Shock profiles for fluid-particles flows

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Abstract A hydrodynamical limit of a coupled kinetic-fluid model describing the interaction between particles and fluid is considered with emphasis on the existence of smooth propagating fronts. We focus on two different types of models based, respectively, on a Burgers' and an Euler description of the dynamics of the fluid equation exhibiting similarities and differences.

1 Fluid-kinetic models for particle-laden flow

A particle-laden flow can be conceived as a class of *two-phase fluid flow* composed by: a *carrier/continuous phase*, and a *disperse/particle phase* made of small, immiscible, dilute particles. A starting significant example is fine aerosol particles in air, where aerosols are the dispersed phase and air is the carrier phase. There are also more applications of such type of phenomena in reality: pollution dispersion in the atmosphere, fluidization in combustion processes, aerosol deposition in spray medication, gravity and turbidity currents, turbulent shallow-waters flows, gas-particle jet, fluidized beds...

Focusing on the case of a 1D spatial domain, $x \in \mathbb{R}$, the *disperse phase* is identified by the kinetic variable $f_\epsilon = f_\epsilon(t, x, v)$, which describes the particles distribution at time t, position x and moving with speed v. The unknown f_ϵ satisfies the kinetic

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equation

$$\partial_t f_\epsilon + v\,\partial_x f_\epsilon = \frac{1}{\epsilon}\,\mathcal{L}_{u_\epsilon} f_\epsilon$$

where $\mathcal{L}_u$ is the Fokker–Planck operator $\mathcal{L}_u f := \partial_v\big\{(v-u)f + \theta\,\partial_v f\big\}$ with u describing the velocity of the carrier phase (to be detailed) and $\theta > 0$ measures random fluctuations, which can be realistically regarded as the *temperature* of the mixture. The coupling due to carrier phase emerges in the friction-type term $\partial_v\big\{(v-u)f\big\}$ expressing the tendency of particles with speed v to coalesce to the carrier speed u.

Somewhat complementary, the *carrier phase* is described by the fluid variable pair (n_ϵ, u_ϵ) for the carrier density and velocity for the fluid, i.e. the momentum $n_\epsilon u_\epsilon$ solves the dynamic relation

$$\partial_t(n_\epsilon u_\epsilon) + \partial_x\big(n_\epsilon u_\epsilon^2 + p\big) = \frac{1}{\epsilon}\,(J_\epsilon - \rho_\epsilon u_\epsilon)\,,$$

where $p = p(n_\epsilon)$ is the value of the *pressure* of the fluid-phase, while the terms ρ_ϵ and J_ϵ are given by

$$\rho_\epsilon(t,x) := \int f_\epsilon(t,x,v)\,\mathrm{d}v \quad \text{and} \quad J_\epsilon(t,x) := \int v\,f_\epsilon(t,x,v)\,\mathrm{d}v\,,$$

where the integration with respect to v is considered in $\mathbb{R}$. Clearly, an additional equation describing the dynamics of n_ϵ is needed to close the system.

As a first option, we consider the case later on called **Burgers–Fokker–Planck system** (in short, BFP), where the carrier is a sort of *incompressible fluid*, i.e. we assume that $n_\epsilon(t,x) = \bar{n}$ for any (t,x). In particular, being null, the term $\partial_x p$ does not affect the dynamics. The incompressibility here is a different concept with respect to the standard incompressibility arising in the case of Navier–Stokes equations.

A more complete and coherent alternative is referred to as **Euler–Fokker–Planck system** (in short, EFP): here, the carrier is an effectively *compressible fluid*, i.e. the function n_ϵ satisfies the dynamical identity

$$\partial_t n_\epsilon + \partial_x\,(n_\epsilon u_\epsilon) = 0$$

In such a case, additional hypotheses on the pressure p are required. Here, following the standard approach, we assume that the function p is such that

$$p'(n),\ p''(n) > 0 \quad \forall\, n \geq 0 \qquad \text{and} \qquad \lim_{n\to+\infty} \frac{p(n)}{n} = +\infty$$

A classical example is given by the γ-law, i.e. $p(n) = C\,n^\gamma$ where $C > 0$ and $\gamma > 1$. Precisely, for diatomic gases (the most familiar case is air) γ equals $7/5$ and for monoatomic ones (such as Helium and Argon) γ equals $5/3$.

2 Behavior as $\epsilon \to 0^+$ and flowing regime

As it is well-known, the *Maxwell distributions* $M_u = M_u(v)$, parametrized by u and defined by

$$M_u(v) := \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{|v-u|^2}{2\theta}\right),$$

span the kernel of the Fokker–Planck operator $\mathcal{L}_u$. Therefore, since

$$\lim_{\epsilon\to 0^+} \mathcal{L}_{u_\epsilon} f_\epsilon = \lim_{\epsilon\to 0^+} \epsilon\,\{\partial_t f_\epsilon + v\partial_x f_\epsilon\} = 0\,,$$

the kinetic term f_ϵ is such that $f_\epsilon \simeq \rho M_u$ as $\epsilon \to 0^+$. Moreover, there formally holds

$$J_\epsilon \simeq \rho u \quad \text{and} \quad \int v^2 f_\epsilon \,\mathrm{d}v \simeq \rho u^2 + \theta\rho\,,$$

leading to a conservation law for the *hybrid density* $r := \rho + n$,

$$\partial_t(ru) + \partial_x\left\{ru^2 + p(n) + \theta\rho\right\} = 0\,,$$

We end up with two possible descriptions fitting in the general form of a system of conservation laws $\partial_t \mathcal{W} + \partial_x F(\mathcal{W}) = 0$ for some vector-valued function $\mathcal{W}$.

The choice BFP leads to $\mathcal{W} = (\rho, ru) = \big(\rho, (\rho+\bar{n})u\big) \in \mathbb{R}^2$ satisfying

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0\,, \\ \partial_t(ru) + \partial_x(ru^2 + \theta\rho) = 0\,. \end{cases} \tag{1}$$

Differently, for the case EFP, the unknown $\mathcal{W} = (r, \rho, ru) = \big(r, \rho, (\rho+n)u\big)$ is a three-dimensional solution of the evolutionary system

$$\begin{cases} \partial_t r + \partial_x(ru) = 0\,, \\ \partial_t\rho + \partial_x(\rho u) = 0\,, \\ \partial_t(ru) + \partial_x\left\{ru^2 + p(n) + \theta\rho\right\} = 0\,. \end{cases} \tag{2}$$

Both systems can be proved to be (strictly) hyperbolic, see [1, 4].

Higher order diffusive terms appear by means of the standard Chapman–Enskog expansion used as a tool to pass from the formal limit $\epsilon = 0$ to the regime of small ϵ. Shallowly entering the details, let us set

$$f_\epsilon(t,x,v) = \rho_\epsilon(t,x) M_{u_\epsilon(t,x)}(v) + \epsilon g_\epsilon(t,x,v)\,,$$

where

$$\int g_\epsilon(t,x,v)\,\mathrm{d}v = 0$$

Looking for terms of order ϵ, we end up with a viscous system of conservation laws of the form

$$\partial_t \mathcal{W} + \partial_x F(\mathcal{W}) = \epsilon\, \partial_x \left\{ \mathbf{D}(\mathcal{W}) \partial_x \mathcal{W} \right\} \tag{3}$$

for some appropriate matrices $\mathbf{D} = \mathbf{D}(\mathcal{W})$. To provide the explicit form of the diffusion correction $\mathbf{D}$ in the two cases BFP and EFP – the detailed derivation can be found in [4] – let us set

$$\alpha := \frac{n^2}{r^2}\,, \qquad \beta := \frac{\rho \cdot np'}{r^2} \quad \text{and} \quad \gamma := \frac{\rho}{r}\,. \tag{4}$$

Note that, in the case $n \equiv \bar{n}$, there holds $\alpha = \bar{n}/r^2$, $\beta = 0$, $\gamma = \rho/r$. Then, the flowing regime for BFP is described by

$$\mathbf{D}(\mathcal{W}) = \alpha\gamma u \begin{pmatrix} u & -1 \\ 0 & 0 \end{pmatrix} + \theta \begin{pmatrix} 1/r^2 & 0 \\ -\gamma u & \gamma \end{pmatrix} \tag{5}$$

Differently, the flowing regime for EFP corresponds to the choice

$$\mathbf{D}(\mathcal{W}) = \beta \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \theta \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ -\gamma u & 0 & \gamma \end{pmatrix} \tag{6}$$

Definition 1 Let $\mathbf{A} := \mathrm{d}F(\mathcal{W}_*)$ and $\mathbf{D} := \mathbf{D}(\mathcal{W}_*)$. The constant coefficient system

$$\partial_t \mathcal{W} + \mathbf{A} \partial_x \mathcal{W} = \epsilon \mathbf{D}\, \partial_x^2 \mathcal{W}\,,$$

obtained by linearization of (3) at some constant state $\mathcal{W}_*$, is

i. *parabolic at* $\mathcal{W}_*$ if the eigenvalues of the symmetric matrix $\mathbf{S_D} := \frac{1}{2}\left(\mathbf{D} + \mathbf{D}^\intercal\right)$ have positive real parts;
ii. *uniformly stable at* $\mathcal{W}_*$ if for any $T > 0$ there exists $C = C(T) > 0$ such that

$$\sup\big\{ \|\mathcal{W}(t,\cdot)\|_{L^2} \,:\, 0 < \epsilon < 1,\, t \in [0,T] \big\} \leq C(T) \|\mathcal{W}_0\|_{L^2}$$

where $\mathcal{W}$ is the solution of the Cauchy problem (1) with initial condition $\mathcal{W}(0,\cdot) = \mathcal{W}_0 \in L^2(\mathbb{R})$.

Note that uniform stability can be stated equivalently as

$$\sup\{ \| \exp\{-t\big(i\xi\mathbf{A} + \epsilon\xi^2\mathbf{D}\big)\}\| \,:\, t \in [0,T],\, \xi \in \mathbb{R}\} \leq C(T).$$

where $\|\cdot\|$ denotes the operator norm from L^2 to L^2.

Incidentally, we observe a major difference between the two models.

Proposition 1 *Both hyperbolic* BFP *and parabolic* BFP *are not invariant under Galilean transformation. Differently, hyperbolic* EFP *and hyperbolic-parabolic* EFP *are invariant under such family of coordinates' changes.*

Proof To check for the absence/presence of this property, let us consider the change of variables $(t,x) \mapsto (s,y) = (t, x - u_0 t)$ for some $u_0 \in \mathbb{R}$. It can be easily verified that conservation of mass is invariant under this transformation: indeed, there holds

$$\partial_t r + \partial_x(ru) = \partial_s r - u_0\partial_y r + \partial_y\left\{r(v+u_0)\right\} = \partial_s r + \partial_y(rv),$$

Then, hyperbolic BFP is not invariant, since

$$0 = \partial_t(ru) + \partial_x(ru^2 + \theta\rho) = \partial_s(rv) + \partial_y(rv^2 + \theta\rho) + u_0\partial_y v\,,$$

with the same fate holding for the parabolic version of BFP. Differently, both hyperbolic and hyperbolic-parabolic EFP are invariant under Galilean transformations, as follows from similar computations. Indeed, focusing on the third equation and introducing the *total pressure* $P(n,\rho) := p(n) + \theta\rho$, there holds

$$\begin{aligned}
\partial_t(ru) + \partial_x(ru^2 + P) &= \partial_s\left\{r(v+u_0)\right\} - u_0\partial_y\left\{r(v+u_0)\right\} + \partial_y\left\{r(v+u_0)^2 + P\right\} \\
&= \partial_s(rv) + u_0\partial_s r - u_0\partial_y(rv) - u_0^2\partial_y r + \partial_y\left\{r(v^2 + 2u_0 v + u_0^2) + P\right\} \\
&= \partial_s(rv) + \partial_y(rv^2 + P) - 2u_0\partial_y(rv) - u_0^2\partial_y r + 2u_0\partial_y(rv) + u_0^2\partial_y r \\
&= \partial_s(rv) + \partial_y(rv^2 + P),
\end{aligned}$$

showing that the hyperbolic system (2) is invariant with respect to Galilean transformations. To check that the same invariance holds for the enriched system including the diffusion term given in (6), we reformulate it with respect to the variable $\mathcal{U} = (r,\rho,u)$. Thus, we end up with

$$\partial_t G(\mathcal{U}) + \partial_x H(\mathcal{U}) = \epsilon\partial_x\left\{\mathbf{E}(\mathcal{U})\partial_x\mathcal{U}\right\}$$

for some appropriate G, H and $\mathbf{E}$ explicitly given by

$$\mathbf{E}(\mathcal{U}) := \sqrt{\alpha}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \theta\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-\alpha & 0 \\ 0 & 0 & \rho \end{pmatrix}.$$

Therefore, EFP is invariant under Galilean transformations, because $\mathbf{E}$ does not depend explicitly on the velocity field u. $\square$

3 Convex entropies for both BFP and EFP

Both models are endowed with a convex entropy. To begin with, let us focus on BFP system. Without loss of generality, we may assume $\bar{n} \equiv 1$. The correspondent kinetic-fluid model possesses an entropy $H_\epsilon^{\mathrm{BFP}}$ defined by

$$H_\epsilon^{\mathrm{BFP}}(u, f_\epsilon) := H_{\mathrm{fl}}^{\mathrm{BFP}}(u) + H_{\mathrm{p}}^{\mathrm{BFP}}(f_\epsilon)\,,$$

where

$$H_{\mathrm{fl}}^{\mathtt{BFP}}(u) := \tfrac{1}{2}u^2 \qquad \text{and} \qquad H_{\mathrm{p}}^{\mathtt{BFP}}(f_\epsilon) := \int f_\epsilon \left\{\tfrac{1}{2}v^2 + \theta \ln f_\epsilon\right\} \mathrm{d}v$$

correspond to the entropy of the fluid and of the particles phase, respectively. Since

$$f_\epsilon \simeq \rho M_u \qquad \text{and} \qquad \int v^2 f_\epsilon \, \mathrm{d}v \simeq \rho u^2 + \theta\rho$$

in the limiting regime $\epsilon \to 0^+$, the function $H_\epsilon^{\mathtt{BFP}}$ converges –up to linear terms in ρ– to the following entropy $\eta^{\mathtt{BFP}}$ for the corresponding limiting system

$$\eta^{\mathtt{BFP}}(\mathcal{W}) := \tfrac{1}{2}\, ru^2 + \theta\rho \ln \rho = \tfrac{1}{2}u^2 + \tfrac{1}{2}\rho u^2 + \theta\rho \ln\rho\,,$$

witth some appropriate associated entropy flux $q^{\mathtt{BFP}}$. Indeed, upon direct computation, the positive definite hessian of the entropy $D^2\eta^{\mathtt{BFP}}$ is a symmetrizer of both $\mathrm{d}F$ and $\mathbf{D}$. In addition, there holds

$$\mathrm{Tr}\big(D^2\eta^{\mathtt{BFP}}\mathbf{D}\big) > 0 \quad \text{and} \quad \det\big(D^2\eta^{\mathtt{BFP}}\mathbf{D}\big) = \det D^2\eta^{\mathtt{BFP}} \det \mathbf{D} > 0\,.$$

for any $\mathcal{W}$ with strictly positive components. Hence, $D^2\eta^{\mathtt{BFP}}\mathbf{D}$ is positive definite and BFP, in such new coordinates, is parabolic.

Similarly, setting

$$\Pi(n) := \int_0^n\int_0^y \frac{p'(x)}{x}\, \mathrm{d}x\,\mathrm{d}y\,,$$

the kinetic-fluid model corresponding to EFP has an entropy $H_\epsilon^{\mathtt{EFP}}$ given by

$$H_\epsilon^{\mathtt{EFP}}(n,u,f_\epsilon) := H_{\mathrm{fl}}^{\mathtt{EFP}}(n,u) + H_{\mathrm{p}}^{\mathtt{EFP}}(f_\epsilon)\,,$$

where

$$H_{\mathrm{fl}}^{\mathtt{EFP}}(n,u) := \tfrac{1}{2}nu^2 + \Pi(n) \quad \text{and} \quad H_{\mathrm{p}}^{\mathtt{EFP}}(f_\epsilon) := \int f_\epsilon \left\{\tfrac{1}{2}v^2 + \theta \ln f_\epsilon\right\} \mathrm{d}v\,.$$

Following the same path of BFP, we infer the entropy $\eta^{\mathtt{EFP}}$ which is given by

$$\eta^{\mathtt{EFP}}(\mathcal{W}) := \tfrac{1}{2}ru^2 + \Pi(n) + \theta\rho\ln\rho = \tfrac{1}{2}nu^2 + \Pi(n) + \tfrac{1}{2}\rho u^2 + \theta\rho\ln\rho\,.$$

As before, the convex function $\eta^{\mathtt{EFP}}$ is a symmetrizer of both $\mathrm{d}F$ and $\mathbf{D}$. Since the first row of $\mathbf{D}$ is composed by zeros, the system cannot be parabolic. Nevertheless, EFP has a parabolic-hyperbolic decomposition in the sense that $\mathbf{D}$ can be written as

$$\begin{pmatrix} 0 & \mathbf{0}_{2\times 1} \\ \mathbf{a}_{1\times 2} & \mathbf{b}_{2\times 2} \end{pmatrix}$$

where $\mathbf{b}_{2\times 2}$ is a 2×2 diagonal matrix with positive entries. Moreover, the *Shizuta–Kawashima condition* –which can be regarded as a stability condition– is satisfied for $\theta > 0$, i.e. no element of the kernel of $\mathbf{D}$ are eigenvalues of $\mathrm{d}F$, see [10].

4 Weak shock profiles for BFP and EFP

Let us consider special solutions to (3) in the form of traveling waves, i.e. $\mathcal{W}(t,x) = \mathrm{W}(x-ct)$, satisfying the asymptotic conditions

$$\lim_{y\to-\infty} \mathrm{W}(y) = \mathcal{W}_*, \qquad \text{and} \qquad \lim_{y\to+\infty} \mathrm{W}(y) = \mathcal{W}_+$$

for given states $\mathcal{W}_*$ and $\mathcal{W}_+$. Such solutions are called *shock waves*, or, simply, *shocks*. The function $y \mapsto \mathrm{W}(y)$ is named *profile* and the parameter c is the *speed*.

Rescaling indepedent variable by setting $y \mapsto z := y/\epsilon$ and plugging into (3), we end up with the system of ordinary differential equations

$$\mathbf{D}(\mathrm{W})\frac{\mathrm{dW}}{\mathrm{d}\,y} = F(\mathrm{W}) - F(\mathcal{W}_*) - c(\mathrm{W} - \mathcal{W}_*).$$

As a consequence of the Galilean invariance, we are allowed to change to a reference frame co-moving with the shock. In such a case –which is possible for EFP and not for BFP– we can limit the analysis to the case $c = 0$ without loss of generality.

A basic role is played by the classical *Rankine–Hugoniot identity*, which reads as

$$c\,[[\mathcal{W}]] = [[F(\mathcal{W})]]\,, \tag{7}$$

where c is the *jump speed* and $[[\mathcal{W}]] := \mathcal{W}_+ - \mathcal{W}_*$. Equality (7) can be regarded as a constraint on c given the asymptotic states $\mathcal{W}_*$ and $\mathcal{W}_+$.

Poorly speaking, we may consider two major possible regimes depending on the strength of the jump $[[\mathcal{W}]]$: *small amplitude* vs. *moderate/large amplitude*. For weak shocks, a fundamental tool for proving existence of shock profiles is the *Liu's entropy criterion*, see [3, 5].

Theorem 1 (Existence of weak shock profiles) *System* (3) *with* $F(\mathcal{W}) = (\rho\,u, ru^2 + \theta\rho)$*,* $\mathbf{D}$ *given by* (5) *(corresponding to* BFP*) and with* $F(\mathcal{W}) = (ru, \rho u, ru^2 + p + \theta\rho)$*,* $\mathbf{D}$ *given by* (6) *(corresponding to* EFP*) support smooth weak shock profiles.*

The complete proofs, contained in [4], essentially consists in the verifications that the assumptions in [9] (for BFP) and [6] (for EFP) hold. Once existence is established, asymptotic stability is also guaranteed (see [7, 8] for L^p-perturbations with sharp decay rates).

5 Moderate/large shocks for EFP at zero temperature

In this final part, we focus on EFP hyperbolic-parabolic model for the case of moderate/large strength. In such a case, as previously stated, we can reduce to $c = 0$ without loss of generality. Hence, we look for a heteroclinic orbit $y \mapsto \mathrm{W} = (r, \rho, w)$ with $w = ru$ for the system

$$\mathbf{D}(\mathrm{W})\frac{\mathrm{dW}}{\mathrm{d}y} = F(\mathrm{W}) - F(\mathcal{W}_*)$$

connecting the asymptotic states $\mathrm{W}(-\infty) = \mathcal{W}_* = (r_*, \rho_*, w_*)$ and $\mathrm{W}(+\infty) = \mathcal{W}_+ = (r_+, \rho_+, w_+)$, the former being fixed and the latter variable.

Upon integration, we reduce to the two dimensional ODE

$$\begin{cases} \beta\dfrac{\mathrm{d}r}{\mathrm{d}y} - (\beta + \theta\alpha)\dfrac{\mathrm{d}\rho}{\mathrm{d}y} = w_*\left(\dfrac{n}{r} - \dfrac{n_*}{r_*}\right), \\ -\dfrac{\theta\gamma w_*}{r}\dfrac{\mathrm{d}r}{\mathrm{d}y} = \dfrac{w_*^2}{r} + p(n) + \theta\rho - \dfrac{w_*^2}{r_*} - p(n_*) - \theta\rho_* \end{cases} \tag{8}$$

with α, β and γ defined in (4). In the singular limit $\theta \to 0^+$, corresponding to the temperature-less regime, system (8) reduces to a differential-algebraic system

$$\begin{cases} \beta\dfrac{\mathrm{d}n}{\mathrm{d}y} = w_*\left(\dfrac{n}{r} - \dfrac{n_*}{r_*}\right) \\ \dfrac{w_*^2}{r} + p(n) = \dfrac{w_*^2}{r_*} + p(n_*) \end{cases} \tag{9}$$

to be considered with the constraint $\rho = r - n \geq 0$.

Theorem 2 (Existence of shock profiles for $\theta = 0$) *Given* $u_* > 0$*, let us set*

$$\kappa_* := \frac{n_* p'(n_*)}{p(n_*)} \quad \textit{and} \quad \kappa := \frac{r_* u_*^2}{p(n_*)}\,.$$

For any $\kappa > 0$ *with* $\kappa \neq \kappa_*$*, there exists* $\tau_\# = \tau_\#(\kappa) \in (0,1)$ *such that there exist heteroclinic solutions* $n = n(y)$ *to* (9) *satisfying the constraint* $r(n) > n$ *if and only if* $\tau < \tau_\#(\kappa) < 1$ *where* $\tau := n_*/r_*$*. Moreover, if* $\kappa < \kappa_*$*,* $\lim\limits_{y\to-\infty} n(y) = n_\times$ *and* $\lim\limits_{y\to+\infty} n(y) = n_*$*; moreover, if* $\kappa > \kappa_*$*,* $\lim\limits_{y\to-\infty} n(y) = n_*$ *and* $\lim\limits_{y\to+\infty} n(y) = n_\times$*.*

The complete proof, contained in [4], is based on a detailed analysis of the differential-algebraic system with special care for the constraint $\rho = r - n > 0$ for a rescaled version of (9). This is performed by introducing the unknowns $a := n/n_*$ and $b := r/r_*$ together with the adimensionalized pressure $q(a) := p(n_* a)/p(n_*)$, which leads to the constrained equation

$$\frac{\mathrm{d}a}{\mathrm{d}z} = \frac{1}{b} - \frac{1}{a} \quad \text{with} \quad b = b(a) := \frac{\kappa}{1 + \kappa - q(a)}\,,$$

defined for $a \in \big(0, q^{-1}(1+\kappa)\big)$, where z is a strictly monotone function of y.

In some special cases, explicit forms for the threshold can be determined. As an illustrative example, let us consider the case of the γ-law with $\gamma = 2$. Then, there holds $\kappa_* = \gamma = 2$ and $q(a) = a^\gamma$, so that

$$\tau_\# = \tau_\#(\kappa) = \frac{3\sqrt{3}}{2} \cdot \frac{\kappa}{(1+\kappa)^{3/2}}$$

In the special case $\gamma = 2$, the function g is a rational function whose factorization is

$$g(a) := \frac{1}{b(a)} - \frac{1}{a} = -\frac{a^3 - (1+\kappa)a + \kappa}{\kappa a} = -\frac{(a+a_-)(a-1)(a-a_\times)}{\kappa a}$$

where a_- and $a_\times$ are given by

$$a_- := \tfrac{1}{2}\left\{(1+4\kappa)^{1/2} + 1\right\}, \qquad a_\times := \tfrac{1}{2}\left\{(1+4\kappa)^{1/2} - 1\right\}$$

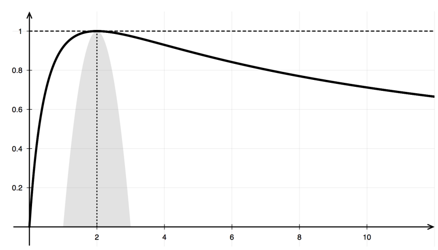

Fig. 1 The function $\tau_\#$ for the case $p(n) = Cn^2$. Weak shocks are concentrated in a neighborhood of $\kappa = \kappa_* = \gamma = 2$ (an example is enhanced in the grey region).

A natural question would be to extend such existence results of moderate/large amplitude (smooth) shock profiles valid for null temperature to the case of small temperature by some singular perturbation argument. Moreover, stability analysis of such profiles is an additional natural issue worthwhile to be investigated analytically and numerically.

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