VANISHING VISCOSITY REGIMES AND NON–STANDARD SHOCK RELATIONS FOR SEMICONDUCTOR SUPERLATTICES MODELS[*]

T. GOUDON[†], J. NIETO[‡], O. SÁNCHEZ[‡], AND J. SOLER[‡]

Abstract. This paper is concerned with the analysis of asymptotic problems from discrete drift-diffusion models describing charge transport in semiconductor superlattices. The regimes we are interested in lead to balance laws. However, the nonconservative structure of the discrete system might produce defect measure terms in the limit process. These defect terms, concentrated on the shock discontinuities, can be related to non standard jump relations (in contrast with the usual Rankine–Hugoniot law) when considering discontinuous solutions and wave fronts.

Key words. High–field limits, semiconductor superlattices, drift-diffusion models, lattice differential equations, conservation laws, shock waves, travelling waves, compensated–compactness.

AMS subject classifications. 35Q99, 82C70, 35L67, 35Q40, 81V10

1. Introduction. A semiconductor superlattice (SL) is an array of layers with different semiconductor materials. In such devices, the period of the structure, that will be denoted by l in what follows, is small compared to the lateral dimension. Typically, depending on the charge density and the bias (voltage) applied at the contacts of these devices, we can observe different charge transport regimes. This work is focused on the so-called high–field regime, where self–sustained oscillations appear. It consists of a sequence of shock profiles that are produced at the contact junctions. The aim is to deduce the velocities of these shocks, or traveling waves solutions, in a continuum limit.

There are different approaches describing, in agreement with physical experiments, the behavior of the electron density in semiconductor superlattices. The models proposed in the literature range from purely quantum to kinetic or drift-diffusion systems, including various combinations of them depending on the basic ingredients to transport: band structure, field strength and scattering. In this work we focus on one of these models, proposed in [3], for weakly coupled superlattices, where the main transport mechanism is the sequential resonant tunneling. We also refer to the extension of these models in [2, 19] and the references therein. The model consists of a discrete drift–diffusion system in which quantum effects are incorporated in the drift velocity and the diffusion coefficient. The purpose of this paper is to study the hierarchy of models which arises when performing the continuum limit for this discrete drift–diffusion system (DDD). A specific attention must be paid to the case where the diffusion effects arise at a lower order compared to transport effects. We wish in this way to shed some light, from a mathematical point of view, on the wave fronts that give rise to the self–sustained oscillations [2, 7]. It turns out that obtaining an expression for the speed of the front leads to non standard mathematical difficulties. Several expressions for this velocity have been proposed in the literature and we wish to discuss their range of validity based on scaling arguments and compactness

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[†] INRIA Project-Team SIMPAF & labo. Paul Painlevé CNRS-USTL, INRIA Lille Nord Europe Research Centre, Park Plazza 40, avenue Halley, F-59650 Villeneuve d'Ascq cedex, France (thierry.goudon@inria.fr).

[‡]Dept. de Matemática Aplicada, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain (jjmnieto@ugr.es, ossanche@ugr.es, jsoler@ugr.es).

techniques.

1.1. The discrete drift-diffusion model. The behavior of the charge in the SL can be described through the DDD system of differential equations satisfied by the electron density $n_i(t)$ and the electric field $F_i(t)$ where the index i refers to the cell we consider in the array. We refer to [3] or the review [2] for a detailed derivation of the equations.

First, we have a discrete Poisson equation relating the field and the density

$$F_i - F_{i-1} = \frac{e}{\varepsilon}(n_i - N_D), \tag{1.1}$$

where $(-e)$ is the elementary electron charge, ε is the average permittivity in the SL and N_D stands for the doping density in the wells. In this paper we restrict our attention to the case where the doping profile is constant so that N_D is a given positive constant.

Second, the evolution of the density in the ith cell is driven by the following charge continuity equation

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} = \mathscr{J}_{i-1\to i} - \mathscr{J}_{i\to i+1}, \tag{1.2}$$

where $\mathscr{J}_{i\to i+1}$ denotes the tunneling current density through the barrier separating the cells $\#i$ and $\#(i+1)$. We remark immediately that

$$\frac{\varepsilon}{e}\frac{\mathrm{d}F_i}{\mathrm{d}t} + \mathscr{J}_{i\to i+1} = \mathscr{J}(t), \tag{1.3}$$

does not depend on the considered cell. This is the so-called Ampère law, where $e\mathscr{J}(t)$ stands for the total current density through the SL. The model is completed by a constitutive law which defines the current density $\mathscr{J}_{i\to i+1}$ by means of the (n_k, F_k)'s. Such a constitutive law has essentially a phenomenological basis; we refer for example to [2] or to [19, §3.3 and §5] for possible derivations. Denoting by l the length of the SL cells, we adopt here the following simple definition

$$\mathscr{J}_{i\to i+1} = \frac{n_i v(F_i)}{l} - \frac{D(F_i)(n_{i+1} - n_i)}{l^2}, \tag{1.4}$$

which involves given non-negative functions of the electric field: v (drift velocity) and D (diffusion "coefficient"). The functions depend on the physical properties of the material used within the SL. Observe that we can use (1.1) and (1.4) to write a system of equations where the unknowns are the electric fields; indeed (1.3) then becomes

$$\frac{\mathrm{d}F_i}{\mathrm{d}t} + v(F_i)\frac{F_i - F_{i-1}}{l} - D(F_i)\frac{(F_{i+1} - 2F_i + F_{i-1})}{l^2} = \frac{e}{\varepsilon}\Big(\mathscr{J}(t) - \frac{v(F_i)N_D}{l}\Big). \tag{1.5}$$

Boundary conditions are needed to complete the problem and to make it well-posed. There exists a large variety of relevant extra conditions to close this system of equations, depending on the physical context, and the discussion of relevant boundary conditions is an important modeling issue. However, the idea of this work is to analyze shock profiles and thus we restrict the framework of our study to infinitely large superlattices. The index i ranges in the whole set of integers $\mathbb{Z}$, and we impose conditions at infinity. It is natural to require quasineutrality at infinity, that is

$$\lim_{i \to \pm\infty} n_i = N_D. \tag{1.6}$$

Consistently, owing to (1.1), we infer that the electric field tends to some constant state

$$\lim_{i \to -\infty} F_i = F_-, \qquad \lim_{i \to +\infty} F_i = F_+,$$

and, coming back to (1.5), we realize that the asymptotic state should verify

$$\frac{v(F_\pm)}{l} N_D = \mathscr{J}. \tag{1.7}$$

Therefore, in what follows, we restrict our study to the case in which the current is controlled: we assume that the total current $\mathscr{J}$ is a given constant. A typical shape of the function $v(F)$ is displayed in Figure 1.1 (see [3, 7] and [2, Appendix A] for exact formulas of v and D, and Appendix B for an explicit approximation based in a numerical fit). It allows to define $0 < F_{\mathrm{s},-} \leq F_{\mathrm{u}} \leq F_{\mathrm{s},+} < \infty$ solutions of (1.7).

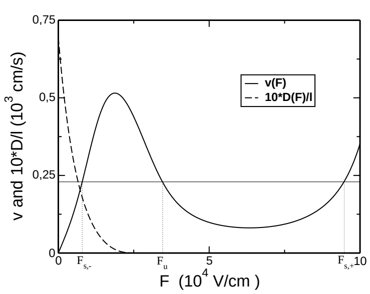

Fig. 1.1. *Drift-velocity $v(F)$ and (amplified) diffusion coefficient $10\frac{D(F)}{l}$. Horizontal line corresponds to a constant value of $\frac{l\mathscr{J}}{N_D}$ and states $0 < F_{\mathrm{s},-} \leq F_{\mathrm{u}} \leq F_{\mathrm{s},+} < \infty$ are solutions of* (1.7).

Throughout the paper we assume that the data are such that there exist three distinct such solutions; in such a case, $F_{\mathrm{s},-}$ and $F_{\mathrm{s},+}$ are stable states, while F_{u} is unstable (according to the sign of $v'(F)$ at these points). Therefore, a physically relevant case consists in dealing with solutions joining the equilibrium states $F_- = F_{\mathrm{s},-}$ and $F_+ = F_{\mathrm{s},+}$ and one is interested in the characteristics of the corresponding front, a question addressed in [19, 2]. The concept of front is associated with the concept of heterocline solution in the framework of dynamical systems, see [1, 11]. Discrete shocks can be defined as well, and the theory of discrete dynamical systems provides the existence of discrete traveling front connecting the stable equilibrium states for the system (1.5), see Theorem 2.3 below. Note that dealing with finite SL, boundary conditions could introduce a new front perturbing the study of the dynamics of discontinuities; we refer to [9] for an analysis of this situation.

1.2. Dimensional issues and asymptotic problems. The aim of this section is to write the system in dimensionless form. The identification of dimensionless physical parameters will lead to discuss asymptotic questions. The presentation below follows [7]. Let T, L be time and length units, respectively. The doping profile N_D, which is assumed to be constant throughout the SL, defines a unit for the charge density. We denote by F a characteristic value of the electric field. Then, we set

$$\begin{cases} \mathrm{F}\ \overline{F_i}(\bar{t}) = F_i(\mathrm{T}\bar{t}), \\ \dfrac{\mathrm{L}}{\mathrm{T}}\ \overline{v}(\overline{F}) = v(\mathrm{F}\overline{F}), \\ \epsilon\dfrac{\mathrm{L}^2}{\mathrm{T}}\ \overline{D}(\overline{F}) = D(\mathrm{F}\overline{F}), \end{cases}$$

where the overline denotes dimensionless quantities. Note that the parameter $\epsilon > 0$ measures the strength of the diffusion effects compared to the convection effects.

Due to (1.7), the total current rescales as $\mathscr{J} = \frac{LN_D}{l\mathrm{T}}\overline{\mathscr{J}}$. We set

$$h = \frac{l}{\mathrm{L}}$$

and

$$\lambda = \sqrt{\frac{\varepsilon\ \mathrm{F}}{e\ N_D}}.$$

The length λL is interpreted as the Debye length of the device. We see on Figure 1.1 that the drift coefficient vanishes as F goes to 0, and thus diffusion effects have to be considered at leading order. This is the "low-field" regime investigated in [7]. However, as F increases, the drift effects become dominant. For this reason, the regime where ϵ goes to 0 is referred to as the "high-field" regime. Hence (1.5) becomes in dimensionless form

$$\frac{\mathrm{d}\overline{F}_i}{\mathrm{d}\bar{t}} + \frac{1}{h}\overline{v}(\overline{F}_i)(\overline{F}_i - \overline{F}_{i-1}) - \frac{\epsilon}{h^2}\overline{D}(\overline{F}_i)(\overline{F}_{i+1} - 2\overline{F}_i + \overline{F}_{i-1}) = \frac{1}{\lambda^2 h}\big(\overline{\mathscr{J}} - \overline{v}(\overline{F}_i)\big). \quad (1.8)$$

Physical constants and properties of the SL are embodied into the three dimensionless quantities $h, \lambda, \epsilon > 0$.

In [7] the theoretical analysis of the situation $h \simeq 1/\lambda^2 \ll 1$ and $\epsilon = 1$ on a finite superlattice was performed. Adapted to the present situation with a given total current, it leads to the following continuous convection-diffusion model

$$\partial_t F + v(F)\partial_x F - \epsilon D(F)\partial^2_{xx} F = \mathscr{J} - v(F), \quad (1.9)$$

which has been already introduced in [19, eq. (174)].

Here, we focus our interest on high-field regimes where in addition to $h \simeq 1/\lambda^2 \ll 1$ we assume $\epsilon \ll 1$. In this case, we face a new difficulty related to the existence of discrete shocks and the non conservative form of the discrete system. As stated in [2], this regime can be intuitively associated to the balance law

$$\partial_t F + \partial_x V(F) = \mathscr{J} - v(F), \qquad V'(F) = v(F). \quad (1.10)$$

However, due to the non conservative structure of the original system (note that $v(F_i)\frac{F_i - F_{i-1}}{h}$ is not in divergence form), we will obtain the balance law (1.10) up to some defect terms which appear as a source supported only by the discontinuities of the solutions. We will show that the solutions of (1.8) converge towards solutions of (1.10) as far as the latter are regular and do not develop discontinuities. When a singularity occurs new terms appear as a consequence of the non conservative structure of the the original system. In particular, we shall see that the propagation speed of the shock wave solutions is not the standard one which can be derived from the Rankine–Hugoniot relations. This is a well-known feature in numerical analysis of hyperbolic equations: non conservative schemes lead to wrong solutions associated with the original conservation laws. This question has been investigated in details in [9]. Similar problems have been dealt with recently in [8], based on the kinetic interpretation of scalar conservation laws, see [13] and [5] for adaptation to equations with zeroth order source terms. As an interesting consequence, the analysis of the defect terms leads to non standard shock relations satisfied by the discontinuous solutions of the limit system. For instance, considering the equilibrium solutions $F_{\mathrm{s},\pm}$ devised above, the function

$$F(t,x) = \begin{cases} F_{\mathrm{s},-} & \text{if } x < \sigma t, \\ F_{\mathrm{s},+} & \text{if } x > \sigma t \end{cases} \tag{1.11}$$

is a discontinuous solution of (1.10), with the speed $\sigma = \sigma_{\mathrm{RH}}$ defined by the Rankine-Hugoniot condition

$$\sigma_{\mathrm{RH}} = \frac{V(F_{\mathrm{s},+}) - V(F_{\mathrm{s},-})}{F_{\mathrm{s},+} - F_{\mathrm{s},-}} = \frac{\displaystyle\int_{F_{\mathrm{s},-}}^{F_{\mathrm{s},+}} v(F)\,\mathrm{d}F}{F_{\mathrm{s},+} - F_{\mathrm{s},-}}. \tag{1.12}$$

As a consequence of the defect term, we shall obtain the same solution (1.11), but involving a different speed $\sigma \neq \sigma_{\mathrm{RH}}$. We summarize our study by the following diagram that shows the different regimes we wish to investigate

$$\begin{array}{ccc} \boxed{\begin{array}{c} F_i^{\epsilon,h}(t) \\ \text{Discrete Drift Diffusion} \end{array}} & \xrightarrow[\text{[7]}]{h \to 0} & \boxed{\begin{array}{c} F_\epsilon(t,x) \\ \text{Continuous Drift Diffusion} \end{array}} \\ \Big\downarrow \epsilon \to 0 & \overset{h=\epsilon\to 0}{\searrow} & \Big\downarrow \epsilon \to 0 \\ \boxed{\begin{array}{c} F_i^{h}(t) \\ \text{Discrete Balance Law} \end{array}} & \xrightarrow[h \to 0]{} & \boxed{\begin{array}{c} F(t,x) \\ \text{Continuous Balance Law} \end{array}} \end{array} \tag{1.13}$$

The remarkable conclusion of our study is that the limits in this diagram do not commute. Performing the limit $(h \to 0)$ first and then $(\epsilon \to 0)$, we will obtain a modified shock speed σ_{diff} which retains information about the (apparently disappeared) discrete diffusion coefficient. However, the limit $(\epsilon \to 0)$ first and then $(h \to 0)$ leads to a different shock speed σ_{disc} which does not depend on D, but still differs from σ_{RH}. Furthermore, our analysis also indicates that the diagonal process where both h and ϵ go to 0 certainly leads to a delicate analysis, where the result might depend on the details of the relation $h = h(\epsilon)$. We give some hints in this direction, obtaining another possible expression for the shock speed. In the sequel we will use the notation summarized in the diagram; the dependence of the models with respect to the parameters are labeled with superindex on the corresponding solutions. For instance, the solutions to the DDD model (1.8) will be denoted by $F_i^{\epsilon,h}$.

The paper is organized as follows. In Section 2 we summarize some a priori estimates for the DDD solutions required for the limiting process, and justify the existence of discrete traveling waves by the theory of discrete dynamical systems. Section 3 is devoted to the asymptotic regime that leads to continuous models from the DDD one by considering the limits ($\epsilon \to 0$) first and next ($h \to 0$), in diagram (1.13). In Section 4, we proceed the other way around, by performing the upper limit, ($h \to 0$) first and ($\epsilon \to 0$) second, in diagram (1.13). Since the continuous limit $h \to 0$ was essentially developed in [7] we mainly deal with the limit $\epsilon \to 0$, that is a vanishing viscosity analysis. Section (5) ends the paper with some comments and conclusions.

2. Analysis of the DDD model. Throughout the paper we assume that v and D are C^1 non-negative functions having a typical shape as displayed in Figure 1.1. In particular, we shall use the property:

$$\text{for any } 0 < F_{s,-} \leq F \leq F_{s,+}, \text{ we have } \quad \begin{cases} 0 < v(F) \leq M, \\ 0 < \delta \leq D(F) \leq M, \\ |v'(F)| \leq M, \quad |D'(F)| \leq M. \end{cases} \tag{2.1}$$

We denote

$$V(F) = \int_0^F v(z)\,\mathrm{d}z.$$

Let us start by establishing some properties of the solutions to the lattice differential equation defined by (1.8). We first show that (1.8) is well-posed and admits bounded solutions, provided that nonlinearities are Lipschitz continuous and sublinear, see (2.1). This fact guarantees that $(F_i^{\epsilon,h}(t),\, n_i^{\epsilon,h}(t))$ and their time derivative are bounded, with bounds depending *a priori* on ϵ and h. We shall show that the field $F_i^{\epsilon,h}(t)$ is uniformly bounded with respect to h and ϵ and, in fact, the sequence takes values between the equilibrium states $F_{\mathrm{s},-}$ and $F_{\mathrm{s},+}$. In agreement with the physical requirements, the densities $n_i^{\epsilon,h}(t)$ remain non-negative. Furthermore, based on a discrete maximum principle, we can also prove a local in time uniform bound, which thus implies a $W^{1,\infty}$ bound for the field. Having in mind the limiting balance law (1.10), this estimate corresponds to the case of an initially regular solution that might develop shock waves in finite time.

THEOREM 2.1. *Let $F_i^{\epsilon,h}(0)$ be an initial condition verifying $0 < F_{\mathrm{s},-} \leq F_i^{\epsilon,h}(0) \leq F_{\mathrm{s},+}$ and, moreover,*

$$\sum_{i=0}^{\infty} |F_i^{\epsilon,h}(0) - F_{\mathrm{s},+}|^2 < \infty, \qquad \sum_{i=-\infty}^{0} |F_i^{\epsilon,h}(0) - F_{\mathrm{s},-}|^2 < \infty. \tag{2.2}$$

Then,

i) There exists a unique and globally defined solution $t \mapsto F_i^{\epsilon,h}(t)$ of (1.8) associated to this initial data which verifies $0 < F_{\mathrm{s},-} \leq F_i^{\epsilon,h}(t) \leq F_{\mathrm{s},+}$. In addition, the sums $\sum_{i=0}^{\infty} |F_i^{\epsilon,h}(t) - F_{\mathrm{s},+}|^2$, $\sum_{i=-\infty}^{0} |F_i^{\epsilon,h}(t) - F_{\mathrm{s},-}|^2$ are finite, for any $0 \leq t < \infty$. In particular the quasineutrality condition (1.6) holds.

ii) Let $n_i^{\epsilon,h}$ be associated to $F_i^{\epsilon,h}$ by the Poisson relation

$$n_i^{\epsilon,h} - 1 = \frac{F_i^{\epsilon,h} - F_{i-1}^{\epsilon,h}}{h}.$$

If $n_i^{\epsilon,h}(0) \geq 0$ initially, then, for any $t \geq 0$ we have $n_i^{\epsilon,h}(t) \geq 0$.

iii) In addition to the hypothesis in i) and ii), assume that $\sup_{\epsilon,h}(\sup_i(n_i^{\epsilon,h}(0))) \leq C < \infty$ *initially. Then there exists* $T^* < \infty$ *such that* $\sup_{\epsilon,h}(\sup_i(n_i^{\epsilon,h}(t))) \leq C(t) < \infty$ *for any* $0 \leq t < T^*$*, where* $C(t)$ *is a nonnegative increasing function.*

Proof. We start by considering a truncated version of the system (1.8). Let $I \in \mathbb{N}$, the index i lies in $\{-I, ... + I\}$ and we impose the value of the boundary terms $F_{-I-1} = F_-$ and $F_{I+1} = F_+$. Basic arguments of the theory of ordinary differential equations guarantee that the system is well-posed, at least on a small enough time interval, for any initial data.

Now, let F_e be an equilibrium state, solution of $\mathscr{J} = v(F)$. Suppose that initially $F_i(0) \geq F_e$ and similarly for the boundary data $F_\pm \geq F_e$. We shall prove that the F_i's remain bounded from below by F_e. To this end, we set $G_i = F_i - F_e$. Assume the existence of $0 \leq t^* < \infty$ being the first time such that there exists some $k \in \{-I, ... + I\}$ verifying $G_k(t^*) = 0$. We can suppose that either $G_{k-1}(t^*) > 0$ or $G_{k+1}(t^*) > 0$, otherwise $G_i(t^*)$ will be identically 0 (or equivalently, the solution will be identically equal to F_e). Then, checking the general equation for G_i:

$$\begin{aligned}\frac{\mathrm{d}G_i}{\mathrm{d}t} &= \left(\frac{v(F_i)}{h}G_{i-1} + \frac{\epsilon D(F_i)}{h^2}(G_{i+1} + G_{i-1})\right) - \left(\frac{v(F_i)}{h} + \frac{2\epsilon D(F_i)}{h^2}\right)G_i \\ &\quad + \Big(\mathscr{J} - v(F_i)\Big).\end{aligned}$$

we obtain $\frac{\mathrm{d}}{\mathrm{d}t}G_k(t^*) > 0$, which shows that G_k remains non-negative. A similar proof can be adapted for the bound from above.

Let us denote by $F_i^{(I)}(t)$ the solution obtained with the boundary data $F_{-I-1}^{(I)} = F_{s,-}$ and $F_{I+1}^{(I)} = F_{s,+}$, extended on $\mathbb{Z}$ by setting $F_i^{(I)}(t) = F_{s,\pm}$ when $\pm i \geq I + 1$. The corresponding sequence satisfies

$$F_{s,-} \leq F_i^{(I)}(t) \leq F_{s,+},$$

and therefore it is globally defined with $F_i^{(I)} \in C^1([0, \infty))$. Furthermore, $|\frac{\mathrm{d}}{\mathrm{d}t}F_i^{(I)}|$ is bounded uniformly with respect to i and I (but the estimate depends on h and ϵ). By combining the Arzela-Ascoli theorem and a diagonal argument we can extract a subsequence such that, for any i, $F_i^{(I)}(t)$ converges to some $F_i^{\epsilon,h}(t)$ as $I \to \infty$, uniformly on any finite time interval $[0, T]$. Clearly, the limit $F_i^{\epsilon,h}$ verifies (1.8) as well as the estimate $F_{s,-} \leq F_i^{\epsilon,h} \leq F_{s,+}$.

Furthermore, using $\mathscr{J} = v(F_{s,+})$ and expanding in (1.8) $F_{i+1}^{(I)} - 2F_i^{(I)} + F_{i-1}^{(I)} = F_{i-1}^{(I)} - F_i^{(I)} + (F_{i+1}^{(I)} - F_{s,+}) + (F_{s,+} - F_i^{(I)})$, we check that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=0}^{\infty}|F_i^{(I)} - F_{s,+}|^2 &= 2\sum_{i=0}^{\infty}(F_i^{(I)} - F_{s,+})\left\{-\Big(\frac{\epsilon D(F_i^{(I)})}{h^2} + \frac{v(F_i^{(I)})}{h}\Big)(F_i^{(I)} - F_{i-1}^{(I)})\right. \\ &\quad \left. + \Big(v(F_{s,+}) - v(F_i^{(I)})\Big) + \frac{\epsilon D(F_i^{(I)})}{h^2}(F_{i+1}^{(I)} - F_{s,+}) - \frac{\epsilon D(F_i^{(I)})}{h^2}(F_i^{(I)} - F_{s,+})\right\},\end{aligned}$$

and then, using (2.1) we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=0}^{\infty}|F_i^{(I)}-F_{s,+}|^2+\delta\frac{\epsilon}{h^2}\sum_{i=0}^{\infty}|F_i^{(I)}-F_{s,+}|^2\leq M\sum_{i=0}^{\infty}|F_i^{(I)}-F_{s,+}|^2$$
$$+\Big(\frac{\epsilon M}{h^2}+\frac{M}{h}\Big)\sum_{i=0}^{\infty}|F_i^{(I)}-F_{s,+}||F_i^{(I)}-F_{i-1}^{(I)}|+\frac{\epsilon M}{h^2}\sum_{i=0}^{\infty}|F_i^{(I)}-F_{s,+}||F_{i+1}^{(I)}-F_{s,+}|,$$

holds. Note that the infinite sums are actually finite, so we have used the finite or infinite notation by convenience. By using the Young inequality and the Gronwall lemma we conclude that, for any $0\leq t\leq T<\infty$,

$$\sum_{i=0}^{\infty}|F_i^{(I)}-F_{s,+}|^2\leq C<\infty$$

holds with a constant C depending on $T,\epsilon,h,M,F_{s,\pm}$ and $\sum_{i=0}^{\infty}|F_i^{(I)}(0)-F_{s,+}|^2$. Letting $I\to\infty$, the Fatou lemma implies that the same property applies to $F_i^{\epsilon,h}(t)$. It proves that $\lim\limits_{i\to\infty}F_i^{\epsilon,h}(t)=F_{s,+}$. The same reasoning applies to show $\lim\limits_{i\to-\infty}F_i^{\epsilon,h}(t)=F_{s,-}$ and then, (1.6) holds. We skip the details but similar arguments prove uniqueness since we can show that if F^1 and F^2 are two solutions of (1.8) with the same behavior at infinity, then

$$\sum_{i=-\infty}^{\infty}\big|(F_i^1-F_i^2)(t)\big|^2\leq C\sum_{i=-\infty}^{\infty}\big|(F_i^1-F_i^2)(0)\big|^2$$

holds, with C depending on T,ϵ,h.

To prove ii) we write the equation for the densities as follows

$$\frac{\mathrm{d}n_i^{\epsilon,h}(t)}{\mathrm{d}t}=n_{i-1}^{\epsilon,h}\Big(\frac{v(F_{i-1}^{\epsilon,h})}{h}+\epsilon\frac{D(F_{i-1}^{\epsilon,h})}{h^2}\Big)+\epsilon\, n_{i+1}^{\epsilon,h}\frac{D(F_i^{\epsilon,h})}{h^2}$$
$$-n_i^{\epsilon,h}\Big(\frac{v(F_i^{\epsilon,h})}{h}+\epsilon\frac{(D(F_i^{\epsilon,h})+D(F_{i-1}^{\epsilon,h}))}{h^2}\Big). \tag{2.3}$$

Let us assume the existence of $0\leq t^*<\infty$ being the first time such that $n_k(t^*)=0$ for some k. We can assume that $n_{k+1}(t^*)$ or $n_{k-1}(t^*)$ is positive, otherwise $n_i(t^*)$ would be identically zero, which contradicts the quasineutrality condition (1.6). Therefore, we get $\frac{\mathrm{d}}{\mathrm{d}t}n_k^{\epsilon,h}(t^*)>0$ which shows that n_k remains non-negative.

The proof of iii) relies on a similar argument together with a nonlinear version of the Gronwall lemma. Define

$$m_i(t)=n_i^{\epsilon,h}(t)-\sup_k(n_k^{\epsilon,h}(0))-2M\int_0^t\big(\sup_k n_k^{\epsilon,h}(s)+1\big)^2\,\mathrm{d}s,$$

where M is the bound on v' arising in the hypothesis (2.1). Then, let us show that $m_i(t)$ are non-positive. Initially we have $m_i(0)\leq 0$. Actually there exist indices such that $m_i(0)<0$ (otherwise $F_i^{\epsilon,h}(0)$ would be constant, which contradicts (2.2)). We have

$$\frac{\mathrm{d}m_i}{\mathrm{d}t}=-v(F_i^{\epsilon,h})\frac{m_i-m_{i-1}}{h}+\frac{\epsilon}{h^2}\Big(D(F_i^{\epsilon,h})(m_{i+1}-m_i)-D(F_{i-1}^{\epsilon,h})(m_i-m_{i-1})\Big)$$
$$-\Big(\frac{v(F_i^{\epsilon,h})-v(F_{i-1}^{\epsilon,h})}{F_i^{\epsilon,h}-F_{i-1}^{\epsilon,h}}\Big)\Big(\frac{F_i^{\epsilon,h}-F_{i-1}^{\epsilon,h}}{h}\Big)n_{i-1}^{\epsilon,h}-2M\big(\sup_k n_k^{\epsilon,h}+1\big)^2.$$

By using the Poisson law, the two last terms can be dominated by

$$\Big|\frac{v(F_i^{\epsilon,h})-v(F_{i-1}^{\epsilon,h})}{F_i^{\epsilon,h}-F_{i-1}^{\epsilon,h}}\Big|\ \big|n_i^{\epsilon,h}-1\big|\ n_{i-1}^{\epsilon,h}-2M\big(\sup_k n_k^{\epsilon,h}+1\big)^2$$
$$\le M\big(n_i^{\epsilon,h}+1\big)\sup_k n_k^{\epsilon,h}-2M\big(\sup_k n_k^{\epsilon,h}+1\big)^2<0.$$

Therefore, we have

$$\frac{\mathrm{d}m_i}{\mathrm{d}t}<m_{i-1}\Big(\frac{v(F_i^{\epsilon,h})}{h}+\epsilon\frac{D(F_{i-1}^{\epsilon,h})}{h^2}\Big)+\epsilon\, m_{i+1}\frac{D(F_i^{\epsilon,h})}{h^2}$$
$$-m_i\Big(\frac{v(F_i^{\epsilon,h})}{h}+\epsilon\frac{(D(F_i^{\epsilon,h})+D(F_{i-1}^{\epsilon,h}))}{h^2}\Big).$$

Since initially we have $m_i(0)\le 0$, arguing as in the proof of ii) we conclude that $m_i(t)\le 0$ for any $t\ge 0$. By ii) we already know that $n_i^{\epsilon,h}$ is non-negative and thus we have

$$0\le n_i^{\epsilon,h}(t)\le\sup_k(n_k^{\epsilon,h}(0))+2M\int_0^t\big(\sup_k n_k^{\epsilon,h}(s)+1\big)^2\,\mathrm{d}s.$$

Let $\tau>0$ and consider the solution of the ODE $\frac{\mathrm{d}}{\mathrm{d}t}z=2M(z+1)^2$ with initial data $z(0)=\sup_k(n_k^{\epsilon,h}(0))+\tau$. It is defined on a finite time interval $[0,T_\star)$. By continuity, for any i, $z(t)-n_i^{\epsilon,h}(t)$ remains positive, at least on a small time interval. Let $\bar t$ be the smallest time such that there exists some $i\in\mathbb{Z}$ verifying $z(\bar t)-n_i^{\epsilon,h}(\bar t)=0$. Since $u\mapsto(u+1)^2$ is non-decreasing, we obtain

$$n_i^{\epsilon,h}(\bar t)-z(\bar t)\le-\tau+2M\int_0^{\bar t}\Big(\big(\sup_k n_k^{\epsilon,h}(s)+1\big)^2-\big(z(s)+1\big)^2\Big)\,\mathrm{d}s\le-\tau,$$

a contradiction. We conclude that $0\le n_i^{\epsilon,h}(t)\le z(t)$ holds on $[0,T_\star)$. □

The estimates in iii) allows to establish the convergence of the fields to a regular solution of (1.10) as ϵ and h go to 0. However, the statement is restricted to a finite time interval where the limiting solution has not yet developed shocks. Let us introduce the following stepwise constant functions

$$F^{\epsilon,h}(t,x)=F_i^{\epsilon,h}(t)\qquad\text{for } x\in[ih,(i+1)h),\qquad i\in\mathbb{Z},\ t\ge 0. \tag{2.4}$$

As it will be detailed in the Appendix, the same arguments as in [7] prove the following

Theorem 2.2. *Let $F_i^{\epsilon,h}(t)$ be the solution of* (1.8) *with initial conditions $F_i^{\epsilon,h}(0)$ verifying $F_{\mathrm{s},-}\le F_i^{\epsilon,h}(0)\le F_{\mathrm{s},+}$, $n_i^{\epsilon,h}(0)>0$ and $\sup\{n_i^{\epsilon,h}(0): i\in\mathbb{Z}, h>0,\epsilon>0\}\le C_0<\infty$. Let $0<T<T^*$ with T^* defined in Theorem 2.1–iii). Assume also that initially the whole sequence $F^{\epsilon,h}(0,\cdot)$ converges uniformly to some function F_0, as $(\epsilon,h)\to(0,0)$. Then, $F^{\epsilon,h}(t,x)$ converges uniformly on compact sets to $F(t,x)$, where $F\in L^\infty(0,T;W^{1,\infty}(\mathbb{R}))$ is the unique solution of* (1.10) *on $[0,T]\times\mathbb{R}$ with initial data F_0.*

However, more interesting phenomena occur beyond the time of formation of singularities, as we will describe later. Let us end this section by establishing the existence of discrete traveling waves, which are particularly relevant solutions of the discrete drift diffusion model. It is a consequence of the following general statement.

Theorem 2.3 (Mallet–Paret [11]). *Consider the lattice differential equation*

$$\frac{dF_i}{dt}(t)=U(F_{i-1},F_i,F_{i+1}),\quad i\in\mathbb{Z}, \tag{2.5}$$

where the nonlinearity U verifies the diffusivity condition:

$$\frac{\partial U}{\partial F_{i-1}} > 0, \quad \frac{\partial U}{\partial F_{i+1}} > 0,$$

at every point. Suppose furthermore that the system is bistable which means that the function $u(F) := U(F, F, F)$ verifies

$$\begin{cases} u(F) < 0 & if \quad F \in (-\infty, F_{\mathrm{s},-}) \cup (F_{\mathrm{u}}, F_{\mathrm{s},+}) \\ u(F) > 0 & if \quad F \in (F_{\mathrm{s},-}, F_{\mathrm{u}}) \cup (F_{\mathrm{s},+}, \infty), \end{cases}$$

with $u'(F_{\mathrm{s},-}), u'(F_{\mathrm{s},+}) > 0$ and $u'(F_{\mathrm{u}}) < 0$. Then, there exists a unique $\sigma_* \in \mathbb{R}$ for which there exists a monotone function φ such that

$$F_i(t) := \varphi(i - \sigma_* t), \qquad \varphi(-\infty) = F_{\mathrm{s},-}, \quad \varphi(+\infty) = F_{\mathrm{s},+}$$

is a solution of (2.5)*.*

Corollary 2.4. *There exists a unique $\sigma \in \mathbb{R}$ and a monotone function φ, both depending on h and ϵ, such that $F_i^{\epsilon,h}(t) := \varphi(i - \sigma t)$ is a solution of* (1.8) *verifying $\varphi(-\infty) = F_{\mathrm{s},-}$ and $\varphi(+\infty) = F_{\mathrm{s},+}$*

Proof. The system (1.8) is embodied into (2.5) by setting

$$U(F, G, H) = -v(G)\frac{G - F}{h} + \epsilon\frac{H - 2G + F}{h^2} + \mathscr{J} - v(G).$$

Accordingly, $u(F) = U(F, F, F) = \mathscr{J} - v(F)$ has three distinct roots $0 < F_{\mathrm{s},-} < F_{\mathrm{u}} < F_{\mathrm{s},+}$. Theorem 2.3 can be applied directly and it proves the existence of a discrete traveling wave. □

3. From discrete to continuous models.

In this section we perform the limits $\epsilon \to 0$ and $h \to 0$ in diagram (1.13). We shall see that the limit model is not exactly the expected equation (1.10) and that the shock speed in (1.11) connecting the equilibrium states $F_{s,-}$ and $F_{s,+}$ is not given by the Rankine-Hugoniot relation.

3.1. From DDD to discrete balance law ($\epsilon \to 0$).

Using (1.8) and Theorem 2.1 we obtain directly, for any fixed $h > 0$, that $\{t \mapsto F_i^{\epsilon,h}(t),\ i \in \mathbb{Z},\ 0 < \epsilon \leq 1\}$ is bounded in $W^{1,\infty}([0,T])$. The Ascoli–Arzela theorem implies compactness of $\left(F_i^{\epsilon,h}\right)_{\epsilon>0}$ in $C^0([0,T])$. Then, a diagonal argument ensures the existence of a subsequence, still denoted by ϵ, such that $F_i^{h,\epsilon}(t)$ converges uniformly as $\epsilon \to 0$ in $C^0([0,T])$ for any $i \in \mathbb{Z}$. The limit F_i^h satisfies the following discrete limit problem

$$\frac{\mathrm{d}F_i^h}{\mathrm{d}t} + v(F_i^h)\frac{F_i^h - F_{i-1}^h}{h} = \mathscr{J} - v(F_i^h). \tag{3.1}$$

Next, we shall consider the limit $h \to 0$ from the last equation (3.1). This limit is related to the balance law (1.10). However (3.1) corresponds to a non conservative upwind scheme for approximating the continuous derivative $\partial_x V(F)$ and it is well known (see for example [9]) that such non conservative schemes produce wrong results if one wants to approximate (1.10). In particular, we will see that the velocity of a traveling wave solution of the limit equation is not given by (1.12).

3.2. From discrete to continuous balance law ($h \to 0$). To deal with the non conservativeness of the scheme (3.1), we adapt here the analysis in [8]. The idea is to find a non linear change of unknown, in the spirit of the Hopf–Cole transformation, which leads (3.1) into a conservative scheme. Accordingly, we are able to identify the defect term that arises when performing the continuum limit. Multiplying (3.1) by $\mathcal{S}'(F_i^h)$ (with $\mathcal{S} : \mathbb{R} \to \mathbb{R}$ to be determined) we get

$$\begin{aligned}\frac{\mathrm{d}\mathcal{S}(F_i^h)}{\mathrm{d}t} - \mathcal{S}'(F_i^h)(\mathscr{J} - v(F_i^h)) &= -\mathcal{S}'(F_i^h)v(F_i^h)\frac{F_i^h - F_{i-1}^h}{h} \\ &= -\frac{W(\mathcal{S}(F_i^h)) - W(\mathcal{S}(F_{i-1}^h))}{h} + \frac{1}{h}\int_{F_{i-1}^h}^{F_i^h} \Big(v(z)S'(z) - S'(F_i)v(F_i)\Big)\,\mathrm{d}z.\end{aligned} \tag{3.2}$$

Here the flux W is defined by $W'(\mathcal{S}(F)) = v(F)$. Then, we choose the function $\mathcal{S}$ so that the integrand in the last term of (3.2) vanishes. It leads to $\mathcal{S}'(F)v(F) = Constant$. Note in particular that this relation implies that $\mathcal{S}$ is increasing since $v(F)$ is supposed positive when $F > 0$. Accordingly, $G_i^h = \mathcal{S}(F_i^h)$ now satisfies a standard conservative upwind scheme

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}G_i^h + \dfrac{W(G_i^h) - W(G_{i-1}^h)}{h} = \mathbb{S}(G_i^h), \qquad W'(G) > 0, \\ \mathbb{S}(G) = \mathcal{S}'(\mathcal{S}^{(-1)}(G))(\mathscr{J} - W'(G)). \end{cases} \tag{3.3}$$

Let us check that this system satisfies the entropy criterion for balance laws. Pick a convex function η and its associated flux q such that $q' = \eta' W'$. We have

$$\begin{aligned}\frac{\mathrm{d}\eta(G_i^h)}{\mathrm{d}t} - \eta'(G_i^h)\mathbb{S}(G_i^h) &= -\eta'(G_i^h)\frac{W(G_i^h) - W(G_{i-1}^h)}{h} \\ &= -\frac{q(G_i^h) - q(G_{i-1}^h)}{h} + \int_{G_{i-1}^h}^{G_i^h} \frac{q'(z) - \eta'(G_i^h)W'(z)}{h}\,\mathrm{d}z \\ &= -\frac{q(G_i^h) - q(G_{i-1}^h)}{h} + \int_{G_{i-1}^h}^{G_i^h} \frac{\eta'(z) - \eta'(G_i^h)}{h}\,W'(z)\,\mathrm{d}z.\end{aligned} \tag{3.4}$$

Since η is convex and $W' > 0$, the last term becomes non-positive and we conclude

$$\frac{\mathrm{d}\eta(G_i^h)}{\mathrm{d}t} - \eta'(G_i^h)\mathbb{S}(G_i^h) \le -\frac{q(G_i^h) - q(G_{i-1}^h)}{h}. \tag{3.5}$$

Analogously to (2.4), let us define the following piecewise constant functions

$$F^h(t,x) = F_i^h(t) \quad \text{for } x \in [ih, (i+1)h), \quad i \in \mathbb{Z},\, t \ge 0. \tag{3.6}$$

We set $G^h = \mathcal{S}(F^h)$. We can then justify the following convergence statement.

THEOREM 3.1. *Let F_i^h be a solution of (3.1) with $0 < F_{s,-} \le F_i^h(t) \le F_{s,+}$. Assume that $v'(F) \neq 0$ for a.e. $F > 0$. Then, the function F^h given by (3.6) converges a.e. towards $F(t,x) = \mathcal{S}^{(-1)}(G(t,x))$, where $\mathcal{S}'(z) = 1/v(z)$ and G is the unique entropy solution of*

$$\partial_t G + \partial_x W(G) = \mathbb{S}(G) \tag{3.7}$$

with initial data $G(0,x) = \lim_{h\to 0} \mathcal{S}(F^h(0,x))$.

Proof. Our proof uses the relation (3.4) and is based on a compensated compactness argument, in the spirit of [6]. First of all, F^h, and thus G^h, is bounded in $L^\infty((0,T)\times\mathbb{R})$. In a second step, we obtain an additional estimate by using (3.4), for a strictly convex and non-negative entropy η. Indeed, let $0<R<\infty$. Summing over the cells yields, with I^h the greatest integer $\le R/h$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(h\sum_{|i|\le R/h}\eta(G_i^h)\Big)+h\sum_{|i|\le R/h}\int_{G_{i-1}^h}^{G_i^h}\Big(\frac{\eta'(G_i^h)-\eta'(z)}{h}\Big)W'(z)\,\mathrm{d}z\\
&=h\sum_{|i|\le R/h}\eta'(G_i^h)\mathbb{S}(G_i^h)-h\sum_{|i|\le R/h}\frac{q(G_i^h)-q(G_{i-1}^h)}{h}\\
&\le CR+q(G_{-I^h-1}^h)-q(G_{I^h}^h)\le C(1+R).
\end{aligned}$$

By using the strict convexity of the entropy η, and the fact that $W'(z)$ remains strictly positive, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(h\sum_{|i|\le R/h}\eta(G_i^h)\Big)+\alpha h\sum_{|i|\le R/h}\frac{|G_i^h-G_{i-1}^h|^2}{h}\le C(1+R),$$

for some constant $0<2\alpha\le W'(G)\,\eta''(G)$. Since η has been chosen non-negative we conclude

$$\int_0^T\sum_{|i|\le R/h}|G_i^h-G_{i-1}^h|^2\,\mathrm{d}t\le C(1+R+T+RT). \tag{3.8}$$

The third step consists in using again (3.4) to deduce the following equation (which must be understood in a distributional sense) for the piecewise constant function G^h and for any entropy-entropy flux pair (η,q):

$$\partial_t\eta(G^h)+\partial_x q(G^h)=\eta'(G^h)\mathbb{S}(G^h)-\mathscr{R}^h+\mathscr{E}^h,$$

where in the right hand side:

- $\eta'(G^h)\mathbb{S}(G^h)$, being bounded in $L^\infty((0,T)\times\mathbb{R})$, lies in a compact set of $H^{-1}((0,T)\times(-R,R))$ for any $0<T,R<\infty$,
- $\mathscr{E}^h(t,x)$ is defined as follows: for any $\varphi\in C_c^\infty(\mathbb{R})$, compactly supported in $(-R,R)$, and $0\le t\le T$, we have

$$\begin{aligned}
\int_{\mathbb{R}}\mathscr{E}^h\varphi\,\mathrm{d}x &= -\sum_i\int_{ih}^{(i+1)h}\Big(\frac{q(G_i^h)-q(G_{i-1}^h)}{h}\varphi(x)+q(G_i^h)\varphi'(x)\Big)\,\mathrm{d}x\\
&= -\sum_i\big(q(G_i^h)-q(G_{i-1}^h)\big)\ \frac{1}{h}\int_{ih}^{(i+1)h}\big(\varphi(x)-\varphi(ih)\big)\,\mathrm{d}x.
\end{aligned}$$

Therefore, using

$$\int_{ih}^{(i+1)h}(\varphi(x)-\varphi(ih))\,\mathrm{d}x=\int_{ih}^{(i+1)h}\int_{ih}^{x}\varphi'(z)\,\mathrm{d}z\,\mathrm{d}x=h\int_{ih}^{(i+1)h}\varphi'(z)\,\mathrm{d}z$$

and applying the Cauchy–Schwartz inequality twice, we get

$$\Big|\int_{\mathbb{R}}\mathscr{E}^h\varphi\,\mathrm{d}x\Big|\le C\Big(\sum_{i\le R/h}|G_i^h-G_{i-1}^h|^2\Big)^{1/2}\ \sqrt{h}\ \Big(\int_{\mathbb{R}}|\varphi'(z)|^2\,\mathrm{d}z\Big)^{1/2}.$$

It means, by using estimate (3.8), that $\mathscr{E}^h$ is of order $\mathscr{O}(\sqrt{h})$ in $H^{-1}((0,T)\times(-R,R))$ for any $0<T,R<\infty$.

- On the one hand, $\mathscr{R}^h$ comes from the last non-positive term in (3.4) and it is thus a sequence of non-negative measures. More precisely, we have

$$0 \le \int_{-R}^{+R} \mathscr{R}^h \,\mathrm{d}x \;=\; h \sum_{|i|\le R/h} \int_{G^h_{i-1}}^{G^h_i} \frac{\eta'(G^h_i) - \eta'(z)}{h} W'(z)\,\mathrm{d}z$$
$$\le\; Ch \sum_{|i|\le R/h} \frac{|G^h_i - G^h_{i-1}|^2}{h}.$$

Therefore by (3.8), $\mathscr{R}^h$ is a bounded sequence of non-negative measures on $(0,T)\times(-R,R)$. On the other hand, $\mathscr{R}^h$ appears as a sum of $\eta'(G^h)\mathbb{S}(G^h)$, $\mathscr{E}^h$ and time and space derivative of functions bounded in $L^\infty((0,T)\times\mathbb{R})$. In fact we have $\mathscr{R}^h = -\partial_t\eta(G^h) - \partial_x q(G^h) + \eta'(G^h)\mathbb{S}(G^h) + \mathscr{E}^h$. We already know that $\mathscr{E}^h$ is compact in $H^{-1}((0,T)\times(-R,R))$, while $\eta'(G^h)\mathbb{S}(G^h)$ being bounded in $L^\infty((0,T)\times\mathbb{R})$, it is also compact in $H^{-1}((0,T)\times(-R,R))$ as a consequence of the Rellich theorem. The following interpolation argument, due to Murat [12] (see also [15, Lemme 1], [16, Lemma 28]), with $d=2$, $Q^h_2 = \mathscr{R}^h$, $\Gamma_h = -\big(\eta(G^h), q(G^h)\big)$, allows to deduce that $\mathscr{R}^h$ belongs to a compact set of $H^{-1}((0,T)\times(-R,R))$, for any $0<T,R<\infty$.

Lemma 3.2. *Let Ω be a bounded and regular domain in $\mathbb{R}^d$, $d>1$. Let $Q^h = \mathrm{div}\Gamma^h$, with Γ^h bounded in $L^\infty(\Omega)$. Suppose moreover that Q^h splits into $Q^h_1 + Q^h_2$ with Q^h_1 compact in $H^{-1}(\Omega)$ and Q^h_2 bounded in $\mathscr{M}^1(\Omega)$. Then, Q^h is compact in $H^{-1}(\Omega)$.*

Proof. As usual, for $1\le q\le\infty$, we denote by $W^{-1,q}(\Omega)$ the space of distributions which writes as finite sums of zeroth and first order derivatives of functions belonging to $L^q(\Omega)$. Given $1\le p<\infty$, for $1/p+1/q=1$, $W^{-1,q}(\Omega)$ identifies with the dual space of $W^{1,p}_0(\Omega)$, the closure of $C^\infty_c(\Omega)$ in $W^{1,p}(\Omega)$, see [18, Definition 31.3 & Proposition 31.3]. We introduce the potential Φ^h_k solution of $-\Delta\Phi^h_k = Q^h_k$ in Ω with Dirichlet boundary condition $\Phi^h_{k,|\partial\Omega} = 0$. We set $\Phi^h = \Phi^h_1 + \Phi^h_2$. Clearly, Φ^h_1 is compact in $H^1_0(\Omega)$. Furthermore, since $W^{1,p}_0(\Omega)$ embeds compactly in $C^0(\bar\Omega)$ for any $p>d$, by duality the embedding $\mathscr{M}^1(\Omega) = \big(C^0(\bar\Omega)\big)' \subset W^{-1,q}(\Omega)$ is compact for any $q<d/(d-1)$. It follows that Q^h_2 is compact in $W^{-1,q}(\Omega)$ and thus Φ^h_2 is compact in $W^{1,q}_0(\Omega)$, for any $q<d/(d-1)$. However, the additional assumption that Q^h is bounded in $W^{-1,\infty}(\Omega)$ also implies that Φ^h is bounded in $W^{1,p}_0(\Omega)$ for any finite p. Accordingly, we know that $\nabla\Phi^h$ is bounded in $L^p(\Omega)$ for $p>2$ and compact in $L^q(\Omega)$, for $q<d/(d-1)$. Hence, it lies in a compact set of $L^2(\Omega)$ which indeed proves that $Q^h = \mathrm{div}\nabla\Phi^h$ is compact in $H^{-1}(\Omega)$. □

We can apply the div-curl lemma [16, Theorem 1, Theorem 11 & Example 3], see also [15, Section 5], [17, Corollary 3 & Section 3]: let (η,q) and $(\eta_\star,q_\star)$ be two entropy-entropy flux pairs. We can assume that the following weak-$\star$ convergences

$$\eta(G^h) \rightharpoonup \overline{\eta} \qquad q(G^h) \rightharpoonup \overline{q}, \qquad \eta_\star(G^h) \rightharpoonup \overline{\eta_\star} \qquad q_\star(G^h) \rightharpoonup \overline{q_\star}$$

hold in $L^\infty((0,T)\times\mathbb{R})$. From the previous discussion,

$$\mathrm{div}_{t,x}\big(\eta(G^h), q(G^h)\big) = \eta'(G^h)\mathbb{S}(G^h) - \mathscr{R}^h + \mathscr{E}^h$$

and

$$\mathrm{curl}_{t,x}\big(-q_\star(G^h), \eta_\star(G^h)\big) = \eta'_\star(G^h)\mathbb{S}(G^h) - \mathscr{R}^h_\star + \mathscr{E}^h_\star$$

both belong to compact sets in $H^{-1}((0,T)\times(-R,R))$ for any $0<T,R<\infty$. Hence the div-curl lemma implies that

$$\eta_\star(G^h)q(G^h)-\eta(G^h)q_\star(G^h)\rightharpoonup \overline{\eta_\star}\ \overline{q}-\overline{\eta}\ \overline{q_\star}$$

in $\mathscr{D}'((0,T)\times\mathbb{R})$ as h goes to 0. From this relation we can deduce as in [15, pp. 15–16], [16, Theorem 26] and [Section 5][6], that $W(G^h)\rightharpoonup W(G)$ weakly-$\star$ in $L^\infty((0,T)\times\mathbb{R})$ as well as the strong convergence of G^h to G in $L^p((0,T)\times(-R,R))$ for any $0<T,R<\infty$, $1\le p<\infty$ under the non-degeneracy condition:

$$W''(z)\neq 0,\ \text{a.e. } z.$$

Since $W'(z)=v(\mathcal{S}^{-1}(z))$, we get $W''(z)=\frac{v'(\mathcal{S}^{-1}(z))}{\mathcal{S}'(\mathcal{S}^{-1}(z))}$. Thus the non-degeneracy condition is satisfied assuming that $v'(F)\neq 0$ a.e. Once the strong compactness has been obtained it is straightforward from (3.3) and (3.5) to finish the proof. □

Traveling waves solutions. The non conservative structure of the model (3.1) induces new terms when passing to the limit $h\to 0$: the limit F verifies (1.10) with an additional defect measure term which is supported by the singularities of the solution G of (3.7). Accordingly, (1.11) is a discontinuous solution of the limit equation where the shock speed is not (1.12) but instead

$$\sigma_{\text{disc}}=\frac{W(G_+)-W(G_-)}{G_+-G_-}=\frac{W(\mathcal{S}(F_{\text{s},+}))-W(\mathcal{S}(F_{\text{s},-}))}{\mathcal{S}(F_{\text{s},+})-\mathcal{S}(F_{\text{s},-})}=\frac{\displaystyle\int_{F_{\text{s},-}}^{F_{\text{s},+}}\mathrm{d}F}{\displaystyle\int_{F_{\text{s},-}}^{F_{\text{s},+}}\frac{1}{v(F)}\,\mathrm{d}F}.$$

Note that the obtained velocity σ_{disc} corresponds to the harmonic mean of the drift velocity meanwhile σ_{RH} (1.12) is the arithmetic mean. Note also that the diffusion coefficient does not enter in the definition of the shock speed.

4. Continuous models: a vanishing viscosity approach. In this section we are interested in the passage to the limit $\epsilon\to 0$ in the following equation

$$\partial_t F_\epsilon+\partial_x V(F_\epsilon)-\epsilon D(F_\epsilon)\partial^2_{xx}F_\epsilon=\mathscr{J}-v(F_\epsilon). \tag{4.1}$$

Equation (4.1) is completed by initial and boundary conditions

$$F_\epsilon(0,x)=F^{\text{Init}}(x)\ge 0 \qquad \lim_{x\to\pm\infty}F_\epsilon(t,x)=F_{\text{s},\pm}.$$

It arises as a model for the so–called Gunn effect, see [10, 19]. According to the analysis in [7], it can be obtained from (1.8) as $h\to 0$ with $\epsilon>0$ fixed. Actually, [7] deals with a finite system but this analysis can be adapted to the present framework by changing the hypothesis on the boundary conditions by the quasineutrality condition at infinity (1.6) proved in Theorem 2.1.

PROPOSITION 4.1. *Let F_ϵ be the solution of* (4.1)*, such that the coefficients verify* (2.1)*. We assume that $F^{\text{Init}}\in BV(\mathbb{R})$. Then, F_ϵ is bounded (uniformly with respect to ϵ) in $L^\infty(0,T;BV(\mathbb{R}))$ for all $T\ge 0$. If moreover the initial data is non-decreasing, then $x\mapsto F_\epsilon(t,\cdot)$ is non-decreasing too:*

$$F_\epsilon(t,x)\le F_\epsilon(t,x')$$

holds for any $\epsilon > 0$, $t \geq 0$ and $x' \geq x$.

Proof. Let η be a small positive real number. We introduce the function

$$S_\eta(z) = \sqrt{\eta + z^2} - \sqrt{\eta} \geq 0, \quad S'_\eta(z) = \frac{z}{(\eta + z^2)^{1/2}}, \quad S''_\eta(z) = \frac{\eta}{(\eta + z^2)^{3/2}} \geq 0.$$

It is convenient to deal with the space derivative of F_ϵ: $m_\epsilon = \partial_x F_\epsilon = n_\epsilon - 1$ satisfying

$$\partial_t m_\epsilon + \partial_x (v(F_\epsilon) m_\epsilon) - \epsilon \partial_x (D(F_\epsilon) \partial_x m_\epsilon) = -v'(F_\epsilon) m_\epsilon. \tag{4.2}$$

Multiplying by $S'_\eta(m_\epsilon)$ we check that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} S_\eta(m_\epsilon)\, \mathrm{d}x + \epsilon \int_{\mathbb{R}} D(F_\epsilon) S''_\eta(m_\epsilon)\ |\partial_x m_\epsilon|^2 \, \mathrm{d}x \\ = - \int_{\mathbb{R}} v'(F_\epsilon)\ m_\epsilon \big(S'_\eta(m_\epsilon) + Z_\eta(m_\epsilon) \big) \, \mathrm{d}x \,, \end{aligned}$$

where $Z_\eta(m) = \int_0^m z S''_\eta(z)\, \mathrm{d}z$. This formula comes from an integration by parts, having in mind that the boundary conditions imply that m tends to 0 at infinity. A simple computation shows that

$$|m\ Z_\eta(m)| = \Big| \frac{\sqrt{\eta}\ m}{\sqrt{\eta + m^2}} (\sqrt{\eta + m^2} - \sqrt{\eta}) \Big| = \Big(\frac{\eta m^2}{\eta + m^2} \Big)^{1/2}\ S_\eta(m) \leq S_\eta(m).$$

In addition, we have

$$0 \leq m S'_\eta(m) \leq 2 S_\eta(m).$$

Therefore, we deduce that

$$\int_{\mathbb{R}} S_\eta(m_\epsilon)(t,x)\, \mathrm{d}x \leq \int_{\mathbb{R}} S_\eta(m_\epsilon)(0,x)\, \mathrm{d}x + C \int_0^t \int_{\mathbb{R}} S_\eta(m_\epsilon)(s,x)\, \mathrm{d}x\, \mathrm{d}s.$$

Hence, the Gronwall lemma leads to

$$\int_{\mathbb{R}} S_\eta(m_\epsilon)(t,x)\, \mathrm{d}x \leq C(T) \int_{\mathbb{R}} S_\eta(m^{\mathrm{Init}})(x)\, \mathrm{d}x,$$

for any $0 \leq t \leq T < \infty$. We conclude the $L^1(\mathbb{R})$ estimate on $\partial_x F_\epsilon = m_\epsilon$ by letting η go to 0, taking into account that $S_\eta(m)$ converges to $|m|$ and $m^{\mathrm{Init}} \in L^1(\mathbb{R})$. Accordingly we also deduce that $F_\epsilon(t,x) = F_{\mathrm{s},-} + \int_{-\infty}^x m_\epsilon(t,y)\, \mathrm{d}y$ is bounded in $L^\infty((0,T) \times \mathbb{R})$ and then in $L^\infty(0,T; BV(\mathbb{R}))$ as was stated.

Similar computations hold with the function $S_1(z) \mathbb{1}_{z \leq 0}$, where $\mathbb{1}_{z \leq 0}$ denotes the characteristic function associated with the set $\{z \leq 0\}$. This allows to prove that $m_\epsilon(t,x) \geq 0$ when the initial data is non-negative. □

Once the a priori estimates are obtained, the next objective is to determine the behavior of the solutions of (4.1) as $\epsilon \to 0$. It is useful to rewrite the equation as follows

$$\partial_t F_\epsilon + \partial_x V(F_\epsilon) - \epsilon \partial_x \big(D(F_\epsilon) \partial_x F_\epsilon \big) + \epsilon D'(F_\epsilon) \big(\partial_x F_\epsilon \big)^2 = \mathscr{J} - v(F_\epsilon). \tag{4.3}$$

We remark that the difficulties come from the non conservative term $\epsilon D'(F_\epsilon) \big(\partial_x F_\epsilon \big)^2$. Using Proposition 4.1, (4.3) shows that this term is the sum of:

- $\mathscr{J} - v(F_\epsilon)$ which is bounded in $L^\infty((0,\infty)\times\mathbb{R})$,
- $\partial_t F_\epsilon$ and $\partial_x v(F_\epsilon)$ which, being the derivatives of bounded functions, are bounded in $W^{-1,\infty}((0,\infty)\times\mathbb{R})$,
- $\epsilon\partial_x(D(F_\epsilon)\partial_x F_\epsilon)$ is the derivative of a bounded function in $L^\infty(0,T;L^1(\mathbb{R}))$, hence it is bounded in $W^{-1,1}((0,T)\times\mathbb{R})$.

Thus, the term $(-\epsilon D'(F_\epsilon)\big(\partial_x F_\epsilon\big)^2)$ up to a subsequence, admits a limit at least in the sense of distributions on $(0,T)\times\mathbb{R}$. This is precisely the defect term which makes the limit equation different from (1.10) and that we wish to identify.

Following the ideas developed in [8]: we propose a non linear change of unknown which eliminates the non conservative terms in (4.3). Multiply (4.1) by $\mathcal{S}'(F_\epsilon)$, where $\mathcal{S}$ is a function to be determined. We get

$$\begin{aligned}\partial_t\mathcal{S}(F_\epsilon) &= -\mathcal{S}'(F_\epsilon)v(F_\epsilon)\partial_x F_\epsilon + \epsilon\mathcal{S}'(F_\epsilon)D(F_\epsilon)\partial^2_{xx}F + \mathcal{S}'(F_\epsilon)(\mathscr{J} - v(F_\epsilon))\\ &= -\partial_x W(\mathcal{S}(F_\epsilon)) + \epsilon\partial_x\big(D(F_\epsilon)\partial_x\mathcal{S}(F_\epsilon)\big)\big) + \mathcal{S}'(F_\epsilon)(\mathscr{J} - v(F_\epsilon))\\ &\quad -\epsilon\Big(\mathcal{S}'(F_\epsilon)D'(F_\epsilon) + \mathcal{S}''(F_\epsilon)D(F_\epsilon)\Big)\big(\partial_x F_\epsilon\big)^2,\end{aligned}$$

where the flux W is defined by

$$W'(\mathcal{S}(F)) = v(F).$$

Therefore, we choose $\mathcal{S}$ so that

$$\mathcal{S}'(F)D'(F) + \mathcal{S}''(F)D(F) = \big(\mathcal{S}'D\big)'(F) = 0.$$

It means that $\mathcal{S}'$ is proportional to $1/D$. In particular, $\mathcal{S}'$ is strictly monotone and therefore invertible. Then, $G_\epsilon = \mathcal{S}(F_\epsilon)$ satisfies

$$\partial_t G_\epsilon + \partial_x W(G_\epsilon) - \epsilon\partial_x\big(\mathbb{D}(G_\epsilon)\partial_x G_\epsilon\big) = \mathbb{S}(G_\epsilon), \tag{4.4}$$

which is a classical balance law written in conservative form with a vanishing viscous term, where

$$\mathbb{D}(G) = D(\mathcal{S}^{(-1)}(G)), \qquad \mathbb{S}(G) = \mathcal{S}'(\mathcal{S}^{(-1)}(G))\ (\mathscr{J} - W'(G)).$$

Then, we can prove the following

THEOREM 4.2. *Let F_ϵ be the solution of* (4.1)*, such that* (2.1) *are satisfied. Assume that $0 < F_{s,-} \le F^{\rm Init}(x) \le F_{s,+}$. Then, F_ϵ converges to $F = \mathcal{S}^{(-1)}(G)$ in $L^p((0,T)\times(-R,+R))$, for any $1 \le p < \infty$, $0 < T, R < \infty$, where $\mathcal{S}'(z) = 1/D(z)$ and G is the unique entropy solution of*

$$\partial_t G + \partial_x W(G) = \mathbb{S}(G), \tag{4.5}$$

with initial data $G_{|t=0} = \mathcal{S}(F^{\rm Init})$.

Proof. The functions $G_\epsilon = \mathcal{S}(F_\epsilon)$ are bounded in $L^\infty((0,\infty)\times\mathbb{R})$ and they are solutions of (4.4). We apply directly the usual arguments of the theory of scalar conservation laws [16, Section 7] and see also [6, Section 3 & 5], [15, Section 5], [17, Section 3] to prove that G_ϵ converges to the unique entropy solution to (4.5) in $L^p((0,T)\times(-R,+R))$ for any $0 < T, R < \infty$, $1 \le p < \infty$. Note also that, assuming a BV bound on the initial data, this bound is inherited by F^ϵ and thus by G^ϵ. In this case, we can get the strong compactness of G^ϵ, and thus of F^ϵ, by using the Simon-Aubin theorem, [14] (see e.g. [15, Section 4]). Of course, the limit F and G will be also BV functions. □

Traveling waves solutions. The non conservative structure of the model (4.1) induces new terms when passing to the limit $\epsilon \to 0$, so that the limit F verifies (1.10) with an additional defect measure term. As a consequence, (1.11) is a discontinuous solution to the limit equation where the shock speed profile is (instead of (1.12))

$$\sigma_{\text{diff}} = \frac{W(G_+) - W(G_-)}{G_+ - G_-} = \frac{W(\mathcal{S}(F_{\text{s},+})) - W(\mathcal{S}(F_{\text{s},-}))}{\mathcal{S}(F_{\text{s},+}) - \mathcal{S}(F_{\text{s},-})} = \frac{\displaystyle\int_{F_{\text{s},-}}^{F_{\text{s},+}} \frac{v(F)}{D(F)}\,\mathrm{d}F}{\displaystyle\int_{F_{\text{s},-}}^{F_{\text{s},+}} \frac{\mathrm{d}F}{D(F)},},$$

which is the Rankine-Hugoniot condition for (4.5). Note that now the shock speed depends on the diffusion coefficient D.

A modified continuous model with vanishing viscosity. For any smooth enough function F, we have

$$v(F(x))\frac{F(x) - F(x-h)}{h} = v(F(x))\,\Big(\partial_x F(x) - \frac{h}{2}\,\partial^2_{xx}F(x)\Big) + \mathscr{O}(h^2).$$

Therefore, if F is a (smooth) solution of

$$\partial_t F + \partial_x V(F) - \big(\epsilon D(F) + hv(F)/2\big)\partial^2_{xx}F = \mathscr{J} - v(F), \tag{4.6}$$

then it is consistent at order $\mathscr{O}(h^2)$ with the discrete model (1.8) which means that $\tilde{F}_i(t) = F(t, ih)$ satisfies

$$\frac{\mathrm{d}\tilde{F}_i}{\mathrm{d}t} = \frac{v(\tilde{F}_i)}{h}(\tilde{F}_i - \tilde{F}_{i-1}) - \Big(\epsilon D(\tilde{F}_i) + h\frac{v(\tilde{F}_i)}{2}\Big)\frac{\tilde{F}_{i+1} - 2\tilde{F}_i + \tilde{F}_{i-1}}{h^2} = \mathscr{J} - v(\tilde{F}_i) + \eta_i^h$$

with $|\eta_i^h| \le Ch^2$. This motivates to consider (4.6) as a valuable model in the continuous limit, that can be expected to retain more information from the discrete original modeling than (4.1). The analysis performed above can be repeated to investigate the "diagonal limit" $\epsilon = kh \to 0$, where $k > 0$ is fixed. We obtain similar results to those in Theorem 4.2 with $D(F)$ replaced by $D(F) + kv(F)/2$. In particular the shock speed obtained that way reads

$$\sigma_{\text{k}} = \frac{\displaystyle\int_{F_{\text{s},-}}^{F_{\text{s},+}} \frac{v(F)}{v(F) + 2k\,D(F)}\,\mathrm{d}F}{\displaystyle\int_{F_{\text{s},-}}^{F_{\text{s},+}} \frac{\mathrm{d}F}{v(F) + 2k\,D(F)}}.$$

This result has to be compared with the formula obtained on formal grounds in [2] for the case where $\epsilon = h \to 0$ in (1.8): it is proposed to define the shock speed by

$$\sigma_{\text{f}} = \frac{\displaystyle\int_{F_{\text{s},-}}^{F_{\text{s},+}} \frac{v(F)}{D(F) + v(F)}\,\mathrm{d}F}{\displaystyle\int_{F_{\text{s},-}}^{F_{\text{s},+}} \frac{\mathrm{d}F}{D(F) + v(F)}}.$$

5. Concluding remarks. We have discussed various aspects of the asymptotic process as ϵ, h go to 0 in discrete drift diffusion models for SL. These questions are related to small period and large Debye length regimes, together with a discussion of the relative strength of diffusion and transport effects. As far as the asymptotic analysis is restricted to short times where solutions of the underlying balance law do not exhibit singularities, we can prove the convergence of the solutions of (1.8) to those of (1.10). However, extending the analysis to more general situations reveals a very intricate interplay between the scaling parameters. In particular, our analysis shows that the limits do not commute in the diagram (1.13). In turn, we also exhibit several relevant formulae for shock speeds, depending on the scaling assumptions. Let us comment further with an example. Figure 5.1 represents the values of the speeds $\sigma_{\rm disc}, \sigma_{\rm diff}, \sigma_{\rm k}$ $(k = 1), \sigma_{\rm f}$ compared to the Rankine-Hugoniot speed $\sigma_{\rm RH}$, using explicit expressions for the coefficients v and D (see Appendix B).

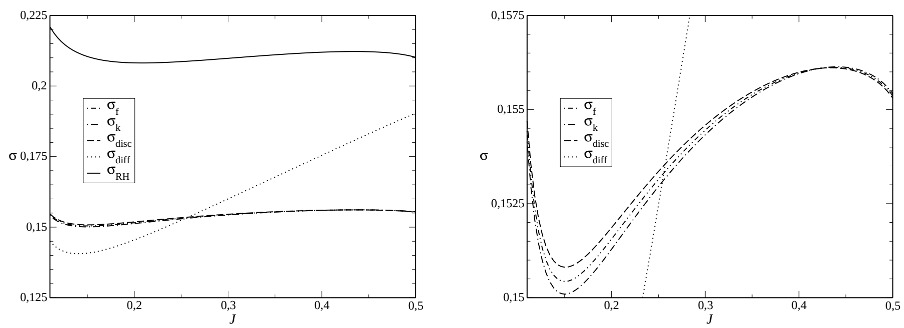

FIG. 5.1. *Values of the different shock speeds as functions of the controlled current $\mathscr{J}$ ($\sigma_{\rm k}$ is represented for $k = 1$). Note that the vertical scale is different in the left and right figures*

We observe in Figure 5.1 left, that the shock speeds we obtained substantially differ from the Rankine–Hugoniot speed. On Figure 5.1 right, we remark that $\sigma_{\rm k}$, and $\sigma_{\rm f}$ are close to the harmonic mean $\sigma_{\rm disc}$. This is because of, in that range of data, the field is high and thus the diffusion coefficient $D(F)$ is small. Indeed, letting D go to 0 in the corresponding formulae we observe they tend to $\sigma_{\rm disc}$. However the difference with $\sigma_{\rm diff}$ remains significant. In any case, it is a remarkable fact that although the limit equation is not diffusive it can retain information from the discrete diffusion of the original model.

Appendix A. Proof of Theorem 2.2. Let us define the auxiliary sequence ($\mathbb{P}^1$ interpolation)

$$G^{\epsilon,h}(t,x) = F_i^{\epsilon,h}(t) + \frac{x-ih}{h}(F_{i+1}^{\epsilon,h}(t) - F_i^{\epsilon,h}(t)) \quad \text{for } x \in [ih,(i+1)h), \quad i \in \mathbb{Z} \tag{A.1}$$

for any $t \geq 0$. By Theorem 2.1–i) the sequences $\left(F^{\epsilon,h}\right)_{\epsilon,h}$ and $\left(G^{\epsilon,h}\right)_{\epsilon,h}$ are uniformly bounded on $[0,T]\times\mathbb{R}$. In addition, the Poisson relation shows that the spatial derivative of $G^{\epsilon,h}$ is a piecewise constant function whose values coincide with $(n_i^{\epsilon,h} - 1)$. Then, Theorem 2.1–iii) proves that $\left(\partial_x G^{\epsilon,h}\right)_{\epsilon,h}$ is bounded in $L^\infty((0,T)\times\mathbb{R})$. Let us now establish, from (1.8), a useful identity. Let $\phi \in C_0^\infty(\mathbb{R})$. We get

$$\big\langle \partial_t F^{\epsilon,h}, \phi \big\rangle = \sum_{i\in\mathbb{Z}} \left(-v(F_i^{\epsilon,h}) \frac{F_{i+1}^{\epsilon,h} - F_i^{\epsilon,h}}{h} + \mathscr{J} - v(F_i^{\epsilon,h}) \right) \int_{ih}^{(i+1)h} \phi(x)\,\mathrm{d}x$$
$$+\epsilon \sum_{i\in\mathbb{Z}} D(F_i^{\epsilon,h}) \frac{F_{i+1}^{\epsilon,h} - 2F_i^{\epsilon,h} + F_{i-1}^{\epsilon,h}}{h^2} \int_{ih}^{(i+1)h} \phi(x)\,\mathrm{d}x.$$

Similar techniques as in [7] for the right hand side term and the fact that ϕ has compact support, lead to

$$\begin{aligned} \big\langle \partial_t F^{\epsilon,h}, \phi \big\rangle &= \big\langle -v(F^{\epsilon,h})\partial_x G^{\epsilon,h}, \phi \big\rangle + \big\langle (\mathscr{J} - v(F^{\epsilon,h})), \phi \big\rangle \\ &-\epsilon \sum_{i\in\mathbb{Z}} \frac{F_{i+1}^{\epsilon,h} - F_i^{\epsilon,h}}{h} D(F_i^{\epsilon,h}) \frac{\int_{(i+1)h}^{(i+2)h} \phi(x)\,\mathrm{d}x - \int_{ih}^{(i+1)h} \phi(x)\,\mathrm{d}x}{h} \\ &-\epsilon \sum_{i\in\mathbb{Z}} \left(\frac{F_{i+1}^{\epsilon,h} - F_i^{\epsilon,h}}{h} \right)^2 \frac{D(F_{i+1}^{\epsilon,h}) - D(F_i^{\epsilon,h})}{F_{i+1}^{\epsilon,h} - F_i^{\epsilon,h}} \int_{(i+1)h}^{(i+2)h} \phi(x)\,\mathrm{d}x \end{aligned} \tag{A.2}$$

In particular, note that

$$\sum_{i\in\mathbb{Z}} \frac{1}{h} \left| \int_{(i+1)h}^{(i+2)h} \phi(x)\,\mathrm{d}x - \int_{ih}^{(i+1)h} \phi(x)\,\mathrm{d}x \right| = \sum_{i\in\mathbb{Z}} \left| \int_{ih}^{(i+1)h} \frac{\phi(x+h) - \phi(x)}{h}\,\mathrm{d}x \right|$$
$$\leq \int_{\mathbb{R}} \left| \frac{1}{h} \int_x^{x+h} \phi'(s) ds \right| \mathrm{d}x \leq \int_{\mathbb{R}} \frac{1}{h} \int_{s-h}^{s} |\phi'(s)|\,\mathrm{d}x\,\mathrm{d}s = \|\phi'\|_{L^1(\mathbb{R})}.$$

The bound obtained in Theorem 2.1 allows to deduce

$$\big\langle \partial_t F^{\epsilon,h}, \phi \big\rangle \leq C(t)(\|\phi\|_{L^1(\mathbb{R})} + \epsilon \|\phi'\|_{L^1(\mathbb{R})}),$$

where $C(t)$ does not depend on (ϵ, h). Using (A.1) we deduce that $\partial_t G^{\epsilon,h}$ is bounded in $L^\infty(0,T; W^{-1,\infty}(\mathbb{R}))$. Then, the Simon-Aubin lemma [14] proves that $G^{\epsilon,h}$ lies in a compact set of $C([0,T]; C^0[-R,R])$, for any $0 < T, R < \infty$. So, up to a subsequence, it converges uniformly on compact sets to some function F. In addition $F \in L^\infty(0,T; W^{1,\infty}(\mathbb{R}))$. We note here that the weak–$\star$ limit in $L^\infty((0,T)\times\mathbb{R})$ of $\partial_x G^{\epsilon,h}$ has to be precisely $\partial_x F \in L^\infty((0,T)\times\mathbb{R})$.

Moreover, using (A.1) and the Poisson relation we get

$$\big| G^{\epsilon,h}(t,x) - F^{\epsilon,h}(t,x) \big| \leq h\ (C(t) + 1).$$

Consequently, $F^{\epsilon,h}$ converges to F uniformly on compact sets. It remains to show that F satisfies (1.10). To this end, given $\phi \in C_0^\infty([0,T)\times\mathbb{R})$, we integrate (A.2) with respect to time and take limit $(\epsilon, h) \to (0,0)$. It yields

$$\int_0^\infty \int_{\mathbb{R}} F \partial_t \phi(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} F_0 \phi(0,x)\,\mathrm{d}x$$
$$= \int_0^\infty \int_{\mathbb{R}} \big(v(F)\partial_x F - (\mathscr{J} - v(F)\big)\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t,$$

where we have used the fact that the last two terms in (A.2) are of order $\mathcal{O}(\epsilon)$. Then F is the unique regular solution of (1.10) with initial data F_0 and the whole sequence converges.

Appendix B. Drift velocity and diffusion coefficient.

In order to make the paper self-contained and the numerical results reproducible (Figure 5.1), we give explicit expressions of the velocity and diffusion in terms of the electric field. In spite of the existence of exact formulas for v and D, due to their complexity, see [3], we have used the following numerical approximations inspired by the results [2, Appendix A] and [4]

$$v(F) = (1 - e^{-1.16F}) \times \left[\frac{0.81}{0.47 + (0.58F - 12.76)^2} + \frac{0.54}{0.47 + (0.58F - 6.88)^2} + \frac{0.27}{0.47 + (0.58F - 1)^2}\right],$$

$$D(F) = e^{-2F} \times \left[\frac{0.279}{0.47 + (0.29F - 12.76)^2} + \frac{0.186}{0.47 + (0.29F - 6.88)^2} + \frac{0.093}{0.47 + (0.29F - 1)^2} + \frac{0.02Fe^{2F}}{1 + (2F)^2}\right].$$

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