DIFFUSION LIMIT FOR NON HOMOGENEOUS AND NON-MICRO-REVERSIBLE PROCESSES [1]

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Abstract

We study the (parabolic) hydrodynamic limit of linear kinetic equations. We deal with collision terms that do not satisfy detailed balance pinciple and where the coefficients depend on the space variable. The possible dependence of the equilibrium upon position induces an additional drift term in the limit equation.

Key words: Boltzmann equation, Diffusion approximation, non-detailed balance property, transport-diffusion equation, moment method, Hilbert expansion.

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1 Introduction

We are interested in the behaviour of the solutions $f_\varepsilon(t, x, v)$ of the following kinetic equation

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$$\begin{cases} \partial_t f_\varepsilon + \dfrac{1}{\varepsilon} a(v) \cdot \nabla_x f_\varepsilon = \dfrac{1}{\varepsilon^2} L(f_\varepsilon) \ \ in \ \mathbb{R}_t^+ \times \mathbb{R}_x^N \times V_v, \\ f_{\varepsilon,|t=0} = f_{\varepsilon,0} \ \ in \ \mathbb{R}_x^N \times V_v, \end{cases} \tag{1}$$

as the small parameter ε goes to 0. Such problems arise in various physical situations such as the description of the evolution of plasmas, semi-conductors, rarefied gases, fluxes of neutrons or photons... The unknown $f_\varepsilon(t, x, v)$ is interpreted as a distribution function of particles (which may be, depending on the context, electrons, holes, gas molecules, ...). The variable v represents all the degrees of freedom of the particles (translational velocity as well as internal degrees of freedom of molecules for instance). It takes its value in a certain measured space V (possibly $\mathbb{R}^N$ itself), equipped with a (σ-finite) measure $d\mu(v)$ in such a way that the integral

$$\int_\Omega f(t, x, v)\, d\mu(v),$$

is the density of the particles whose parameter v lies in the (measurable) domain $\Omega \subset V$. The function $a : V \to \mathbb{R}^N$ is the kinetic velocity of particles. The variables t, x are the time and position variables as usual. The right-hand side in (1) is intended to describe interaction phenomena. Here, we deal with a linear situation where the operator L reads

$$\begin{cases} L(f) = K(f) - \Sigma(x, v) f, \\ K(f) = \displaystyle\int_V \sigma(x, v, w) f(w)\, d\mu(w). \end{cases}$$

In this definition the functions σ and Σ, defined on $\mathbb{R}^N \times V \times V$ and $\mathbb{R}^N \times V$ respectively, are nonnegative. It is rather natural to assume that L is conservative, which means, forgetting the question of integrability for the time being, that

$$\int_V L(f)\, d\mu(v) = 0 \tag{2}$$

holds. This leads to the following relation between the kernel σ and the collision frequency Σ

$$\int_V \sigma(x, w, v)\, d\mu(w) = \Sigma(x, v).$$

The small parameter $\varepsilon > 0$ arising in (1) is related to the particle mean free path. When it tends to 0, the penalization of the collision term forces the solution f_ε to be an equilibrium state, i.e. to belong to the null space of L, which thus has to be precised. On the other hand, the speed of propagation $a(v)/\varepsilon$ tends to ∞ and, in the limit $\varepsilon \to 0$, one expects to obtain a diffusion equation for the macroscopic quantities such as the density

$$\rho(t,x) = \int_V f(t,x,v)\, d\mu(v)\,.$$

The above modeling corresponds to many different physical situation. The classical framework is given by $V = \mathbb{R}^N$, $d\mu$ the Lebesgue measure and $a(v) = v$. But we can also consider relativistic particles $a(v) = v/\sqrt{1+v^2}$ (in convenient units) or even semi-classical transport phenomena like in semiconductors physics. In this last case v is a wave vector, V is a torus (the Brillouin zone) equipped with its Haar measure $d\mu$ and $a(v)$ is the gradient of the so called band diagramm. One also encounters, in neutronic applications, cases where V is a union of coronae in $\mathbb{R}^N$ and $d\mu(v)$ can be a weighted Lebesgue measure see [8]. An other framework of applications of our theory consists in discrete velocities models. Then V is a set of indices and $d\mu$ is a discrete measure. The set of discrete velocities is given by $\{a(v),\ v \in V\}$.

Concerning the collision operator, a classical example is given by the linear (semiconductor) Boltzmann equation where

$$\begin{cases} \Sigma(x,v) = \displaystyle\int_V b(x,v,w) M(w)\, d\mu(w), \\ K(f)(v) = \displaystyle\int_V b(x,w,v) M(v) f(w)\, d\mu(w), \end{cases} \tag{3}$$

with $M(v) = \exp(-v^2/2)$ and b is symmetric with respect to v and w. We refer to R. Petterson [19] or F. Poupaud [20] and references therein for details on the problem (1), (3). Note however that the symmetry of b yields the constraint $L(M) = 0$, also referred to as the global balance condition. Actually, in previous works, it is always assumed a more stringent condition, referred to as the detailed balance condition, or microreversibility condition, which reads, for all v, w in V

$$b(v,w)M(w)M(v) = b(w,v)M(v)M(w) = \sigma(v,w)M(w) = \sigma(w,v)M(v).$$

Note that, even if b depends on x, the normalized equilibrium state M does not. The limit problem associated to (1), (3) consists of a simple diffusion equation for ρ (with a diagonal diffusion matrix D if one assumes properties of invariance by rotations for b) as shown in [20].

The approximation of kinetic equations by diffusion equations has been investigated for a long time, starting with the pioneering works of E. Wigner [24], A. Bensoussan-J.-L. Lions-G. Papanicolaou [7] and E. Larsen-J. Keller [16]. A recent review of results and methods related to this problem can be found in the notes of F. Golse [11]. Results include certain non linear collision terms, as for the justification of the Rosseland approximation by C. Bardos-F. Golse-B. Perthame [3], C. Bardos-F. Golse-B. Perthame-R. Sentis [4] or the study of Pauli's semiconductors equations by F. Golse-F. Poupaud [14] as well as discrete velocity models in P. L. Lions-G. Toscani [17] or combination to homogeneization effects by G. Allaire-G. Bal [1], G. Allaire-Y. Capdeboscq [2] and T. Goudon-F. Poupaud [15]. For applications to semi-conductors and analysis of boundary-layers we refer to N. Ben Abdallah-P. Degond-S-Génieys [6], F. Poupaud [20], N. Ben Abdallah-P. Degond [5], P. Degond-C. Schmeiser [9].

Our aim in this paper is to deal with the linear operator (2), allowing space dependence and without requiring a detailed balance relation. Let us now set up some notations and assumptions before we give the statement of our main results.

2 Preliminaries and main result

As mentioned above, our first task is a discussion of the properties of the collisions operator L. When no confusion can arise, we drop the $x-$dependence in our notation. On the differential cross section, we assume:

$$\text{(A1)} \qquad \begin{cases} \sigma(x,v,w) \text{ is } dx \otimes d\mu(v) \otimes d\mu(w) \text{ measurable and positive,} \\ \Sigma(x,v) = \displaystyle\int_V \sigma(x,w,v)\, d\mu(w) < +\infty, \; dx \otimes d\mu(v) \text{ a.e.} \end{cases}$$

where 'a.e.' means 'almost everywhere'. This assumption is very general. Notice that it implies that $\Sigma > 0$ a.e., and, as mentioned in the introduction, that the operator L satisfies the conservation property (2). Our second assumption concerns the existence of a (non a priori unique) equilibrium:

$$\text{(A2)} \quad \begin{cases} \text{There exists an almost everywhere positive measurable function } F \text{ and a} \\ \text{positive constant } M \text{ such that} \\ \Sigma(x,v)\ F(x,v) = \displaystyle\int_V \sigma(x,v,w)\ F(x,w)\, d\mu(w), \quad dx \otimes d\mu(v) \text{ a.e.,} \\ \displaystyle\int_V \left(\frac{1}{\Sigma(x,v)} + \Sigma(x,v) \right)\ F(x,v)\, d\mu(v) \le M, \quad \int_V F(x,v)\, d\mu(v) = 1, \; dx \text{ a.e.} \end{cases}$$

Remark that the condition that F has an integral equal to 1 is only a normalization condition. The assumption (A2) is for instance fulfilled if we can apply the Krein Rutmann theorem. In particular, in the appendix below, we shall see that a sufficient condition is

$$\int_V \sup_{w\in V} t(v,w)\, d\mu(v) < \infty,$$

where $t(v,w)$ is defined by

$$t(v,w) \;=\; \frac{1+\Sigma(v)}{1+\Sigma(w)} \frac{\sigma(v,w)}{\Sigma(v)}\,.$$

However we will show that a compactness property of the collision operator is not necessary to obtain a coercivity property of the operator. To prove the closedness of the image of the collision operator we need a more technical assumption. We assume:

$$\text{(A3)} \quad \begin{cases} \text{There exists a positive constant } \kappa \text{ such that} \\ F(x,v) \le \kappa \left(\Sigma(x,w) + \dfrac{1}{\Sigma(x,w)} \right) \dfrac{1}{\Sigma(x,v)}\, \sigma(x,v,w), \quad dx\otimes d\mu(v) \otimes d\mu(w) \text{ a.e.} \end{cases}$$

The other set of assumptions we need is related with the diffusion limit. The equilibrium function F should have a vanishing mean velocity and in some sense the collision term has to control the drift term. We are thus led to assume that there exists two positive constants C_1 and C_2 such that

$$\text{(B1)} \qquad |a(v)|\,|\nabla_x F(x,v)| \le C_1\, \Sigma(x,v)\, F(x,v), \quad dx \otimes d\mu(v) \text{ a.e.},$$

$$\text{(B2)} \qquad \int_V |a(v)|^2 \frac{F(x,v)}{\Sigma(x,v)}\, d\mu(v) \le C_2, \quad dx \text{ a.e.},$$

$$\text{(B3)} \qquad \int_V a(v)\, F(x,v)\, d\mu(v) = 0, \quad dx \text{ a.e.}$$

We point out that $a(v)F(x,v)$ is integrable for x a.e. because of (A2) and (B2), so that (B3) makes sense. Indeed by Cauchy Schwarz inequality we get

$$\int_V |a(v)|\, F(x,v)\, d\mu(v) \le \left(\int_V |a(v)|^2 \frac{F(x,v)}{\Sigma(x,v)}\, d\mu(v) \right)^{1/2} \left(\int_V F(x,v)\; \Sigma(x,v)\, d\mu(v) \right)^{1/2}.$$

Finally, the last hypothesis we need is useful to obtain compactness of the concentration of particles. It is a geometrical assumption on the velocities, weaker than the usual one that is necessary to apply averaging lemma of [12]. This point was already remarked in the previous works [17, 15].

$$\text{(C)} \qquad \text{For any } \xi \in \mathbb{R}^N \backslash \{0\}, \ \mu(\{v \in V, \text{ such that } a(v) \cdot \xi \neq 0\}) > 0.$$

This assumption has to be compared to the following one which arises with averaging lemma methods

$$\text{For any } \xi \in \mathbb{R}^N \backslash \{0\}, \ \mu(\{v \in V, \text{ such that } a(v) \cdot \xi = 0\}) = 0. \tag{4}$$

Since $\mu(V) \neq 0$ the assumption (4) clearly implies (C). But the converse is not true. If we consider discrete velocities models, (C) means nothing but the fact that the subspace spanned by $\{a(v), \ v \in V\}$ coincides with the whole space $\mathbb{R}^N$ while the assumption (4) is never satisfied in this case: take a velocity v_0 such that $\mu(v_0) \neq 0$, then (4) does not hold for ξ orthogonal to $a(v_0)$. We mention however that asymptotic results, as the number of velocities grow, have been obtained by S. Mischler [18], with application to the proof of convergence of numerical schemes.

We already mentioned in the introduction that one of our main contribution is to remove the microreversibility condition, which would read:

$$\sigma(x, v, w) \, F(x, w) = \sigma(x, w, v) \, F(x, v).$$

It has to be compared with the integrated relationship (A2). In order to state our result on the collision operator in this context we introduce the space (where we skip the x-dependence)

$$H = \{f : V \to \mathbb{R}, \quad d\mu \ \text{ measurable such that } f^2 \, \frac{\Sigma}{F} \in L^1(d\mu)\}.$$

The space H is a Hilbert space equipped with the scalar product

$$(f, g)_H := \int_V f(v) \ g(v) \, \frac{\Sigma(v)}{F(v)} \, d\mu(v).$$

Remark that $F \in H$ and that we have the continuous embedding $H \subset L^1(d\mu)$ because of (A2). Indeed we have

$$\|F\|_H = \left(\int_V \Sigma(v) \ F(v) \, d\mu(v) \right)^{1/2}; \quad \int |f(v)| \, d\mu(v) \leq \|f\|_H \ \left(\int_V \frac{F(v)}{\Sigma(v)} \, d\mu(v) \right)^{1/2}.$$

Therefore it makes sense to introduce the following closed subspace

$$H_0 := \{f \in H \text{ such that } \int_V f(v)\, d\mu(v) = 0\}.$$

We are thus led to the following statement.

Proposition 1 *Assume (A1) and (A2). Then , the operators* $(\frac{1}{\Sigma})K$ *and* $(\frac{1}{\Sigma})L$ *are continuous operator on* H *(i.e. belong to* $\mathcal{L}(H)$*) and the nullspace of* L *(or* $(\frac{1}{\Sigma})L$*) has dimension* 1 *and is spanned by* F*. Moreover, the bilinear form*

$$B(f,g) := -\int_V L(f)\; g\, \frac{1}{F}\, d\mu$$

is non negative and continuous on $H \times H$ *and satisfies the following dissipative entropy inequality*

$$B(f,f) = \frac{1}{2}\int_V \int_V \left(\frac{f(v)}{F(v)} - \frac{f(w)}{F(w)}\right)^2 \sigma(v,w)\, F(w)\, d\mu(v)\, d\mu(w) \geq \frac{1}{2}\|(\frac{1}{\Sigma})L(f)\|_H^2 \geq 0. \quad (5)$$

With the technical assumption (A3) we can precise the preceding result and obtain a Fredholm alternative.

Proposition 2 *Assume (A1), (A2) and (A3). Let* $f \in H$ *and* $\rho_f := \int_V f(v)\, d\mu(v)$*, then we have the coercivity inequality*

$$B(f,f) \geq \frac{1}{2M\kappa}\, \|f - \rho_f F\|_H^2. \quad (6)$$

Moreover for any $h \in L^2(\frac{1}{\Sigma\, F}\, d\mu)$ *the problem to find* $f \in H$ *such that* $L(f) = h$ *has a solution if and only if* $\int_V h(v)\, d\mu(v) = 0$*. The solution is unique in* H_0 *and satisfies*

$$\|f\|_H \leq 2M\kappa\, \left(\int_V h^2(v) \frac{1}{\Sigma(v)\, F(v)}\, d\mu(v)\right)^{1/2}.$$

Note that, by (A2), $L^2(\frac{1}{\Sigma\, F}\, d\mu)$ embeds into $L^1(d\mu)$, so that the null integral condition on h makes sense. Thanks to this proposition and the assumptions (B) we can introduce the special functions $\chi \in [L^\infty(\mathbb{R}^N; L^2(\frac{\Sigma(x,v)}{F(x,v)}\, d\mu(v))]^N$ and $\varphi \in L^\infty(\mathbb{R}^N; L^2(\frac{\Sigma(x,v)}{F(x,v)}\, d\mu(v))$ which are the solutions of

$$L(\chi) = a(v)\, F(x,v), \qquad L(\varphi) = a(v) \cdot \nabla_x F(x,v). \quad (7)$$

(The right-hand sides belong to $L^\infty(\mathbb{R}^N; L^2(\frac{1}{\Sigma F}\,d\mu(v))$ with vanishing $v-$integral). These functions give the coefficients in the diffusive limit equation, namely let us set

$$\begin{cases} A(x) = \displaystyle\int_V a(v)\varphi(x,v)\,d\mu(v) \ \in [L^\infty(\mathbb{R}^N)]^N, \\ \\ D(x) = -\displaystyle\int_V a(v)\otimes\chi(x,v)d\mu(v) \ \in [L^\infty(\mathbb{R}^N)]^{N\times N}. \end{cases} \tag{8}$$

We will show that the diffusion matrix D has a positive definite symmetric part (but D is not necessarily symmetric). We can now state our main result.

THEOREM 1 *Assume (A), (B) and (C). Let $f_{\varepsilon,0}$ satisfy $\displaystyle\int_{\mathbb{R}^N\times V} f_{\varepsilon,0}^2/F\,d\mu(v)\,dx < \infty$ and $\rho_{\varepsilon,0} = \displaystyle\int_V f_{\varepsilon,0}\,d\mu(v) \rightharpoonup \rho_0$ in $H^{-1}_{loc}(\mathbb{R}^N)$. Let f_ε be the associated sequence of solutions of (1). Then, up to a subsequence, $\rho_\varepsilon(t,x) = \displaystyle\int_V f_\varepsilon(t,x,v)\,d\mu(v) \to \rho$, strongly in $L^2(0,T;L^2_{loc}(\mathbb{R}^N))$, where ρ satisfies the following drift-diffusion equation*

$$\begin{cases} \partial_t\rho + \mathrm{div}_x(A(x)\rho - D(x)\nabla_x\rho) = 0 \ \text{ in } \mathbb{R}^+\times\mathbb{R}^N, \\ \rho_{|t=0} = \rho_0 \ \text{ in } \mathbb{R}^N, \end{cases}$$

where the coefficients are defined by (7) and (8).

In the next section we focus on the properties of the collision operator and we prove Propositions 1 and 2. In Section 4, we guess the limit equation by inserting a formal ansatz in (1). Section 5 is devoted to the derivation of a priori estimates satisfied by the sequence of solutions f_ε. Finally, in Section 6, we pass to the limit $\varepsilon\to 0$ and we prove the main Theorem.

3 H-Theorem and Coercivity of the Collision Operator

In this section we omit the dependence with respect to x because it plays only the role of a parameter. Our first goal is to prove Proposition 1. We shall combine the conservation relation (A1) to the equilibrium relation

$$K(F) = \int_V \sigma(v,w)F(w)\,d\mu(w) = \Sigma(v)F(v), \tag{9}$$

to establish some dissipativity property for the operator L. We first compute $\|f\|_H^2$ in two ways. We use (A1) to obtain

$$\begin{aligned}\|f\|_H^2 &= \int_V \int_V \frac{f^2(v)}{F(v)} \sigma(w,v)\, d\mu(v)\; d\mu(w) \\ &= \int_V \int_V \frac{f^2(w)}{F(w)} \sigma(v,w)\, d\mu(v)\, d\mu(w). \end{aligned} \tag{10}$$

But (9) leads to

$$\Sigma(v) = \int_V \frac{F(w)}{F(v)} \sigma(v,w)\, d\mu(w).$$

Therefore we also have

$$\|f\|_H^2 = \int_V \int_V \frac{f^2(v)}{F^2(v)} \sigma(v,w) F(w)\, d\mu(v)\, d\mu(w). \tag{11}$$

For non negative f and g, we compute

$$\begin{aligned}\int_V K(f)(v)\, g(v)\, \frac{1}{F(v)}\, d\mu(v) &= \int_V \int_V \sigma(v,w) f(w)\, g(v)\, \frac{1}{F(v)}\, d\mu(v)\, d\mu(w) \\ &\le \left(\int_V \int_V \sigma(v,w) f^2(w) \frac{1}{F(w)}\, d\mu(v)\, d\mu(w) \right)^{1/2} \\ &\quad \times \left(\int_V \int_V \sigma(v,w) g^2(v) \frac{F(w)}{F^2(v)}\, d\mu(v)\, d\mu(w) \right)^{1/2}. \end{aligned}$$

In view of (10) and (11) it gives

$$\int_V K(f)(v)\, g(v)\, \frac{1}{F(v)}\, d\mu(v) \le \|f\|_H\, \|g\|_H.$$

Since $B(f,g) = (f,g)_H - \displaystyle\int_V K(f)(v)\, g(v)\, \frac{1}{F(v)}\, d\mu(v)$ it proves the continuity of B on $H \times H$. By choosing $g = \frac{1}{\Sigma} K(f)$, it also yields the continuity of the linear operator $\frac{1}{\Sigma} K$ with norm $\|\frac{1}{\Sigma} K\|_{\mathcal{L}(H)} \le 1$. However, (9) says that we actually have

$$\|\frac{1}{\Sigma} K\|_{\mathcal{L}(H)} = 1.$$

We also have the identity

$$\begin{aligned} B(f,f) = \|f\|_H^2 - (\frac{1}{\Sigma} K(f), f)_H &= \frac{1}{2} \|\frac{1}{\Sigma} L(f)\|_H^2 + \frac{1}{2} \left(\|f\|_H^2 - \|\frac{1}{\Sigma} K(f)\|_H^2 \right) \\ &\ge \frac{1}{2} \|\frac{1}{\Sigma} L(f)\|_H^2 \ge 0. \end{aligned}$$

In order to obtain the entropy dissipation term we combine the two equalities (10) and (11) to get

$$\|f\|_H^2 = \frac{1}{2}\int_V\int_V \left(\frac{f^2(v)}{F^2(v)} + \frac{f^2(w)}{F^2(w)}\right)\sigma(v,w)F(w)\,d\mu(v)\,d\mu(w).$$

But we have

$$(\frac{1}{\Sigma}K(f),f)_H = \int_V\int_V \frac{f(v)}{F(v)}\,\frac{f(w)}{F(w)}\sigma(v,w)F(w)\,d\mu(v)\,d\mu(w),$$

therefore we finally obtain

$$\begin{aligned} B(f,f) &= \|f\|_H^2 - (\frac{1}{\Sigma}K(f),f)_H \\ &= \frac{1}{2}\int_V\int_V \left(\frac{f(v)}{F(v)} - \frac{f(w)}{F(w)}\right)^2 \sigma(v,w)F(w)\,d\mu(v)\,d\mu(w) \end{aligned}$$

which is the desired formula. In particular, if $L(f)=0$ then, by the definition, $B(f,f)=0$ and, because of (A1) (σ is positive), the previous formula leads to

$$\frac{f(v)}{F(v)} = \frac{f(w)}{F(w)}\quad d\mu(v)\otimes d\mu(w) \quad \text{a.e.}$$

It gives the result about the nullspace of L and ends the proof of Proposition 1. □

Remark 1 *If the following microreversibility relation*

$$\sigma(v,w)F(w) = \sigma(w,v)F(v)$$

holds, similar arguments yield

$$\begin{aligned} &\int_V L(f)\phi(f/F)\,d\mu(v) \\ &= -1/2\int_V\int_V \sigma(v,w)F(w)((f/F)(w)-(f/F)(v))(\Phi(w)-\Phi(v))\,d\mu(w)\,d\mu(v) \le 0, \end{aligned}$$

where $\Phi(v) = \phi((f/F)(v))$, $\phi : \mathbb{R}^+ \longmapsto \mathbb{R}$ *being non decreasing and bounded.*

Remark 2 *The mass conservation identity reads*

$$\forall f\in H, \qquad \int_V L(f)\,d\mu(v) = (\frac{1}{\Sigma}\,L(f),F)_H = 0.$$

Therefore the range of $\frac{1}{\Sigma}L$ *is orthogonal in* H *to* $Vect(F)$: $R(\frac{1}{\Sigma}L) \subset Vect(F)^\perp$. *Conversely if* $g \in R(\frac{1}{\Sigma}L)^\perp$ *we have*

$$B(g,g) = -(\frac{1}{\Sigma}L(g), g)_H = 0 \geq \frac{1}{2}\|\frac{1}{\Sigma}L(g)\|_H^2.$$

Then $R(\frac{1}{\Sigma}L)^\perp \subset N(\frac{1}{\Sigma}L) = Vect(F)$. *We conclude that* $\overline{R(\frac{1}{\Sigma}L)} = Vect(F)^\perp$. *The additional assumption (A3) allows us to prove that the range is closed.*

Inequality (5) can be precised with the following coercivity result, which requires the bound from below in (A3). We set $W(v,w) = (f/F)(w) - (f/F)(v)$. Then, we have $f(v)F(w) = f(w)F(v) - F(v)F(w)W(v,w)$ and integration with respect to w gives

$$f(v) - \rho_f F(v) = -F(v)\int_V F(w)W(v,w)\,d\mu(w).$$

The right hand side is estimated by using Cauchy-Schwarz's inequality and (A2) as follows

$$\begin{aligned}\left|F(v)\int_V F(w)W(v,w)\,d\mu(w)\right|^2 &\leq \int_V (\Sigma(w) + \frac{1}{\Sigma(w)})\,F(w)\,d\mu(w) \\ &\quad \times \int_V W^2(v,w)F^2(v)\frac{\Sigma(w)}{1+\Sigma^2(w)}\,F(w)\,d\mu(w) \\ &\leq M\int_V W^2(v,w)F^2(v)\frac{\Sigma(w)}{1+\Sigma^2(w)}\,F(w)\,d\mu(w).\end{aligned}$$

Then the norm of $f(v) - \rho_f F(v)$ can be estimated as follows

$$\|f - \rho_f F\|_H^2 \leq M\int_V\int_V W^2(v,w)F(v)\,\frac{\Sigma(v)\Sigma(w)}{1+\Sigma^2(w)}\,F(w)\,d\mu(w)\,d\mu(v).$$

After using (A3), we recognize (5) and it leads to (6). To solve the equation $\frac{1}{\Sigma}L(f) = \frac{1}{\Sigma}h$ we have already seen in Remark 2 that a necessary condition is $\frac{1}{\Sigma}h \in Vect(F)^\perp$, which is exactly the null average condition in Proposition 2. The equation is equivalent to the variational formulation

$$\forall g \in H, \qquad B(f,g) = (\frac{1}{\Sigma}h, g)_H.$$

But this is automatically satisfied for $g = F$. Then it is only necessary to satisfy the variational equality for H_0 since $H = H_0 + Vect(F)$. On H_0 the bilinear form B is coercive because of (6). Therefore by using the Lax Milgram theorem, there is a unique solution in H_0 which can be estimated with the inverse of the coercivity constant. This completes the proof of Proposition 2. □

4 Formal asymptotic

As usual, we can guess the limit problem by inserting a formal expansion of the solution in equation (1) as follows

$$f_\varepsilon = f^{(0)} + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \ldots$$

and, then, identifying the terms having the same power of ε. We get

$$1/\varepsilon^2 \text{ term:} \qquad L(f^{(0)}) = 0,$$

which yields $f^{(0)}(t,x,v) = \rho(t,x)F(x,v)$. Next, we have

$$1/\varepsilon \text{ term:} \qquad L(f^{(1)}) = a(v)\cdot\nabla_x f^{(0)} = \mathrm{div}_x(\rho a(v)\ F) = \mathrm{div}_x(a(v)\ F)\rho + a(v)F\cdot\nabla_x\rho. \tag{12}$$

By (B3), the $v-$integral of the right hand side vanishes, so that we are led to

$$f^{(1)} = f_0^{(1)} + \alpha(t,x)F,$$

where $f_0^{(1)}$ is the solution of (12) in H_0 and $\alpha : \mathbb{R}^+ \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is unknown. Finally, the ε^0 term is

$$L(f^{(2)}) = \partial_t f^{(0)} + a(v)\cdot\nabla_x f^{(1)},$$

and integration with respect to v yields

$$\partial_t\rho + \mathrm{div}_x\left(\int_V a(v)f^{(1)}\,d\mu(v)\right) = 0 = \partial_t\rho + \mathrm{div}_x\left(\int_V a(v)f_0^{(1)}\,d\mu(v)\right), \tag{13}$$

by (2) and (B3). Hence, we realize that it is not necessary to determine α. Let $\chi = (\chi_1,...,\chi_N)$ and φ the functions solutions of (7). The formula

$$f_0^{(1)}(t,x,v) = \varphi(x,v)\rho(t,x) + \chi(x,v)\cdot\nabla_x\rho(t,x)\,,$$

provides a solution of (12). It remains to compute

$$\begin{aligned}\int_V a(v)f_0^{(1)}\,d\mu(v) &= \left(\int_V a(v)\varphi\,d\mu(v)\right)\rho + \left(\int_V a(v)\otimes\chi\,d\mu(v)\right).\nabla_x\rho \\ &= A(x)\rho(t,x) - D(x).\nabla_x\rho(t,x),\end{aligned}$$

where $A(x)\in\mathbb{R}^N$ and $D(x)\in\mathcal{M}^{N\times N}$. Finally, (13) becomes

$$\partial_t\rho + \mathrm{div}_x(A(x)\rho(t,x) - D(x).\nabla_x\rho(t,x)) = 0. \tag{14}$$

Lemma 1 *Let (A), (B) and (C) hold. The symmetric part of the matrix*

$$D(x) = -\int_V a(v) \otimes \chi(x,v)\, d\mu(v)\,,$$

is positive definite. Precisely, for any $\xi \in \mathbb{R}^N \backslash \{0\}$*, one has*

$$0 < D(x)\xi \cdot \xi \leq 2\kappa M C_2 \,|\xi|^2\,,$$

while

$$|A(x)| = \left| \int_V a(v)\, \varphi(x,v)\, d\mu(v) \right| \leq 2M\kappa\sqrt{C_1 C_2}.$$

Proof. Let $\xi \in \mathbb{R}^N$. Using Proposition 1 we have

$$D(x)\xi\cdot\xi = -(\frac{1}{\Sigma}\, a(v)\cdot\xi\, F, \chi\cdot\xi)_H = B(\chi\cdot\xi, \chi\cdot\xi) \geq \|\frac{1}{\Sigma}\, a(v)\cdot\xi\, F\|_H^2 = \int_V (a(v)\cdot\xi)^2\, \frac{F}{\Sigma}\, d\mu(v).$$

The assumption (C) together with the positivity of F and Σ implies this quantity is positive. To obtain the bound above we use the last estimate of the Proposition 2. We have

$$\|\chi\cdot\xi\|_H \leq 2M\kappa \|\frac{1}{\Sigma}\, a(v)\cdot\xi\ F\|_H \leq 2M\kappa\sqrt{C_2}\,|\xi|\,,$$

where we use (B2). Therefore

$$D(x)\xi\cdot\xi = -(\frac{1}{\Sigma}\, a(v)\cdot\xi\ F, \chi\cdot\xi)_H \leq \|\frac{1}{\Sigma}\, a(v)\cdot\xi\ F\|_H \|\chi\cdot\xi\|_H \leq 2M\kappa C_2\,|\xi|^2.$$

The estimate on $A(x)$ is obtained in the same way.

In most of the applications it can actually be shown that D is uniformly coercive, which in turn leads easily to uniqueness for the limit problem; in such a case, obviously, the convergence stated in Theorem 1 applies to the whole sequence.

5 A priori Estimates

In this section, we shall use the coercivity of the collision operator to derive some estimates satisfied by solutions f_ε of (1). We shall not detail the existence theory for (1). Instead, we refer for instance to Petterson's works [19]. Note that in Section 3 we only used the assumptions (A); now it is also necessary to set up a control on the behaviour of the equilibrium F with respect to the space variable (see (B1), (B2)). We also require the flux condition (B3) to hold true. Hence, from now on, we assume that the assumptions (B) are fulfilled. We will have to use many weighted L^2 spaces. Therefore we introduce the following notation for the sake of clarity. The space $L^p(\Omega, d\nu)$ denotes the space of function whose p-th power is integrable on Ω for the measure $d\nu$. When no measure is specified, it means that we consider the Lebesgue measure. We recall the Cauchy problem which reads

$$\begin{cases} \partial_t f_\varepsilon + \dfrac{1}{\varepsilon} a(v) \cdot \nabla_x f_\varepsilon = \dfrac{1}{\varepsilon^2} L(f_\varepsilon) \;\; in\ \mathbb{R}_t^+ \times \mathbb{R}_x^N \times V_v, \\ f_{\varepsilon,|t=0} = f_{\varepsilon,0} \;\; in\ \mathbb{R}_x^N \times V_v, \end{cases}$$

We point out that there is no topology on the space V. Therefore the above equation has to be understood as follows. It holds in $\mathcal{D}'((0,+\infty)\times\mathbb{R}^N)$ for $v \in V$ μ-a.e. The solution f_ε belongs to $\mathcal{C}([0,\infty); L^1(\mathbb{R}^N \times V,\ d\mu(v)\,dx))$ and the Cauchy data equation is an equality in $L^1(\mathbb{R}^N \times V,\ d\mu\,dx)$. The solution is non negative if $f_{\varepsilon,0}$ is non negative and it also belongs to the weighted spaces $L^\infty((0,T); L^2(\mathbb{R}^N \times V, \frac{1}{F}\,d\mu\,dx))$ and $L^2((0,T); L^2(\mathbb{R}^N \times V, \frac{\Sigma}{F}\,d\mu\,dx))$.

We now derive formally the uniform estimates which we need for the asymptotics. The first one expresses the mass conservation. Integrating (1) and using (2) yields

$$\int_{\mathbb{R}^N}\int_V f_\varepsilon(t,x.v)\,d\mu(v)\,dx = \int_{\mathbb{R}^N}\int_V f_{\varepsilon,0}(x.v)\,d\mu(v)\,dx. \tag{15}$$

To obtain the second one we multiply (1) by f_ε/F and we integrate. It leads to

$$\begin{aligned} 1/2\,\frac{d}{dt}\int_{\mathbb{R}^N}\int_V f_\varepsilon^2/F\,d\mu(v)\,dx \;&+\; 1/\varepsilon\int_{\mathbb{R}^N}\int_V a(v)\cdot\nabla_x f_\varepsilon\, f_\varepsilon/F\,d\mu(v)\,dx \\ &-\; 1/\varepsilon^2\int_{\mathbb{R}^N}\int_V L(f_\varepsilon)\,f_\varepsilon/F\,d\mu(v)\,dx = 0. \end{aligned} \tag{16}$$

Let us expand f_ε as follows

$$\begin{cases} f_\varepsilon(t,x,v) = \rho_\varepsilon(t,x)F(x,v) + \varepsilon g_\varepsilon(t,x,v), \\ \rho_\varepsilon(t,x) = \displaystyle\int_V f_\varepsilon(t,x,v)\,d\mu(v), \qquad \int_V g_\varepsilon(t,x,v)\,d\mu(v) = 0. \end{cases} \tag{17}$$

Then, Proposition 2 gives

$$-1/\varepsilon^2 \int_{\mathbb{R}^N}\int_V L(f_\varepsilon)\, f_\varepsilon/F\, d\mu(v)\, dx = \int_{\mathbb{R}^N} B(g_\varepsilon, g_\varepsilon)\, dx \geq \frac{1}{2M\kappa}\int_{\mathbb{R}^N}\int_V g_\varepsilon^2\, \Sigma/F\, d\mu(v)\, dx, \tag{18}$$

by using (2). We this in mind, we treat the second integral in (16):

$$\begin{aligned}
1/\varepsilon \int_{\mathbb{R}^N}\int_V a(v)\cdot\nabla_x f_\varepsilon\, f_\varepsilon/F\, d\mu(v)\, dx &= 1/\varepsilon \int_{\mathbb{R}^N}\int_V \mathrm{div}_x\left(a(v)\,\frac{f_\varepsilon^2}{2F}\right) d\mu(v)\, dx \\
&\quad +1/(2\varepsilon)\int_{\mathbb{R}^N}\int_V a(v)\cdot\nabla_x F\,\frac{f_\varepsilon^2}{F^2}\, d\mu(v)\, dx \\
&= 0 + 1/(2\varepsilon)\int_{\mathbb{R}^N}\int_V a(v)\cdot\nabla_x F(\rho_\varepsilon^2 + 2\varepsilon\rho_\varepsilon\frac{g_\varepsilon}{F} + \varepsilon^2\frac{g_\varepsilon^2}{F^2})\, d\mu(v)\, dx \\
&= 1/(2\varepsilon)\int_{\mathbb{R}^N}\rho_\varepsilon^2 \mathrm{div}_x\left(\int_V a(v)\ F\, d\mu(v)\right) dx + \int_{\mathbb{R}^N}\int_V \rho_\varepsilon\ g_\varepsilon\,\frac{a(v)\cdot\nabla_x F}{F}\, d\mu(v)\, dx \\
&\quad +\varepsilon/2\int_{\mathbb{R}^N}\int_V g_\varepsilon^2\,\frac{a(v)\cdot\nabla_x F}{F^2}\, d\mu(v)\, dx \\
&= 0 + \int_{\mathbb{R}^N}\rho_\varepsilon\int_V g_\varepsilon\,\frac{a(v)\cdot\nabla_x F}{F}\, d\mu(v)\ dx + \varepsilon/2\int_{\mathbb{R}^N}\int_V g_\varepsilon^2\,\frac{a(v)\cdot\nabla_x F}{F^2}\, d\mu(v)\, dx\,,
\end{aligned}$$

where we used the decomposition (17) and the flux hypothesis (B3). By (A2) and (B1), we get

$$\begin{aligned}
1/\varepsilon &\left|\int_{\mathbb{R}^N}\int_V a(v)\cdot\nabla_x f_\varepsilon\, f_\varepsilon/F\, d\mu(v)\, dx\right| \\
&\leq C_1\int_{\mathbb{R}^N}\rho_\varepsilon\int_V g_\varepsilon\,\Sigma\, d\mu(v)\, dx + C_1\varepsilon/2\int_{\mathbb{R}^N}\int_V g_\varepsilon^2\,\frac{\Sigma}{F}\, d\mu(v)\, dx \\
&\leq \frac{C_1}{4\nu}\int_{\mathbb{R}^N}\int_V \rho_\varepsilon^2\,\Sigma\ F\, d\mu(v)\, dx + C_1(\nu+\varepsilon/2)\int_{\mathbb{R}^N}\int_V g_\varepsilon^2\,\frac{\Sigma}{F}\, d\mu(v)\, dx \\
&\leq \frac{C_1 M}{4\nu}\int_{\mathbb{R}^N}\rho_\varepsilon^2\, dx + C_1(\nu+\varepsilon/2)\int_{\mathbb{R}^N}\int_V g_\varepsilon^2\,\frac{\Sigma}{F}\, d\mu(v)\, dx\,,
\end{aligned}$$

with $\nu > 0$ to be determined later. Using Cauchy Schwarz inequality we have

$$0 \leq \rho_\varepsilon \leq \left(\int_V \frac{f_\varepsilon^2}{F}\, d\mu(v)\right)^{1/2}. \tag{19}$$

Then the preceding inequality becomes

$$\begin{aligned}
1/\varepsilon &\left|\int_{\mathbb{R}^N}\int_V a(v)\cdot\nabla_x f_\varepsilon\, f_\varepsilon/F\, d\mu(v)\, dx\right| \\
&\leq \frac{C_1 M}{4\nu}\int_{\mathbb{R}^N}\int_V \frac{f_\varepsilon^2}{F}\, d\mu(v)\, dx + C_1(\nu+\varepsilon/2)\int_{\mathbb{R}^N}\int_V g_\varepsilon^2\,\frac{\Sigma}{F}\, d\mu(v)\, dx\,.
\end{aligned} \tag{20}$$

By (18) and (20) and time-integration, (16) becomes

$$1/2 \int_{\mathbb{R}^N} \int_V f_\varepsilon^2(t)/F \, d\mu(v)\, dx + (\frac{1}{2M\kappa} - (\nu + \varepsilon/2)C_1) \int_0^t \int_{\mathbb{R}^N} \int_V g_\varepsilon^2(s) \, \frac{\Sigma}{F} \, d\mu(v)\, dx\, ds$$
$$\leq 1/2 \int_{\mathbb{R}^N} \int_V f_\varepsilon^2(0)/F \, d\mu(v)\, dx + C_1 M/(4\nu) \int_0^t \int_{\mathbb{R}^N} \int_V f_\varepsilon^2(s)/F \, d\mu(v)\, dx\, ds.$$

Thus, we choose $\nu > 0$ so that (for instance) $\frac{1}{2M\kappa} - \nu C_1 \geq \frac{1}{4M\kappa}$; and, then, we are concerned with values of ε in a small interval $0 < \varepsilon \leq \varepsilon_0$, so that $\frac{1}{2M\kappa} - (\nu + \varepsilon/2)C_1 \geq \frac{1}{8M\kappa}$. It follows that $I_\varepsilon(t) = 1/2 \int_{\mathbb{R}^N} \int_V f_\varepsilon^2(t,x,v)/F(x,v)\, d\mu(v)\, dx$ satisfies

$$I_\varepsilon(t) + \frac{1}{8M\kappa} \int_0^t \int_{\mathbb{R}^N} \int_V g_\varepsilon^2(s,x,v) \frac{\Sigma(x,v)}{F(x,v)} \, d\mu(v)\, dx\, ds \leq I_\varepsilon(0) + C_0 \int_0^t I_\varepsilon(s)\, ds,$$

where $C_0 > 0$ only depends on C_1 and $M\kappa$. By applying Gronwall's lemma, one deduces that

$$I_\varepsilon(t) + \frac{1}{8M\kappa} \int_0^t \int_{\mathbb{R}^N} \int_V g_\varepsilon^2(s,x,v) \, \frac{\Sigma(x,v)}{F(x,v)} \, d\mu(v)\, dx\, ds \leq C(T) I_\varepsilon(0)\,, \tag{21}$$

holds for $t \in (0,T)$ and $\varepsilon \in (0, \varepsilon_0)$, $C(T)$ depending on C_1, $M\kappa$ and $0 \leq T < \infty$. This leads to the following statement.

Proposition 3 *Assume (A) and (B). Let $f_{\varepsilon,0}$ be the initial data for (1) which satisfies*

$$\sup_\varepsilon \int_{\mathbb{R}^N} \int_V f_{\varepsilon,0}^2/F \, d\mu(v)\, dx = I_0 < \infty.$$

Then, there exists $\varepsilon_0 > 0$ such that, for $0 \leq T < \infty$ and $0 < \varepsilon < \varepsilon_0$,

i) the sequence f_ε is bounded in $L^\infty(0,T; L^2(\mathbb{R}^N \times V, 1/F \, d\mu(v)\, dx))$,

ii) the sequence g_ε is bounded in $L^2((0,T) \times \mathbb{R}^N \times V, \Sigma/F \, d\mu(v)\, dx\, dt)$,

iii) the sequence ρ_ε is bounded in $L^\infty(0,T; L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N))$,

iv) The current $j_\varepsilon(t,x) = 1/\varepsilon \int_V a(v) f_\varepsilon \, d\mu(v)$ is bounded in $L^2((0,T) \times \mathbb{R}^N)$.

Proof. The first estimates $i)$ and $ii)$ follow from (21) while the bounds on ρ_ε are direct consequences of (15), (19) and $i)$. Finally, we estimate the current by using (B2) and

(B3),

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^N} j_\varepsilon^2 \, dx\, dt &= \int_0^T \int_{\mathbb{R}^N} \left| \int_V \frac{1}{\varepsilon} \; a(v) \; f_\varepsilon \, d\mu(v) \right|^2 \, dx\, dt \\
&= \int_0^T \int_{\mathbb{R}^N} \left| \int_V a(v) g_\varepsilon \, d\mu(v) \right|^2 \, dx\, dt \\
&\leq \int_0^T \int_{\mathbb{R}^N} \left(\int_V a(v)^2 \frac{F}{\Sigma} \, d\mu(v) \right) \left(\int_V g_\varepsilon^2 \, \frac{\Sigma}{F} \, d\mu(v) \right) \, dx\, dt \\
&\leq C_2 \int_0^T \int_{\mathbb{R}^N} \int_V g_\varepsilon^2 \, \frac{\Sigma}{F} \, d\mu(v)\, dx\, dt \leq C_2 C(T) I_0,
\end{aligned}$$

which achieves the proof. □

6 Convergence to Drift-Diffusion Equation

This Section is devoted to the proof of the convergence result: up to a subsequence, ρ_ε converges to ρ satisfying the drift-diffusion equation (14) formally obtained in Section 4. Following the remark at the beginning of Section 5, the solution f_ε belongs to $C^0([0, +\infty[; L^1(\mathbb{R}^N \times V, \, d\mu(v)\, dx))$, with $L(f_\varepsilon) \in L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}^N; L^1(V \, d\mu(v)\, dx))$ and the equation (1) holds $\mu-$a.e. in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^N)$. This means that, for all $\phi \in L^\infty(V)$,

$$\partial_t \left(\int_V f_\varepsilon \phi \, d\mu(v) \right) + \frac{1}{\varepsilon} \mathrm{div}_x \left(\int_V a(v) \; f_\varepsilon \phi \, d\mu(v) \right) = \frac{1}{\varepsilon^2} \int_V L(f_\varepsilon) \phi \, d\mu(v),$$

holds in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^N)$. Still by using the decomposition $f_\varepsilon(t, x, v) = \rho_\varepsilon(t, x) F(x, v) + \varepsilon \; g_\varepsilon(t, x, v)$, this relation can be rewritten, for all $\zeta \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$ and $\phi \in L^\infty(V)$,

$$\begin{aligned}
&\varepsilon \left(\int_D f_\varepsilon \phi \partial_t \zeta \, d\mu(v)\, dx\, dt + \int_D a(v) g_\varepsilon \phi \cdot \nabla_x \zeta \, d\mu(v)\, dx\, dt \right) \\
&+ \int_D a(v) \rho_\varepsilon F \phi \cdot \nabla_x \zeta \, d\mu(v)\, dx\, dt + \int_D L(g_\varepsilon) \phi \zeta \, d\mu(v)\, dx\, dt = 0,
\end{aligned} \tag{22}$$

where we set $D = (0, T) \times \mathbb{R}^N \times V$. We wish to pass to the limit as $\varepsilon \to 0$ in (22). It is also worth writing immediately the following mass conservation equation

$$\partial_t \rho_\varepsilon + \mathrm{div}_x j_\varepsilon = 0, \tag{23}$$

which holds in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^N)$, obtained by using the test function $\phi(v) = 1$. The remainder of the proof is decomposed in several steps.

Step 1: Mass conservation, density and current
By Proposition 3, we can assume, possibly at the cost of extracting subsequences, that

$$\rho_\varepsilon \rightharpoonup \rho \text{ in } L^\infty((0,T);L^2(\mathbb{R}^N)) \text{ weak-}*, \tag{24}$$

$$j_\varepsilon \rightharpoonup j \text{ in } L^2((0,T)\times\mathbb{R}^N) \text{ weak}, \tag{25}$$

With (24) and (25), we can pass to the limit in (23); we get

$$\partial_t\rho + \mathrm{div}_x j = 0. \tag{26}$$

We can relate these limits with the behaviour of the kinetic quantities f_ε and g_ε as $\varepsilon \to 0$. Indeed, Proposition 3 also allows us to suppose that

$$g_\varepsilon \rightharpoonup g \text{ in } L^2(D, \Sigma/F\, d\mu(v)\, dx\, dt) \text{ weak-} * . \tag{27}$$

The convergence (27) means that

$$\lim_{\varepsilon\to 0}\int_D g_\varepsilon\psi\, d\mu(v)\, dx\, dt = \int_D g\psi\, d\mu(v)\, dx\, dt\,, \tag{28}$$

provided the test function ψ satisfies

$$\int_D \psi^2\,\frac{F}{\Sigma}\, d\mu(v)\, dx\, dt < \infty. \tag{29}$$

In particular, notice that

$$\begin{cases} \psi(t,x,v) = \phi(v)\zeta(t,x), & \phi\in L^\infty(V), \zeta\in L^2((0,T)\times\mathbb{R}^N), \\ \psi(t,x,v) = a(v)\phi(v)\zeta(t,x), & \phi\in L^\infty(V), \zeta\in L^2((0,T)\times\mathbb{R}^N), \end{cases} \tag{30}$$

are admissible test functions by (A2) and (B2), respectively.

Then, by (A2), $f_\varepsilon = \rho_\varepsilon F + \varepsilon g_\varepsilon$ appears as the sum of two sequences of the space $L^2(D, \Sigma/F\, d\mu(v)\, dx\, dt)$, the second one converging strongly to 0. Since, by (A2), $\int_V F\psi\, d\mu(v)$ belongs to $L^2((0,T)\times\mathbb{R}^N)$ for ψ satisfying (29), (24) yields $f_\varepsilon \rightharpoonup f = \rho F$ weakly-$*$ in $L^2(D, \Sigma/F\, d\mu(v)\, dx\, dt)$. Moreover, using (28) with $\psi = 1\zeta(t,x)$, we obtain

$$\rho_\varepsilon = \int_V f_\varepsilon\, d\mu(v) \rightharpoonup \rho = \int_V f\, d\mu(v) \text{ weakly in } L^2((0,T)\times\mathbb{R}^N).$$

Similarly, for the current, we use (28) and (30) with $\psi = a(v)\zeta(t,x)$. Therefore, the limit current j is clearly related to g by

$$j(t,x) = \int_V a(v) g(t,x,v)\, d\mu(v). \tag{31}$$

Then, we will obtain a diffusion equation for ρ by relating g to ρ when $\varepsilon \to 0$ in (22).

Step 2: Passing to the limit in the kinetic equation
Let us go back to the kinetic equation (22). In view of the L^1 bound on f_ε, the integral involving time derivative in (22) vanishes in the limit $\varepsilon \to 0$. Furthermore, as mentioned in (30), $a(v)\phi(v) \cdot \nabla_x \zeta(t,x)$ fulfills (29) for $\phi \in L^\infty(V)$ and $\zeta \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$, so that the second term in (22) also tends to 0. We thus have

$$\lim_{\varepsilon \to 0} \left(- \int_V L(g_\varepsilon) \phi \zeta \, d\mu(v)\, dx\, dt \right) = \lim_{\varepsilon \to 0} \int_V \rho_\varepsilon a(v) F \phi \cdot \nabla_x \zeta \, d\mu(v)\, dx\, dt.$$

The left hand side is nothing but

$$\int_0^T \int_{\mathbb{R}^N} B(g_\varepsilon, \phi\zeta F)\, dx\, dt = \widetilde{B}(g_\varepsilon, \phi\zeta F)\,,$$

where, in view of Proposition 1, $\widetilde{B}$ defines a continuous bilinear map on $\left[L^2(D, \frac{\Sigma}{F}\, d\mu(v)\, dx\, dt)\right]^2$. Therefore we can pass to the limit with (27) and we obtain

$$\lim_{\varepsilon \to 0} \int_D L(g_\varepsilon) \phi \zeta \, d\mu(v)\, dx\, dt = \int_D L(g) \phi \zeta \, d\mu(v)\, dx\, dt.$$

Next, the right hand side reads

$$\int_0^T \int_{\mathbb{R}^N} \rho_\varepsilon(t,x) \left(\int_V a(v) F(x,v) \phi(v)\, d\mu(v) \right) \cdot \nabla_x \zeta(t,x)\, dx\, dt,$$

where, by (A2) and (B2), the integral with respect to v defines a $L^\infty((0,T) \times \mathbb{R}^N)$ function. It suffices to apply (24), and we are led to

$$\int_D L(g) \phi \zeta \, d\mu(v)\, dx\, dt + \int_D \rho a(v) F \phi \cdot \nabla_x \zeta \, d\mu(v)\, dx\, dt = 0. \tag{32}$$

Step 3: Regularity of ρ

Let us set

$$\eta(v) = \begin{cases} \dfrac{a(v)}{|a(v)|} & \text{on} \quad \Omega = \{v \in V, a(v) \neq 0\}, \\ 0 & \text{on} \quad \complement\Omega. \end{cases}$$

Note that $\eta(v)$ is an admissible test function. This choice yields

$$\int_V L(g)\eta(v)\zeta \, d\mu(v) \, dx \, dt + \int_V \theta(x)\rho\nabla_x\zeta \, dx \, dt = 0, \tag{33}$$

where θ stands for the following (symmetric) matrix

$$\theta(x) = \int_V \frac{a(v) \otimes a(v)}{|a(v)|} F(x,v) \, d\mu(v).$$

By (A2), (B2) and (B1), θ and $D_x\theta$ belong to bounded sets in $\left[L^\infty(\mathbb{R}^N)\right]^{N\times N}$ and $\left[L^\infty(\mathbb{R}^N)\right]^{N\times N\times N}$, respectively. Furthermore, (C) implies that $\theta(x)$ is definite positive. By continuity, it follows that for any compact $K \subset \mathbb{R}^N$, there exists $\alpha_K > 0$ such that, for all $x \in K$,

$$\theta(x) \geq \alpha_K I.$$

Then, (33) leads to (denoting $\mathrm{Div}_x(A_{ij}(x)) = \sum_{j=1}^N \partial_{x_j} A_{ij}(x)$)

$$|\langle \mathrm{Div}_x(\theta\rho), \zeta\rangle_{\mathcal{D}',\mathcal{D}}| = \left|\widetilde{B}(g, \eta(v)F\zeta)\right| \leq \sqrt{M}\|\zeta\|_{L^2((0,T)\times\mathbb{R}^N)}\|g\|_{L^2(D,\Sigma/F\,d\mu(v)\,dx\,dt)},$$

by using (A2) and the continuity of $\widetilde{B}$. This proves that $\mathrm{Div}_x(\theta\rho) = \rho\mathrm{Div}_x(\theta) + \theta\nabla_x\rho$ lies in $L^2((0,T)\times\mathbb{R}^N)$. The regularity properties of θ discussed above allow us to conclude that $\nabla_x\rho \in L^2_{loc}((0,T)\times\mathbb{R}^N)$. Moreover, (32) becomes

$$\int_D L(g)\phi\zeta \, d\mu(v) \, dx \, dt - \int_D \mathrm{div}_x(a(v)F\rho)\phi\zeta \, d\mu(v) \, dx \, dt = 0.$$

Since this relation holds for all ϕ, ζ, we obtain finally the following pointwise relation

$$L(g) = \mathrm{div}_x(a(v)F\rho) = a(v)F \cdot \nabla_x\rho + \rho a(v) \cdot \nabla_x F,$$

$d\mu(v) \otimes dx \otimes dt$-a.e.

Step 4: Limit equation
We check that $a(v)F$ and $a(v)\cdot\nabla_x F$ define $L^\infty((0,T)\times\mathbb{R}^N; L^2(V, \frac{1}{\Sigma F}\, d\mu(v)))$ functions by (A2) and (B1) and their average on V vanish by (B3). Therefore we can apply Proposition 2 to define $\chi = (\chi_1, ..., \chi_N)$ and φ, solution of (7). Estimates in Proposition 2 show that these functions belong to $L^\infty((0,T)\times\mathbb{R}^N; L^2(V, \frac{\Sigma}{F}\, d\mu(v)))$. Furthermore, taking $\psi(t,x,v) = \zeta(t,x)$ in (28) gives

$$\int_0^T \int_{\mathbb{R}^N} \zeta \left(\int_V g_\varepsilon \, d\mu(v) \right) dx\, dt = 0 \to \int_0^T \int_{\mathbb{R}^N} \zeta \left(\int_V g \, d\mu(v) \right) dx\, dt,$$

as $\varepsilon \to 0$, which proves that g has a null $V-$average. Uniqueness in Proposition 2 yields

$$g(t,x,v) = \chi(x,v)\cdot\nabla_x\rho(t,x) + \varphi(x,v)\rho(t,x).$$

It follows that (31) becomes

$$\begin{cases} j(t,x) = A\rho - D\nabla_x\rho, \\ A(x) = \displaystyle\int_V a(v)\varphi(x,v)\, d\mu(v), \qquad D(x) = -\int_V a(v)\otimes\chi(x,v)\, d\mu(v). \end{cases}$$

Inserting this into (26) finally leads to the announced drift-diffusion equation

$$\partial_t\rho + \mathrm{div}_x(A\rho - D\nabla_x\rho) = 0.$$

Notice also that estimates in Proposition 3 combined to (23) and a application of Ascoli's theorem, ensures that ρ_ε lies in a compact set of $C^0([0,T]; H^{-1}_{loc}(\mathbb{R}^N))$ which garantees that the initial data for the limit problem is ρ_0, the weak limit of $\rho_{\varepsilon,0} = \int_V f_{\varepsilon,0}\, d\mu(v)$.

Step 5: Strong convergence
The proof of the strong convergence of ρ_ε relies on a compensated-compactness argument introduced by Lions-Toscani in [17] and also used in [15]. This argument avoids the use of the averaging lemma [13], [12]. Indeed, with $\phi(v) = \eta(v)$, as defined in Step 3 above, in (22), we get the following relation in $\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^N)$

$$\begin{aligned} \theta(x)\nabla_x\rho_\varepsilon \;\; = \;\; & -\mathrm{Div}_x(\theta(x))\rho_\varepsilon + \int_V \eta(v)\, L(g_\varepsilon)\, d\mu(v) \\ & -\varepsilon\left(\partial_t \left[\int_V \eta(v)\, f_\varepsilon\, d\mu(v) \right] + \mathrm{Div}_x \left[\int_V \frac{a(v)\otimes a(v)}{|a(v)|}\, g_\varepsilon\, d\mu(v) \right] \right). \end{aligned} \tag{34}$$

By using the bounds in Proposition 3 and Rellich's theorem it can be checked that the right hand side in (34) lies in a compact set of $H^{-1}_{loc}((0,T)\times\mathbb{R}^N)$. Since θ is invertible and $\nabla_x\theta$, θ^{-1} are locally bounded one deduces that $\nabla_x\rho_\varepsilon$ is (strongly) compact in $H^{-1}_{loc}((0,T)\times\mathbb{R}^N)$. Let us introduce the following vector fields (having $N+1$ components)

$$U_\varepsilon = (\rho_\varepsilon, j_\varepsilon), \qquad V_\varepsilon = (\rho_\varepsilon, 0, \cdots, 0)\,,$$

which satisfy

$$\begin{aligned}
&\mathrm{div}_{t,x} U_\varepsilon = \partial_t \rho_\varepsilon + \mathrm{div}_x j_\varepsilon = 0 \in \text{Compact set of } H^{-1}_{loc},\\
&\mathrm{curl}_{t,x} V_\varepsilon = \begin{pmatrix} 0 & -{}^t\nabla_x\rho_\varepsilon \\ \nabla_x\rho_\varepsilon & 0 \end{pmatrix} \in \text{Compact set of } (H^{-1}_{loc})^{(N+1)\times(N+1)}.
\end{aligned}$$

It suffices now to apply the div-curl lemma of L. Tartar [22], [23] to obtain

$$U_\varepsilon \cdot V_\varepsilon = \rho_\varepsilon^2 \to \begin{pmatrix} \rho \\ j \end{pmatrix} \cdot \begin{pmatrix} \rho \\ 0 \end{pmatrix} = \rho^2 \ in\ \mathcal{D}'((0,T)\times\mathbb{R}^N)$$

which finally gives, with the bound on ρ_ε in $L^\infty(0,T;L^2(\mathbb{R}^N))$, the srong convergence $\rho_\varepsilon \longrightarrow \rho$ in $L^2(0,T;L^2_{loc}(\mathbb{R}^N))$. □

A Existence of equilibrium

We search for an integrable function F satisfying $F > 0$, $\mu - a.e.$ and $K(F) = \Sigma F$. Furthermore, in order to exploit the conservation property, we require that $\Sigma F \in L^1(V)$. Hence, we should have $K(F) = \Sigma F \in L^1(V)$ and $\frac{1}{\Sigma}K(F) = F \in L^1(V)$. Thus, K has to be interpreted as on operator from $L^1((1+\Sigma)\,d\mu(v))$ into $L^1((1+1/\Sigma)\,d\mu(v))$. To avoid the use of these weight functions, let us set $G = (1+\Sigma)F$ so that the relation $K(F) = \Sigma F$ becomes

$$G = T(G)$$

where the operator T is defined by

$$T(G)(v) = \int_V \frac{1+\Sigma(v)}{1+\Sigma(w)}\frac{\sigma(v,w)}{\Sigma(v)}\,G(w)\,d\mu(w) := \int_V t(v,w)\,G(w)\,d\mu(w)$$

and should act from $L^1(V)$ into itself.

Assume that T has an eigenvalue $\lambda > 0$ associated to a positive eigenfunction $G > 0$ $\mu - a.e.$. Then, integrating the corresponding relation $F = G/(1+\Sigma) > 0$ yields

$$(\lambda - 1)\int_V \Sigma F \, d\mu(v) = 0,$$

which forces $\lambda = 1$ since $\Sigma F > 0$ a.e.

Hence, it suffices to prove that T has a positive eigenvalue associated to a positive eigenfunction. Such a result follows from applying the Krein-Rutman theorem. Referring for instance to [21] (see Theorem V.6.6) since the kernel σ is positive $d\mu(v) \otimes d\mu(w)$ a.e., it remains to show that T^p is compact for some power $p \in \mathbb{N}$. We can refer to Eveson's work [10] for a characterization of compactness for such an integral operator. Note however that the simple condition

$$\int_V \sup_{w \in V} t(v, w) \, d\mu(v) < \infty$$

implies $T \in \mathcal{L}(L^1(V))$ with T^2 compact.

Notice that Krein-Rutman's criterion also gives the uniqueness of a normalized positive eigenfunction F which is much more than necessary: in Sections 2-3 existence of a normalized equilibrium was sufficient to prove Proposition 2.

Let us also mention that a simple example of our results is given by the linear Boltzmann equation (3) where the Maxwellian M is defined with a space-dependent temperature $\mathcal{T}(x)$. Precisely, we have

$$\begin{cases} N = 3, \qquad V = \mathbb{R}^3, \qquad a(v) = v, \\ M(x, v) = (2\pi \mathcal{T}(x))^{-3/2} \exp\left(-\dfrac{v^2}{2\mathcal{T}(x)}\right). \end{cases}$$

Then, roughly speaking our set of assumptions means that $\mathcal{T} \in W^{1,\infty}(\mathbb{R}^N)$, with $\mathcal{T}(x) \geq \mathcal{T}_0 > 0$ and the collision frequency Σ is larger than $|v|(1+v^2)$.

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