VANISHING PRESSURE
IN GAS DYNAMICS EQUATIONS

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Abstract. This work is devoted to the analysis of the behaviour of solutions of gas dynamics equations as the pressure goes to 0 in the context of regular solutions. We obtain in this way a first justification of the connection to pressureless gases model.

Key Words. One-dimensional degenerate hyperbolic system, Zero pressure gas dynamics

AMS Subject Classification. 35L65, 76P05, 35Q35, 76N15

1 Introduction

This work is a first attempt to deal with vanishing pressure in the following system of gas dynamics equations

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x(\rho u) = 0 \;\; in \; \mathbb{R}_t^+ \times \mathbb{R}_x \,, \\ \partial_t(\rho u) + \partial_x(\rho u^2/2 + \varepsilon^2 p(\rho)) = 0 \;\; in \; \mathbb{R}_t^+ \times \mathbb{R}_x \,. \end{array} \right. \tag{1}$$

In (1), the pressure law $p : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is required to satisfy

$$p'(\rho) > 0 \; for \; \rho > 0, \tag{2}$$

$$p"(\rho) \geq 0, \tag{3}$$

$$2p'(\rho) \geq \rho p"(\rho). \tag{4}$$

In the sequel, we will denote by $P(\rho)$ a function verifying

$$P'(\rho) = \sqrt{p'(\rho)}/\rho.$$

Notice in particular that the power laws $p(\rho) = \rho^\gamma$, with $\gamma > 1$ as well as the isothermal law $p(\rho) = \rho$ fulfill (2-3), but (4) restricts to $1 \leq \gamma \leq 3$. We obtain in these cases $P(\rho) = 2\sqrt{\gamma}/(\gamma - 1)\,\rho^{(\gamma-1)/2}$ for $\gamma > 1$ and $P(\rho) = \ln(\rho)$ for $\gamma = 1$. The problem is supplemented by prescribying Cauchy data for the density $\rho(0, x) = \rho_0(x) \geq 0$ and the velocity $u(0, x) = u_0(x)$. Formally, as ε vanishes, we are led to the following pressureless gases equations

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x(\rho u) = 0 \;\; in \; \mathbb{R}_t^+ \times \mathbb{R}_x \,, \\ \partial_t(\rho u) + \partial_x(\rho u^2/2) = 0 \;\; in \; \mathbb{R}_t^+ \times \mathbb{R}_x \,. \end{array} \right. \tag{5}$$

The system (5) was introduced as a simplified model of the dynamics of galaxies by Zeldovich, see [9] and a first mathematical analysis is due to Bouchut in [1]. Existence of solutions and connection to scalar conservation laws are studied by

Grenier [5], Brenier-Grenier [4] and, also by Bouchut-James who used in [3] the notion of duality solutions of [2] (see also Poupaud-Rascle [8] where an equivalent formulation is proposed). Regular solutions, even in the multidimensional context, are investigated by Poupaud [7].

In this work, we restrict to the case of regular solutions: crucial bounds are obtained by considering the Riemann invariants associated to (1), and, then we pass to the limit as ε goes to 0. In this way, we obtain the following

Theorem 1 *Let $\rho_0^\varepsilon, u_0^\varepsilon$ be a sequence of initial data for (1) satisfying*

$$\left\{\begin{array}{l} \rho_0^\varepsilon \geq d_\varepsilon > 0, \quad \partial_x(u_0^\varepsilon \pm \varepsilon P(\rho_0^\varepsilon)) \geq 0, \\ \rho_0^\varepsilon \rightharpoonup \rho_0 \, weakly * \, in \, L^\infty_{loc}(\mathbb{R}), \\ u_0^\varepsilon \rightharpoonup u_0 \, in \, W^{1,\infty}_{loc}(\mathbb{R}). \end{array}\right. \tag{6}$$

where d_ε is a sequence of positive reals, possibly tending to 0. Then the associated solutions of (1) satisfy

$$\begin{array}{l} \rho^\varepsilon \rightharpoonup \rho \, weakly * \, in \, L^\infty_{loc}(\mathbb{R}^+ \times \mathbb{R}), \\ u^\varepsilon \to u \, in \, C^0_{loc}(\mathbb{R}^+ \times \mathbb{R}), \end{array}$$

where (ρ, u) is the unique solution to (5) with initial data (ρ_0, u_0).

Remark 1 *Note that (6) implies that $\partial_x u_0^\varepsilon \geq \varepsilon |\partial_x(P(\rho_0^\varepsilon))|$, thus $\partial_x u_0^\varepsilon$ is non-negative, and, passing to the limit, $\partial_x u_0$ is too. Moreover, in terms of ρ_0^ε, this yields $|\partial_x(\rho_0^\varepsilon)| \leq 1/\varepsilon \, \rho_0^\varepsilon / \sqrt{p'(\rho_0^\varepsilon)} \, \partial_x u_0^\varepsilon$ which essentially means that $\partial_x \rho_0^\varepsilon$ may blow up as $(\varepsilon\sqrt{p'(d_\varepsilon)})^{-1}$.*

In Section 2, we perform some computations on the Riemann invariants $u \pm \varepsilon P(\rho)$ associated to solutions of (1), following essentially ideas of P. D. Lax [6]. This allows us to discuss crucial bounds on the solutions in Section 3 and we conclude in Section 4.

2 Riemann Invariants

Let $c = \sqrt{p'(\rho)}$. We set $w = u + \varepsilon P(\rho)$ and $z = u - \varepsilon P(\rho)$, where (ρ, u) satisfies (1). Then, assuming that ρ are u are regular enough, some elementary computations leads to

$$\left\{\begin{array}{l} \partial_t w + (u + \varepsilon c)\partial_x w = 0, \\ \partial_t z + (u - \varepsilon c)\partial_x z = 0. \end{array}\right. \tag{7}$$

Following P. D. Lax, [6], possible loss of regularity of the solutions can be observed by considering the equations satisfied by the first derivative of w and z. Let $W = \partial_x w$, $Z = \partial_x z$ and set $\alpha = \dfrac{\rho p"(\rho)}{2p'(\rho)} = c'(\rho)/P'(\rho)$. By deriving (7) we get

$$\left\{\begin{array}{l} \partial_t W + (u + \varepsilon c)\partial_x W + (1+\alpha)/2\, W^2 + (1-\alpha)/2\, ZW = 0, \\ \partial_t Z + (u - \varepsilon c)\partial_x Z + (1+\alpha)/2\, Z^2 + (1-\alpha)/2\, ZW = 0, \end{array}\right. \tag{8}$$

since $\partial_x u = \partial_x(\frac{w+z}{2}) = 1/2(W+Z)$ and $\partial_x(\varepsilon P(\rho)) = 1/2(W-Z)$. We introduce the characteristics associated to the velocities $u \pm \varepsilon c$, namely

$$\begin{cases} \frac{d}{dt}X(t,x) = u(t,X(t,x)) + \varepsilon c(t,X(t,x)), \\ \frac{d}{dt}Y(t,x) = u(t,Y(t,x)) - \varepsilon c(t,Y(t,x)), \end{cases}$$

and $X(0,x) = x = Y(0,x)$. For ϕ a real-valued function depending on time and position, we set $\phi^\sharp(t,x) = \phi(t,X(t,x))$ and $\phi^\flat(t,x) = \phi(t,Y(t,x))$. Now, we can rewrite (8) as the following ode

$$\frac{d}{dt}S + AS^2 + BS = 0, \tag{9}$$

where S stands for $W^\sharp$ and $A = (1+\alpha^\sharp)/2$, $B = (1-\alpha^\sharp)/2\,Z^\sharp$ or $S = Z^\flat$, $A = (1+\alpha^\flat)/2$, $B = (1-\alpha^\flat)/2\,W^\flat$, respectively. In (9), the function B can be viewed as the time derivative of another function. We introduce h and k as functions of the two variables (w,z) verifying $\partial_z h(w,z) = (1-\alpha(\rho))/(2\varepsilon c(\rho)) = \partial_w k(w,z)$ with $\rho = P^{-1}((w-z)/2\varepsilon)$, P^{-1} being the inverse of P. Indeed, we remark that

$$\begin{aligned} \frac{d}{dt}(h^\sharp(t,x)) &= \frac{d}{dt}(h(w^\sharp(t,x), z^\sharp(t,x)) \\ &= (\partial_z h)^\sharp(t,x)\,(2\varepsilon c\partial_x z)(t,X(t,x)) \\ &= ((1-\alpha)Z)^\sharp = +2B \end{aligned} \tag{10}$$

by (7). Similarly, we get

$$\frac{d}{dt}(k^\flat) = (\partial_w k)^\flat(-2\varepsilon c(\partial_x w)(t,Y(t,x)) = -((1-\alpha)W)^\flat = -2B. \tag{11}$$

Combining (10) and (11) to (9) leads to

$$\frac{d}{dt}S + AS^2 + (\frac{d}{dt}C)S = 0 \tag{12}$$

with $C = h^\sharp/2$ or $-k^\flat/2$. This Ricatti-like equations can be integrated easily; we obtain

$$\begin{aligned} S(t,x) = \Bigg(& 1/S(0,x)\,\exp((C(t,x) - C(0,x)) \\ & + \int_0^t A(\tau,x)\exp(C(t,x) - C(\tau,x))\,d\tau\Bigg)^{-1}. \end{aligned} \tag{13}$$

Since $A = (1+\alpha)/2 \geq 1/2 > 0$ by (2-3), for $S(0,x) \geq 0$ the solution of (13) exists globally in time which in turn leads to global existence for (1).

3 Estimates

The computations made in the previous Section allow us to derive some estimates on the solution of (1). We have

Proposition 1 *Let $\rho_0^\varepsilon, u_0^\varepsilon$ be the initial data of (1) satisfying the requirements of Theorem 1. Then there exists a global regular solution $(\rho^\varepsilon, u^\varepsilon)$ of (1) and the sequences $(u_\varepsilon, \varepsilon P(\rho^\varepsilon))_{\varepsilon>0}$ are bounded in $L^\infty(\mathbb{R}^+; W^{1,\infty}_{loc}(\mathbb{R}))$ while $(\rho_\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty_{loc}(\mathbb{R}^+ \times \mathbb{R})$.*

We turn to the Riemann invariant. Assumption (6) means that $W(0,x)$ and $Z(0,x)$ are non-negative. By (13), one deduces that W and Z remain non-negative and are defined for all time. In turn, this implies that $t \longmapsto h^\sharp(t,x)$ is non-decreasing by (10) and $t \longmapsto k^\flat(t,x)$ is non-increasing by (11), since (4) means that $1-\alpha \geq 0$. Thus $t \longmapsto C(t,x)$ is non-decreasing and it follows that

$$0 \leq S(t,x) \leq \left(1/S(0,x) + t/2\right)^{-1}. \tag{14}$$

Therefore $W = \partial_x w^\varepsilon$ and $Z = \partial_x z^\varepsilon$ are bounded in $L^\infty(\mathbb{R}^+ \times \mathbb{R}_{loc})$ while integrating (7) along the characteristics provides a $L^\infty(\mathbb{R}^+ \times \mathbb{R}_{loc})$ bound on $w^\varepsilon, z^\varepsilon$. Thus, $u^\varepsilon = 1/2(w^\varepsilon + z^\varepsilon)$ and $\varepsilon P(\rho^\varepsilon) = 1/2(w^\varepsilon - z^\varepsilon)$ are bounded in $L^\infty(\mathbb{R}; W^{1,\infty}(-R,+R))$ for all $0 < R < \infty$. Finally, coming back to (1) we get

$$\partial_t \rho^\varepsilon + u^\varepsilon \partial_x \rho^\varepsilon + \rho^\varepsilon \partial_x u^\varepsilon = 0$$

which can be integrated along the characteristic curves associated to the velocity u^ε and provides the estimate on ρ^ε in $L^\infty((0,T)\times(-R,+R))$ for all $0 < T, R < \infty$.

Remark 2 *One needs assumption (4) to obtain a uniform bound on $\partial_x u^\varepsilon$ and $\partial_x(\varepsilon P(\rho^\varepsilon))$. Indeed, for $\varepsilon > 0$ given, by (13) W, Z belongs to $L^\infty(\mathbb{R} \times \mathbb{R}_{loc})$, but this estimate is not uniform wrt. ε. For instance, considering power law with $\gamma > 3$, W, Z blow up as $e^{1/\varepsilon}$.*

4 Passage to the limit

We are ready to end the proof of Theorem 1. Since u^ε satisfies

$$\begin{aligned} &\partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + \varepsilon^2/\rho^\varepsilon \partial_x(p(\rho^\varepsilon)) = 0 \\ &= \partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + \varepsilon\sqrt{p'(\rho^\varepsilon)}\partial_x(\varepsilon P(\rho^\varepsilon)) = 0 \end{aligned} \tag{15}$$

by (1), we deduce from Proposition 1 that $\partial_t u^\varepsilon$ lies in a bounded set of $L^\infty_{loc}(\mathbb{R}^+ \times \mathbb{R})$. Indeed, ρ^ε is bounded in $L^\infty_{loc}(\mathbb{R}^+ \times \mathbb{R})$ and by (2), p' is non decreasing, thus $p'(\rho^\varepsilon)$ is bounded in $L^\infty_{loc}(\mathbb{R}^+ \times \mathbb{R})$. Therefore, Proposition 1 implies that the last term in (15) is bounded (and actually tends to 0!). One deduces that u^ε belongs to a bounded set in $W^{1,\infty}_{loc}(\mathbb{R}^+ \times \mathbb{R})$ and by Ascoli's theorem, we can assume

$$u^\varepsilon \to u \text{ in } C^0((0,T)\times(-R,+R)),\ 0 < T, R < \infty, \tag{16}$$

at least for a subsequence. Furthermore, we can also suppose, say, $\rho^\varepsilon \rightharpoonup_* \rho$ in $L^\infty((0,T)\times(-R,+R))$. These convergence properties permit us to pass to the limit in the $\mathcal{D}'$ sense in (1) and we are led to (5) as ε goes to 0. We check easily that the initial data for (ρ, u) are the limits (ρ_0, u_0) of $(\rho_0^\varepsilon, u_0^\varepsilon)$. Since ρ_0 has no atomic part, uniqueness for (5) follows, see [3], which ensures that the whole sequence converges and achieves the proof of Theorem 1. Note that we can also pass to the limit in (15) which shows that u is a regular solution of the Burgers equation.

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