Multidimensional high-field limit of the electrostatic Vlasov-Poisson-Fokker-Planck system

Thierry GOUDON[*], Juanjo NIETO[†], Frédéric POUPAUD[‡], Juan SOLER[§]

April 19, 2008

Abstract

In this paper, the high-field limit of the Vlasov-Poisson-Fokker-Planck system for charged particles is rigorously derived. The first result is obtained in any space dimension by using modulated energy techniques. It requires the smoothness of the solutions of the limit problem. In dimension 2, it is possible to handle more general data by using methods developed for a diagonal defect measures theory. The convergence of the concentration of particles is obtained in the space of bounded measures. In both cases, the limit of the sequence of densities of distribution functions is shown to solve a non linear system of partial differential equations which is related to Ohm's law.

Keywords. High-field limit. Vlasov-Poisson-Fokker-Planck. Modulated Energy. Relative entropy. Diagonal defect measures.

AMS. 35Q99. 35B25

[*]Labo. Paul Painlevé, UMR 8524 CNRS-Université des Sciences et Technologies de Lille, Cité Scientifique, F-59655 Villeneuve d'Ascq cedex, France E-mail: thierry.goudon@math.univ-lille1.fr

[†]Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain E-mail: jjmnieto@ugr.es

[‡]Labo. J. A. Dieudionné CNRS-Université Nice-Sophia Antipolis, Parc Valrose, F-06108 Nice cedex 02, France E-mail: poupaud@math.unice.fr

[§]Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain E-mail: jsoler@ugr.es

1 Introduction

1.1 The physical model

We consider a plasma with one species of particles, electrons for instance, whose evolution is described by its distribution function f. The particles interact with a thermal bath and with themselves through the Coulomb interaction. The distribution function f solves the Vlasov-Poisson-Fokker-Planck (VPFP) system

$$\partial_t f + v \cdot \nabla_x f - \frac{q}{m_e} \nabla \Phi \cdot \nabla_v f = L_{FP}[f] \tag{1}$$

$$:= \frac{1}{\tau} \operatorname{div}_v (vf + \theta_e \nabla_v f), \qquad (x, v) \in \mathbb{R}^N \times \mathbb{R}^N, \ t \geq 0, \tag{2}$$

$$-\Delta_x \Phi = \frac{q}{\epsilon_0}(n - D), \qquad n(t, x) = \int_{\mathbb{R}^N} f \, \mathrm{d}v. \tag{3}$$

The function $D = D(x)$ is the density of a background of positive charges and is assumed to be given. It can be for instance the doping profile in the framework of semiconductors theory. The density of electrons, n, is coupled with the potential Φ via the Poisson equation (3). The space variable is denoted by $x \in \mathbb{R}^N$ and $v \in \mathbb{R}^N$ is the velocity. The physical constants involved in the problem are:

- ϵ_0 the vacuum permittivity,
- q the elementary charge of the electrons,
- m_e the mass of the electrons,
- τ_e the relaxation time due to collisions of the particles with the thermal bath,
- k_B the Boltzmann constant,
- T_{th} the temperature of the thermal bath,
- $\sqrt{\theta_e} = \sqrt{\frac{k_B T_{th}}{m_e}}$ the thermal velocity.

After a dimensionless analysis (see below in this Section) the system can be written in terms of a small parameter, ϵ, consisting in the square of the quotient between the mean free path and the Debye length. Thus, the system (1)-(3) takes the dimensionless form

$$\varepsilon \Big(\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon \Big) - \nabla_x \Phi_\varepsilon \cdot \nabla_v f_\varepsilon = \operatorname{div}_v (v f_\varepsilon + \nabla_v f_\varepsilon), \tag{4}$$

$$-\Delta_x \Phi_\varepsilon = (n_\varepsilon - D_\varepsilon), \qquad n_\varepsilon(t, x) = \int_{\mathbb{R}^N} f_\varepsilon \, \mathrm{d}v. \tag{5}$$

1.2 Formal analysis

We can rewrite the rescaled kinetic equation in (4) as follows

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon} \operatorname{div}_v \Big(e^{-\frac{|v+\nabla\Phi_\varepsilon|^2}{2}} \nabla_v (f_\varepsilon e^{\frac{|v+\nabla\Phi_\varepsilon|^2}{2}}) \Big).$$

From this formula, it is tempting to guess that, as ε tends to 0,

$$f_\varepsilon \simeq n(t,x) \; \frac{1}{(2\pi)^{N/2}} \; e^{-\frac{|v+\nabla\Phi|^2}{2}}, \tag{6}$$

coupled to the Poisson equation

$$-\Delta_x \Phi = n - D. \tag{7}$$

Then, the question is to determine the evolution equation satisfied by the macroscopic density $n(t,x)$. By using (6), we get for the macroscopic current

$$j_\varepsilon(t,x) = \int_{\mathbb{R}^N} v \; f_\varepsilon(t,x,v) \, \mathrm{d}v \simeq -n(t,x) \, \nabla_x \Phi.$$

On the other hand, integration of the kinetic equation with respect to the variable v leads to the conservation equation

$$\partial_t n_\varepsilon + \operatorname{div}_x j_\varepsilon = 0. \tag{8}$$

Then, the formal asymptotics (6) leads to the following limiting system

$$\partial_t n - \operatorname{div}_x (n \nabla_x \Phi) = 0, \qquad -\Delta_x \Phi = n - D. \tag{9}$$

We remark that the relationship $j = -n\nabla\Phi$ is the infinitesimal version in dimensionless form of the well–known Ohm's law which tells that the current is proportional to the electric field.

1.3 Mathematical results

In (4)-(5) the nonlinear term is of the same order of magnitude that the diffusion Fokker-Planck term, which is due to the hypothesis that the mean free path of the electrons is much smaller than the Debye length. In this case, the limit procedure $\varepsilon \to 0$ is called high-field limit to distinguish it from the other possibility in which the diffusion dominates the behavior, i.e. the low-field limit (see [18]). High-field asymptotics has been first introduced in the context of semiconductors kinetic theory in [19]. In this last paper the linear Fokker-Planck operator $f \to \operatorname{div}_v(vf + \nabla_v f)$ in (4) is replaced by a linear Boltzmann operator. When the electric field $-\nabla\Phi$ is assumed to be given, the limit equation is rigorously derived. The corresponding boundary conditions have been discussed in [10, 11]. Also numerical simulations and comparisons with other models have been performed in [9, 10]. Various extension of these results for different physical models including quantum transport have been obtained in [1, 3, 13, 15]. In [2], a rigorous proof of the derivation of the high–field limit in a nonlinear context is performed. The field is not coupled via the Poisson equation but the Boltzmann operator is non linear due to the modeling of degeneracy effects.

When the electric field is coupled via the Poisson equation to the distribution function, the only rigorous result concerning the high–field limit has been obtained in [17], in the one-dimensional case, by using a cancelation of the singularity of the 1D Poisson kernel. The aim of this paper is to extend the analysis of the high-field limit to the multidimensional case.

In a first approach, by using modulated energy (relative entropy) methods (see [8]) we are able to pass to the limit in any space dimension. The drawback is that the solutions of the limit problem have to be assumed very smooth.

In dimension 2, we can improve this result by considering general weak solutions of the limit system. For that, we combine the cancelation properties of the singularities due to the antisymmetry of the Poisson kernel (see [14, 17, 21]) together with mathematical tools developed in the framework of a diagonal defect measure theory (see [20]).

In the following we consider only smooth solution of the VPFP system (4)-(5). We refer to [5, 6, 12] for the existence and uniqueness of strong solutions of the VPFP system. For the derivatives with respect to the space variables we use the notations

$$\nabla = \nabla_x = \Big(\frac{\partial}{\partial x_i}\Big)_{i=1,\dots,N}, \qquad \operatorname{div} = \operatorname{div}_x = \sum_{i=1,\dots,N} \frac{\partial}{\partial x_i},$$

$$\Delta = \Delta_x = \sum_{i=1,\dots,N} \frac{\partial^2}{(\partial x_i)^2}, \qquad \mathrm{D}^2 = \Big(\frac{\partial^2}{\partial x_i \partial x_j}\Big)_{i,j=1,\dots,N}.$$

For the Laplace equation $-\Delta\Phi(x) = \rho(x)$ together with appropriate conditions at infinity (see [4]), we consider solutions given by the convolution with the elementary solution E_N of the operator $-\Delta$:

$$-\nabla\Phi = -\nabla\Delta^{-1}\rho = \nabla E_N * \rho, \quad \text{with} \quad \nabla E_N(x) = \frac{1}{\omega_{N-1}} \frac{x}{|x|^N},$$

where ω_{N-1} is the measure of the sphere of dimension $N-1$.

Throughout the paper, we denote by $\mathcal{M}^1(\mathbb{R}^N)$ the set of bounded Radon measures on $\mathbb{R}^N$, while $\mathcal{M}^1_+(\mathbb{R}^N)$ stands for its positive cone. Also, as we will specify in Definition 1 below, the convergence in measure is said tightly if the dual functions are continuous and bounded.

Our main results are the followings

Theorem 1 *Let $n^I \in W^{1,1}(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$, $D \in W^{1,1}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$, $p > N$, and let n be the unique solution of the system (9) in $L^\infty\big(0,\infty; L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\big)$, with initial condition $n(t=0) = n^I$. Let f^I_ε be a sequence of smooth nonnegative distribution functions verifying*

$$\varepsilon \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (1 + |v|^2 + |x| + |\log f^I_\varepsilon|) f^I_\varepsilon \; \mathrm{d}v \; \mathrm{d}x \to 0, \qquad \int_{\mathbb{R}^N} |\nabla \Phi^I_\varepsilon - \nabla \Phi^I|^2 \, \mathrm{d}x \to 0,$$

with $-\Delta \Phi^I = n^I - D$ and $-\Delta \Phi^I_\varepsilon = \int_{\mathbb{R}^N} f^I_\varepsilon \; \mathrm{d}v - D_\varepsilon$. We also assume that the background charge density verifies $D_\varepsilon \geq 0$, $D_\varepsilon \to D$ in $L^2(\mathbb{R}^N)$ and that the global neutrality relation

$$\int_{\mathbb{R}^{2N}} f^I_\varepsilon(x,v) \, \mathrm{d}v \, \mathrm{d}x = \int_{\mathbb{R}^N} D_\varepsilon(x) \, \mathrm{d}x = M_\varepsilon \tag{10}$$

holds. Let f_ε be the strong solutions of the VPFP system (4)-(5) with initial data f^I_ε. Then, the corresponding field sequence verifies that $\nabla \Phi_\varepsilon$ converges to $\nabla \Phi$ in $L^\infty(0,\infty; L^2(\mathbb{R}^N))$, the density n_ε converges to n in $C^0([0,T], \mathcal{M}^1_+(\mathbb{R}^N)-tight)$ and the current $j_\varepsilon = \int_{\mathbb{R}^N} v \, f_\varepsilon \, \mathrm{d}v$ converges to $-n \, \nabla \Phi$ in $\mathcal{M}^1([0,T] \times \mathbb{R}^N)^N - tight$.

The previous result has been obtained under the assumption that the solution of the limit problem is smooth. In dimension two we can consider general weak solution of the limit problem:

Theorem 2 *Let f^I_ε be a sequence of smooth nonnegative functions. More precisely, we assume that the sequence of initial data satisfies*

$$f^I_\varepsilon \in W^{1,1}(\mathbb{R}^4), \qquad (1 + |v|^2 + |x|^2)^{\gamma/2}\big(|f^I_\varepsilon| + |\nabla_{x,v} f^I_\varepsilon| + |\nabla^2_{x,v} f^I_\varepsilon|\big) \in L^\infty(\mathbb{R}^4), \tag{11}$$

for some $\gamma > 3$, and that f_ε are the corresponding strong solutions of the VPFP system (4)-(5). We also assume that there is a positive constant $C > 0$, independent on ε such that

$$\int_{\mathbb{R}^4} f^I_\varepsilon(x,v) \, \mathrm{d}v \, \mathrm{d}x + \int_{\mathbb{R}^2} |\nabla \Phi^I_\varepsilon(x)|^2 \; \mathrm{d}x + \varepsilon \int_{\mathbb{R}^4} (|x|^2 + |v|^2) \; f^I_\varepsilon(x,v) \, \mathrm{d}v \, \mathrm{d}x \leq C \tag{12}$$

with $-\Delta\Phi^I_\varepsilon = \int_{\mathbb{R}^N} f^I_\varepsilon \,\mathrm{d}v - D_\varepsilon$*. Let the background charge densities* D_ε *be given functions which verify*

$$D_\varepsilon \geq 0, \quad (1+|x|)D_\varepsilon \in L^1(\mathbb{R}^2), \quad D_\varepsilon \to D \text{ in } L^1(\mathbb{R}^2), \quad \|D_\varepsilon\|_{L^p(\mathbb{R}^2)} \leq C, \tag{13}$$

for some constants $C > 0$ *and* $p > 2$*, independent on* ε*. We also assume that the global neutrality relation (10) holds. Then, up to a subsequence,* n_ε *converges in* $C^0([0,T], \mathcal{M}^1(\mathbb{R}^2)-$*tight) to* n*, a weak solution of (9) with initial data* $n^I = \lim_{\varepsilon\to 0} n_\varepsilon(0,\cdot)$ *(in the tight sense). For the current we have that* $j_\varepsilon = \int_{\mathbb{R}^N} v\, f_\varepsilon \,\mathrm{d}v$ *converges to* $-n\,\nabla\Phi$ *in* $\mathcal{M}^1([0,T]\times\mathbb{R}^N)^N-$*tight.*

Remark 1 *The above results can be easily extended to the case of time-relaxation interaction kernels (see for example [2, 3, 11, 13])*

$$L_{TR}[f] := \frac{1}{\tau}\int_{\mathbb{R}^N} \left(f(v)\, e^{\frac{|v'|^2}{2}} - e^{\frac{|v|^2}{2}}\, f(v') \right) \mathrm{d}v'$$

instead of using the Fokker-Planck operator L_{FP} *defined in (1), even in its quantum partner. Other more complex and realistic interaction kernels will be dealt with in a forthcoming paper of the authors.*

1.4 Dimensionless analysis

Let us give here the dimensionless analysis which allows to identify the small parameter of our problem. Let us denote by $\mathcal{N}$ a typical value for the concentration of the particles. Two important physical quantities which characterize the plasma are the following length scales:

$$\text{the mean free path:} \qquad \ell_e = \sqrt{\theta_e}\tau_e,$$

which is the average distance traveled by a particle between two successive collisions, and

$$\text{the Debye length:} \qquad \Lambda = \sqrt{\frac{\epsilon_0 k_B T_{th}}{q^2\mathcal{N}}},$$

which is the typical length of the perturbations of a quasi-neutral plasma. In this paper, we are interested in asymptotic regimes where the mean free path of the electrons is much smaller than the Debye length. More precisely, we assume:

$$\Lambda \gg \ell_e, \quad \text{which means} \quad \mathcal{N}{\tau_e}^2 \ll \frac{\epsilon_0 m_e}{q^2}$$

(this universal physical constant has the value 3.1×10^{-4} s^2/m^3, where the units s and m are seconds and meters, respectively). Then we set

$$\varepsilon = \left(\frac{\ell_e}{\Lambda}\right)^2,$$

which is a small parameter intended to tend to 0.

We choose now the macroscopic units of time, space and velocity as follows:

$$T = \frac{1}{\varepsilon}\tau_e, \quad L = \frac{1}{\varepsilon}\ell_e, \quad V = \sqrt{\theta_e}.$$

Let $U_{th} = \frac{k_B T_{th}}{q}$ be the thermal potential. Then, we use the following changes of variables and unknowns

$$\begin{cases} t = Tt', \qquad x = Lx', \qquad v = Vv', \qquad f(t,x,v) = \dfrac{\mathcal{N}}{V^3} f'_\varepsilon(t/T, x/L, v/V), \\ D(t,x) = \mathcal{N}\ D'_\varepsilon(t/T, x/L), \qquad \Phi(t,x) = \dfrac{U_{th}}{\varepsilon}\ \Phi'_\varepsilon(t/T, x/L), \end{cases}$$

where primed quantities are dimensionless. After some straightforward manipulations, we are finally led to the dimensionless VPFP system described by (4)-(5), where we drop the primes.

The paper is structured as follows. Section 2 is devoted to uniform estimates and compactness results which are needed in the next Section. The proof of Theorem 1 is given in Section 3 using a modulated energy method. Section 4 deals with the proof of Theorem 2.

2 Apriori estimates

The goal of this Section is to derive uniform estimates needed for the proofs of Theorems 1 and 2. The two dimensional case requires some specific arguments which will be detailed aside.

2.1 General case

We first remark that assumption (12) of Theorem 2 also holds true under the conditions of Theorem 1. We have

Proposition 1 *Let f_ε^I be a nonnegative initial data verifying (11). Let also D_ε satisfy (13) and the global neutrality condition (10). Then, the classical solution of the VPFP system (4)-(5) satisfies*

$$\int_{\mathbb{R}^{2N}} f_\varepsilon(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = M_\varepsilon, \tag{14}$$

for all $t \geq 0$. Assuming furthermore that the uniform estimate (12) holds, then there exists a constant C_T independent of ε such that the solutions of VPFP satisfy

$$\begin{cases} \sup_{t\in[0,T]} \displaystyle\int_{\mathbb{R}^N} |\nabla\Phi_\varepsilon(t,x)|^2\ \mathrm{d}x \leq C_T, \\ \displaystyle\int_0^T\int_{\mathbb{R}^{2N}} |v|^2\ f_\varepsilon(t,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \leq C_T, \\ \sup_{t\in[0,T]} \displaystyle\int_{\mathbb{R}^{2N}} |x|^2\ f_\varepsilon(t,x,v)\,\mathrm{d}v\,\mathrm{d}x \leq C_T, \end{cases} \tag{15}$$

for any $T > 0$.

These estimates come out by multiplying the system (4)-(5) by $|x|^2$ and $|v|^2$ and integrating by parts. These integrations by parts are justified because the distribution function is smooth and fastly decreasing at infinity. In dimension $N \geq 3$ the same properties are true for the potential. However, one has to notice that generally, in 2D, $\Delta^{-1}\rho \to \infty$ as $|x| \to \infty$ and $\nabla\Delta^{-1}\rho$ does not belong to $L^2(\mathbb{R}^2)$ when $\int \rho\,\mathrm{d}x \neq 0$ (see [22], Lemma 2.5). This remark motivates the global neutrality assumption (10).

Proof of Proposition 1. Let us set $\rho_\varepsilon = n_\varepsilon - D_\varepsilon$. The charge conservation relations (8) and (14) are derived in a standard way. Now, we want to derive the energy equation. Taking into account that $\Phi_\varepsilon = E_N * \rho_\varepsilon$ and $E(-x) = E(x)$ we have

$$\begin{aligned} \int_{\mathbb{R}^N} \Phi_\varepsilon(t,x)\ \partial_t\rho_\varepsilon(t,x)\,\mathrm{d}x &= \int_{\mathbb{R}^{2N}} E_N(x-y)\ \rho_\varepsilon(t,y)\ \partial_t\rho_\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2N}} E_N(x-y)\ \rho_\varepsilon(t,y)\ \rho_\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} \Phi_\varepsilon(t,x)\ \rho_\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

where we have used that ρ_ε is smooth with respect to time. We want to combine $\rho_\varepsilon = -\Delta\Phi_\varepsilon$ with an integration by parts. This is totally justified in dimension $N \geq 3$, but as we pointed out before, the case $N = 2$ must be studied independently (see Lemma 1 below). So in any dimension we get

$$\int_{\mathbb{R}^N} \Phi_\varepsilon(t,x)\ \partial_t\rho_\varepsilon(t,x)\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} \left|\nabla\Phi_\varepsilon(t,x)\right|^2\mathrm{d}x.$$

On the other hand, since $j_\varepsilon \Phi_\varepsilon$ vanishes at infinity, from (8) we get

$$\int_{\mathbb{R}^N} \Phi_\varepsilon(t,x)\ \partial_t \rho_\varepsilon(t,x)\,\mathrm{d}x = -\int_{\mathbb{R}^N} \Phi_\varepsilon(t,x)\ \nabla\cdot j_\varepsilon(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^N} j_\varepsilon(t,x)\cdot\nabla\Phi_\varepsilon(t,x)\ \mathrm{d}x.$$

Hence, we conclude that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} \left|\nabla\Phi_\varepsilon(t,x)\right|^2 \mathrm{d}x = \int_{\mathbb{R}^{2N}} v\ f_\varepsilon\cdot\nabla\Phi_\varepsilon(t,x)\,\mathrm{d}v\,\mathrm{d}x. \tag{16}$$

Now we multiply equation (4) by $|v|^2/2$ and integrate against x and v. We obtain

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2N}} \frac{|v|^2}{2}\ f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^{2N}} v\ f_\varepsilon\cdot\nabla\Phi_\varepsilon(t,x)\,\mathrm{d}v\,\mathrm{d}x = -\int_{\mathbb{R}^{2N}} |v|^2\ f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x + 2\,M_\varepsilon.$$

This equation together with (16) yields the following equation for the total energy of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon\int_{\mathbb{R}^{2N}} \frac{|v|^2}{2}\ f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} \left|\nabla\Phi_\varepsilon(t,x)\right|^2 \mathrm{d}x\right) = -\int_{\mathbb{R}^{2N}} |v|^2\ f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x + 2N. \tag{17}$$

Assuming (12) and integrating this relation with respect to time prove the two first estimates in (15).

To bound the second space moment, we multiply equation (4) by $|x|^2/2$ and integrate against x and v. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2N}} \frac{|x|^2}{2}\ f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{R}^{2N}} x\cdot v\ f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x \le \Big(\int_{\mathbb{R}^{2N}} |x|^2\ f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\ \int_{\mathbb{R}^{2N}} |v|^2\ f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\Big)^{1/2}.$$

Thus we have the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{R}^{2N}} |x|^2\ f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\Big)^{1/2} \le \Big(\int_{\mathbb{R}^{2N}} |v|^2\ f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\Big)^{1/2}.$$

Integration with respect to time combined with the bound on the kinetic energy ends the proof. ∎

Let us introduce in the following corollaries two consequences of the above Proposition.

Corollary 1 *Assuming the hypotheses on f_ε^I and D_ε of Proposition 1, for any $T>0$ the following properties are verified:*

i) $(1+|x|^2)\, n_\varepsilon$ *is a bounded sequence in* $L^\infty(0,T;L^1(\mathbb{R}^N))$,

ii) j_ε *is a bounded sequence in* $\big(L^2(0,T;L^1(\mathbb{R}^N))\big)^N$ *and* $(1+|x|)j_\varepsilon$ *is a bounded sequence in* $\big(L^1([0,T]\times\mathbb{R}^N)\big)^N$,

iii) $\int_{\mathbb{R}^N} v\otimes v f_\varepsilon \,\mathrm{d}v$ *is bounded in* $\big(L^1((0,T)\times\mathbb{R}^N)\big)^{2N}$.

Proof. Claims *i)* and *iii)* are direct consequences of (15). The current is estimated by using the Cauchy-Schwartz inequality as follows

$$\int_{\mathbb{R}^N} |j_\varepsilon(t,x)|\,\mathrm{d}x \le \left(\int_{\mathbb{R}^{2N}} f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x\right)^{1/2} \left(\int_{\mathbb{R}^{2N}} |v|^2\; f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x\right)^{1/2}.$$

In the right-hand side, the first term is bounded in $L^\infty(0,T)$ by using (14), while the second term is bounded in $L^2(0,T)$ by using (15). The first moment of j_ε is also bounded since

$$\begin{aligned}\int_0^T\!\!\int_{\mathbb{R}^N} |x|\;|j_\varepsilon(t,x)|\,\mathrm{d}x\,\mathrm{d}t \;&\le\; \int_0^T\!\!\int_{\mathbb{R}^{2N}} |x|\;|v|\;f_\varepsilon(t,x)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\ &\le\; \int_0^T\!\!\int_{\mathbb{R}^{2N}} \left(\frac{|x|^2+|v|^2}{2}\right) f_\varepsilon(t,x)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \le C_T,\end{aligned}$$

as consequence of (15). ∎

These estimates imply compactness properties in measure spaces. We first recall some definitions and basic facts from measure theory (an as complete as possible reference is for instance [7]).

Definition 1 *Considering sequences* $\big(\rho_n\big)_{n\in\mathbb{N}}$ *in* $\mathcal{M}^1(\mathbb{R}^N)$*, we say that:*

a) ρ_n *converges vaguely to* ρ *if*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} \varphi\,\rho_n\,\mathrm{d}x = \int_{\mathbb{R}^N} \varphi\,\rho\,\mathrm{d}x \tag{18}$$

for any continuous function with compact support $\varphi\in C_c(\mathbb{R}^N)$ *(actually, the convergence (18) holds for any function* φ *which is continuous and vanishes at infinity).*

b) The convergence is said to hold tightly if (18) is satisfied for any function which is continuous and bounded: $\varphi\in C^0\cap L^\infty(\mathbb{R}^N)$.

c) Let $\left(\rho_n\right)_{n\in\mathbb{N}}$ be a sequence of nonnegative measures which satisfies a) and, furthermore, $\lim\limits_{n\to\infty} \rho_n(\mathbb{R}^N) = \rho(\mathbb{R}^N)$. Then, ρ_n converges to ρ tightly.

d) Let $\left(\rho_n\right)_{n\in\mathbb{N}}$ be a sequence of measures (without sign assumption) which satisfies $\sup_{n\in\mathbb{N}} |\rho_n|(\mathbb{R}^N) < \infty$ and furthermore for any $\eta > 0$ there exists a compact set $K_\eta \subset \mathbb{R}^N$ such that $\sup_{n\in\mathbb{N}} |\rho_n|(\mathbb{R}^2 \setminus K_\eta) \le \eta$. Then, $\left(\rho_n\right)_{n\in\mathbb{N}}$ is relatively compact for the tight topology.

The compactness result is the following.

Corollary 2 *The sequence $\left(n_\varepsilon\right)_{\varepsilon>0}$ is relatively compact in $C^0([0,T], \mathcal{M}^1_+(\mathbb{R}^N) - tight)$ and $\left(j_\varepsilon\right)_{\varepsilon>0}$ is relatively compact in $\mathcal{M}^1([0,T]\times\mathbb{R}^N)^N - tight$.*

Proof. First, we deduce that n_ε satisfies an equicontinuity property by using the charge conservation equation (8). Indeed, let $\eta \in C_c^\infty(\mathbb{R}^N)$. We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} n_\varepsilon(t,x)\ \eta(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} j_\varepsilon(t,x)\cdot\nabla\eta(x)\,\mathrm{d}x.$$

By using Corollary 1 *ii)*, integration from s to t yields

$$\begin{aligned}\Big|\int_{\mathbb{R}^N} n_\varepsilon(t,x)\ \eta(x)\,\mathrm{d}x &- \int_{\mathbb{R}^N} n_\varepsilon(s,x)\ \eta(x)\,\mathrm{d}x\Big| \\ &\le \|\nabla\eta\|_{L^\infty(\mathbb{R}^N)}\left(\int_s^t\Big(\int_{\mathbb{R}^N}|j_\varepsilon(\tau,x)|\,\mathrm{d}x\Big)^2\mathrm{d}\tau\right)^{1/2}\ \sqrt{t-s} \\ &\le C_T\ \|\nabla\eta\|_{L^\infty(\mathbb{R}^N)}\ \sqrt{t-s}.\end{aligned}$$

Besides, the sequence $\left(n_\varepsilon\right)_{\varepsilon>0}$ satisfies the Prokhorof tightness criterion (see e.g. [7]) uniformly with respect to time:

$$\sup_{\varepsilon>0}\ \sup_{t\in[0,T]}\int_{\mathbb{R}^N} n_\varepsilon(t,x)\,\mathrm{d}x < \infty,$$

and *i)* leads to

$$\sup_{\varepsilon>0}\ \sup_{t\in[0,T]}\int_{|x|>R} n_\varepsilon(t,x)\,\mathrm{d}x \xrightarrow[R\to\infty]{} 0.$$

Thus, a standard density argument proves the equicontinuity of the sequence $\left(n_\varepsilon\right)_{\varepsilon>0}$ in $C^0([0,T];\mathcal{M}^1(\mathbb{R}^N) - tight)$.

When dealing with the current, the estimates *ii)* allow to extract a subsequence which converges tightly to some j in $\mathcal{M}^1([0,T]\times\mathbb{R}^N) - tight$. ∎

2.2 The case of the 2 space dimension

The aim of this subsection is to justify the integration by parts

$$-\int_{\mathbb{R}^2} \Phi_\varepsilon \Delta \Phi_\varepsilon \, \mathrm{d}x = \int_{\mathbb{R}^2} |\nabla \Phi_\varepsilon|^2 \, \mathrm{d}x.$$

To start with, we need some results about the solutions of (4), when the parameter $\varepsilon > 0$ is fixed. The techniques used by Degond [12] to obtain global existence of smooth solutions of the VPFP system (with one species) in 2 space dimensions, apply directly to our problem. We recall the following statement.

Theorem 3 *(Degond [12]) Let D satisfy (13) and let f_ε^I be a nonnegative initial data which satisfies*

$$f_\varepsilon^I \in W^{1,1}(\mathbb{R}^4); \quad (1+|v|^2)^{\gamma/2}\big(|f_\varepsilon^I| + |\nabla_{x,v} f_\varepsilon^I|\big) \in L^\infty(\mathbb{R}^4), \tag{19}$$

for some $\gamma > 2$. Then, there is a unique classical solution $(f_\varepsilon, \Phi_\varepsilon)$ of the VPFP system (4)-(5) satisfying

$$\left\{ \begin{array}{l} f_\varepsilon \geq 0, \qquad f_\varepsilon \in L^\infty(0,T; W^{1,1}(\mathbb{R}^4)), \\ (1+|v|^2)^{\gamma/2}\big(|f_\varepsilon| + |\nabla_{x,v} f_\varepsilon|\big) \in L^\infty((0,T)\times\mathbb{R}^4), \\ \partial_t f_\varepsilon,\ \Delta_v f_\varepsilon \in L^2((0,T)\times\mathbb{R}^4), \qquad \nabla\Phi_\varepsilon \in L^\infty(0,T; W^{1,\infty}(\mathbb{R}^2)), \end{array} \right. \tag{20}$$

for every $0 < T < \infty$.

Moreover, still by adapting the arguments in [12] we can easily strengthen the previous claim.

Proposition 2 *(Degond [12]) Let f_ε^I be a nonnegative initial data which satisfies (11). Let D satisfy (13). Then the classical solution $(f_\varepsilon, \Phi_\varepsilon)$ of the VPFP system (4)-(5) given by Theorem 3 satisfies*

$$\left\{ \begin{array}{l} (1+|v|^2+|x|^2)^{\gamma/2}\big(|f_\varepsilon| + |\nabla_{x,v} f_\varepsilon| + |\nabla^2_{x,v} f_\varepsilon|\big) \in L^\infty([0,T]\times\mathbb{R}^4), \\ (1+|v|^2+|x|^2)^{(\gamma-1)/2}\big|\partial_t f_\varepsilon\big| \in L^\infty([0,T]\times\mathbb{R}^4), \end{array} \right. \tag{21}$$

for every $0 < T < \infty$.

This result is not at all optimal because it does not take into account the smoothing effect of the VPFP system proved by Bouchut [5, 6]. However, it is enough for our purpose. Actually, we will use that $(1+|x|)n_\varepsilon \in L^\infty([0,T]\times\mathbb{R}^2)$ and $n_\varepsilon \in W^{1,\infty}([0,T]\times\mathbb{R}^2)$, which are obvious consequences of (20) and (21).

Of course, the estimates in this statement are not uniform with respect to the small parameter ε.

Lemma 1 *Let $\rho \in L^2(\mathbb{R}^2)$ be such that*

$$\int_{\mathbb{R}^2} (1+|x|)|\rho(x)|\,\mathrm{d}x < \infty, \qquad \int_{\mathbb{R}^2} \rho(x)\,\mathrm{d}x = 0.$$

Consider the potential Φ given by

$$\Phi(x) = E * \rho(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|x-y|\ \rho(y)\,\mathrm{d}y.$$

Then, Φ is a continuous and bounded function such that $\lim_{|x|\to\infty}\Phi(x) = 0$. Furthermore, we have $\nabla\Phi \in L^2(\mathbb{R}^2)$. In particular, the identity

$$\int_{\mathbb{R}^2} \rho(x)\ \Phi(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} |\nabla\Phi(x)|^2\,\mathrm{d}x$$

holds.

Proof. Let us denote by $\widehat{\cdot}$ the Fourier transform. The function E being the fundamental solution of $-\Delta$, we have $|\xi|^2\widehat{\Phi}(\xi) = \widehat{\rho}(\xi)$. Since the integral of ρ vanishes, we get

$$|\widehat{\rho}(\xi)| = |\widehat{\rho}(\xi) - \widehat{\rho}(0)| \le |\xi|\ \|\nabla\widehat{\rho}\|_{L^\infty(\mathbb{R}^2)} \le |\xi|\ \|x\rho\|_{L^1(\mathbb{R}^2)}.$$

We denote by χ_A the characteristic function of the set A. Then, we notice that

$$\begin{aligned}\left|\widehat{\Phi}(\xi)\right| &= \left|\frac{\widehat{\rho}(\xi)}{|\xi|^2}\right| \le \left|\frac{\widehat{\rho}(\xi)-\widehat{\rho}(0)}{|\xi|^2}\right|\chi_{\{|\xi|\le 1\}} + \left|\frac{\widehat{\rho}(\xi)}{|\xi|^2}\right|\chi_{\{|\xi|>1\}}\\ &\le \|x\rho\|_{L^1(\mathbb{R}^2)}\ \frac{1}{|\xi|}\chi_{\{|\xi|\le 1\}} + \left|\frac{\widehat{\rho}(\xi)}{|\xi|^2}\right|\chi_{\{|\xi|>1\}}.\end{aligned}$$

In the right-hand side, $1/|\xi| \in L^1(B_{\mathbb{R}^2}(0,1))$ while an application of the Cauchy-Schwartz inequality proves that the second term is also integrable. Hence, $\widehat{\Phi} \in L^1(\mathbb{R}^2)$. It follows that $x \mapsto \Phi(x)$ is a continuous and bounded function which tends to 0 at infinity. Similarly, we obtain

$$\left|\widehat{\nabla\Phi}(\xi)\right| = \left|\frac{\xi}{|\xi|^2}\widehat{\rho}(\xi)\right| \le \|x\rho\|_{L^1(\mathbb{R}^2)}\chi_{\{|\xi|\le 1\}} + |\widehat{\rho}(\xi)|\chi_{\{|\xi|>1\}}.$$

Since ρ lies in $L^2(\mathbb{R}^2)$, we conclude that $\nabla\Phi \in L^2(\mathbb{R}^2)$ by the Plancherel theorem. With these properties in mind, it is easy to justify the formula

$$\int_{\mathbb{R}^2} |\nabla\Phi|^2\,\mathrm{d}x = -\int_{\mathbb{R}^2} \Phi\Delta\Phi\,\mathrm{d}x$$

via standard approximation and truncation arguments. ∎

3 The modulated energy method

The aim of this Section is to prove Theorem 1. We first introduce the modulated energy (see [8]), which allows us to show that the field $-\nabla\Phi_\varepsilon$ converges in $L^2(\mathbb{R}^N)^N$ to the limiting field $-\nabla\Phi$. The proof requires some regularity properties of the limit solutions as well as the convergence of the initial electric field. We define the modulated energy $H(t,\varepsilon)$ and the positive modulated energy $H_p(t,\varepsilon)$ as follows

$$H(t,\varepsilon) := \varepsilon\int_{\mathbb{R}^{2N}} f_\varepsilon\Big(\log f_\varepsilon + \frac{1}{2}|v+\nabla\Phi|^2\Big)\,\mathrm{d}x\,\mathrm{d}v + \frac{1}{2}\int_{\mathbb{R}^N}|\nabla(\Phi_\varepsilon-\Phi)|^2\,\mathrm{d}x, \quad (22)$$

$$0 \le H_p(t,\varepsilon) := \varepsilon\int_{\mathbb{R}^{2N}} f_\varepsilon\Big(|\log f_\varepsilon| + \frac{1}{2}|v+\nabla\Phi|^2\Big)\,\mathrm{d}x\,\mathrm{d}v + \frac{1}{2}\int_{\mathbb{R}^N}|\nabla(\Phi_\varepsilon-\Phi)|^2\,\mathrm{d}x. \quad (23)$$

The idea is to study the dynamics of the modulated energy $H(t,\varepsilon)$ (this quantity is also named relative entropy). It allows to control the positive modulated energy $H_p(t,\varepsilon)$ thanks to Gronwall's Lemma. Under convenient assumptions on the initial data, $H_p(t,\varepsilon)$ converges to zero as ε goes to 0. Then, the convergence of the field is obtained.

To do that, it is necessary to have smooth properties for the limiting quantities n and $\nabla\Phi$. Also, the convergence of the density n_ε is obtained thanks to the compactness result of Corollary 2. Then we are able to determine the limit j of the current j_ε by using the entropy production. The advantage of this method is that the result in Theorem 1 holds in any dimension.

Proposition 3 *(Smoothness of solutions of the limit system). Under the assumptions of Theorem 1, the strong solution $(n,\nabla\Phi)$ of (9) satisfies*

$$\forall T>0, \quad \nabla\Phi \in W^{1,\infty}([0,T]\times\mathbb{R}^N).$$

Proof. Using Theorem 3 of [17], the unique strong solution $(n,\nabla\Phi)$ of (9) satisfies n and $\rho = n - D \in W^{1,\infty}([0,T]\times\mathbb{R}^N)$, $\forall T>0$. Therefore, Lemma 8 of [17] implies that $\nabla\Phi \in L^\infty\big([0,T];W^{1,\infty}(\mathbb{R}^N)\big)^N$. Then, by differentiating the first equation of (9), it is easy to check that $\nabla\rho$ and ∇n are in $\in L^\infty\big([0,T];L^1(\mathbb{R}^N)\big)^N$ and, taking into account that $\partial_t n = -\nabla\Phi\cdot\nabla n + n(D-n)$, we have $\partial_t\rho = \partial_t n \in L^\infty\big([0,T];L^1(\mathbb{R}^N)\big)$. ∎

3.1 The modulated energy equation

In the N-dimensional case we can prove the following result concerning the evolution of the modulated energy. Throughout this section we repeatedly apply integration by parts, which are justified in the Subsection 2.2 for $N=2$ and is a consequence of the regularity properties at infinity of the Poisson equation for $N>2$.

Lemma 2 *Let f_ε be a solution of the VPFP system and n a solution of the limiting equation (9). Then, the balance of the modulated energy is*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}H(t,\varepsilon) \;=\; & -\int_{\mathbb{R}^{2N}} |(v+\nabla\Phi)\sqrt{f_\varepsilon}+2\nabla_v\sqrt{f_\varepsilon}|^2 \;\mathrm{d}x\;\mathrm{d}v \\
& +\varepsilon\int_{\mathbb{R}^{2N}} \Big(f_\varepsilon\,(v+\nabla\Phi)\cdot\frac{\partial}{\partial t}\nabla\Phi + f_\varepsilon\,(v+\nabla\Phi)\cdot\,\mathrm{D}^2\Phi\cdot v\Big)\;\mathrm{d}x\;\mathrm{d}v \\
& +\int_{\mathbb{R}^N}\nabla(\Phi_\varepsilon-\Phi)\cdot\,\mathrm{D}^2\Phi\cdot\nabla(\Phi_\varepsilon-\Phi)\;\mathrm{d}x \\
& +\int_{\mathbb{R}^N}\nabla(\Phi_\varepsilon-\Phi)\cdot\nabla\Phi\,(D-D_\varepsilon)\;\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^N}(n-D)\,|\nabla(\Phi_\varepsilon-\Phi)|^2\;\mathrm{d}x.
\end{aligned}$$

Proof. We first add $\nabla\Phi\cdot\nabla_v f_\varepsilon$ in both terms of equation (4) to find

$$\varepsilon\Big(\frac{\partial}{\partial t}f_\varepsilon+v\cdot\nabla f_\varepsilon\Big)-\mathrm{div}_v\Big(e^{-\frac{|v+\nabla\Phi|^2}{2}}\nabla_v\Big(e^{\frac{|v+\nabla\Phi|^2}{2}}f_\varepsilon\Big)\Big)=(\nabla\Phi_\varepsilon-\nabla\Phi)\cdot\nabla_v f_\varepsilon. \tag{24}$$

Then, the result is obtained by multiplying this equation by $(1+\log f_\varepsilon+\frac{1}{2}|v+\nabla\Phi|^2)$ and integrating. We first remark that the left–hand side of the above equation is nothing but the Vlasov-Fokker-Planck equation with the field $\nabla\Phi$. Thus, the following computations are classical:

$$\begin{aligned}
\varepsilon\;&\int_{\mathbb{R}^{2N}}\Big(\frac{\partial}{\partial t}f_\varepsilon+v\cdot\nabla f_\varepsilon\Big)\times\Big(1+\log f_\varepsilon+\frac{1}{2}|v+\nabla\Phi|^2\Big)\;\mathrm{d}x\;\mathrm{d}v \\
&=\;\varepsilon\;\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{R}^{2N}} f_\varepsilon\log f_\varepsilon+\frac{1}{2}|v+\nabla\Phi|^2 f_\varepsilon\;\mathrm{d}x\;\mathrm{d}v\Big) \\
&\quad-\varepsilon\;\int_{\mathbb{R}^{2N}}(v+\nabla\Phi)\cdot(\frac{\partial}{\partial t}\nabla\Phi+v\cdot\,\mathrm{D}^2\Phi)f_\varepsilon\;\mathrm{d}x\;\mathrm{d}v.
\end{aligned}$$

Also,

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}}\Big(\mathrm{div}_v\big(e^{-\frac{|v+\nabla\Phi|^2}{2}}\nabla_v\big(e^{\frac{|v+\nabla\Phi|^2}{2}}f_\varepsilon\big)\big)\Big)\times\Big(1+\log\big(e^{\frac{|v+\nabla\Phi|^2}{2}}f_\varepsilon\big)\Big)\;\mathrm{d}x\;\mathrm{d}v \\
&=\int_{\mathbb{R}^{2N}} e^{-\frac{|v+\nabla\Phi|^2}{2}}|\nabla_v\big(e^{\frac{|v+\nabla\Phi|^2}{2}}f_\varepsilon\big)|^2\;\frac{1}{e^{\frac{|v+\nabla\Phi|^2}{2}}f_\varepsilon}\;\mathrm{d}x\;\mathrm{d}v \\
&=\int_{\mathbb{R}^{2N}}\Big|(v+\nabla\Phi)\sqrt{f_\varepsilon}+2\nabla_v\sqrt{f_\varepsilon}\Big|^2\;\mathrm{d}x\;\mathrm{d}v\geq 0.
\end{aligned}$$

Finally, the right–hand side of (24) gives the following contribution

$$\int_{\mathbb{R}^{2N}} \frac{1}{2}|v+\nabla\Phi|^2\,\nabla(\Phi_\varepsilon-\Phi)\cdot\nabla_v f_\varepsilon\ \,\mathrm{d}x\ \mathrm{d}v = -\int_{\mathbb{R}^{2N}} (v+\nabla\Phi)\cdot\nabla(\Phi_\varepsilon-\Phi) f_\varepsilon\ \,\mathrm{d}x\ \mathrm{d}v$$

$$= -\int_{\mathbb{R}^N} (j_\varepsilon + n_\varepsilon\nabla\Phi)\cdot\nabla(\Phi_\varepsilon-\Phi)\ \mathrm{d}x$$

$$= -\int_{\mathbb{R}^N} (j_\varepsilon + n\nabla\Phi)\cdot\nabla(\Phi_\varepsilon-\Phi)\ \mathrm{d}x + \int_{\mathbb{R}^N} (n-n_\varepsilon)\nabla(\Phi_\varepsilon-\Phi)\cdot\nabla\Phi\ \mathrm{d}x = I_1 + I_2.$$

The term I_1 is treated in the same way that we have obtained the energy equation in Section 2. Using (8), the Poisson equation and the limiting equation (9) we can rewrite this first term as

$$I_1 = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} |\nabla\Phi_\varepsilon - \nabla\Phi|^2\ \,\mathrm{d}x.$$

For the second term, we use integration by parts which are justified as in Section 2 and the Poisson equation to find

$$\begin{aligned} I_2 \quad &= \int_{\mathbb{R}^N} (D-\Delta\Phi - D_\varepsilon + \Delta\Phi_\varepsilon)\,\nabla(\Phi_\varepsilon-\Phi)\cdot\nabla\Phi\ \mathrm{d}x \\ &= \int_{\mathbb{R}^N} (D-D_\varepsilon)\,\nabla(\Phi_\varepsilon-\Phi)\cdot\nabla\Phi\ \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} |\nabla(\Phi_\varepsilon-\Phi)|^2(n-D)\ \mathrm{d}x \\ &+ \int_{\mathbb{R}^N} \nabla(\Phi_\varepsilon-\Phi)\cdot\ \mathrm{D}^2\Phi\cdot\nabla(\Phi_\varepsilon-\Phi)\ \mathrm{d}x. \end{aligned}$$

These computations easily lead to the announced result. ∎

3.2 Asymptotics

Let $T>0$ be an arbitrary positive constant. In the following, C_T denotes various positive constants independent on ε. By assuming the hypotheses of Theorem 1 and using the uniform estimates of Propositions 1 and 3 we can bound the modulated energy

$$\begin{aligned} \frac{d}{dt}H(t,\varepsilon) \quad &\leq \quad C_T\Big(\int_{\mathbb{R}^{2N}} \varepsilon(f_\varepsilon + \frac{1}{2}|v+\nabla\Phi|^2 f_\varepsilon) + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\Phi_\varepsilon - \nabla\Phi|^2\ \,\mathrm{d}x\ \mathrm{d}v\Big) \\ &\qquad + C_T\int_{\mathbb{R}^{2N}} (D-D_\varepsilon)^2\,\mathrm{d}x \leq C_T\Big(H_p(t,\varepsilon) + \|D-D_\varepsilon\|^2_{L^2(\mathbb{R}^N)} + \varepsilon\Big), \\ H(t,\varepsilon) \quad &\leq \quad H(0,\varepsilon) + C_T\int_0^t H_p(s,\varepsilon)\ \mathrm{d}s + C_T\big(\|D-D_\varepsilon\|^2_{L^2(\mathbb{R}^N)} + \varepsilon\big). \end{aligned} \tag{25}$$

Since we cannot assure that $H(t,\varepsilon)$ is nonnegative due to the $f\log f$ term, we study the positive modulated energy $H_p(t,\varepsilon)$. We observe that both modulated energies are related in the following sense

$$H(t,\varepsilon) \leq H_p(t,\varepsilon) \leq H(t,\varepsilon) + \varepsilon \int_{\mathbb{R}^{2N}} |x| f_\varepsilon + 2\varepsilon \int_{\mathbb{R}^{2N}} e^{\frac{-|v+\nabla\Phi|^2 - |x|}{16}},$$

where we have used the standard bound

$$f_\varepsilon |\log f_\varepsilon| \leq f_\varepsilon \log f_\varepsilon + \frac{1}{4}\Big(|v+\nabla\Phi|^2 + |x|\Big) f_\varepsilon + 2e^{\frac{-|v+\nabla\Phi|^2 - |x|}{16}}.$$

Then, combining these inequalities with (25), the conservation of the total mass and the bounds of the second moments of Proposition 1 , we conclude

$$H_p(t,\varepsilon) \leq H_p(0,\varepsilon) + C_T \int_0^t H_p(s,\varepsilon) ds + C_T(\varepsilon + \|D - D_\varepsilon\|^2_{L^2(\mathbb{R}^N)}).$$

Finally, Gronwall's lemma ensures that

$$0 \leq \; H_p(t,\varepsilon) \leq \; C_T\big(H_p(0,\varepsilon) + \varepsilon + \|D - D_\varepsilon\|^2_{L^2(\mathbb{R}^N)}\big) \to 0, \quad \text{as } \varepsilon \to 0$$

from which the convergence of the field $\nabla\Phi_\varepsilon$ follows. Then, $n_\varepsilon \to n$ in $C^0\big([0,T]; H^{-1}(\mathbb{R}^N)\big)$. Actually, this convergence holds in $C^0([0,T], \mathcal{M}^1_+(\mathbb{R}^N) - tight)$ thanks to Corollary 2.

In order to obtain the convergence of the current j_ε we turn back to the relative energy equation of Lemma 2. Integrating with respect to time and using the above convergence we obtain

$$\int_0^T \int_{\mathbb{R}^{2N}} \big|(v+\nabla\Phi)\sqrt{f_\varepsilon} + 2\nabla_v \sqrt{f_\varepsilon}\big|^2 \; \mathrm{d}x \; \mathrm{d}v \; \mathrm{d}t \to 0, \quad \text{as } \varepsilon \to 0.$$

It implies that

$$\int_0^T \int_{\mathbb{R}^N} |j_\varepsilon + n_\varepsilon \nabla\Phi| \; \mathrm{d}x \; \mathrm{d}t = \int_0^T \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} (v+\nabla\Phi) f_\varepsilon \; \mathrm{d}v \right| \mathrm{d}x \; \mathrm{d}t$$
$$\leq \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| \Big((v+\nabla\Phi)\sqrt{f_\varepsilon} + 2\nabla_v\sqrt{f_\varepsilon}\Big)\sqrt{f_\varepsilon} \right| \; \mathrm{d}v \; \mathrm{d}x \; \mathrm{d}t \to 0, \quad \text{as } \varepsilon \to 0.$$

Since the limiting field $-\nabla\Phi$ is a bounded continuous function, we have $n_\varepsilon\nabla\Phi \to n\nabla\Phi$ in $C^0([0,T], \mathcal{M}^1_+(\mathbb{R}^N) - tight)^N$. We conclude that $j_\varepsilon \to n\nabla\Phi$ tightly in $\mathcal{M}^1([0,T]\times\mathbb{R}^N)^N$. It ends the proof of Theorem 1.

4 Compactness method in space dimension 2

The aim of this Section is to prove Theorem 2. Indeed, in dimension 2, we are able to improve the result of Theorem 1 by considering general weak solutions of the limiting problem.

Thanks to Corollary 2 (at the cost of extracting subsequences) we have for any $T > 0$

$$\begin{aligned} n_\varepsilon(t) \to n(t), &\qquad \text{tightly in } \mathcal{M}^1_+(\mathbb{R}^2), \text{ uniformly with respect to } t \in [0,T], \\ j_\varepsilon \to j, &\qquad \text{tightly in } \mathcal{M}^1([0,T]\times\mathbb{R}^2)^N. \end{aligned} \tag{26}$$

In virtue of these convergences, we can pass to the limit in the charge conservation equation. We get

$$\partial_t n + \operatorname{div} j = 0.$$

Using Proposition 1, it is easy to obtain that $j_\varepsilon + n_\varepsilon \nabla\Phi_\varepsilon \to 0$ in the distributional sense (see below in this Section). Then, the difficulty is to pass to the limit in the nonlinear term $n_\varepsilon\nabla\Phi_\varepsilon$. Actually, it is far to be obvious that the product $n\nabla\Phi$ is well defined. Indeed, we have seen in Section 2 that we have only estimates on f_ε and its moments in a L^1-setting. Therefore, we only know that n is a measure with respect to the space variable. On the other hand, Φ is defined through the convolution of $n - D$ with the fundamental solution of $-\Delta$. Hence, it is not obvious at all how the product $n\nabla\Phi$ makes sense.

This is dealt with by exploiting the symmetry properties of the fundamental solution of $-\Delta$, denoted in this Section by $E(= E_2)$. Namely, we use the following (formal) identities, where φ stands for a smooth test function and ρ for a measure:

$$\begin{aligned} \int_{\mathbb{R}^N} \rho(x)\nabla(E*\rho)(x)\ \varphi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \nabla E(x-y)\varphi(x)\ \rho(x)\rho(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \nabla E(x-y)\big(\varphi(x)-\varphi(y)\big)\ \rho(x)\rho(y)\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}$$

This formula uses that $\nabla E(x)$ is an odd function. Since $\varphi(x)-\varphi(y)$ vanishes as $x-y\to 0$, we can expect that it will compensate the singularity of the interaction kernel ∇E. This symmetrized expression was introduced by Schochet [21] and is a particular formulation of compensated compactness arguments used by Delort [14] to obtain the existence of solutions with vortex sheets of the 2D incompressible Euler equation.

In a one-dimensional framework the definition does not present any trouble when ρ is a measure, since the function

$$(x,y) \longmapsto \nabla E_1(x-y)\big(\varphi(x)-\varphi(y)\big) = -\frac{1}{2}\operatorname{sgn}(x-y)\ \big(\varphi(x)-\varphi(y)\big)$$

is continuous on $\mathbb{R}\times\mathbb{R}$. This has been used by Nieto-Poupaud-Soler [17] for the high-field limit analysis of the 1D VPFP system. In a two-dimensional framework, however,

$$(x,y)\longmapsto \nabla E(x-y)\big(\varphi(x)-\varphi(y)\big)=-\frac{1}{2\pi}\frac{x-y}{|x-y|^2}\ \big(\varphi(x)-\varphi(y)\big)$$

is continuous only on $\mathbb{R}^2\times\mathbb{R}^2\setminus\mathbb{D}$, where $\mathbb{D}$ is the diagonal $\mathbb{D}=\{(x,x),\ x\in\mathbb{R}^2\}$. Therefore, in order to prove stability results it is necessary to introduce an additional diagonal defect measure. In particular, the defect measure vanishes when the set of atoms of the measure ρ is empty. This has been discussed by Poupaud in [20], with application to global existence for the 2D Euler equation. Another application is related to adhesion dynamics as for presureless equation or problems arising from biology, see [16].

We can now introduce the concept of solutions on which our study in dimension 2 is based.

Definition 2 *We will say that n is a weak solution of the system*

$$\left\{\begin{array}{ll} \partial_t n-\operatorname{div}(n\nabla\Phi)=0 & \textit{in }\ \mathbb{R}^+\times\mathbb{R}^2,\\ -\Delta\Phi=n-D & \textit{in }\ \mathbb{R}^+\times\mathbb{R}^2,\\ n_{t=0}=n^I & \textit{in }\ \mathbb{R}^2, \end{array}\right. \tag{27}$$

if the following assertions are fulfilled

- $n\in C^0([0,\infty),\mathcal{M}^1_+(\mathbb{R}^2)-tight)$ *and* $n(t=0)=n^I$.
- *For any $t\in[0,\infty)$, the measure $n(t)$ is diffuse (which means that, for any $x_0\in\mathbb{R}^2$, $n(t)(\{x_0\})=0$),*
- *For any test function $\varphi\in C_c^\infty(\mathbb{R}^2)$, we have*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\varphi(x)n(t,x)\,\mathrm{d}x \;&=\; -\int_{\mathbb{R}^2} n(t,x)\nabla\varphi(x)\cdot\nabla E*D(x)\,\mathrm{d}x\\ &\quad -\int_{\mathbb{R}^4}\frac{1}{4\pi}\frac{x-y}{|x-y|^2}\cdot(\nabla\varphi(x)-\nabla\varphi(y))n(t,y)n(t,x)\,\mathrm{d}y\,\mathrm{d}x\end{aligned}$$

in $\mathcal{D}'([0,\infty))$.

4.1 Asymptotics in dimension 2

The asymptotic analysis is based on the conservation equations

$$\left\{\begin{array}{l} \partial_t n_\varepsilon+\operatorname{div} j_\varepsilon=0,\\ \varepsilon\Big(\partial_t j_\varepsilon+\operatorname{div}\displaystyle\int_{\mathbb{R}^2} v\otimes v f_\varepsilon\,\mathrm{d}v\Big)=-n_\varepsilon\nabla\Phi_\varepsilon-j_\varepsilon. \end{array}\right. \tag{28}$$

We use the convergence result (26) and the uniform estimates on the second moments of f_ε given in Proposition 1. Passing to the limit in (28) in the sense of distributions, we get

$$\begin{cases} \partial_t n + \mathrm{div} j = 0, \\ j = -\lim\limits_{\varepsilon\to 0} n_\varepsilon \nabla\Phi_\varepsilon, \end{cases} \qquad \text{(in the distributional sense on } [0,\infty)\times\mathbb{R}^2). \tag{29}$$

Therefore, we are left with the task of determining the limit of the product $n_\varepsilon\nabla\Phi_\varepsilon$ in the sense of distributions. To this end, let us split the potential as follows

$$\begin{cases} \Phi_\varepsilon = E * (n_\varepsilon - D_\varepsilon) = \Psi_\varepsilon - U_\varepsilon, \qquad E(x) = -\dfrac{1}{2\pi}\ln(|x|), \\ U_\varepsilon(x) = E * D_\varepsilon(x) = -\dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}^2} \ln|x-y|\ D_\varepsilon(y)\,\mathrm{d}y, \\ \Psi_\varepsilon(t,x) = E * n_\varepsilon(t,x) = -\dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}^2} \ln|x-y|\ n_\varepsilon(t,y)\,\mathrm{d}y. \end{cases}$$

First of all, we remark that $\nabla U_\varepsilon = \nabla E * D_\varepsilon$ is continuous and bounded. Indeed, by using the Hardy–Littlewood–Sobolev theorem of fractional integration (see [24]) we have

$$|\nabla U| \le C\frac{1}{|x|} * |D(x)| \le C\|D\|^{\alpha}_{L^p(\mathbb{R}^2)}\|D\|^{1-\alpha}_{L^1(\mathbb{R}^2)}, \quad \text{with } \alpha = \frac{1}{3-p'} \ \text{ and } \ p>2.$$

In the same way we have $\|\nabla U_\varepsilon - \nabla U\|_{L^\infty(\mathbb{R}^2)} \le C\|D-D_\varepsilon\|^{\alpha}_{L^p(\mathbb{R}^2)}\|D-D_\varepsilon\|^{1-\alpha}_{L^1(\mathbb{R}^2)}$. Using (13) proves that $\nabla U_\varepsilon(x) \to \nabla U(x)$ uniformly on $\mathbb{R}^2$. Therefore

$$\lim_{\varepsilon\to 0} n_\varepsilon(t)\ \nabla U = n(t)\ \nabla U, \quad \text{tightly in } \mathcal{M}^1(\mathbb{R}^2),\ \forall t\in[0,+\infty). \tag{30}$$

Of course, the convergence analysis for $n_\varepsilon\nabla\Psi_\varepsilon$ is much more involved. As announced in Definition 2, we use the following trick (for which we refer to Schochet [21], see also [17], [20]): for any $\eta \in C_c^\infty(\mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^2} n_\varepsilon(t,x)\ \nabla\Psi_\varepsilon(t,x)\eta(x)\,\mathrm{d}x = \frac{1}{4\pi}\int_{\mathbb{R}^4} \frac{(x-y)(\eta(x)-\eta(y))}{|x-y|^2}\ n_\varepsilon(t,x)\ n_\varepsilon(t,y)\,\mathrm{d}y\,\mathrm{d}x. \tag{31}$$

On the one hand, using the uniform convergence of n_ε with respect to time, we get for all $t\in[0,T]$,

$$n_\varepsilon(t,x)\ n_\varepsilon(t,y) \to n(t,x)\ n(t,y), \quad \text{tightly in } \mathcal{M}^1(\mathbb{R}^4).$$

On the other hand, the function

$$(x,y)\in\mathbb{R}^2\times\mathbb{R}^2 \longmapsto \frac{(x-y)(\eta(x)-\eta(y))}{|x-y|^2}$$

is bounded and continuous on $\mathbb{R}^4 \setminus \mathbb{D}$, where $\mathbb{D}$ is the diagonal $\mathbb{D} = \{(x,x); x \in \mathbb{R}^2\}$. By using a classical result of measure theory, we can pass to the limit in (31) provided that $\mathbb{D}$ is a negligible set for the product measure: $n(t) \otimes n(t)(\mathbb{D}) = 0$, see Lemma 4 of the Appendix. Actually, this condition is equivalent to the fact that $n(t)$ is a diffuse measure. This property will be obtained as a consequence of the following

Lemma 3 *For any $T > 0$, the distributional limit of $-\nabla\Phi_\varepsilon$ is $F = \nabla E * (n - D)$ uniformly with respect to $t \in [0,T]$. The function F belongs to $C^0([0,T]; L^2(\mathbb{R}^2))^2$ and $n(t,x) = D + \mathrm{div}F$.*

Proof. Let $\eta \in C_c^\infty(\mathbb{R}^2)$. For any $t \geq 0$, we have

$$\int_{\mathbb{R}^2} \nabla\Psi_\varepsilon(t,x)\ \eta(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} n_\varepsilon(t,x) \left(\frac{x-y}{2\pi|x-y|}\eta(y)\,\mathrm{d}y \right) \mathrm{d}x = \int_{\mathbb{R}^2} n_\varepsilon(t,x) \nabla E * \eta(x)\,\mathrm{d}x.$$

It is a simple matter to observe that $\nabla E * \eta$ is a continuous and bounded function. Therefore, we conclude that, uniformly with respect to $t \in [0,T]$,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \nabla\Psi_\varepsilon(t,x)\ \eta(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} n(t,x)\nabla E * \eta(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} \nabla E * n\ \eta(x)\,\mathrm{d}x.$$

By (15), we can assume that $-\nabla\Phi_\varepsilon$ has a limit, denoted by F, in $C^0([0,T]; L^2(\mathbb{R}^2)-weak)$. Note that the convergence is uniform with respect to time due to the uniform convergence of n_ε. Then, we pass to the limit in $-\mathrm{div}(\nabla\Phi_\varepsilon) = -\mathrm{div}(\nabla\Psi_\varepsilon - \nabla U)$ and obtain

$$\mathrm{div}F(t,x) = -\mathrm{div}(\nabla E * n(t,x) - \nabla U(x)) = n(t,x) - D(x),$$

for any $t \in [0,T]$. ∎

Corollary 3 *For any $x_0 \in \mathbb{R}^2$, we have $n(t)(\{x_0\}) = 0$.*

Proof. Our proof follows the argument in [14], [21]. Let $\chi \in C_c^\infty(\mathbb{R}^2)$ such that

$$\begin{cases} 0 \leq \chi(x) \leq 1, \\ \chi(x) = 1 \quad \text{for } |x| \leq 1, \\ \chi(x) = 0 \quad \text{for } |x| \geq 2. \end{cases}$$

Let also $x_0 \in \mathbb{R}^2$. We get

$$
\begin{aligned}
0 \le n(t)(\{x_0\}) &\le \int_{\mathbb{R}^2} \chi\Big(\frac{x-x_0}{r}\Big) n(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^2} \chi\Big(\frac{x-x_0}{r}\Big)(\nabla\cdot F(t,x) + D(x))\,\mathrm{d}x \\
&\le -\frac{1}{r}\int_{\mathbb{R}^2} \nabla\chi\big(\frac{x-x_0}{r}\big)\cdot F(t,x)\,\mathrm{d}x + \int_{\{|x|\le 2r\}} D(x)\,\mathrm{d}x \\
&\le \frac{1}{r}\Big(\int_{\mathbb{R}^2} \Big|\nabla\chi\big(\frac{x-x_0}{r}\big)\Big|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_{\{|x|\le 2r\}} |F(t,x)|^2\,\mathrm{d}x\Big)^{1/2} \\
&\quad + \int_{\{|x|\le 2r\}} D(x)\,\mathrm{d}x \\
&\le \|\nabla\chi\|_{L^2(\mathbb{R}^2)}\Big(\int_{\{|x|\le 2r\}} F(t,x)\,\mathrm{d}x\Big)^{1/2} + \int_{\{|x|\le 2r\}} D(x)\,\mathrm{d}x.
\end{aligned}
$$

We conclude by observing that the right-hand side tends to 0 as $r \to 0$. ∎

5 Appendix : a lemma from measure theory

For the sake of completeness we give a proof of the following Lemma which was used in the previous Section.

Lemma 4 *Let $\big(\mu_n\big)_{n\in\mathbb{N}}$ be a sequence of nonnegative bounded measures on a set X. We suppose that it converges tightly to $\mu \in \mathcal{M}^1(X)$. Let $f : X \to \mathbb{R}$ be a bounded borelian function such that*

$$\mu(\{x \in X,\ f \text{ is not continuous at the point } x\}) = 0.$$

Then, one has

$$\lim_{n\to\infty} \int f\,\mathrm{d}\mu_n = \int f\,\mathrm{d}\mu.$$

Various proofs of this classical statement are possible. We refer among others to [25] (Ch. IV, Th. 62 & 63) and also [14], [21]. Here, for the sake of completeness we present some hints, following arguments in [7].

First, consider a lower semi-continuous (lsc) function φ, verifying $a \le \varphi(x) \le b$. It can be approached by a nondecreasing sequence of continuous functions φ_k:

$$a \le \varphi_k(x) \le \varphi_{k+1}(x) \le \varphi(x) \le b, \qquad \lim_{k\to\infty} \varphi_k(x) = \sup_{k\in\mathbb{N}} \varphi_k(x) = \varphi(x).$$

Since the μ_n's are nonnegative measures, we get

$$0 \leq \int \frac{\varphi_k - a}{b-a} \, \mathrm{d}\mu_n \leq \int \frac{\varphi - a}{b-a} \, \mathrm{d}\mu_n.$$

Then, the tight convergence $\mu_n \rightharpoonup \mu$ implies that

$$\begin{aligned} 0 \ &\leq \lim_{n\to\infty} \int \frac{\varphi_k - a}{b-a} \, \mathrm{d}\mu_n = \int \frac{\varphi_k - a}{b-a} \, \mathrm{d}\mu \ &&\leq \liminf_{n\to\infty} \int \frac{\varphi - a}{b-a} \, \mathrm{d}\mu_n \\ & && \leq \frac{1}{b-a} \liminf_{n\to\infty} \int \varphi \, \mathrm{d}\mu_n - \int \frac{a}{b-a} \, \mathrm{d}\mu, \end{aligned}$$

holds for any $k \in \mathbb{N}$. However, the nonnegative sequence $\frac{\varphi_k(x)-a}{b-a}$ converges pointwise to $\frac{\varphi(x)-a}{b-a}$ in a nondecreasing way. Hence, the monotone convergence theorem applies and letting $k \to \infty$ leads to

$$\int \frac{\varphi - a}{b-a} \, \mathrm{d}\mu \leq \frac{1}{b-a} \liminf_{n\to\infty} \int \varphi \, \mathrm{d}\mu_n - \int \frac{a}{b-a} \, \mathrm{d}\mu.$$

Since $\int a/(b-a) \, \mathrm{d}\mu$ is finite, we conclude that

$$\int \varphi \, \mathrm{d}\mu \leq \liminf_{n\to\infty} \int \varphi \, \mathrm{d}\mu_n$$

holds for any bounded lsc function φ. If we are dealing with a bounded upper semicontinuous function (usc) φ, we obtain

$$\int \varphi \, \mathrm{d}\mu \geq \limsup_{n\to\infty} \int \varphi \, \mathrm{d}\mu_n$$

(consider the bounded lsc function $b - \varphi(x)$...).

Next, let f be a bounded borelian function. We set

$$\begin{aligned} \underline{\varphi}(x) &= \sup \big\{ \varphi(x), \, \varphi : X \to \mathbb{R}\text{continuous s.t. } \forall y \in X, \, \varphi(y) \leq f(y) \big\}, \\ \overline{\varphi}(x) &= \inf \big\{ \varphi(x), \, \varphi : X \to \mathbb{R}\text{continuous s.t. } \forall y \in X, \, \varphi(y) \geq f(y) \big\}. \end{aligned}$$

We check that $\underline{\varphi}$ (resp. $\overline{\varphi}$) is a measurable bounded lsc (resp. usc) function. Furthermore, we have

$$\underline{\varphi}(x) = \liminf_{y\to x} f(y) \leq f(x) \leq \overline{\varphi}(x) = \limsup_{y\to x} f(y).$$

In particular, we realize that the set $\mathbb{D}$ of discontinuity points of f is nothing but

$$\mathbb{D} = \{x \in X,\, \underline{\varphi}(x) < \overline{\varphi}(x)\}.$$

Applying the previous consideration for lsc and usc functions, we obtain

$$\begin{aligned}\int \underline{\varphi}\,\mathrm{d}\mu \quad &\le \liminf_{n\to\infty} \int \underline{\varphi}\,\mathrm{d}\mu_n \quad \le \liminf_{n\to\infty} \int f\,\mathrm{d}\mu_n \\ &\qquad\qquad\qquad \le \limsup_{n\to\infty} \int f\,\mathrm{d}\mu_n \le \limsup_{n\to\infty} \int \overline{\varphi}\,\mathrm{d}\mu_n \le \int \overline{\varphi}\,\mathrm{d}\mu.\end{aligned}$$

Therefore, when $\mathbb{D}$ is a $\mu-$negligeable set we are led to

$$\int \underline{\varphi}\,\mathrm{d}\mu = \int \overline{\varphi}\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu = \lim_{n\to\infty} \int f\,\mathrm{d}\mu_n.$$

■

Acknowledgements

This research was partially supported by the EU financed network IHP-HPRN-CT-2002-00282 and by MCYT (Spain), Proyecto BFM2002–00831.

References

[1] Arnold A., Carillo J.A., Gamba I. and Shu C.-W., *Low and high field scaling limits for the Vlasov- and Wigner-Poisson-Fokker-Planck system*, Transport Theory Statist. Phys., **30** (2001), no 2-3, 121–153.

[2] Ben Abdallah N. and Chaker H., *The high field asymptotics for degenerate semiconductors*, Math. Models Methods Appl. Sci., **11** (2001), no 7, 1253–1272.

[3] Ben Abdallah N., Degond P., Markowich P. and Schmeiser C., *High field approximations of the spherical harmonics expansion model for semiconductors*, Z. Angew. Math. Phys., **52** (2001), no 2, 201–230.

[4] Bénilan Ph., Vol 2 of Dautray R., Lions J. L., **Analyse mathématique et calcul numérique**, Masson, Paris 1988.

[5] BOUCHUT F., *Existence and uniqueness of a global smooth solution for the VPFP system in three dimensions*, J. Func. Anal., **111** (1993), 239–258.

[6] BOUCHUT F., *Smoothing effect for the non-linear VPFP system*, J. Diff. Equations, **122** (1995), 225–238.

[7] BOURBAKI N., **Eléments de Mathématiques**, Fascicule XXXV, Livre VI, Chapitre IX: Intégration, Herrmann, Paris 1969.

[8] BRENIER Y. *Convergence of the Vlasov-Poisson system to the incompressible Euler equations*, Comm. P.D.E., **25** (2000), 737-754.

[9] CERCIGNANI C., GAMBA I. M., JEROME J. W. AND SHU C.-W., *Device benchmark comparisons via kinetic, hydrodynamic, and high-field models*, Comput. Methods Appl. Mech. Eng. **181** (2000), no. 4, 381–392.

[10] CERCIGNANI C., GAMBA I. M. AND LEVERMORE C. D., *A drift-collision balance for a Boltzmann-Poisson system in bounded domains*, SIAM J. Appl. Math. **61** (2001), no. 6, 1932–1958.

[11] CERCIGNANI C., GAMBA I. M. AND LEVERMORE C. D., *High field approximations to a Boltzmann-Poisson system and boundary conditions in a semiconductor*, Appl. Math. Lett. **10** (1997), no. 4, 111–117.

[12] DEGOND P., *Global existence of smooth solutions for the Vlasov-Fokker-Planck equation in 1 and 2 space dimension*, Ann. Scient. Ecole Normale Sup., **19** (1986), 519–542.

[13] DEGOND P. AND JÜNGEL A., *High field approximations of the energy-transport model for semicondustors with non-parabolic band structure*, Z. Angew. Math. Phys., **52** (2001), no 6, 1053–1070.

[14] DELORT J.-M., *Existence de nappes de tourbillon en dimension deux*, J. Amer. Math. Soc., **4** (1991), 553–586.

[15] MARKOWICH P. AND RINGHOFER C., *Quantum hydrodynamics for semiconductors in the high field case*, Appl. Math. Lett., **7** (1994), no 5, 37–41.

[16] NIETO J., *Hydrodynamical limit for a drift-diffusion system modeling large-population dynamics*, J. Math. Anal. Appl. **291**, (2004), 716–726.

[17] Nieto J., Poupaud F. and Soler J., *High-field limit for the Vlasov-Poisson-Fokker-Planck system*, Arch. Rat. Mech. Anal., **158** (2001), 29–59.

[18] Poupaud F. and Soler J., *Parabolic limit and stability of the Vlasov-Poisson-Fokker-Planck system*, Math. Models Methods Appl. Sci., Vol. **10**, No **7**, (2000), 1027-1045.

[19] Poupaud F., *Runaway phenomena and fluid approximation under high fields in semiconductor kinetic theory*, Z. angew. Math. Mech., **72** (1992), 359–372.

[20] Poupaud F., *Diagonal deffect measures, adhesion dynamics and Euler equations*, Meth. Appl. Analysis, **9** (2002), 533-561.

[21] Schochet S., *The weak vorticity formulation of the 2D Euler equations and concentration-cancellation*, Comm. PDE, **20** (1995), 1077–1104.

[22] Semmes S., *A primer on Hardy spaces, and some remarks on a theorem of Evans and Müller*, Comm. PDE, **19** (1994), 277–314.

[23] Soler J., *Asymptotic behaviour for the Vlasov-Poisson-Fokker-Planck system*, Nonlinear Anal. TMA. **30** (1997), 5217-5228.

[24] Stein E. M., **Singular integrals and differentiability properties of functions**. Princeton University Press, Princeton, New Jersey, 1970.

[25] Schwartz, L., **Cours d'analyse** Vol. 1. Hermann, Paris, 1981.