}v\otimes v\ f_i(t,x,v)\,\mathrm{d}v.$$

Integrating (3.1) with respect to $i\,\mathrm{d}v$ and $iv\,\mathrm{d}v$, we obtain

$$i\partial_t\rho_i+\nabla_x\cdot J_i=\frac{1}{\tau_c}\int_{\mathbb{R}^3}iQ_i(f)\,\mathrm{d}v, \tag{5.2}$$

and, still using the scaling assumption $\eta = \beta$

$$\frac{1}{\beta^2}\partial_t J_i + \mathrm{Div}_x \mathbb{P}_i + i\rho_i \nabla_x \Phi = -\frac{1}{\beta^2 \varepsilon}\frac{1}{i^{2/3}}(J_i - i\rho_i u) + \frac{1}{\beta \tau_c}\int_{\mathbb{R}^3} ivQ_i(f)\,\mathrm{d}v. \tag{5.3}$$

As a matter of fact, we remark that the system (3.1) conserves the total momentum since we have

$$\partial_t\Big(nu + \frac{1}{\beta^2}\sum_{i=1}^{\infty} J_i\Big) + \mathrm{Div}_x\Big(nu\otimes u + \sum_{i=1}^{\infty}\mathbb{P}_i\Big) + \chi\nabla_x p + \Big(\alpha\beta^2 n + \sum_{i=1}^{\infty} i\rho_i\Big)\nabla_x\Phi = \mu\Delta_x u. \tag{5.4}$$

Let us start by deducing from Theorem 4.2 the following a priori estimate.

PROPOSITION 5.1. *Assume that* (4.7) *holds and that the conditions* **(HP1)**-**(HP2)** *and* **(HC0)**-**(HC8)** *are satisfied. Moreover, we assume*

(HC9) $(1+\Phi)\exp(-\frac{1}{2}\Phi(x)) \in \mathrm{L}^1(\Omega)$.

(HP3) *If* $h(0+) = -\infty$ *we assume there exists* $0 < s_1 < 1$ *such that*

$$\sup\left\{\frac{\Pi(n)}{nh(n)}, 0 < n < s_1\right\} = m < +\infty.$$

We consider an equilibrium $\mathscr{M}_i = \mathscr{Q}_i \omega^i$ *where* $0 < \omega < \omega_\star$ *is such that*

$$\sum_{i=1}^{\infty}\left(\int_\Omega\int_{\mathbb{R}^3}\Big(1 + i\frac{v^2}{2} + i\Phi(x)\Big)e^{-i(v^2/4+\Phi(x)/2)}\,\mathrm{d}v\,\mathrm{d}x\right)\mathscr{M}_i = K < \infty.$$

We suppose that the initial data $(f_{\mathrm{init},i}, n_{\mathrm{init}}, u_{\mathrm{init}})$ *satisfies* $f_{\mathrm{init},i} \geq 0$, $n_{\mathrm{init}} \geq 0$ *and that the quantities*

$$\sum_{i=1}^{\infty}\int_\Omega\int_{\mathbb{R}^3} f_{\mathrm{init},i}\Big(1 + i + \Big|\ln\Big(\frac{f_{\mathrm{init},i}}{\mathscr{M}_i}\Big)\Big| + i\frac{v^2}{2} + i|\Phi(x)|\Big)\,\mathrm{d}v\,\mathrm{d}x$$

$$\int_\Omega \Big(n_{\mathrm{init}} + n_{\mathrm{init}}\frac{|u_{\mathrm{init}}|^2}{2} + |\Pi(n_{\mathrm{init}})| + n_{\mathrm{init}}\beta^2|\alpha\Phi|\Big)\,\mathrm{d}x$$

are finite and bounded uniformly with respect to all the parameters $\varepsilon, \beta, \eta, \alpha, \rho_P/\rho_F$. *Then, we have:*

(i) The quantity

$$\sum_{i=1}^{\infty}\int_\Omega\int_{\mathbb{R}^3} f_i\Big(1 + i + \Big|\ln\Big(\frac{f_i}{\mathscr{M}_i}\Big)\Big| + i\frac{v^2}{2} + i|\Phi(x)|\Big)\,\mathrm{d}v\,\mathrm{d}x$$

is bounded uniformly for $t \geq 0$,

(ii) n, $|\Pi(n)|$ *and* $\beta^2|\alpha\Phi|n$ *are bounded in* $L^\infty(\mathbb{R}^+; L^1(\Omega))$.

(iii) $\sqrt{n}\ u$ *is bounded in* $L^\infty(\mathbb{R}^+; L^2(\Omega))$.

(iii) $\sqrt{\mu}\nabla_x u$ *is bounded in* $L^2(\mathbb{R}^+\times\Omega)$,

(v) Denoting $D_i(t,x,v) = (v - \beta^{-1}u(t,x))\sqrt{i^{1/3}f_i(t,x,v)} + 2\nabla_v\sqrt{i^{-5/3}f_i(t,x,v)}$, *the quantity*

$$\sum_{i=1}^{\infty}\int_0^\infty\int_\Omega\int_{\mathbb{R}^3}\Big|\frac{D_i}{\sqrt{\varepsilon}}\Big|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t$$

is bounded.

Here "bounded" means "bounded uniformly with respect to all the parameters $\varepsilon, \beta, \eta, \alpha, \frac{\rho_P}{\rho_F}$*".*
The proof is a consequence of the dissipation estimate in Theorem 4.2. The point consists in controlling the negative parts of the free energy. To this end, let us recall the following elementary claim, see e. g. [16, 22].

LEMMA 5.2. *Let* X *be a subset of* $\mathbb{R}^D$*, possibly* $\mathbb{R}^D$ *itself. Let* $U : X \to \mathbb{R}^+$ *such that* $(1+U)e^{-\nu U} \in L^1(X)$ *for some* $0 < \nu < 1$*. Let* $g : X \to \mathbb{R}^+$*. Then, we have*

$$0 \le \int_X g \ln^-(g)\,\mathrm{d}y \le \nu \int_X Ug\,\mathrm{d}y + \int_X (1+\nu U)e^{-\nu U}\,\mathrm{d}y.$$

Proof. We use the following decomposition

$$g\ln^-(g) = \Big[e\,\frac{g}{e}\Big(-\ln\Big(\frac{g}{e}\Big)\Big) - g\ln(e)\Big]\mathbf{1}_{0\le g\le e^{-\nu U}} + g(-\ln(g))\mathbf{1}_{e^{-\nu U}\le g\le 1}.$$

Since $s \mapsto s(-\ln(s))$ is non decreasing on $(0, 1/e)$ and $s \mapsto (-\ln(s))$ is non increasing, we infer

$$\int g\ln^-(g)\,\mathrm{d}y \le \int_{0\le g\le e^{-\nu U}} e^{-\nu U}\Big(-\ln\Big(\frac{e^{-\nu U}}{e}\Big)\Big)\,\mathrm{d}y + \nu\int_{e^{-\nu U}\le g\le 1} Ug\,\mathrm{d}y.$$

□

We apply this result with $X = \mathbb{N}\times\Omega\times\mathbb{R}^3$, endowed with the measure $\mathscr{M}_i\,\mathrm{d}i\,\mathrm{d}v\,\mathrm{d}x$, where $\mathrm{d}i$ stands for the counting measure on $\mathbb{N}$, $U = \frac{i}{2}(v^2/2 + \Phi(x))$, $\nu = 1/2$ and $g = f_i(t,x,v)/\mathscr{M}_i$. It allows to control the negative part of $f_i\ln(f_i/\mathscr{M}_i)$. This statement also permits to treat the isothermal case where $p(n) = n$, and $\Pi(n) = n\ln(n) - n$. The case of polytropic laws $p(n) = n^\gamma$ does not contain more difficulty because we can add a conserved quantity to define a non negative free energy $\tilde\Pi(n) = \Pi(n) + \frac{\gamma}{\gamma-1}n$. This reasoning applies as far as $h(0^+) < 0$ is finite. The remaining cases are treated with the following claim.

LEMMA 5.3. *We suppose that* $h(0^+) = -\infty$*. Then*

$$\int \Pi^-(n)\,\mathrm{d}x \le C\Big(1 + \int n\,\mathrm{d}x\Big) + \nu\int \alpha\Phi n\,\mathrm{d}x$$

holds for some constants C*, and* $0 < \nu < 1$*.*

Proof. The proof is directly inspired from [18, Theorem 18, Step 7]. Let us start with a couple of simple, but useful, remarks. We remind that $p(0) = 0$ and p is non decreasing. Then $s \mapsto h(s)$ is a non decreasing function, which is non negative for $s \ge 1$, and non positive for $s \le 1$. Integrating by parts we get $\Pi(n) = nh(n) - p(n)$, which is therefore a convex function: it reaches its minimum for $n = 1$, Π is non increasing and non positive on $(0,1)$, non decreasing and non positive on $(1, s^\star)$ for some $s^\star > 1$ and non decreasing and non negative on $(s^\star, \infty)$. We also notice that $\frac{\mathrm{d}}{\mathrm{d}n}\big(\frac{\Pi(n)}{n}\big) = \frac{p(n)}{n^2} > 0$: we shall use the fact that $n \mapsto \frac{\Pi^-(n)}{n}$ is non increasing.

We define a reference function

$$\tilde n(x) = \Upsilon\Big(-\Big(K + \delta\frac{\alpha\beta^2}{\chi}\Phi(x)\Big)\Big).$$

It depends on the two parameters $K > 0$ and $\delta > 0$, that will be made precise along the proof. By **(HC4)**, the potential $\alpha\Phi$ is bounded from below and we can pick K large enough, depending on δ, so that $\tilde n(x) \le 1$. By using **(HC7)-(HC8)**, we can further assume that both $\tilde n$ and $\Pi^-(\tilde n)$ are integrable. Finally, we also require that $\tilde{n}(x) < s_1$, the constant that appears in **(HP3)**. We split the integral to be evaluated into three pieces, according to the following sets

$$\Omega_\star = \{x \in \Omega,\ 0 \le n(x) \le \tilde{n}(x)\},\ \Omega_0 = \{x \in \Omega,\ \tilde{n}(x) \le n(x) \le 1\},\ \Omega^\star = \{x \in \Omega,\ 1 \le n(x) \le s^\star\}.$$

Since $s \mapsto \Pi^-(s)$ is non decreasing on $(0,1)$ and $s \mapsto \frac{\Pi^-(s)}{s}$ is non increasing, we observe that $\Pi^-(n(x)) \le \Pi^-(\tilde{n}(x))$ on $\Omega_\star$ and $\Pi^-(n(x)) \le \Pi^-(1)n(x)$ on $\Omega^\star$. It follows that

$$\int_{\Omega_\star} \Pi^-(n)\,\mathrm{d}x + \int_{\Omega^\star} \Pi^-(n)\,\mathrm{d}x \le \int_\Omega \Pi^-(\tilde{n})\,\mathrm{d}x + \Pi^-(1)\int_\Omega n\,\mathrm{d}x.$$

Next, by using the fact that $s \mapsto \frac{\Pi^-(s)}{s}$ is non increasing again, we obtain

$$\begin{aligned}\int_{\Omega_0} \Pi^-(n)\,\mathrm{d}x &= \int_{\Omega_0} n\frac{\Pi^-(n)}{n}\,\mathrm{d}x \le \int_{\Omega_0} n\frac{\Pi^-(\tilde{n})}{\tilde{n}}\,\mathrm{d}x \\ &\le \int_{\Omega_0} n \times (-mh(\tilde{n}))\,\mathrm{d}x = \int_{\Omega_0} nm\Big(\delta\frac{\alpha\beta^2}{\chi}\Phi + K\Big)\,\mathrm{d}x.\end{aligned}$$

We conclude by imposing $\delta m\beta^2/\chi = \nu < 1$. □

With these properties, we already have estimates on the macroscopic quantities, but we can go a step further, identifying leading terms owing to the dissipation term D. To this end, we need an estimate on a higher moment with respect to the size variable, namely, we can suppose that

$$\sum_{I=1}^\infty \int_\Omega\int_{\mathbb{R}^3} i^2\ (1+v^2)\ f_i\,\mathrm{d}v\,\mathrm{d}x \quad \text{is bounded uniformly on } [0,T] \tag{5.5}$$

(uniformly with respect to the scaling parameters) for any $0 < T < \infty$. This condition, the physical interpretation of which is not direct, is however somehow classical when dealing with coagulation-fragmentation problem. It is likely that it can be satisfied under appropriate hypothesis on the initial data and the kinetic coefficients, see [2].

Lemma 5.4. *Let the assumptions of Proposition 5.1 be fulfilled together with (5.5). Then, we have*

$$J_i = i\rho_i u + \beta\sqrt{\varepsilon}K_i, \qquad \mathbb{P}_i = \rho_i\mathbb{I} + \beta^{-2}J_i\otimes u + \sqrt{\varepsilon}\mathbb{K}_i$$

where the remainders K and $\mathbb{K}$ are bounded in $L^2(0,T;L^1(\Omega\times\mathbb{N}))$.

Proof. We rewrite

$$J_i = i\rho_i u + \beta i\int_{\mathbb{R}^3}\Big((v-\beta^{-1}u)f_i + \frac{1}{i}\nabla_v f_i\Big)\,\mathrm{d}v = i\rho_i u + \beta\sqrt{\varepsilon}\int_{\mathbb{R}^3}\frac{D_i}{\sqrt{\varepsilon}}\sqrt{\frac{f_i}{i^{1/3}}}\,\mathrm{d}v.$$

The last integral, that is denoted K_i, can be dominated by using the Cauchy-Schwarz inequality

$$\sum_{i=1}^\infty\int_\Omega |K_i|\,\mathrm{d}x \le \beta\left(\sum_{i=1}^\infty\int_\Omega\int_{\mathbb{R}^3}\Big|\frac{D_i}{\sqrt{\varepsilon}}\Big|^2\,\mathrm{d}v\,\mathrm{d}x\right)^{1/2}\left(\sum_{i=1}^\infty\int_\Omega\int_{\mathbb{R}^3}\frac{f_i}{i^{1/3}}\,\mathrm{d}v\,\mathrm{d}x\right)^{1/2}.$$

We proceed similarly with

$$\mathbb{P}_i = \beta^{-2}J_i\otimes u + \rho_i\mathbb{I} + \sqrt{\varepsilon}\int_{\mathbb{R}^3}\frac{D_i}{\sqrt{\varepsilon}}\otimes v\sqrt{i^{5/3}f_i}\,\mathrm{d}v.$$

Denoting $\mathbb{K}_i$ the last integral, we have the following estimate

$$\sum_{i=1}^{\infty}\int_\Omega |\mathbb{K}_i|\,\mathrm{d}x \le \left(\sum_{i=1}^{\infty}\int_\Omega\int_{\mathbb{R}^3}\left|\frac{D_i}{\sqrt{\varepsilon}}\right|^2 \mathrm{d}v\,\mathrm{d}x\right)^{1/2}\left(\sum_{i=1}^{\infty}\int_\Omega\int_{\mathbb{R}^3} v^2 i^{5/3} f_i\,\mathrm{d}v\,\mathrm{d}x\right)^{1/2}.$$

□

5.1. Flowing regime. For this regime we suppose that both $\rho_P/\rho_F = \beta^{-2}$ and $\eta = \beta$ are fixed, as well as α; we only let ε go to 0. According to (5.1) and the conclusions in Lemma 5.4, we expect that

$$J_i \simeq i\rho_i u, \qquad \mathbb{P}_i \simeq \rho_i \mathbb{I} + \beta^{-2} i\rho_i u\otimes u.$$

Plugging this ansatz into (5.2) and (5.4), we deduce that the dynamics in this regime is described by the following system

$$\left\{\begin{array}{l}\partial_t\bar\rho_i + \nabla_x\cdot(\bar\rho_i\bar u) = \dfrac{1}{\tau_c}Q_i(\bar\rho),\\ \partial_t\bar n + \nabla_x\cdot(\bar n\bar u) = 0,\\ \partial_t\big((\bar n+\beta^{-2}\nu)\bar u\big) + \mathrm{Div}_x\big((\bar n+\beta^{-2}\nu)\bar u\otimes\bar u + (\chi\bar p+\nu)\mathbb{I}\big) + (\alpha\beta^2\bar n^2+\nu)\nabla_x\Phi = \mu\Delta_x\bar u,\end{array}\right. \tag{5.6}$$

where we have set

$$\nu = \sum_{i=1}^{\infty} i\bar\rho_i.$$

In other words, we expect that n, u and f_i tend to $\bar n$, $\bar u$ and $\frac{\bar\rho_i}{(2\pi/i)^{3/2}}\exp\big(-i\frac{|v-\bar u/\beta|^2}{2}\big)$ respectively as $\varepsilon\to 0$, with $\bar n$,$\bar u$ and the $\bar\rho_i$'s solution of (5.6). As a matter of fact, we observe that

$$\partial_t\nu + \nabla_x\cdot(\nu\bar u) = 0.$$

We obtain a multiphase flow system where particles distribution function are subject to exchanges through coagulation and breakup and advection with the fluid velocity $\bar u$. The motion of the fluid can be interpreted as a Navier-Stokes (or Euler if $\mu = 0$) system for the composite density $\bar n + \beta^{-2}\nu$ and the velocity $\bar u$, involving a complex pressure law.

The macroscopic model derived so far inherits the dissipative properties of the kinetic model. Such a property is important: by contrast many models for multiphase flows are known for not having nice structure properties, see [44] and the references therein. Indeed, here we can check that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Bigg\{&\sum_{i=1}^{\infty}\int\Big(\bar\rho_i\ln\Big(\frac{\bar\rho_i}{\mathscr{M}_i}\Big) + i\bar\rho_i\Phi\Big)\mathrm{d}x\\ &+\int\Big((\bar n+\beta^{-2}\nu)\frac{\bar u^2}{2} + \alpha\beta^2\bar n\Phi + \chi\Pi(\bar n) + \nu\ln(\nu) - \nu\Big)\mathrm{d}x\Bigg\}\\ = \frac{1}{\tau_c}&\sum_{i=1}^{\infty}\int Q_i(\bar\rho)\ln\Big(\frac{\bar\rho_i}{\mathscr{M}_i}\Big) - \int\mu|\nabla_x\bar u|^2\,\mathrm{d}x \le 0.\end{aligned}$$

5.2. Bubbling regime. For this regime, the scaling assumptions cast as follows

$$\beta = \frac{1}{\sqrt{\varepsilon}} = \eta, \qquad \frac{\rho_P}{\rho_F} = \varepsilon,$$

while we set $\alpha = \mathrm{sgn}(\alpha)\varepsilon$. Coming back to (5.1) with these assumptions, we guess that, at leading order, the f_i's look like centered Maxwellians $f_i(t,x,v) \simeq \frac{\rho_i(t,x)}{(2\pi/i)^{3/2}}\ e^{-iv^2/2}$. Therefore, we infer that

$$\mathbb{P}_i \simeq i\int_{\mathbb{R}^3} v\otimes v \frac{\rho_i(t,x)}{(2\pi/i)^{3/2}}\ e^{-iv^2/2}\,\mathrm{d}v = \rho_i \mathbb{I}.$$

Relation (5.3) allows to obtain the limiting particles current: letting $\varepsilon = \beta^{-2}$ go to 0 in (5.3) yields

$$J_i \simeq i\rho_i u_i - i^{5/3}\rho_i\nabla_x\Phi - i^{2/3}\nabla_x\rho_i.$$

Coming back to (5.2) and the fluid equations, we arrive at the following system

$$(5.7)\qquad \left\{\begin{array}{l} i\partial_t\bar{\rho}_i + i\nabla_x\cdot\big(\bar{\rho}_i(u - i^{2/3}\nabla_x\Phi)\big) = i^{2/3}\Delta_x\bar{\rho}_i + \dfrac{i}{\tau_c}Q_i(\bar{\rho}),\\ \partial_t\bar{n} + \nabla_x\cdot(\bar{n}\bar{u}) = 0,\\ \partial_t(\bar{n}\bar{u}) + \mathrm{Div}_x(\bar{n}\bar{u}\otimes\bar{u}) + \nabla_x\Big(\chi\bar{p} + \displaystyle\sum_{i=1}^{\infty}\bar{\rho}_i\Big) + (\mathrm{sgn}(\alpha)\bar{n} + \nu)\nabla_x\Phi = \mu\Delta_x\bar{u}, \end{array}\right.$$

still with the notation $\nu = \sum_{i=1}^{\infty} i\bar{\rho}_i$. We thus expect that n, u and f_i behave like $\bar{n}$, $\bar{u}$ and $\frac{\bar{\rho}_i}{(2\pi/i)^{3/2}}e^{-iv^2/2}$ respectively when $\varepsilon\to 0$. Now particles densities are driven by convection–diffusion equations, with a size-dependent diffusion coefficient (considering the evolution equation satisfied by $\bar{\rho}_i$, the diffusion coefficient is proportional to the inverse of the radius of the grain, and in particular it degenerates as the size becomes large, see [12]). The particles influence weakly the fluid through the pressure term (which can be incorporated in a common Lagrange multiplier for incompressible flows) and the external force.

Again, we observe that the macroscopic model satisfies a dissipation property, which states

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\left\{\sum_{i=1}^{\infty}\int\Big(\bar{\rho}_i\ln\Big(\frac{\bar{\rho}_i}{\mathscr{M}_i}\Big) + i\bar{\rho}_i\Phi\Big)\,\mathrm{d}x + \int\Big(\bar{n}\frac{\bar{u}^2}{2} + \chi\Pi(\bar{n}) + \mathrm{sgn}(\alpha)\bar{n}\Phi\Big)\,\mathrm{d}x\right\}\\ &= \frac{1}{\tau_c}\sum_{i=1}^{\infty}\int Q_i(\bar{\rho})\ln\Big(\frac{\bar{\rho}_i}{\mathscr{M}_i}\Big) - \int\mu|\nabla_x\bar{u}|^2\,\mathrm{d}x - \sum_{i=1}^{\infty}\int\Big|2i^{-1/3}\nabla_x\sqrt{\bar{\rho}_i} + i^{5/6}\sqrt{\bar{\rho}_i}\nabla_x\Phi\Big|^2\,\mathrm{d}x \le 0.\end{aligned}$$

6. Conclusion. We consider a simple model describing a suspension of polydisperse particles. Particles size is subject to variations through binary coagulation and fragmentation described by the discrete Smoluchowski operator. Particles interact with the surrounding fluid through the effect of mutual drag forces. We bring out several properties of the system, when assuming that gelation does not occur. In particular we exhibit a family of relevant stationary solutions, and we show that the dissipative structure known for a single specie of particles extends to the polydisperse case. In turn, we justify the stability of the equilibrium states. Furthermore, the a priori estimates we have established can also be used to discuss hydrodynamic regimes. We obtain this way new multi-fluid models, two-way coupling the behavior of the dense and the disperse phases.

REFERENCES

[1] M. J. Andrews and P. J. O'Rourke. The multiphase particle-in-cell (MP-PIC) method for dense particulate flows. *Int. J. Multiphase Flow*, 22(2):379–402, 1996.
[2] J. M. Ball, J. Carr, and O. Penrose. The Becker–Döring cluster equations: Basic properties and asymptotic behaviour of solutions. *Comm. Math. Phys.*, 104:657–692, 1986.
[3] C. Baranger. *Modélisation, étude mathématique et simulation des collisions dans les fluides complexes*. PhD thesis, ENS Cachan, 2004.
[4] C. Baranger, L. Boudin, P.-E. Jabin, and S. Mancini. A modeling of biospray for the upper airways. *ESAIM:Proc*, 14:41–47, 2005.
[5] C. Baranger and L. Desvillettes. Coupling Euler and Vlasov equations in the context of sprays: local smooth solutions. *Journal of Hyperbolic Differential Equations*, 3(1):1–26, 2006.
[6] R. Becker and W. Döring. Kinetische behandlung der keimbildung in übersättigten dämpfern. *Ann. Phys.*, 24:719–752, 1935.
[7] L. Boudin, B. Boutin, B. Fornet, T. Goudon, P. Lafitte, F. Lagoutière, and B. Merlet. Fluid-particles flows: A thin spray model with energy exchanges. *ESAIM: Proc.*, 28:195–210, 2009.
[8] L. Boudin, L. Desvillettes, C. Grandmont, and A. Moussa. Global existence of solutions for the coupled Vlasov and Navier-Stokes equations. *Differential and Integral Equations*, 22(11–12), 2009.
[9] L. Boudin, L. Desvillettes, C. Grandmont, and A. Moussa. Global existence of solutions for the coupled Vlasov and Navier-Stokes equations. *Differential Integral Equations*, 22(11-12):1247–1271, 2009.
[10] M. J. Caceres, J. A. Carrillo, and J. Dolbeault. Nonlinear stability in L^p for a confined system of charged particles. *SIAM J. Math. Anal.*, 34:478–494, 2002.
[11] C. Calgaro, E. Creusé, and T. Goudon. Simulation of mixture flows: Pollution spreading and avalanches. Technical report, INRIA, 2011-12.
[12] J.-A. Canizo, L. Desvillettes, and K. Fellner. Absence of gelation for models of coagulation-fragmentation with degenerate diffusion. *Nuovo Cimento C*, 33(1):79–86, 2010.
[13] J. Carr. Asymptotic behaviour of solutions to the coagulation-fragmentation equations. I. The strong fragmentation case. *Proc. Roy. Soc. Edinburgh Sect. A*, 121(3-4):231–244, 1992.
[14] J. Carr and F. P. da Costa. Asymptotic behavior of solutions to the coagulation-fragmentation equations. II. Weak fragmentation. *J. Statist. Phys.*, 77(1-2):89–123, 1994.
[15] J.-A. Carrillo, R. Duan, and A. Moussa. Global classical solutions close to equilibrium to the vlasov-euler-fokker-planck system. *AIMS–Kinetic and Related Models*, 4:227–258, 2011.
[16] J. A. Carrillo and T. Goudon. Stability and asymptotics analysis of a fluid-particles interaction model. *Comm. PDE*, 31:1349–1379, 2006.
[17] J. A Carrillo, T. Goudon, and P. Lafitte. Simulation of fluid & particles flows: Asymptotic preserving schemes for bubbling and flowing regimes. *J. Comput. Phys.*, 227(16):7929–7951, 2008.
[18] J. A. Carrillo, A. Jüngel, P. A. Markowich, G. Toscani, and A. Unterreiter. Entropy dissipation methods for degenerate parabolic systems and generalized Sobolev inequalities. *Monatshefte für Mathematik*, 133:1–82, 2001.
[19] J.-A. Carrillo and K. Karper, T. abd Trivisa. On the dynamics of a uid-particle interaction model: The bubbling regime. *Nonlinear Analysis TMA*, 74(8):2778–2801, 2011.
[20] J.-F. Collet. Some modelling issues in the theory of fragmentation-coagulation systems. *Commun. Math. Sci.*, 2(suppl. 1):35–54, 2004.
[21] M. De Luca. *Contribution à la modélisation de la pulvérisation dun liquide phytosanitaire en vue de réduire les pollutions*. PhD thesis, Univ. Aix–Marseille 2, 2007.
[22] J. Dolbeault and G. Rein. Time-dependent rescalings and Lyapunov functionals for the Vlasov-Poisson and Euler-Poisson systems, and for related models of kinetic equations, fluid dynamics and quantum physics. *Math. Models Methods Appl. Sci.*, 11:407–432, 2001.
[23] T Elperin, N. Kleeorin, M. A Liberman, V. S L'vov, A. Pomyalov, and I Rogachevskii. Clustering of fuel droplets and quality of spray in Diesel engines. 2003. `http://arxiv.org/nlin.CD/0305017v1`.
[24] M. Escobedo, S. Mischler, and B. Perthame. Gelation in coagulation and fragmentation models. *Comm. Math. Phys.*, 231:157–188, 2002.
[25] S. K. Friedlander. *Smoke, Dust and Haze*. Wiley, 1977.
[26] T. Goudon, P.-E. Jabin, and A. Vasseur. Hydrodynamic limit for the Vlasov-Navier-Stokes equations. I. Light particles regime. *Indiana Univ. Math. J.*, 53(6):1495–1515, 2004.
[27] T. Goudon, P.-E. Jabin, and A. Vasseur. Hydrodynamic limit for the Vlasov-Navier-Stokes equations. II. Fine particles regime. *Indiana Univ. Math. J.*, 53(6):1517–1536, 2004.
[28] T. Goudon, S. Jin, and B. Yan. Simulation of fluid–particles flows: Heavy particles, flowing regime and AP–

schemes. *Comm. Math. Sci.*, 10(1):355–385, 2012.
[29] T. Goudon, P. Lafitte, and M. Rousset. Modeling and simulation of fluid-particles flows. In T.-T Li, Y.-J. Peng, and B. Rao, editors, *Some Problems on Nonlinear Hyperbolic Equations and Applications, The French-Chinese Summer Institute on Applied Mathematics held at Fudan University, Shanghai, September 1-21, 2008*, volume 15 of *Series in Contemporary Applied Mathematics, CAM*. World Scientific, 2010.
[30] T. Goudon, J.-G. Liu, S. Jin, and B. Yan. Asymptotic-preserving schemes for kinetic–fluid modeling of disperse two-phase flows. Technical report, INRIA, 2011.
[31] T. Goudon, A. Moussa, L. He, and P. Zhang. The Navier–Stokes–Vlasov–Fokker–Planck system near equilibrium. *SIAM J. Math. Anal.*, 42(5):2177–2202, 2010.
[32] T. Goudon and F. Poupaud. On the modeling of the transport of particles in turbulent flows. *Math. Modelling and Numerical Analysis (M2AN)*, 38:673–690, 2004.
[33] K. Hamdache. Global existence and large time behaviour of solutions for the Vlasov-Stokes equations. *Japan J. Indust. Appl. Math.*, 15:51–74, 1998.
[34] P.-E. Jabin and B. Niethammer. On the rate of convergence to equilibrium in the Becker–Döring equations. *J. Differential Equations*, 191:518–543, 2003.
[35] P. Laurençot and S. Mischler. Convergence to equilibrium for the continuous coagulation-fragmentation equation. *Bull. Sci. Math.*, 127(3):179–190, 2003.
[36] H. Liu, Z. Wang, and R. Fox. A level set approach for dilute non-collisional fluid-particle flows. *J. Comput. Phys.*, 230:920–936, 2011.
[37] J. Mathiaud. *Etude de systèmes de type gaz-particules*. PhD thesis, ENS Cachan, 2006.
[38] A. Mellet and A. Vasseur. Global weak solutions for a Vlasov-Fokker-Planck/Navier-Stokes system of equations. *Math. Mod. Meth. Appl. Sci*, 17(7):1039–1063, 2007.
[39] A. Mellet and A. Vasseur. Asymptotic analysis for a Vlasov–Fokker–Planck/compressible Navier–Stokes system of equations. *Comm. Math. Phys.*, 281(3):573–596, 2008.
[40] A. Moussa. *Etude mathématique et numérique du transport d'aérosols dans le poumon humain*. PhD thesis, ENS Cachan, 2009.
[41] J. R. Norris. Smoluchowskis coagulation equation: uniqueness, non-uniqueness and hydrodynamic limit for the stochastic coalescent. *Ann. Appl. Probab.*, 9:78–109, 1999.
[42] N. A. Patankar and D. D. Joseph. Lagrangian numerical simulation of particulate flows. *Int. J. Multiphase Flow*, 27:1685–1706, 2001.
[43] N. A. Patankar and D. D. Joseph. Modeling and numerical simulation of particulate flows by the Eulerian–Lagrangian approach. *Int. J. Multiphase Flow*, 27:1659–1684, 2001.
[44] A. Prosperetti and G. Tryggvason. *Computational methods for multiphase flows*. Cambridge University Press, 2007.
[45] G. Rein. Non-linear stability of gaseous stars. *Arch. Rat. Mech. Anal.*, 168:115–130, 2003.
[46] L. Saint-Raymond. *Hydrodynamic Limits of the Boltzmann Equation*, volume 1971 of *Lecture Notes in Math.* Springer, 2009.
[47] M. Slemrod. Trend to equilibrium in the Becker–Döring cluster equation. *Nonlinearity*, 2:429–443, 1989.
[48] B. Sportisse. *Modélisation et simulation de la pollution atmosphérique*. PhD thesis, Université Pierre et Marie Curie, 2007. Habilitation à Diriger les Recherches, Sciences de l'Univers.
[49] I. Vinkovic. *Dispersion et mélange turbulents de particules solides et de gouttelettes par une simulation des grandes échelles et une modélisation stochastique lagrangienne. Application à la pollution de l'atmosphère*. PhD thesis, Ecole Centrale de Lyon, 2005.
[50] F. A. Williams. *Combustion theory*. Benjamin Cummings Publ., 1985. (2nd edition).