Consistency analysis of a 1D Finite Volume scheme for barotropic Euler models

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Abstract

This work is concerned with the consistency study of a 1D (staggered kinetic) finite volume scheme for barotropic Euler models. We prove a Lax-Wendroff-like statement: the limit of a converging (and uniformly bounded) sequence of stepwise constant functions defined from the scheme is a weak entropic-solution of the system of conservation laws.

1   Introduction

The model. This work is concerned with the consistency study of a (staggered kinetic) Finite Volume (FV) scheme for barotropic Euler models

$$\partial_t \rho + \partial_x(\rho V) = 0, \tag{1}$$

$$\partial_t(\rho V) + \partial_x(\rho V^2 + p(\rho)) = 0. \tag{2}$$

The unknowns are the density ρ and the velocity V. The pressure $(\rho \mapsto p(\rho))$ is assumed to be $\mathcal{C}^2([0,\infty))$ with $p(\rho) > 0$, $p'(\rho) > 0$, $p''(\rho) \geqslant 0$, $\forall \rho > 0$. Thus, the sound speed $c : \rho \mapsto \sqrt{p'(\rho)}$ is well defined and is an increasing function.

We consider the problem (1)-(2) on the bounded domain $(0, L) \times [0, T]$ with the boundary conditions $V(0, t) = 0 = V(L, t)$, $\forall t > 0$ and the initial conditions $\rho(x, 0) = \rho_0(x)$, $V(x, 0) = V_0(x)$, $\forall x \in (0, L)$ with ρ_0, $V_0 \in L^\infty(0, L)$.

Let $\Phi : \rho > 0 \mapsto \Phi(\rho)$ such that $\rho \Phi'(\rho) - \Phi(\rho) = p(\rho)$, $\forall \rho > 0$. The quantity $\mathcal{S} = \frac{1}{2}\rho|V|^2 + \Phi(\rho)$ is an entropy of the system: entropy solutions to (1)-(2) are required to satisfy: for any $\varphi \in \mathcal{C}_c^\infty((0, L) \times [0, T))$ such that $\varphi \geqslant 0$,

$$-\int_0^T \int_0^L \left[\mathcal{S}\partial_t\varphi + (\mathcal{S} + p(\rho))V\partial_x\varphi \right](x, t)\, \mathrm{d}x\, \mathrm{d}t - \int_0^L \mathcal{S}_0(x)\varphi(x, 0)\, \mathrm{d}x \leqslant 0. \tag{3}$$

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The meshes. We consider a set of $J+1$ points $0 = x_1 < x_2 < ... < x_J < x_{J+1} = L$. The x_j are the edges of the so-called primal mesh $\mathcal{T}$. We set $\delta x_{j+1/2} = x_{j+1} - x_j$. The centers of the primal cells, $x_{j+1/2} = (x_j + x_{j+1})/2$ for $j \in \{1,..,J\}$, realize the dual mesh $\mathcal{T}^\star$. We set $\delta x_j = (\delta x_{j-1/2} + \delta x_{j+1/2})/2$ for $j \in \{2,..,J-1\}$ and $\delta x = \text{size}(\mathcal{T}) = \max_j \delta x_{j+1/2}$. The adaptive time step is δt^k and we set $\delta t = \max_k \delta t^k$.

The scheme. We analyze the scheme introduced in [1]. It works on staggered grids: the densities, $\rho_{j+1/2}$, $j \in \{1,..,J\}$, are evaluated at centers whereas the velocities, V_j, $j \in \{1,..,J+1\}$, are evaluated at edges. We set, for $j \in \{1,..,J\}$ and $i \in \{2,..,J\}$

$$\rho_{j+1/2}^0 = \frac{1}{\delta x_{j+1/2}} \int_{x_j}^{x_{j+1}} \rho_0(x)\,\mathrm{d}x, \quad V_i^0 = \frac{1}{\delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} V_0(x)\,\mathrm{d}x. \tag{4}$$

The density is first updated with a FV approximation on the primal mesh

$$\delta x_{j+1/2}(\rho_{j+1/2}^{k+1} - \rho_{j+1/2}^k) + \delta t^k(\mathscr{F}_{j+1}^k - \mathscr{F}_j^k) = 0, \quad \forall j \in \{1,..,J\}. \tag{5}$$

Then, the velocity is updated with a FV approximation on the dual mesh:

$$\delta x_j(\rho_j^{k+1} V_j^{k+1} - \rho_j^k V_j^k) + \delta t^k\left(\mathscr{G}_{j+1/2}^k - \mathscr{G}_{j-1/2}^k + \pi_{j+1/2}^{k+1/2} - \pi_{j-1/2}^{k+1/2}\right) = 0, \tag{6}$$

for $j \in \{2,..,J\}$, while $V_1^{k+1} = V_{J+1}^{k+1} = 0$. The density on the edges ρ_j^k is defined by

$$2\delta x_j \rho_j^k = \delta x_{j+1/2}\rho_{j+1/2}^k + \delta x_{j-1/2}\rho_{j-1/2}^k, \quad \forall j \in \{2,..,J\}.$$

The definition of the fluxes relies on the kinetic framework. We refer the reader to [1] for details. Let us introduce the two following functions $\mathscr{F}^+$ and $\mathscr{F}^-$

$$\mathscr{F}^\pm(\rho, V) = \frac{\rho}{2c(\rho)} \int_{\xi \gtrless 0} \xi\, \mathbb{1}_{|\xi - V| \leqslant c(\rho)}\mathrm{d}\,\xi.$$

We adopt the following formulas for mass fluxes: $\mathscr{F}_1^k = \mathscr{F}_{J+1}^k = 0$,

$$\mathscr{F}_j^k = \mathscr{F}^+(\rho_{j-1/2}^k, V_j^k) + \mathscr{F}^-(\rho_{j+1/2}^k, V_j^k), \quad \forall j \in \{2,..,J\}, \tag{7}$$

and, for momentum fluxes: $\mathscr{G}_{3/2}^k = \dfrac{V_2^k}{2}\mathscr{F}^-(\rho_{5/2}^k, V_2^k)$, $\mathscr{G}_{J+1/2}^k = \dfrac{V_J^k}{2}\mathscr{F}^+(\rho_{J-1/2}^k, V_J^k)$,

$$\begin{aligned}
\mathscr{G}_{j+1/2}^k = &\frac{V_j^k}{2}\left(\mathscr{F}^+(\rho_{j-1/2}^k, V_j^k) + \mathscr{F}^+(\rho_{j+1/2}^k, V_{j+1}^k)\right) \\
&+ \frac{V_{j+1}^k}{2}\left(\mathscr{F}^-(\rho_{j+1/2}^k, V_j^k) + \mathscr{F}^-(\rho_{j+3/2}^k, V_{j+1}^k)\right), \ \forall j \in \{2,..,J-1\}.
\end{aligned} \tag{8}$$

The discrete pressure gradient combines a space centered scheme and time semi implicit discretization, namely it uses

$$\pi_{j+1/2}^{k+1/2} = \rho_{j+1/2}^k \Phi'(\rho_{j+1/2}^{k+1}) - \Phi(\rho_{j+1/2}^k).$$

Properties of the scheme. The analysis is driven by the shapes of the functions $\mathscr{F}^{\pm}$, see [1, Lemma 3.2]. Here, we shall use the following properties

(i) Smoothness: $(\rho, V) \in (0, \infty) \times \mathbb{R} \mapsto \mathscr{F}^{\pm}(\rho, V)$ are of class C^1,

(ii) Consistency: $\mathscr{F}^{+}(\rho, V) + \mathscr{F}^{-}(\rho, V) = \rho V, \quad \forall V \in \mathbb{R}, \; \forall \rho \geqslant 0.$

(9)

Under CFL conditions, see [1], the scheme preserves the positivity of the discrete density and discrete kinetic and internal energies evolution equations hold.

Lemma 1.1 *Let $N \in \mathbb{N}$. Assume $\min_i \left(\rho_{i+1/2}^0\right) > 0$. For all $k \in \{0, .., N-1\}$, there exists $\mathcal{V}^k > 0$, which depends only on the state (ρ^k, V^k), such that if*

$$\frac{\delta t^k}{\min_j \left(\delta x_{j+1/2}\right)} \mathcal{V}^k \leqslant 1,$$

(10)

then, $\min_i \left(\rho_{i+1/2}^k\right) > 0, \forall k \in \{0, .., N\}$ and

$$0 \leqslant \sum_{k=0}^{N-1} \sum_{j=2}^{J} D_j^k \leqslant C, \quad \text{with} \quad D_j^k = \frac{1}{4} \delta x_j \rho_j^{k+1} (V_j^{k+1} - V_j^k)^2,$$

(11)

$$\frac{\delta x_{j+1/2}}{\delta t^k} \left[e_{j+1/2}^{k+1} - e_{j+1/2}^k \right] + \overline{G}_{j+1}^k - \overline{G}_j^k + \pi_{j+1/2}^{k+1/2} \left[V_{j+1}^{k+1} - V_j^{k+1} \right] \leqslant \frac{D_j^k}{\delta t^k},$$

(12)

$$\frac{\delta x_j}{\delta t^k} \left[E_{K,j}^{k+1} - E_{K,j}^k \right] + \Gamma_{j+1/2}^k - \Gamma_{j-1/2}^k + \left[\pi_{j+1/2}^{k+1/2} - \pi_{j-1/2}^{k+1/2} \right] V_j^{k+1} + \frac{D_j^k}{\delta t^k} \leqslant 0,$$

(13)

where $E_{K,j}^k = \frac{1}{2} \rho_j^k (V_j^k)^2$ and $e_{j+1/2}^k = \Phi(\rho_{j+1/2}^k)$ are the kinetic and internal energies. The fluxes are defined by $\overline{G}_1^k = \overline{G}_{J+1}^k = 0$ and

$$\overline{G}_j^k = \Phi(\rho_{j-1/2}^k) V_j^{k+1} - \frac{\delta x_{j-1/2}}{2 \delta t^k} \left[\bar{\Phi}\left(\overline{\rho_{j-1/2}^{k+1}} \right) - \bar{\Phi}(\rho_{j-1/2}^k) \right], \; \forall j \in \{2, .., J\},$$

$$\Gamma_{j+1/2}^k = \frac{1}{2} V_j^k V_{j+1}^k \frac{\mathscr{F}_j^k + \mathscr{F}_{j+1}^k}{2} + \frac{1}{2} (V_j^k - V_{j+1}^k)^2 \frac{\mathscr{F}_j^{k,|\cdot|} + \mathscr{F}_{j+1}^{k,|\cdot|}}{2}, \; \forall j \in \{1, .., J\},$$

$$\overline{\overline{\rho_{j-1/2}^{k+1}}} = \rho_{j-1/2}^k - \frac{2 \delta t^k}{\delta x_{j-1/2}} \left(\mathscr{F}^-(\rho_{j+1/2}^k, V_j^k) - \mathscr{F}^-(\rho_{j-1/2}^k, V_j^k) - \rho_{j-1/2}^k (V_j^{k+1} - V_j^k) \right),$$

and $\mathscr{F}_1^{k,|\cdot|} = \mathscr{F}_{J+1}^{k,|\cdot|} = 0$, $\mathscr{F}_j^{k,|\cdot|} = \mathscr{F}^{+}(\rho_{j-1/2}^k, V_j^k) - \mathscr{F}^{-}(\rho_{j+1/2}^k, V_j^k)$, $\forall j \in \{2, .., J\}$. The function $\bar{\Phi}$ is a C^2 extension of the function Φ (see [1, Section 4.3]).

Results. As in [2], we prove a Lax-Wendroff-like statement: the limit of a converging (and uniformly bounded) sequence of stepwise constant functions defined from the scheme is a weak entropic-solution of the system of conservation laws.

2 Consistency analysis

Notation. Assuming that $\sum_{k=0}^{N-1} \delta t^k = T$, we define the reconstructions $(i = 0, 1)$

$$\rho_\delta^{(i)} = \sum_{k=0}^{N-1} \sum_{j=1}^{J} \rho_{j+1/2}^{k+i} \chi_{j+1/2}^{k+1/2}, \quad \pi_\delta = \sum_{k=0}^{N-1} \sum_{j=1}^{J} \pi_{j+1/2}^{k+1/2} \chi_{j+1/2}^{k+1/2}, \quad V_\delta = \sum_{k=0}^{N-1} \sum_{j=2}^{J} V_j^k \chi_j^{k+1/2},$$

where $\chi_j^{k+1/2} = \chi_{[x_{j-1/2},\, x_{j+1/2}[\times[t^k,\, t^{k+1}[}, \quad \chi_{j+1/2}^{k+1/2} = \chi_{[x_j,\, x_{j+1}[\times[t^k,\, t^{k+1}[}.$

We also introduce the following discrete norms

$$||\rho_\delta||_{\infty,\mathcal{T}} = \max_{0\leqslant k\leqslant N} \max_{1\leqslant j\leqslant J} |\rho_{j+1/2}^k|, \qquad ||V_\delta||_{\infty,\mathcal{T}^\star} = \max_{0\leqslant k\leqslant N} \max_{2\leqslant j\leqslant J} |V_j^k|,$$

$$||\rho_\delta||_{1;\mathrm{BV},\mathcal{T}} = \sum_{k=0}^N \delta t^k \sum_{j=2}^J |\rho_{j+1/2}^k - \rho_{j-1/2}^k|, \quad ||V_\delta||_{1;\mathrm{BV},\mathcal{T}^\star} = \sum_{k=0}^N \delta t^k \sum_{j=1}^J |V_{j+1}^k - V_j^k|,$$

$$||\rho_\delta||_{\mathrm{BV};1,\mathcal{T}} = \sum_{j=1}^J \delta x_{j+1/2} \sum_{k=0}^{N-1} |\rho_{j+1/2}^{k+1} - \rho_{j+1/2}^k|.$$

For $\varphi \in \mathcal{C}_c^\infty((0,L)\times[0,T))$, we set $\varphi_{j+1/2}^k = \varphi(x_{j+1/2}, t^k)$ and $\varphi_j^k = \varphi(x_j, t^k)$. The interpolate $\varphi_\mathcal{T}$ of φ on the primal mesh and its discrete derivatives are defined by

$$\varphi_\mathcal{T}(\cdot,0) = \sum_{j=1}^J \varphi_{j+1/2}^0 \chi_{j+1/2}^{1/2}(\cdot,0), \qquad \varphi_\mathcal{T}(\cdot,t) = \sum_{k=0}^{N-1}\sum_{j=1}^J \varphi_{i+1/2}^{k+1} \chi_{j+1/2}^{k+1/2}(\cdot,t), \ \forall t>0,$$

$$\eth_t\varphi_\mathcal{T} = \sum_{k=0}^{N-1}\sum_{j=1}^J \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j+1/2}^k}{\delta t^k} \chi_{j+1/2}^{k+1/2}, \quad \eth_x\varphi_\mathcal{T} = \sum_{k=0}^{N-1}\sum_{j=2}^J \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j} \chi_j^{k+1/2}.$$

Similarly, the interpolate $\varphi_{\mathcal{T}^\star}$ of φ on $\mathcal{T}^\star$ and its discrete derivatives are given by

$$\varphi_{\mathcal{T}^\star}(\cdot,0) = \sum_{j=2}^J \varphi_j^0 \chi_j^{1/2}(\cdot,0), \qquad \varphi_{\mathcal{T}^\star}(\cdot,t) = \sum_{k=0}^{N-1}\sum_{j=2}^J \varphi_j^{k+1} \chi_j^{k+1/2}(\cdot,t), \ \forall t>0,$$

$$\eth_t^\star\varphi_{\mathcal{T}^\star} = \sum_{k=0}^{N-1}\sum_{j=2}^J \frac{\varphi_j^{k+1} - \varphi_j^k}{\delta t^k} \chi_j^{k+1/2}, \qquad \eth_x^\star\varphi_{\mathcal{T}^\star} = \sum_{k=0}^{N-1}\sum_{j=1}^J \frac{\varphi_{j+1}^{k+1} - \varphi_j^{k+1}}{\delta x_{j+1/2}} \chi_{j+1/2}^{k+1/2}.$$

Assumptions. Let $(\mathcal{T}_m)_{m\geqslant 1}$ be a sequence of meshes s. t. size$(\mathcal{T}_m) \to 0$ and a familly of time steps $(\delta t_m^k)_{k\geqslant 0,\, m\geqslant 1}$ verifying $\delta t_m \to 0$ and (10). Assume that there exists $N_m \in \mathbb{N}$ s. t. $\sum_{k=0}^{N_m-1} \delta t_m^k = T$. The scheme defines $(\rho_{\delta_m}^{(0)}, V_{\delta_m})_{m\geqslant 1}$. Suppose that

$$||\rho_{\delta_m}^{(0)}||_{\infty,\mathcal{T}} + ||V_{\delta_m}||_{\infty,\mathcal{T}^\star} \leqslant C_\infty, \qquad ||\rho_{\delta_m}^{(0)}||_{1;\mathrm{BV},\mathcal{T}} + ||V_{\delta_m}||_{1;\mathrm{BV},\mathcal{T}^\star} \leqslant C_{\mathrm{BV}} \qquad (14)$$

holds and, in the case $(\rho \mapsto \frac{p'(\rho)}{\rho}) \notin L_{\mathrm{loc}}^1(0,\infty)$, $||(\rho_{\delta_m}^{(0)})^{-1}||_{\infty,\mathcal{T}} \leqslant C$. We assume that there exists $(\bar\rho, \bar V) \in L^\infty((0,T)\times(0,L))^2$ such that

$$(\rho_{\delta_m}^{(0)}, V_{\delta_m}) \longrightarrow (\bar\rho, \bar V) \text{ in } L^r((0,T)\times(0,L))^2, \ 1\leqslant r < \infty. \qquad (15)$$

Main results. The uniform bounds imply that there exists constants such that

$$\sup_{0\leqslant\rho,|V|\leqslant C_\infty} |\mathscr{A}(\rho,V)| \ \leqslant \ C_\mathscr{A}, \quad \text{with } \mathscr{A} = \mathscr{F}^\pm, \ \partial_\rho\mathscr{F}^\pm \text{ and } \partial_V\mathscr{F}^\pm,$$

$$\sup_{0\leqslant\rho\leqslant C_\infty+4(C_\infty^2+C_{\mathscr{F}^\pm})} |\mathscr{B}(\rho)| \ \leqslant \ C_\mathscr{B}, \quad \text{with } \mathscr{B} = \Phi, \ \Phi', \text{ and } \bar\Phi'.$$

Note also that we have $|\Phi(\rho_{j+1/2}^k)| \leqslant C_{\Phi,\rho}\rho_{j+1/2}^k$, $\forall j,k$. Furthermore, we show that $||\rho_{\delta_m}^{(0)}||_{\mathrm{BV};1,\mathcal{T}} \leqslant C$ by using (5) which allows to dominate $\delta x_{j+1/2}|\rho_{j+1/2}^{k+1} - \rho_{j+1/2}^k|$ by

$$\delta t^k \Big[C_{\partial_\rho \mathscr{F}^\pm}(|\rho_{j+1/2}^k - \rho_{j-1/2}^k| + |\rho_{j+3/2}^k - \rho_{j+1/2}^k|) + 2C_{\partial_V \mathscr{F}^\pm}|V_{j+1}^k - V_j^k| \Big].$$

Consequently, $\rho_{\delta_m}^{(1)} \to \bar{\rho}$ and $\pi_{\delta_m} \to p(\bar{\rho})$ in $L^r((0,T) \times (0,L))$; with (4) and since $\rho_0, V_0 \in L^\infty(0,L)$, we get $\rho_{\delta_m}^{(0)}(\cdot,0) \to \rho_0$ and $V_{\delta_m}(\cdot,0) \to V_0$ in $L^r((0,L))$, $1 \leqslant r < \infty$.

Finally, in the sequel, when a function $\varphi \in \mathcal{C}_c^\infty((0,L) \times [0,T))$ is given, we assume that δt_m and δx_m are sufficiently small so that $\varphi(x,\cdot) \equiv 0$, $\forall x \in [0,x_{3/2}] \cup [x_{J+1/2}, L]$ and $\varphi(\cdot,t) \equiv 0$, $\forall t \in [t^{N-1}, t^N]$. Moreover, since φ is smooth, $\varphi_{\mathcal{T}_m}$, $\varphi_{\mathcal{T}_m^\star} \to \varphi$, $\eth_t\varphi_{\mathcal{T}_m}$, $\eth_t^\star\varphi_{\mathcal{T}_m^\star} \to \partial_t\varphi$, and $\eth_x\varphi_{\mathcal{T}_m}$, $\eth_x^\star\varphi_{\mathcal{T}_m^\star} \to \partial_x\varphi$, in $L^r((0,T) \times (0,L))$, $1 \leqslant r \leqslant \infty$.

Theorem 2.1 *Assume* (14) *and* (15). *Then,* $(\bar{\rho}, \bar{V})$ *satisfies* (1)-(2) *in the distribution sense in* $(\mathcal{C}_c^\infty((0,L) \times [0,T)))'$, *that is*

$$-\int_0^T \int_0^L \Big[\bar{\rho}\partial_t\varphi + \bar{\rho}\bar{V}\partial_x\varphi\Big](x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_0^L \rho_0(x)\varphi(x,0)\,\mathrm{d}x = 0, \quad (16)$$

$$-\int_0^T \int_0^L \Big[\bar{\rho}\bar{V}\partial_t\varphi + (\bar{\rho}\bar{V}^2 + p(\bar{\rho}))\partial_x\varphi\Big](x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_0^L \rho_0(x)V_0(x)\varphi(x,0)\,\mathrm{d}x = 0. \quad (17)$$

Moreover, $(\bar{\rho}, \bar{V})$ *satisfies the entropy inequality* (3).

Proof. Let $\varphi \in \mathcal{C}_c^\infty((0,L) \times [0,T))$. For the sake of simplicity, the index m is dropped.

Mass balance. We multiply (5) by $\varphi_{j+1/2}^{k+1}$ and sum the results for $0 \leqslant k \leqslant N-1$ and $1 \leqslant j \leqslant J$ to obtain

$$\underbrace{\sum_{k=0}^{N-1}\sum_{j=1}^J \delta x_{j+1/2}(\rho_{j+1/2}^{k+1} - \rho_{j+1/2}^k)\varphi_{j+1/2}^{k+1}}_{:=T_1} + \underbrace{\sum_{k=0}^{N-1}\delta t^k\sum_{j=1}^J (\mathscr{F}_{j+1}^k - \mathscr{F}_j^k)\varphi_{j+1/2}^{k+1}}_{:=T_2} = 0.$$

For T_1, since $\varphi_{j+1/2}^N = 0$, a discrete integration by part w.r.t. time yields

$$T_1 = -\sum_{k=0}^{N-1}\sum_{j=1}^J \delta x_{j+1/2}\rho_{j+1/2}^k(\varphi_{j+1/2}^{k+1} - \varphi_{j+1/2}^k) - \sum_{j=1}^J \delta x_{j+1/2}\rho_{j+1/2}^0\varphi_{j+1/2}^0.$$

Noting that

$$\int_{t^k}^{t^{k+1}} \int_{x_j}^{x_{j+1}} \rho_\delta^{(0)}\eth_t\varphi_\mathcal{T}\,\mathrm{d}x\,\mathrm{d}t = \delta x_{j+1/2}\rho_{j+1/2}^k(\varphi_{j+1/2}^{k+1} - \varphi_{j+1/2}^k)$$

for $k \in \{0,..,N-1\}$, $j \in \{1,..,J\}$, we get

$$T_1 = -\int_0^T \int_0^L \rho_\delta^{(0)}\eth_t\varphi_\mathcal{T}\,\mathrm{d}x\,\mathrm{d}t - \int_0^L \rho_\delta^{(0)}(x,0)\varphi_\mathcal{T}(x,0)\,\mathrm{d}x.$$

For T_2, by integrating by part w.r.t. space, we readily obtain

$$T_2 = -\sum_{k=0}^{N-1}\delta t^k\sum_{j=2}^J \mathscr{F}_j^k(\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}).$$

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Bearing in mind that $2\delta x_j = \delta x_{j-1/2} + \delta x_{j+1/2}$, we then combine the two following expressions of mass fluxes (see (9)-(ii))

$$\mathscr{F}_j^k = \rho_{j\pm 1/2}^k V_j^k \mp R_j^{k,\pm} \text{ with } R_j^{k,\pm} = \mathscr{F}^{\pm}(\rho_{j+1/2}^k, V_j^k) - \mathscr{F}^{\pm}(\rho_{j-1/2}^k, V_j^k)$$

to write

$$\mathscr{F}_j^k = \left[\frac{\delta x_{j-1/2}}{2\delta x_j} \rho_{j-1/2}^k V_j^k + \frac{\delta x_{j+1/2}}{2\delta x_j} \rho_{j+1/2}^k V_j^k \right] + \left[\frac{\delta x_{j-1/2}}{2\delta x_j} R_j^{k,-} - \frac{\delta x_{j+1/2}}{2\delta x_j} R_j^{k,+} \right].$$

This expression of the mass fluxes leads to $T_2 = -T_{2,1} - T_{2,2}$ with

$$T_{2,1} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^{J} \frac{1}{2} \left[\delta x_{j-1/2} \rho_{j-1/2}^k + \delta x_{j+1/2} \rho_{j+1/2}^k \right] V_j^k \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j},$$

$$T_{2,2} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^{J} \frac{1}{2} \left[\delta x_{j-1/2} R_j^{k,-} - \delta x_{j+1/2} R_j^{k,+} \right] V_j^k \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j}.$$

We now observe that, for $k \in \{0, .., N-1\}$, $j \in \{2, .., J\}$,

$$\int_{t^k}^{t^{k+1}} \int_{x_{j-1/2}}^{x_j} \rho_\delta^{(0)} V_\delta \eth_x \varphi_{\mathcal{T}} \, \mathrm{d}x \, \mathrm{d}t = \delta t^k \frac{\delta x_{j-1/2}}{2} \left[\rho_{j-1/2}^k V_j^k \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j} \right],$$

and

$$\int_{t^k}^{t^{k+1}} \int_{x_j}^{x_{j+1/2}} \rho_\delta^{(0)} V_\delta \eth_x \varphi_{\mathcal{T}} \, \mathrm{d}x \, \mathrm{d}t = \delta t^k \frac{\delta x_{j+1/2}}{2} \left[\rho_{j+1/2}^k V_j^k \frac{\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}}{\delta x_j} \right].$$

Summing these equalities yields

$$T_{2,1} = \int_0^T \int_{x_{3/2}}^{x_{J+1/2}} \rho_\delta^{(0)} V_\delta \eth_x \varphi_{\mathcal{T}} \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_0^L \rho_\delta^{(0)} V_\delta \eth_x \varphi_{\mathcal{T}} \, \mathrm{d}x \, \mathrm{d}t,$$

since $\eth_x \varphi_{\mathcal{T}}(x, \cdot) \equiv 0$ for $x \in [0, x_{3/2}] \cup [x_{J+1/2}, L]$.

With (15), we pass to the limit in T_1 and $T_{2,1}$. We prove that $(\bar{\rho}, \bar{V})$ satisfies the mass conservation equation (16) by showing that $T_{2,2} \to 0$ since

$$|T_{2,2}| \leqslant C_{\partial_\rho \mathscr{F}^{\pm}} \ |\partial_x \varphi|_{L^\infty} \ ||V_\delta||_{\infty, \mathcal{T}^\star} \ ||\rho_\delta||_{1;\mathrm{BV}, \mathcal{T}} \ \delta x \lesssim \delta x.$$

Momentum balance. We multiply (6) by φ_j^{k+1} and sum for $0 \leqslant k \leqslant N-1$ and $2 \leqslant j \leqslant J$ to obtain $T_3 + T_4 + T_5 = 0$ with

$$T_3 = \sum_{k=0}^{N-1} \sum_{j=2}^{J} \delta x_j (\rho_j^{k+1} V_j^{k+1} - \rho_j^k V_j^k) \varphi_j^{k+1},$$

$$T_4 = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^{J} (\mathscr{G}_{j+1/2}^k - \mathscr{G}_{j-1/2}^k) \varphi_j^{k+1}, \quad T_5 = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^{J} (\pi_{j+1/2}^{k+1/2} - \pi_{j-1/2}^{k+1/2}) \varphi_j^{k+1}.$$

For T_3, integrating by part w.r.t time yields

$$T_3 = -\sum_{k=0}^{N-1} \sum_{j=2}^{J} \delta x_j \rho_j^k V_j^k (\varphi_j^{k+1} - \varphi_j^k) - \sum_{j=2}^{J} \delta x_j \rho_j^0 V_j^0 \varphi_j^0.$$

Next, we observe that

$$\int_{t^k}^{t^{k+1}}\int_{x_{j-1/2}}^{x_{j+1/2}}\rho_\delta^{(0)}V_\delta\eth_t^\star\varphi_{\mathcal{T}^\star}\,\mathrm{d}x\,\mathrm{d}t = V_j^k\frac{\varphi_j^{k+1}-\varphi_j^k}{\delta t^k}\int_{t^k}^{t^{k+1}}\int_{x_{j-1/2}}^{x_{j+1/2}}\rho_\delta^{(0)}\,\mathrm{d}x\,\mathrm{d}t$$

$$= \delta x_j\rho_j^k V_j^k(\varphi_j^{k+1}-\varphi_j^k).$$

Summing these equalities for $k\in\{0,..,N-1\}$, $j\in\{2,..,J\}$ yields

$$T_3 = -\int_0^T\int_{x_{3/2}}^{x_{J+1/2}}\rho_\delta^{(0)}V_\delta\eth_t^\star\varphi_{\mathcal{T}^\star}\,\mathrm{d}x\,\mathrm{d}t - \int_{x_{3/2}}^{x_{J+1/2}}\rho_\delta^{(0)}(x,0)V_\delta(x,0)\varphi_{\mathcal{T}^\star}(x,0)\,\mathrm{d}x,$$

$$= -\int_0^T\int_0^L\rho_\delta^{(0)}V_\delta\eth_t^\star\varphi_{\mathcal{T}^\star}\,\mathrm{d}x\,\mathrm{d}t - \int_0^L\rho_\delta^{(0)}(x,0)V_\delta(x,0)\varphi_{\mathcal{T}^\star}(x,0)\,\mathrm{d}x.$$

For T_4 and T_5, we first integrate by part w.r.t space and obtain

$$T_4 = -\sum_{k=0}^{N-1}\delta t^k\sum_{j=1}^{J}\mathscr{G}_{j+1/2}^k(\varphi_{j+1}^{k+1}-\varphi_j^{k+1}),\quad T_5 = -\sum_{k=0}^{N-1}\delta t^k\sum_{j=1}^{J}\pi_{j+1/2}^{k+1/2}(\varphi_{j+1}^{k+1}-\varphi_j^{k+1}).$$

We then use the following expression of the momentum flux

$$\begin{cases}\mathscr{G}_{j+1/2}^k &= \frac{1}{2}\rho_{j+1/2}^k\Big[(V_j^k)^2+(V_{j+1}^k)^2\Big]+Q_{j+1/2}^k,\\[2mm]Q_{j+1/2}^k &= -\frac{1}{2}V_j^k R_j^{k,+}+\frac{1}{2}V_{j+1}^k R_{j+1}^{k,-}\\[2mm]&\quad -\frac{1}{2}(V_{j+1}^k-V_j^k)\Big[\mathscr{F}^+(\rho_{j+1/2}^k,V_{j+1}^k)-\mathscr{F}^-(\rho_{j+1/2}^k,V_j^k)\Big].\end{cases}$$

to write $T_4 = -T_{4,1}-T_{4,2}$ with

$$T_{4,1} = \sum_{k=0}^{N-1}\delta t^k\sum_{j=1}^{J}\frac{1}{2}\rho_{j+1/2}^k\Big[(V_j^k)^2+(V_{j+1}^k)^2\Big](\varphi_{j+1}^{k+1}-\varphi_j^{k+1})$$

$$T_{4,2} = \sum_{k=0}^{N-1}\delta t^k\sum_{j=1}^{J}Q_{j+1/2}^k(\varphi_{j+1}^{k+1}-\varphi_j^{k+1}).$$

Next, we observe that

$$\int_{t^k}^{t^{k+1}}\int_{x_j}^{x_{j+1}}\rho_\delta^{(0)}(V_\delta)^2\eth_x^\star\varphi_{\mathcal{T}^\star}\,\mathrm{d}x\,\mathrm{d}t = \delta t^k\rho_{j+1/2}^k\frac{(V_j^k)^2+(V_{j+1}^k)^2}{2}(\varphi_{j+1}^{k+1}-\varphi_j^{k+1})$$

and, consequently, summing for $k\in\{0,..,N-1\}$, $j\in\{1,..,J\}$,

$$T_{4,1} = \int_0^T\int_0^L\rho_\delta^{(0)}(V_\delta)^2\eth_x^\star\varphi_{\mathcal{T}^\star}\,\mathrm{d}x\,\mathrm{d}t.$$

Similarly, for T_5, we get

$$T_5 = -\int_0^T\int_0^L\pi_\delta\eth_x^\star\varphi_{\mathcal{T}^\star}\,\mathrm{d}x\,\mathrm{d}t.$$

With (15), we pass to the limit in T_3, $T_{4,1}$ and T_5. We obtain that $(\bar\rho,\bar V)$ satisfies the momentum balance equation (17) by showing that $T_{4,2}\to 0$. Indeed, we have

$$|T_{4,2}|\leqslant ||V_\delta||_{\infty,\mathcal{T}^\star}|\partial_x\varphi|_{L^\infty}\Big(C_{\partial_\rho\mathscr{F}^\pm}||\rho_\delta||_{1;\mathrm{BV},\mathcal{T}}+C_{\mathscr{F}^\pm}||V_\delta||_{1;\mathrm{BV},\mathcal{T}^\star}\Big)\delta x \lesssim \delta x.$$

Entropy inequality. We now assume that $\varphi \geqslant 0$.

• Kinetic energy. We multiply (13) by $\delta t^k \varphi_j^{k+1}$ and sum for $0 \leqslant k \leqslant N-1$ and $2 \leqslant j \leqslant J$. We obtain to get $T_6 + T_7 + T_8 \leqslant 0$ with

$$T_6 = \sum_{k=0}^{N-1} \sum_{j=2}^{J} \delta x_j \Big[E_{K,j}^{k+1} - E_{K,j}^{k} \Big] \varphi_j^{k+1}, \quad T_7 = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^{J} \Big[\Gamma_{j+1/2}^{k} - \Gamma_{j-1/2}^{k} \Big] \varphi_j^{k+1},$$

$$T_8 = \sum_{k=0}^{N-1} \delta t^k \sum_{j=2}^{J} \Big[\pi_{j+1/2}^{k+1/2} - \pi_{j-1/2}^{k+1/2} \Big] V_j^{k+1} \varphi_j^{k+1} + \sum_{k=0}^{N-1} \sum_{j=2}^{J} D_j^k \varphi_j^{k+1}.$$

Integrating by part w.r.t. time yields

$$T_6 = - \sum_{k=0}^{N-1} \sum_{j=2}^{J} \delta x_j E_{K,j}^{k} \Big[\varphi_j^{k+1} - \varphi_j^{k} \Big] - \sum_{j=1}^{J} \delta x_j E_{K,j}^0 \varphi_j^0$$

$$= - \int_0^T \int_0^L \frac{1}{2} \rho_\delta^{(0)} (V_\delta)^2 \eth_t^\star \varphi_{\mathcal{T}^\star} \, dx \, dt - \int_0^L \frac{1}{2} \rho_\delta^{(0)}(x,0) \big(V_\delta(x,0) \big)^2 \varphi_{\mathcal{T}^\star}(x,0) \, dx.$$

For T_7, we write $\Gamma_{j+1/2}^{k} = \frac{1}{4} \rho_{j+1/2}^{k} \big[(V_j^k)^3 + (V_{j+1}^k)^3 \big] + \frac{1}{4} S_{j+1/2}^{k}$ where

$$S_{j+1/2}^{k} = V_j^k V_{j+1}^k \Big[R_{j+1}^{k,-} - R_j^{k,+} \Big] + (V_{j+1}^k - V_j^k)^2 \Big[\mathscr{F}_j^{k,|\cdot|} + \mathscr{F}_{j+1}^{k,|\cdot|} - \rho_{j+1/2}^{k}(V_j^k + V_{j+1}^k) \Big].$$

Integration by part w.r.t space leads to $T_7 = -T_{7,1} - T_{7,2}$ with

$$T_{7,1} = \int_0^T \int_0^L \frac{1}{2} \rho_\delta^{(0)} (V_\delta)^3 \eth_x^\star \varphi_{\mathcal{T}^\star} \, dx \, dt, \quad T_{7,2} = \frac{1}{4} \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^{J} S_{j+1/2}^{k} \Big[\varphi_{j+1}^{k+1} - \varphi_j^{k+1} \Big].$$

Finally $|T_{7,2}| \lesssim \delta x$ since it is dominated by

$$\delta x |\partial_x \varphi|_{L^\infty} \|V_\delta\|_{\infty,\mathcal{T}^\star} \Bigg[\frac{C_{\partial_\rho \mathscr{F}^\pm}}{2} \|V_\delta\|_{\infty,\mathcal{T}^\star} \|\rho_\delta\|_{1;\mathrm{BV},\mathcal{T}}$$

$$+ (2 C_{\mathscr{F}^\pm} + \|V_\delta\|_{\infty,\mathcal{T}^\star} \|\rho_\delta\|_{\infty,\mathcal{T}}) \|V_\delta\|_{1;\mathrm{BV},\mathcal{T}^\star} \Bigg].$$

• Internal energy. Multiply (12) by $\delta t^k \varphi_{j+1/2}^{k+1}$ and sum for $0 \leqslant k \leqslant N-1$ and $1 \leqslant j \leqslant J$ to get $T_9 + T_{10} + T_{11} \leqslant 0$ with

$$T_9 = \sum_{k=0}^{N-1} \sum_{j=1}^{J} \delta x_{j+1/2} \Big[e_{j+1/2}^{k+1} - e_{j+1/2}^{k} \Big] \varphi_{j+1/2}^{k+1}, \ T_{10} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^{J} \Big[\overline{G}_{j+1}^k - \overline{G}_j^k \Big] \varphi_{j+1/2}^{k+1},$$

$$T_{11} = \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^{J} \pi_{j+1/2}^{k+1/2} (V_{j+1}^{k+1} - V_j^{k+1}) \varphi_{j+1/2}^{k+1} - \sum_{k=0}^{N-1} \sum_{j=1}^{J} D_j^k \varphi_{j+1/2}^{k+1}.$$

Owing to integration by part w.r.t. time, we get

$$T_9 = - \sum_{k=0}^{N-1} \sum_{j=1}^{J} \delta x_{j+1/2} \Phi(\rho_{j+1/2}^{k})(\varphi_{j+1/2}^{k+1} - \varphi_{j+1/2}^{k}) - \sum_{j=1}^{J} \delta x_{j+1/2} \Phi(\rho_{j+1/2}^0) \varphi_{j+1/2}^0,$$

$$= - \int_0^T \int_0^L \Phi(\rho_\delta^{(0)}) \eth_t \varphi_{\mathcal{T}} \, dx \, dt - \int_0^L \Phi(\rho_\delta^{(0)}(x,0)) \varphi_{\mathcal{T}}(x,0) \, dx.$$

For T_{10}, we rewrite the flux as follows

$$
\begin{cases}
\overline{G}_j^k = \dfrac{1}{2\delta x_j}\Big[\delta x_{j-1/2}\Phi(\rho_{j-1/2}^k) + \delta x_{j+1/2}\Phi(\rho_{j+1/2}^k)\Big]V_j^k + U_{1,j}^k + U_{2,j}^k + U_{3,j}^k,\\[2mm]
U_{1,j}^k = e_{j-1/2}^k(V_j^{k+1} - V_j^k), \qquad U_{2,j}^k = -\dfrac{\delta x_{j+1/2}}{2\delta x_j}\Big[e_{j+1/2}^k - e_{j-1/2}^k\Big]V_j^k,\\[2mm]
U_{3,j}^k = -\dfrac{\delta x_{j-1/2}}{2\delta t^k}\Big[\bar{\Phi}(\overline{\rho_{j-1/2}^{k+1}}) - \bar{\Phi}(\rho_{j-1/2}^k)\Big].
\end{cases}
$$

It leads to $T_{10} = -T_{10,0} - T_{10,1} - T_{10,2} - T_{10,3}$ with

$$
\begin{cases}
T_{10,0} = \displaystyle\int_0^T\int_0^L \Phi(\rho_\delta^{(0)})V_\delta \eth_x\varphi_{\mathcal{T}}\,\mathrm{d}x\,\mathrm{d}t,\\[2mm]
T_{10,i} = \displaystyle\sum_{k=0}^{N-1}\delta t^k\sum_{j=2}^J U_{i,j}^k(\varphi_{j+1/2}^{k+1} - \varphi_{j-1/2}^{k+1}), \quad i=1,2,3.
\end{cases}
$$

The term $T_{10,1}$ can be bounded as follows

$$
|T_{10,1}| \leqslant C_{\Phi,\rho}|\partial_x\varphi|\sum_{k=0}^{N-1}\delta t^k\sum_{j=2}^J \delta x_j\rho_{j-1/2}^k|V_j^{k+1} - V_j^k|. \tag{18}
$$

Since $a \leqslant \min(a,b) + |b-a|$, we get $\rho_{j-1/2}^k \leqslant \rho_j^k + |\rho_{j+1/2}^k - \rho_{j-1/2}^k|$. This leads to

$$
|T_{10,1}| \leqslant C_{\Phi,\rho}|\partial_x\varphi|_{L^\infty}\bigg(\underbrace{\sum_{k=0}^{N-1}\delta t^k\sum_{j=2}^J \delta x_j\rho_j^k|V_j^{k+1} - V_j^k| + 2||V_\delta||_{\infty,\mathcal{T}^\star}||\rho_\delta||_{1;\mathrm{BV},\mathcal{T}}\delta x}_{:=T^\star}\bigg).
$$

Writing $\rho_j^k = \rho_j^{k+1} - (\rho_j^{k+1} - \rho_j^k)$ and using the Cauchy-Schwarz inequality yields

$$
T^\star \leqslant 2\Big(TL||\rho_\delta||_{\infty,\mathcal{T}}\Big)^{1/2}\Big(\delta t\sum_{k=0}^{N-1}\sum_{j=2}^J D_j^k\Big)^{1/2} + 2||V_\delta||_{\infty,\mathcal{T}^\star}||\rho_\delta||_{\mathrm{BV};1,\mathcal{T}}\delta t \lesssim \delta t^{1/2}.
$$

It finally leads to $|T_{10,1}| \lesssim \delta t^{\frac{1}{2}} + \delta x$. The term $T_{10,2}$ can be bounded as follows

$$
|T_{10,2}| \leqslant C_{\Phi'}||V_\delta||_{\infty,\mathcal{T}^\star}|\partial_x\varphi|_{L^\infty}||\rho_\delta||_{1;\mathrm{BV},\mathcal{T}}\,\delta x \lesssim \delta x.
$$

We now turn to $T_{10,3}$. We remark that

$$
\Big|\overline{\rho_{j-1/2}^{k+1}} - \rho_{j-1/2}^k\Big| \leqslant \frac{2\delta t^k}{\delta x_{j-1/2}}\Big(C_{\partial_\rho\mathscr{F}^\pm}|\rho_{j+1/2}^k - \rho_{j-1/2}^k| + \rho_{j-1/2}^k|V_j^{k+1} - V_j^k|\Big).
$$

Hence, using the same bound as for $T_{10,1}$ yields

$$
|T_{10,3}| \leqslant C_{\bar{\Phi}'}|\partial_x\varphi|_{L^\infty}\bigg(\Big(C_{\partial_\rho\mathscr{F}^\pm} + 2||V_\delta||_{\infty,\mathcal{T}^\star}\Big)||\rho_\delta||_{1;\mathrm{BV},\mathcal{T}}\delta x + T^\star\bigg) \lesssim \delta t^{1/2} + \delta x.
$$

• Pressure terms. It remains to get the limit of $T_8 + T_{11} = -T_{12,0} - T_{12,1} - T_{12,2} - T_{12,3}$

with

$$T_{12,0} = \int_0^T \int_0^L \pi_\delta V_\delta \bar{\eth}_x^\star \varphi_{\mathcal{T}^\star} \, dx \, dt, \qquad T_{12,1} = \sum_{k=0}^{N-1} \sum_{j=2}^{J} D_j^k (\varphi_{j+1/2}^{k+1} - \varphi_j^{k+1}),$$

$$T_{12,2} = \frac{1}{2} \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^{J} \pi_{j+1/2}^{k+1/2} (V_j^{k+1} - V_j^k + V_{j+1}^{k+1} - V_{j+1}^k)(\varphi_{j+1}^{k+1} - \varphi_j^{k+1}),$$

$$T_{12,3} = -\frac{1}{2} \sum_{k=0}^{N-1} \delta t^k \sum_{j=1}^{J} \pi_{j+1/2}^{k+1/2} (V_{j+1}^{k+1} - V_j^{k+1})(2\varphi_{j+1/2}^{k+1} - \varphi_{j+1}^{k+1} - \varphi_j^{k+1}).$$

We bound $T_{12,1}$ and $T_{12,3}$ as follows

$$T_{12,1} \leqslant \frac{|\partial_x \varphi|_{L^\infty}}{2} \left(\sum_{k=0}^{N-1} \sum_{j=2}^{J} D_j^k \right) \delta x \lesssim \delta x, \quad T_{12,3} \leqslant \frac{C_\pi}{4} |\partial_{xx} \varphi|_{L^\infty} ||V_\delta||_{1;\mathrm{BV},\mathcal{T}^\star} (\delta x)^2 \lesssim (\delta x)^2.$$

Note that $|\pi_{j+1/2}^{k+1/2}| \leqslant (C_{\Phi'} + C_{\Phi,\rho})\rho_{j+1/2}^k$. It readily leads to

$$|T_{12,2}| \leqslant (C_{\Phi'} + C_{\Phi,\rho})|\partial_x \varphi|_{L^\infty} T^\star \lesssim \delta t^{\frac{1}{2}}.$$

With (15), we pass to the limit in T_6, $T_{7,1}$, T_9, $T_{10,0}$ and $T_{12,0}$. We arrive at (3) since the other terms tend to 0. ∎

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