APPROXIMATION BY HOMOGENEIZATION AND DIFFUSION OF KINETIC EQUATIONS *

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Abstract. This work is devoted to the approximation by the diffusion of kinetic equations. The approximation is justified by homogenization and in the limit of vanishing mean free path. Our analysis includes the grey transfer model but also discrete velocity model. The technics of proofs are based on compensated compactness and double scale limit.

1. INTRODUCTION

In this work, we are concerned with kinetic equations

$$\begin{cases} \partial_t u + \dfrac{1}{\varepsilon}\ v \cdot \nabla_x u = \dfrac{1}{\varepsilon^2}\ \sigma_\delta\ \rho^\alpha\ (\rho - u)\ \ in\ \mathbb{R}_t^+ \times \mathbb{R}_x^N \times V_v \\ \\ u_{|t=0} = u_0\ \ in\ \mathbb{R}_x^N \times V_v. \end{cases} \tag{1}$$

In (1), one denotes

$$\rho(t,x) = \int_V u(t,x,v)dv. \tag{2}$$

Throughout this paper, V is a bounded domain in $\mathbb{R}^N$ (for instance the sphere

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S^{N-1}) equipped with a measure dv satisfying

$$\int_V dv = 1 \ \ ; \ \ \int_V v \ dv = 0. \tag{3}$$

Furthermore, we require that the matrix

$$\theta = \int_V v \otimes v \ dv \tag{4}$$

is positive definite (or, equivalently, $\int_V (v \cdot \xi)^2 dv > 0$ holds for all $\xi \in \mathbb{R}^N \backslash \{0\}$).

Such equations may be seen as a simplified grey transfer model describying the flux of photons in a plasma containing opaque scatterers [7], [8], [15]. This kind of problem also arises in modeling flux of neutrons in nuclear engineering [23]. We point out that our study includes discrete velocity model. Indeed, if the finite set $V = \{v^1, \ldots, v^M\}$ is equipped with the measure $dv = \sum_{i=1}^{M} \omega_i \delta_{v^i}$, then assumption (3) becomes $\sum_{i=1}^{M} \omega_i = 1$ and $\sum_{i=1}^{M} \omega_i v^i = 0$. The fact that the matrix $\theta = \sum_{i=1}^{M} \omega_i v^i \otimes v^i$ is positive definite means nothing but $\{v^i, \ i \in \{1, \ldots, M\} \ \textit{such that} \ \omega_i \neq 0\}$ is generator. Remark that necessarily $M \geq N+1$ since V is both generator and linearly dependent. In this case, we denote $u_i = u_i(t,x) = u(t,x,v^i)$ and (1) is replaced by the following system of equations

$$\left\{ \begin{array}{l} \partial_t u_i + \dfrac{1}{\varepsilon} \ v^i \cdot \nabla_x u_i = \dfrac{1}{\varepsilon^2} \ \sigma_\delta \ \rho^\alpha (\rho - u_i) \ \ for \ i \in \{1, \ldots, M\}, \\ \\ \rho = \displaystyle\sum_{i=1}^{M} \omega_i u_i. \end{array} \right. \tag{5}$$

The precise form of the opacity σ_δ will be described later (see (6) below); we just notice that it may depend on the position x and on the averaged quantity ρ. However, we emphasize here the dependance on a parameter δ. In (1), it appears two parameters ε and δ which are related to characteristic length scales: ε represents the mean free path, i.e. the mean distance of free flight of the particles, while δ describes the size of the heterogeneities in the medium. As ε goes to 0, with δ fixed, we are led to a diffusion equation as shown in E.

Larsen-J. Keller [21], C. Bardos-R. Santos-R. Sentis [9], C. Bardos-F. Golse-B. Perthame [7], C. Bardos-F. Golse-B. Perthame-R. Sentis [8] or P.L. Lions-G. Toscani [22]. Similar questions also arise in semi-conductors theory and the problem is treated by F. Poupaud [28], F. Golse-F. Poupaud [18]. On the other hand, if one considers the case $\delta \longrightarrow 0$, ε fixed, we keep a kinetic description, involving effective coefficients as in the work by F. Golse [15], [16] and L. Dumas-F. Golse [13].

Our aim in this work is to study the critical case where the characteristic lengths have the same order $\varepsilon = \delta$. Then, one replaces σ_δ in (1) by a sequence $\sigma_\varepsilon(t,x) = \Sigma_\varepsilon(x, \rho_\varepsilon(t,x))$ where Σ_ε satisfies

$$\begin{cases} 0 < \underline{\sigma} \leq \Sigma_\varepsilon \leq \overline{\sigma}, \\ \{\lambda \longmapsto \Sigma_\varepsilon(x,\lambda),\ \varepsilon \in (0,\varepsilon_0),\ |x| < R\} \text{ is equicontinuous for some } \epsilon_0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and any } R > 0, \\ \forall \lambda \in [0,\infty), \quad \Sigma_\varepsilon(.,\lambda) \rightharpoonup \Sigma(.,\lambda) \ \ in\ L^\infty(\mathbb{R}^N_x)\ weakly\ *. \end{cases} \tag{6}$$

Then, we are interested in the behaviour as ε tends to 0 of the related solution $u_\varepsilon(t,x,v)$ of (1), possibly associated to a sequence of initial data u_0^ε. One expects to obtain a diffusion equation, involving an effective diffusion matrix which depends on the limit of σ_ε. It is worth pointing out that such kind of approximations is often used in practice, in particular for numerical simulations [12], [23], [27], [11]. A first approach to the linear problem ($\alpha = 0$) is due to A. Bensoussan-J.L. Lions-G. Papanicolaou [10], R. Sentis [29], F. Golse [16] with periodicity assumptions. Homogeneization of the transport equation, without penalized terms is dealt with in [15], [16], [13]. We also refer to the work of R. Alexandre [1] and also C. Bardos-L. Dumas-P. Gérard-F. Golse [6] where related problems are considered.

In this work we will not deal with existence proof for the problems with ε fixed. We will already assume that there is a solution u_ε which belongs to $C^0([0,\infty); L^1(\mathbb{R}^N_x \times V)) \bigcap L^\infty([0,\infty) \times \mathbb{R}^N_x \times V)$. This is enough to justify the derivation of the uniform estimates of the next Sections. Let us only mention that the existence proof is an easy consequence of the fact that the right hand side in (1) is Lipschitz with respect to u_ε in L^1 in the case where $\alpha \geq 0$ and

$\Sigma_\varepsilon(x,\lambda)$ is Lipschitz with respect to λ. In the other case the proof can be more involved and the use of averaging lemma of F. Golse-P.L. Lions-B. Perthame-R. Sentis [17] can be necessary, see [8].

Our first result applies to the monodimensional case, $N=1$ and does not require periodicity assumption. Our strategy follows ideas in P.L. Lions-G. Toscani [22] and uses entropy dissipation properties and compensated compactness arguments (see L. Tartar [31], [32] or consult the survey [14]).

Theorem 1 *Assume $N=1$ and $|\alpha|<1$. Let u_0^ε be a bounded sequence of $L^\infty(I\!\!R\times V)$ which weak * converges towards u_0 and which satisfies $u_0^\varepsilon>0$ and*

$$\int_{I\!\!R\times V}(1+\omega(x)+|\ln(u_0^\varepsilon)|)u_0^\varepsilon dxdv<C, \tag{7}$$

with $\omega(x)=(1+x^2)^{\beta/2}$; $0<\beta<1/2$. Then, up to subequences, u_ε and ρ_ε converge to ρ in $L^p((0,T)\times I\!\!R\times V)$ or $L^p((0,T)\times I\!\!R)$ respectively, for $1\le p<\infty$ (strongly), where ρ is a solution of

$$\left\{\begin{array}{l}\partial_t\rho-\partial_x\left(\dfrac{\theta}{(1-\alpha)\Sigma(x,\rho)}\partial_x(\rho^{1-\alpha})\right)=0\ \ in\ (0,T)\times I\!\!R,\\[2ex] \rho_{|t=0}=\rho_0=\displaystyle\int_V u_0dv\ \ in\ I\!\!R,\end{array}\right. \tag{8}$$

with $\theta=\displaystyle\int_V v^2dv>0$.

In the case where $\Sigma_\varepsilon(x,\lambda)=\Sigma_\varepsilon(x)$, since the limit equations have a unique solution for a non negative data $\rho_0\in L^1\cap L^\infty(I\!\!R)$, see [5] for the case $\alpha<0$ and [20] for the case $\alpha>0$, the convergence actually applies to the whole sequence $(u_\varepsilon,\ \rho_\varepsilon)_{\varepsilon>0}$. The same result should also holds when $\Sigma(x,\lambda)$ is Lipschitz with respect to λ.

Our next results concern the multidimensional context, assuming a periodicity assumption. In this case we are not able to deal with singularities of the opacity when the concentration vanishes, thus we set $\alpha=0$ and the sequence σ_ε in (6) is defined by

$$\sigma_\varepsilon(x)=\Sigma(\frac{x}{\varepsilon},x,\rho_\varepsilon(x)), \tag{9}$$

where $y \longmapsto \Sigma(y,x,\rho)$ is a $Y-$periodic function, with $Y = [0,1]^N$. The proof relies on double scale convergence arguments developped in G. Allaire [2], G. N'Guetseng [26].

Theorem 2 *Assume $\alpha = 0$. Let $\Sigma = \Sigma(y,x,\lambda) \in L^2(Y; C^0([0,\infty)_\lambda \times I\!R^N_x))$ satisfy*

$$\begin{cases} 0 < \underline{\sigma} \leq \Sigma(y,x,\lambda) \leq \overline{\sigma}, \\ \forall x \in I\!R^N, \quad \forall \lambda \in [0,\infty), \qquad y \longmapsto \Sigma(y,x,\lambda) \text{ is } Y-\text{periodic} \\ \{\lambda \longmapsto \Sigma(y,x,\lambda),\ |x| < R,\ y \in Y\} \text{ is equicontinuous for any} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad R > 0, \end{cases} \tag{10}$$

Let u_0^ε be a bounded sequence of $L^\infty(I\!R^N \times V) \bigcap L^2(I\!R^N \times V)$ converging to u_0 in $L^\infty(I\!R^N \times V)$ weak$*$ and which satisfies $u_0^\varepsilon \geq 0$. Then, up to subsequences, u_ε and ρ_ε converge to ρ in $L^p_{loc}((0,T) \times I\!R^N \times V)$ and $L^p_{loc}((0,T) \times I\!R^N)$ respectively, for $1 \leq p < \infty$ (strongly), where ρ is a solution of

$$\begin{cases} \partial_t \rho - \nabla_x \cdot (Q(x,\rho)\nabla_x \rho) = 0 \ \ in\ (0,T) \times I\!R^N, \\ \rho_{|t=0} = \rho_0 = \displaystyle\int_V u_0 dv \ \ in\ I\!R^N, \end{cases} \tag{11}$$

Q being the matrix defined by (50) and the auxilliary equation (48), whose symmetric part is positive definite.

As in the preceeding theorem we obtain that the whole sequence converges in the case where $\Sigma(y,x,\lambda)$ is Lipschitz with respect to λ. Indeed in this case we have uniqueness of the solution of the limit problem, see Proposition 2.

Remark 1 *This result is rather close to those obtained by G. Allaire-G. Bal in a linear framework [3], [4] where the resolution of a spectral problem is considered. However, Theorem 2 includes nonlinear aspects as well as the possibility to deal with discrete velocities models.*

Remark 2 *Of course Theorem 2 includes the case $\alpha \neq 0$ if we assume moreover that $u_0^\varepsilon \geq \underline{u} > 0$, considering the problem in a bounded domain with periodic boundary conditions for instance. Indeed in that case the solution u_ε remains uniformly bounded from below. But the restriction $u_0^\varepsilon \geq \underline{u} > 0$ (or, equivalently the bound from below on Σ in (10)) seems difficult to be removed ; it allows to avoid difficulties related to degeneracies effects due to the vacuum in the definition of the associated homogenized matrix Q.*

Remark 3 *The assumptions (3) and (4) are quite general and in particular allow to treat the case of discrete velocities. It is worth pointing out that the essential property (4) yet appears in P.L. Lions-G. Toscani [22] as an alternative to the usual assumption required on V in order to apply average lemma of [17].*

This paper is organized as follows. Section 2 is devoted to the monodimensional problem. First, we discuss various estimates on the sequence u_ε which lead to compactness properties. Then, we will pass to the limit as ε goes to 0 in the equations obtained by integrating (1) with the measures dv and vdv respectively. In Section 3, we are concerned with the non linear periodic multidimensional problem. First, we guess the limit equation by plugging a formal ansatz in (1). This naturally leads to an auxilliary problem which holds in the cell Y. This cell problem is then studied in details. Section 4 is devoted to the proof of Theorem 2 and justify the previous derivation. Essential estimates are deduced by slight modifications of the arguments of Section 2 and, then, we apply double scale convergence methods and compensated compactness arguments to obtain the limit equation.

2. MONODIMENSIONAL PROBLEM

Throughout this Section, it is assumed $N = 1$ and we will prove Theorem 1.

2.1. A priori estimates

We look for estimates on the solutions u_ε of (1). Obviously, (3) gives

$$\int_V (\rho_\varepsilon - u_\varepsilon)\, dv = 0. \tag{12}$$

Hence, integrating (1) with respect to x, v yields the following "mass conservation" property

$$\frac{d}{dt}\int_{\mathbb{R}\times V} u_\varepsilon\, dvdx = 0. \tag{13}$$

Let ϕ be a convex regular fonction ($\phi'' \geq 0$). One has

$$\begin{aligned}
&\int_{I\!R\times V} \phi'(u_\varepsilon)(\partial_t + \frac{1}{\varepsilon}\partial_x)u_\varepsilon \; dvdx = \int_{I\!R\times V} (\partial_t + \frac{1}{\varepsilon}\partial_x)\phi(u_\varepsilon) \; dvdx \\
&= \frac{1}{\varepsilon^2}\int_{I\!R\times V} \sigma_\varepsilon \rho_\varepsilon^\alpha (\rho_\varepsilon - u_\varepsilon)\phi'(u_\varepsilon) \; dvdx \\
&= -\frac{1}{\varepsilon^2}\int_{I\!R\times V} \sigma_\varepsilon \rho_\varepsilon^\alpha (\rho_\varepsilon - u_\varepsilon)(\phi'(\rho_\varepsilon) - \phi'(u_\varepsilon)) \; dvdx,
\end{aligned} \tag{14}$$

by using (12). Since ϕ' is non decreasing, one gets

$$\begin{aligned}
&\frac{d}{dt}\int_{I\!R\times V} \phi(u_\varepsilon) \; dvdx \\
&= -\frac{1}{\varepsilon^2}\int_{I\!R\times V} \sigma_\varepsilon \rho_\varepsilon^\alpha (\rho_\varepsilon - u_\varepsilon)(\phi'(\rho_\varepsilon) - \phi'(u_\varepsilon)) \; dvdx \leq 0.
\end{aligned} \tag{15}$$

In particular choosing $\phi(s) = s^p$, $p > 1$ leads to

$$\|u_\varepsilon\|_{L^p(I\!R\times V)} \leq \|u_0^\varepsilon\|_{L^p(I\!R\times V)}$$

and letting p go to $+\infty$ gives

$$\|u_\varepsilon\|_{L^\infty(I\!R^+\times I\!R\times V)} \leq \|u_0^\varepsilon\|_{L^\infty(I\!R^+\times I\!R\times V)}. \tag{16}$$

By (3) one deduces that

$$\|\rho_\varepsilon\|_{L^\infty(I\!R^+\times I\!R)} \leq \|u_0^\varepsilon\|_{L^\infty(I\!R^+\times I\!R\times V)}. \tag{17}$$

Finally, according to [22], an additional entropy bound is crucial to our approach. For the sake of simplicity we derive more or less formally this estimate, using $\phi(s) = s\ln(s)$ (but the argument can be written rigorously with some suitable truncation and regularization). Then, (15) reads as follows

$$\frac{d}{dt}\int_{I\!R\times V} u_\varepsilon \ln(u_\varepsilon) \; dvdx + \frac{1}{\varepsilon^2}\int_{I\!R\times V} \sigma_\varepsilon \rho_\varepsilon^\alpha (\rho_\varepsilon - u_\varepsilon)\ln(\frac{\rho_\varepsilon}{u_\varepsilon}) \; dvdx = 0. \tag{18}$$

Now let us multiply (1) by $\omega(x)$ and integrate with respect to x, v. We get

$$\begin{aligned}
&\frac{d}{dt}\int_{I\!R\times V} \omega(x)u_\varepsilon \; dvdx - \frac{1}{\varepsilon}\int_{I\!R\times V} \omega'(x)vu_\varepsilon \; dvdx \\
&= \frac{1}{\varepsilon^2}\int_{I\!R\times V} \omega(x)\sigma_\varepsilon \rho_\varepsilon^\alpha (\rho_\varepsilon - u_\varepsilon) \; dvdx = 0,
\end{aligned} \tag{19}$$

by (3). Let us denote

$$\begin{aligned}\mathcal{M}(u_\varepsilon)(t) &= \int_{I\!R\times V} (\omega(x)+\ln(u_\varepsilon))u_\varepsilon \; dxdv,\\ \mathcal{H}(u_\varepsilon)(t) &= \frac{1}{\varepsilon^2}\int_0^t \int_{I\!R\times V} \sigma_\varepsilon \rho_\varepsilon^\alpha (\rho_\varepsilon - u_\varepsilon)\ln(\frac{\rho_\varepsilon}{u_\varepsilon}) \; dxdvds.\end{aligned} \tag{20}$$

Add (18) and (19). Then, integrating with respect to t yields

$$\mathcal{M}(u_\varepsilon)(t) + \mathcal{H}(u_\varepsilon)(t) \leq \mathcal{M}(u_0^\varepsilon) + \frac{1}{\varepsilon}\int_0^t \int_{I\!R\times V} \omega'(x) v u_\varepsilon \; dvdxds. \tag{21}$$

We introduce the current

$$j_\varepsilon(t,x) = \frac{1}{\varepsilon}\int_V v u_\varepsilon \; dv. \tag{22}$$

We remark that

$$\begin{aligned}|j_\varepsilon(t,x)|^2 &= |\frac{1}{\varepsilon}\int_V v(u_\varepsilon - \rho_\varepsilon) \; dv|^2 \\ &\leq \frac{M^2}{\varepsilon^2}\Big(\int_V |\sqrt{u_\varepsilon} - \sqrt{\rho_\varepsilon}|(\sqrt{u_\varepsilon}+\sqrt{\rho_\varepsilon}) \; dv\Big)^2,\end{aligned} \tag{23}$$

by (3), with $M = \sup\{|v|, v \in V\}$. We use the classical inequalities $(a+b)^2 \leq 2(a^2+b^2)$ and $(\sqrt{a}-\sqrt{b})^2 \leq C(a-b)\ln(\frac{a}{b})$; therefore Holder's inequality gives the following estimate

$$\begin{aligned}|j_\varepsilon(t,x)|^2 &\leq \frac{M^2}{\varepsilon^2}\int_V (\sqrt{u_\varepsilon}-\sqrt{\rho_\varepsilon})^2 \; dv \int_V (\sqrt{u_\varepsilon}+\sqrt{\rho_\varepsilon})^2 \; dv \\ &\leq \frac{2CM^2}{\varepsilon^2}\int_V (\rho_\varepsilon + u_\varepsilon) \; dv \int_V (\rho_\varepsilon - u_\varepsilon)\ln(\frac{\rho_\varepsilon}{u_\varepsilon}) \; dv \\ &\leq \frac{4CM^2}{\varepsilon^2}\rho_\varepsilon \int_V (\rho_\varepsilon - u_\varepsilon)\ln(\frac{\rho_\varepsilon}{u_\varepsilon}) \; dv.\end{aligned} \tag{24}$$

One deduces that

$$\int_0^t \int_{I\!R} \rho_\varepsilon^{\alpha-1}|j_\varepsilon|^2 \; dxds \leq \frac{4CM^2}{\underline{\sigma}}\mathcal{H}(u_\varepsilon)(t) \tag{25}$$

holds. Since $\alpha < 1$, (17) implies $\rho_\varepsilon^{1-\alpha} \leq \|u_0^\varepsilon\|_{L^\infty}^{1-\alpha}$ and we also have $|j_\varepsilon|^2 \leq \|u_0^\varepsilon\|_{L^\infty}^{1-\alpha}\rho_\varepsilon^{\alpha-1}|j_\varepsilon|^2$, then it follows that

$$\int_0^t \int_{I\!R} |j_\varepsilon|^2 \; dxds \leq \frac{4CM^2}{\underline{\sigma}}\|u_0^\varepsilon\|_{L^\infty}^{1-\alpha}\mathcal{H}(u_\varepsilon)(t) = C_0\mathcal{H}(u_\varepsilon)(t). \tag{26}$$

holds by (25). By using (26), (21) becomes

$$\begin{aligned}
&\mathcal{M}(u_\varepsilon)(t) + \frac{1}{C_0}\int_0^t \int_{I\!R} |j_\varepsilon|^2 \, dxds \le \mathcal{M}(u_0^\varepsilon) + \int_0^t \int_{I\!R} \omega'(x) j_\varepsilon \, dxds \\
&\le \mathcal{M}(u_0^\varepsilon) + \sqrt{C_0}\left(\int_0^t \int_{I\!R} |\omega'(x)|^2 \, dxds\right)^{1/2} \frac{1}{\sqrt{C_0}}\left(\int_0^t \int_{I\!R} |j_\varepsilon|^2 \, dxds\right)^{1/2} \\
&\le \mathcal{M}(u_0^\varepsilon) + \frac{C_0 \Omega T}{2} + \frac{1}{2C_0}\int_0^t \int_{I\!R} |j_\varepsilon|^2 \, dxds,
\end{aligned} \tag{27}$$

where we used Holder's and Young's inequalities and denoted $\Omega = \displaystyle\int_{I\!R} |\omega'(x)|^2 \, dx$ $< \infty$ since $0 < \beta < 1/2$. Let us write (27) as follows

$$\mathcal{M}(u_\varepsilon)(t) + \frac{1}{2C_0}\int_0^t \int_{I\!R} |j_\varepsilon|^2 \, dxds \le \mathcal{M}(u_0^\varepsilon) + C_1 T. \tag{28}$$

Since $s|\ln(s)| = s\ln(s) - 2s\ln(s)\chi_{s\le 1}$, one has

$$\mathcal{N}(u_\varepsilon)(t) \le \mathcal{N}(u_0^\varepsilon) + C_1 T + \int_{I\!R\times V} (-2u_\varepsilon \ln(u_\varepsilon))\chi_{u_\varepsilon \le 1} \, dvdx, \tag{29}$$

with $\mathcal{N}(u_\varepsilon)(t) = \displaystyle\int_{I\!R\times V} u_\varepsilon(\omega(x) + |\ln(u_\varepsilon)|) \, dvdx$. However, the usual trick

$$\begin{aligned}
-2s\ln(s)\chi_{s\le 1} &= -2s\ln(s)\chi_{\exp(-\omega/4)<s\le 1} - 2s\ln(s)\chi_{s\le \exp(-\omega/4)} \\
&\le \frac{s\omega}{2} + 2C\exp(-\omega/8)
\end{aligned}$$

(since $-s\ln(s) \le C\sqrt{s}$ on $(0,1)$) allows us to deduce that

$$\begin{aligned}
\mathcal{N}(u_\varepsilon)(t) \le \mathcal{N}(u_0^\varepsilon) &+ C_1 T + \frac{1}{2}\int_{I\!R\times V} \omega(x) u_\varepsilon \, dvdx \\
&+ 2C\int_{I\!R\times V} \exp(-\omega(x)/8) \, dvdx
\end{aligned}$$

holds and finally

$$\mathcal{N}(u_\varepsilon)(t) \le C_T. \tag{30}$$

Furthermore, by combining (28) with (30), we get

$$\int_0^T \int_{I\!R} |j_\varepsilon|^2 \, dxds \le C_T. \tag{31}$$

Then, (21) gives

$$\begin{aligned}
\mathcal{H}(u_\varepsilon)(t) &\le \mathcal{N}(u_0^\varepsilon) + \mathcal{N}(u_\varepsilon)(t) + \frac{1}{\varepsilon}\left|\int_0^T \int_{I\!R\times V} \omega'(x) v u_\varepsilon \, dvdxds\right| \\
&\le \mathcal{N}(u_0^\varepsilon) + \mathcal{N}(u_\varepsilon)(t) + \sqrt{\Omega T}\left(\int_0^T \int_{I\!R} |j_\varepsilon|^2 \, dxds\right)^{1/2} \le C_T,
\end{aligned}$$

by using Holder's inequality, (30) and (31). Therefore, by (25), we are led to

$$\int_0^T \int_{I\!R} \rho_\varepsilon^{\alpha-1} |j_\varepsilon|^2 \, dxds \le C_T. \tag{32}$$

We also remark that

$$\begin{aligned} \int_0^T \int_{I\!R} |\rho_\varepsilon^\alpha j_\varepsilon|^2 \, dxds &= \int_0^T \int_{I\!R} |\rho_\varepsilon^{\frac{\alpha-1}{2}} j_\varepsilon \rho_\varepsilon^{\frac{\alpha+1}{2}}|^2 \, dxds \\ &\le \|u_0^\varepsilon\|_{L^\infty}^{\alpha+1} \int_0^T \int_{I\!R} \rho_\varepsilon^{\alpha-1} |j_\varepsilon|^2 \, dxds \le C_T. \end{aligned} \tag{33}$$

holds since $\alpha > -1$. We summarize what we showed as follows

Proposition 1 *Let the assumptions of Theorem 1 be fulfilled. Then, one has (uniformly in ε)*

$$\begin{cases} \rho_\varepsilon \textit{ is bounded in } L^1((1+\omega(x))dx) \cap L^\infty((0,T)\times I\!R), \\ \mathcal{N}(u_\varepsilon) \textit{ is bounded in } L^\infty(0,T), \\ \mathcal{H}(u_\varepsilon) \textit{ is bounded in } L^\infty(0,T), \\ j_\varepsilon,\ \rho_\varepsilon^{\frac{\alpha-1}{2}} j_\varepsilon \textit{ and } \rho_\varepsilon^\alpha j_\varepsilon \textit{ are bounded in } L^2((0,T)\times I\!R). \end{cases} \tag{34}$$

2.2. Limit equations

We wish to pass to the limit $\varepsilon \to 0$ in the following equations

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x j_\varepsilon = 0, \\ \varepsilon^2 \partial_t j_\varepsilon + \partial_x \left(\int_V v^2 u_\varepsilon dv \right) = -\sigma_\varepsilon \rho_\varepsilon^\alpha j_\varepsilon, \end{cases} \tag{35}$$

obtained by integrating (1) with the measures dv and vdv respectively. Obviously, we note that

$$\int_V v^2 u_\varepsilon \, dv = \left(\int_V v^2 \, dv \right) \rho_\varepsilon + \int_V v^2 (u_\varepsilon - \rho_\varepsilon) \, dv = \theta \rho_\varepsilon + \kappa_\varepsilon$$

holds, where $\theta = \int_V v^2 dv$. Then, we claim

Lemma 1 *The sequence κ_ε converges to 0 in $L^2((0,T)\times I\!R)$.*

Proof. We have

$$\begin{aligned} |\kappa_\varepsilon(t,x)| &\le \varepsilon M^2 \int_V \frac{|u_\varepsilon - \rho_\varepsilon|}{\varepsilon} \, dv \\ &\le \varepsilon M^2 \int_V \frac{|\sqrt{u_\varepsilon} - \sqrt{\rho_\varepsilon}|}{\varepsilon} (\sqrt{u_\varepsilon} + \sqrt{\rho_\varepsilon}) \, dv \\ &\le \varepsilon M^2 \sqrt{4C} \left(\frac{1}{\varepsilon^2} \rho_\varepsilon \int_V (\rho_\varepsilon - u_\varepsilon) \ln(\frac{\rho_\varepsilon}{u_\varepsilon}) \, dv \right)^{1/2}, \end{aligned}$$

reproducing (23-24). By (34), it follows that

$$\int_0^T \int_{I\!R} \rho_\varepsilon^{\alpha-1} |\kappa_\varepsilon|^2 dxds \le \varepsilon^2 M^4 \frac{4C}{\underline{\sigma}} \mathcal{H}(u_\varepsilon)(T) \le \varepsilon^2 C_T$$

holds. Then, recalling that $\alpha < 1$, we get

$$\int_0^T \int_{I\!R} |\kappa_\varepsilon|^2 dxds \le \varepsilon^2 C_T \|u_0^\varepsilon\|_{L^\infty}^{1-\alpha}$$

which proves Lemma 1. $\square$

By (34), extracting subsequences if necessary, we can assume that

$$\begin{cases} \rho_\varepsilon \rightharpoonup \rho \textit{ in } L^2((0,T)\times I\!R) \textit{ weakly and in } L^\infty((0,T)\times I\!R) \textit{ weakly } *, \\ j_\varepsilon \rightharpoonup j \textit{ in } L^2((0,T)\times I\!R) \textit{ weakly}, \\ \sigma_\varepsilon \rightharpoonup \sigma \textit{ in } L^\infty((0,T)\times I\!R) \textit{ weakly } *. \end{cases}$$

Let us show that ρ_ε converges strongly. We set

$$U_\varepsilon = (\rho_\varepsilon, j_\varepsilon),\ V_\varepsilon = (-\theta\rho_\varepsilon - \kappa_\varepsilon, \varepsilon^2 j_\varepsilon)$$

so that (35) becomes

$$\begin{cases} div_{t,x} U_\varepsilon = 0, \\ curl_{t,x} V_\varepsilon = -\sigma_\varepsilon \rho_\varepsilon^\alpha j_\varepsilon, \end{cases}$$

where we note that the right hand side terms are bounded in $L^2((0,T)\times I\!R)$ and, then, compact in $H^{-1}((0,T)\times I\!R)$. We can apply the *Div-Curl Lemma* (see [31], [32] or also [14]) which yields

$$U_\varepsilon \cdot V_\varepsilon = -\theta\rho_\varepsilon^2 - \rho_\varepsilon\kappa_\varepsilon + \varepsilon^2 j_\varepsilon^2 \rightharpoonup U \cdot V = -\theta\rho^2$$

in $\mathcal{D}'((0,T)\times I\!R)$, by using Lemma 1 and the bounds (34). One deduces that $\rho_\varepsilon^2 \rightharpoonup \rho^2$ holds, which actually gives the strong convergence (see [24])

$$\rho_\varepsilon \to \rho \tag{36}$$

in $L^p((0,T)\times I\!R)$, $1 \le p < \infty$ (the Div-Curl Lemma gives a convergence in L^p_{loc}; however, the weight ω in (34) allows us to extend this convergence in the whole domain).

Next, we want now to prove that $\sigma_\varepsilon j_\varepsilon \rightharpoonup \Sigma(x,\rho)j$. The proof falls into two steps: first, we show that the product passes to the limit $\sigma_\varepsilon j_\varepsilon \rightharpoonup \sigma j$ where

σ is the weak limit of σ_ε and next, we identify this limit σ with $\Sigma(x,\rho(t,x))$. The two steps require the introduction of a suitable regularization of the sequence Σ_ε. Indeed, since the familly $\{\lambda \in [0,\overline{\rho}] \longmapsto \Sigma_\varepsilon(x,\lambda) \in [\underline{\sigma},\overline{\sigma}],\ \varepsilon \in (0,\varepsilon_0),\ |x| \leq R\}$ is equicontinuous and then compact in $\mathcal{C}^0([0,\overline{\rho}];[\underline{\sigma},\overline{\sigma}])$ there exists a sequence Σ^n_ε such that

$$\left\{ \begin{array}{l} |\partial_\rho \Sigma^n_\varepsilon(x,\lambda)| \leq \sqrt{n}, \\ |\Sigma^n_\varepsilon(x,\lambda) - \Sigma_\varepsilon(x,\lambda)| \leq \alpha(n),\ \forall \varepsilon \in (0,\varepsilon_0),\ \forall x \in [-R,R] \end{array} \right. \tag{37}$$

with $\alpha(n) \to 0$.

To achieve the first step, we also have to replace ρ_ε by smoother function. Since $(\rho_\varepsilon)_{\varepsilon>0}$ is compact in $L^2((0,T)\times I\!R)$, there is a sequence of smooth functions $(\rho^n_\varepsilon)_{\varepsilon>0}$ which satisfies :

$$\left\{ \begin{array}{l} |\partial_t \rho^n_\varepsilon(t,x)| \leq K(n), \\ \|\rho^n_\varepsilon - \rho_\varepsilon\|_{L^2((0,T)\times I\!R)} \leq \dfrac{1}{n} \end{array} \right.$$

where $K(n)$ is a positive constant independent of ε. Let us define $\sigma^n_\varepsilon(t,x) = \Sigma^n_\varepsilon(x,\rho^n_\varepsilon(t,x))$. Note that the regularization of Σ_ε and ρ_ε are not choosen independently. Again, we use a compensated compactness argument. We introduce $W^n_\varepsilon = (\rho_\varepsilon,\sigma^n_\varepsilon)$ which satisfies

$$curl_{t,x} W^n_\varepsilon = \partial_\rho \Sigma^n_\varepsilon(x,\rho^n_\varepsilon)\ \partial_t \rho^n_\varepsilon - \partial_x \rho_\varepsilon.$$

For fixed n, the first term is bounded in $L^2((0,T)\times(-R,R))$ and the second one is compact in $H^{-1}((0,T)\times(-R,R))$ because $(\rho_\varepsilon)_{\varepsilon>0}$ is compact in $L^2((0,T)\times(-R,R))$. Using once more time the *Div-Curl Lemma* of [31], [32] leads to

$$U_\varepsilon \cdot W^n_\varepsilon = \rho^2_\varepsilon + \sigma^n_\varepsilon j_\varepsilon \rightharpoonup U \cdot W^n = \rho^2 + \sigma^n j$$

in $\mathcal{D}'((0,T)\times I\!R)$ where σ^n is the weak limit of σ^n_ε. Taking into account (36), this actually implies

$$\sigma^n_\varepsilon j_\varepsilon \rightharpoonup \sigma^n j$$

in $L^2_{loc}((0,T)\times I\!R)$ weak. But we have

$$\begin{array}{rcl} |\sigma^n_\varepsilon - \sigma_\varepsilon| & \leq & |\Sigma^n_\varepsilon(x,\rho^n_\varepsilon) - \Sigma^n_\varepsilon(x,\rho_\varepsilon)| + |\Sigma^n_\varepsilon(x,\rho_\varepsilon) - \Sigma_\varepsilon(x,\rho_\varepsilon)| \\ & \leq & \sqrt{n}|\rho^n_\varepsilon - \rho_\varepsilon| + \alpha(n). \end{array}$$

Therefore we have $\|\sigma_\varepsilon^n - \sigma_\varepsilon\|_{L^2((0,T)\times(-R,R))} \le 1/\sqrt{n} + \alpha(n)$. Since this estimate is uniform with respect to ε it is easy to conclude that

$$\sigma_\varepsilon j_\varepsilon \rightharpoonup \sigma j \tag{38}$$

In the second step, it remains to identify the weak limit σ with $\Sigma(x, \rho(t,x))$. This will be a consequence of the strong convergence (36) combined to the equicontinuity property of Σ_ε. We still use the regularization Σ_ε^n given in (37) and we define a suitable approximation of ρ as follows. Let χ_k be the characteristic function of the borelian set $\{(t,x),\ k/n \le \rho(t,x) < (k+1)/n\}$. We set $\rho_n = \sum_{k\ge 0} \frac{k}{n}\chi_k$. Let us remark that $|\rho_n - \rho| \le 1/n$. We have

$$\begin{aligned}|\sigma_\varepsilon(x) - \Sigma_\varepsilon(x,\rho_n)| &\le |\sigma_\varepsilon(x) - \Sigma_\varepsilon^n(x,\rho_\varepsilon)| + |\Sigma_\varepsilon^n(x,\rho_\varepsilon) - \Sigma_\varepsilon^n(x,\rho_n)| \\ &+ |\Sigma_\varepsilon^n(x,\rho_n) - \Sigma_\varepsilon(x,\rho_n)| \le \sqrt{n}|\rho_\varepsilon - \rho_n| + 2\alpha(n).\end{aligned}$$

However, we can estimate $|\rho_\varepsilon - \rho_n| \le |\rho_\varepsilon - \rho| + |\rho - \rho_n| \le |\rho_\varepsilon - \rho| + 1/n$ so that we are led to

$$|\sigma_\varepsilon(x) - \Sigma_\varepsilon(x,\rho_n)| \le \sqrt{n}|\rho_\varepsilon - \rho| + \frac{1}{\sqrt{n}} + 2\alpha(n).$$

Let $\varphi \in C_0^\infty((0,T)\times I\!R)$. We obtain

$$\begin{aligned}&\int_0^T \int_{I\!R} \Sigma_\varepsilon(x,\rho_n)\varphi\ dx\ dt = \sum_{k\ge 0} \int_0^T \int_{I\!R} \Sigma_\varepsilon(x,k/n)\chi_k(t,x)\varphi\ dx\ dt \\ &\to \sum_{k\ge 0} \int_0^T \int_{I\!R} \Sigma(x,k/n)\chi_k(t,x)\varphi\ dx\ dt = \int_0^T \int_{I\!R} \Sigma(x,\rho_n)\varphi\ dx\ dt\end{aligned}$$

as ε goes to 0 (actually the sum is finite since ρ is bounded). Therefore, passing to the limit $\varepsilon \to 0$ in the following inequality

$$\begin{aligned}\left|\int_0^T \int_{I\!R} (\sigma_\varepsilon(x) - \Sigma_\varepsilon(x,\rho_n))\varphi\ dx\ dt\right| &\le \sqrt{n}\int_0^T \int_{I\!R} |\rho_\varepsilon - \rho||\varphi|\ dx\ dt \\ &+ (\frac{1}{\sqrt{n}} + 2\alpha(n))\|\varphi\|_{L^1((0,T)\times I\!R)}\end{aligned}$$

yields

$$\left|\int_0^T \int_{I\!R} (\sigma - \Sigma(x,\rho_n))\varphi\ dx\ dt\right| \le (\frac{1}{\sqrt{n}} + 2\alpha(n))\|\varphi\|_{L^1((0,T)\times I\!R)},$$

by using the strong convergence of ρ_ε to ρ and the weak convergences $\sigma_\varepsilon \rightharpoonup \sigma$, $\Sigma_\varepsilon(x, \rho_n) \rightharpoonup \Sigma(x, \rho_n)$. It remains to let $n \to \infty$ to conclude that $\sigma(t, x) = \Sigma(x, \rho(t, x))$, by using the continuity of $\lambda \longmapsto \int_0^T \int_{\mathbb{R}} \Sigma(x, \lambda)\phi \, dx \, dt$. Indeed, we can easily check that this map is continuous with a modulus of continuity proportional to those of the equicontinuous familly $\Sigma_\varepsilon(x, \cdot)$ on the support of φ. We proved the following

Lemma 2 *There exists a subsequence such that ρ_ε converges to ρ, strongly in $L^p((0, T) \times \mathbb{R})$, for all $1 \leq p < \infty$, and a.e. (with, moreover, ρ_ε bounded in $L^\infty((0, T) \times \mathbb{R})$), while j_ε and $\sigma_\varepsilon j_\varepsilon$ converge weakly in $L^2((0, T) \times \mathbb{R})$ to j and σj respectively with $\sigma = \Sigma(x, \rho)$.*

Having obtained these preliminary convergence properties, we are interested in the passage to the limit in (35). We can also assume that

$$\nu_\varepsilon = -\sigma_\varepsilon \rho_\varepsilon^\alpha j_\varepsilon \rightharpoonup \nu \tag{39}$$

in $L^2((0, T) \times \mathbb{R})$ since, by Proposition 1, ν_ε is bounded in $L^2((0, T) \times \mathbb{R})$. As ε goes to 0, (35) becomes

$$\begin{cases} \partial_t \rho + \partial_x j = 0, \\ \theta \partial_x \rho = \nu, \end{cases} \tag{40}$$

by using Lemma 1-2. Let us remark that this relationship shows that $\rho \in L^2((0, T); H^1(\mathbb{R}))$ (while it seems that the ρ_ε do not have this regularity). It remains to clarify the relation between ν, σj and ρ and we still follow strategy in [22].

The conclusion is obvious for the *linear case*, $\alpha = 0$. Indeed, (38) gives $\nu = -\sigma j = \theta \partial_x \rho$ by (40).Thus, ρ satisfies the following diffusion equation

$$\partial_t \rho - \partial_x \left(\frac{\theta}{\sigma} \partial_x \rho\right) = 0.$$

Next, we consider the case $\alpha < 0$. We remark that $\sigma_\varepsilon j_\varepsilon = -\nu_\varepsilon \rho_\varepsilon^{-\alpha}$ is the product of a sequence ν_ε weakly convergent in $L^2((0, T) \times \mathbb{R})$ with a sequence bounded in $L^\infty((0, T) \times \mathbb{R})$ which converges a.e. $\rho_\varepsilon^{-\alpha} \to \rho^{-\alpha}$. Therefore, by

(38), the weak limit is $\sigma j = -\nu\rho^{-\alpha}$. We are thus led to the following equation for ρ

$$\partial_t\rho - \partial_x(\frac{\theta}{\sigma}\rho^{-\alpha}\partial_x\rho) = \partial_t\rho - \partial_x(\frac{\theta}{(1-\alpha)\sigma}\partial_x\rho^{1-\alpha}) = 0.$$

Indeed the function $\rho \longmapsto \rho^{1-\alpha}$ is $\mathcal{C}^1$ and $\rho(t)$ lies in $H^1(I\!R)$ which give the equality $\rho^{-\alpha}\partial_x\rho = \partial_x(\rho^{1-\alpha})$ by Stampacchia's lemma.

It remains to study the case $\alpha > 0$. We note that ν_ε is the product of $-\sigma_\varepsilon j_\varepsilon \rightharpoonup -\sigma j$, by (38), with $\rho_\varepsilon^\alpha \to \rho^\alpha$ a.e. and which lies in a bounded set of $L^\infty((0,T)\times I\!R)$. Hence, we get

$$\partial_x(\theta\rho) = \nu = -\sigma j\rho^\alpha. \tag{41}$$

We wish to write this relation involving the derivative of $\rho^{1-\alpha}$ in order to determine j on the set $\{\rho \neq 0\}$. Setting $\phi_n(s) = \frac{1}{1-\alpha}(\frac{1}{n}+s)^{1-\alpha}$, one has

$$\partial_x(\phi_n(\rho)) = (1/n+\rho)^{-\alpha}\ \partial_x\rho = -\left(\frac{\rho}{1/n+\rho}\right)^\alpha\ \frac{\sigma}{\theta}\ j$$

by (41). Letting n goes to $+\infty$ in this last relation gives

$$-\frac{\theta}{\sigma(1-\alpha)}\ \partial_x(\rho^{1-\alpha}) = j\chi_{\rho>0},$$

in $\mathcal{D}'((0,T)\times I\!R)$. To conclude, it remains to show that j vanishes on the set $\{\rho = 0\}$. However, by Proposition 1, we can assume that $g_\varepsilon = \rho_\varepsilon^{\frac{\alpha-1}{2}} j_\varepsilon \rightharpoonup g$ in $L^2((0,T)\times I\!R)$, where $\alpha - 1 < 0$. Therefore, $j_\varepsilon = g_\varepsilon\rho_\varepsilon^{\frac{1-\alpha}{2}}$ converges weakly in $L^2((0,T)\times I\!R)$ to $j = g\rho^{\frac{1-\alpha}{2}}$. This forces $j = 0$, *a.e.* on $\{\rho = 0\}$. Hence, we can write $-\dfrac{\theta}{\sigma}\partial_x\left(\dfrac{\rho^{1-\alpha}}{1-\alpha}\right) = j$. Finally, we are led to the following equation satisfied by ρ

$$\partial_t\rho - \partial_x\left(\frac{\theta}{\sigma(1-\alpha)}\partial_x(\rho^{1-\alpha})\right) = 0.$$

To obtain the Cauchy data, we observe that $\partial_t\rho_\varepsilon = -\partial_x j_\varepsilon$ is bounded in $L^2(0,T;H^{-1}(I\!R))$ while ρ_ε is bounded in $L^\infty(0,T;L^2(I\!R))$. These bounds ensure the compactness of ρ_ε in $C^0([0,T];H^{-1}_{loc}(I\!R))$ (see [30]) and in particular we have $\rho_\varepsilon(0) = \displaystyle\int_V u_0^\varepsilon\, dv \to \rho(0) = \int_V u_0\, dv$ in $H^{-1}_{loc}(I\!R)$ as $\varepsilon \to 0$ (with $\rho_0 \in L^2(I\!R)$). This achieves the proof of Theorem 1. $\square$

3. FORMAL ASYMPTOTICS AND THE CELL PROBLEM IN THE MULTI-DIMENSIONAL CASE

This Section is devoted to the non singular $\alpha = 0$, and multi-dimensional, $N > 1$, problem. Moreover, a periodicity assumption is required on the heterogeneities of the medium, namely : $\sigma_\varepsilon(t,x) = \Sigma(\frac{x}{\varepsilon}, x, \rho_\varepsilon(t,x))$, Σ being Y-periodic in the $y = \frac{x}{\varepsilon}$ variable. As usual in this kind of problem (see [10]) we expand the solution in double scale power series

$$u_\varepsilon(t,x,v) = u^{(0)}(t, \frac{x}{\varepsilon}, x, v) + \varepsilon\ u^{(1)}(t, \frac{x}{\varepsilon}, x, v) + \varepsilon^2\ u^{(2)}(t, \frac{x}{\varepsilon}, x, v) + ... \quad (42)$$

We denote $\rho^{(k)} = \int_V u^{(k)}\ dv$ and $j^{(k)} = \int_V u^{(k)}\ v\ dv$. Putting the expansion (42) in (1) and identifying the terms of order $1/\varepsilon^2$ and $1/\varepsilon$, we get

$$v \cdot \nabla_y u^{(0)} + \Sigma(y, x, \rho^{(0)}) \left(u^{(0)} - \rho^{(0)}\right) = 0, \quad (43)$$

$$\begin{aligned} v \cdot \nabla_y u^{(1)} + \Sigma(y, x, \rho^{(0)}) \left(u^{(1)} - \rho^{(1)}\right) &= -v.\nabla_x u^{(0)} \\ &\quad + \partial_\rho \Sigma(y, x, \rho^{(0)}) \rho^{(1)} (u^{(0)} - \rho^{(0)}). \end{aligned} \quad (44)$$

Then integrating (43) and (44) with respect to v gives

$$div_y j^{(0)} = div_x j^{(0)} + div_y j^{(1)} = 0, \quad (45)$$

while integration of the ε^0 relation leads to

$$\partial_t \rho^{(0)} + div_x j^{(1)} + div_y j^{(2)} = 0. \quad (46)$$

In particular, in (43) and (44), the variables (t,x) are only parameters. In order to solve these cell problems we need the following

Theorem 3 *Assume that $\sigma \in L^\infty(\mathbb{R}^N)$ is a periodic function which satisfies for some positive constants $\underline{\sigma}$ and $\overline{\sigma}$: $\underline{\sigma} \le \sigma(y) \le \overline{\sigma}$. Let g be a $Y-$periodic function which belongs to $L^2(Y \times V)$. Then the problem : find $f \in H_\# = \{f \in L^2(K \times V),\ v \cdot \nabla_y f \in L^2(K \times V),\ for\ any\ compact\ set\ K \subset \mathbb{R}^N,\ f\ Y\text{-}periodic\}$ such that :*

$$v \cdot \nabla_y f(y,v) + \sigma(y) \left(f(y,v) - \int_V f(y,w)\ dw\right) = g(y,v), \quad y \in Y,\ v \in V \quad (47)$$

has a solution if and only if $\int_Y \int_V g(y,v)\ dv\ dy = 0$. *This solution is unique up to a constant.*

For fixed parameters t, x, let us define $\sigma(y) = \Sigma(y, x, \rho^{(0)}(t, x, y))$ and $g(y, v) = 0$. Applying this Theorem, the solutions of the corresponding problem (47) do not depend on y and v (since 0 is an obvious solution). But, in view of (43), $u^{(0)}$ is also a solution so that we have $u^{(0)} = \rho^{(0)}(t, x)$ (which does not depend on y). Then, the solvabilility condition for (44) is satisfied since $\int_V v\ dv = 0$ and Theorem 3 provides existence-uniqueness of the vector valued solution χ of

$$\begin{cases} v \cdot \nabla_y \chi(y,x,\rho,v) \\ \qquad +\Sigma(y,x,\rho)\left(\chi(y,x,\rho,v) - \int_V \chi(y,x,\rho,w)\ dw\right) = v, \\ \\ \int_Y \int_V \chi(y,x,\rho,v)\ dv\ dy = 0. \end{cases} \tag{48}$$

Then, it follows that $u^{(1)}(t, y, x, v) = -\chi(y, x, \rho^{(0)}(t, x), v) \cdot \nabla_x \rho^{(0)}(t, x) + q(t, x)$. Now, we obtain an equation for $\rho^{(0)}$ by integrating (46) with respect to y

$$\partial_t \rho^{(0)} + div_x \left(\int_Y j^{(1)}\ dy \right) = 0, \tag{49}$$

where $\int_Y j^{(1)}\ dy$ is given by

$$\int_Y j^{(1)}\ dy = -\int_Y \chi(y,x,\rho^{(0)},v) \cdot \nabla_x \rho^{(0)}\ v\ dy\ dv.$$

Let us define

$$Q(x,\rho) = \int_Y \int_V v \otimes \chi(y,x,\rho,v)\ dv\ dy, \tag{50}$$

$$D(x,\rho) = \frac{1}{2} \int_Y \int_V (v \otimes \chi(y,x,\rho,v) + \chi(y,x,\rho,v) \otimes v)\ dv\ dy. \tag{51}$$

Then (49) becomes

$$\partial_t \rho^{(0)} - div_x (Q(x,\rho^{(0)}) \nabla_x \rho^{(0)}) = 0. \tag{52}$$

Of course the corresponding Cauchy problem is wellposed if D, the symmetric part of the matrix Q is positive definite. Indeed we have

Theorem 4 *Let $\sigma \in L^\infty(I\!R^N)$ be a periodic function which verifies for some positive constant $\underline{\sigma}$ and $\overline{\sigma}$: $\underline{\sigma} \leq \sigma(y) \leq \overline{\sigma}$. Let $\chi_\sigma \in (H_\#)^N$ be the unique solution of*

$$\begin{cases} v \cdot \nabla_y \chi_\sigma(y,v) + \sigma(y) \left(\chi_\sigma(y,v) - \int_V \chi_\sigma(y,w)\ dw \right) = v, \\ \int_Y \int_V \chi_\sigma(y,v)\ dv\ dy = 0. \end{cases} \tag{53}$$

Let us define

$$Q_\sigma = \int_Y \int_V v \otimes \chi_\sigma(y,v)\ dv\ dy,$$

$$D_\sigma = \frac{1}{2} \int_Y \int_V (v \otimes \chi_\sigma(y,v) + \chi_\sigma(y,v) \otimes v)\ dv\ dy.$$

Then the matrix D_σ is positive definite. More precisely we have

$$\frac{N\ tr(\theta)}{\underline{\sigma}}\ I \geq D_\sigma \geq \frac{1}{\overline{\sigma}}\ \theta > 0$$

where I is the identity matrix. Moreover Q_σ and D_σ are Lipschitz with respect to σ : if σ' satisfies the same hypothesis as σ

$$|D_\sigma - D_{\sigma'}| \leq |Q_\sigma - Q_{\sigma'}| \leq k\ \|\sigma - \sigma'\|_{L^\infty(Y)} \tag{54}$$

with $k = N\ tr(\theta)/\underline{\sigma}^2$.

Remark 4 *In dimension $N = 1$ we have by integration of (53)*

$$\partial_y \int_V \chi_\sigma\ v\ dv = 0,$$

and $Q_\sigma = \int_V \chi_\sigma\ v\ dv$ does not depend on y. Now multiplying (53) by v and integrating give

$$\begin{aligned} \int_Y \sigma \int_V (\chi_\sigma - \rho_\sigma)\ v\ dv\ dy &= \int_Y \sigma \int_V \chi_\sigma\ v\ dv\ dy \\ &= Q_\sigma \int_Y \sigma\ dy = \int |v|^2\ dv = \theta. \end{aligned}$$

This is exactly the result of the previous Section (Theorem 1) since in the periodic case $\int_Y \sigma\ dy$ is the weak limit of the function $\sigma_\varepsilon(x) = \sigma(x/\varepsilon)$.

Remark 5 *It seems that in general the matrix Q is not symmetric. However, if σ is invariant under all rotation R exchanging the axis, then Q is symmetric. Indeed, in this case one has $\chi_i(Ry, Rv) = \chi_k$ where R is the rotation interverting the axis i and k and it follows that $\int_{Y\times V} v \cdot \nabla_y \chi_i \; \chi_k \; dydv = \int_{Y\times V} v \cdot \nabla_y \chi_k \; \chi_i \; dydv$ holds. An example of such a situation is given by $\Sigma(y,\lambda) = \beta(\lambda)\alpha(y_1)...\alpha(y_N)$, α being $1-$periodic.*

The Lipschitz property satisfied by the matrix Q (or its symmetric part) allows us to establish the uniqueness of the limit problem, at least when Σ is uniformly lipschitzian, which in turn implies that the whole sequence ρ_ε converges.

Proposition 2 *Assume that $\lambda \longmapsto \Sigma(y,x,\lambda)$ is uniformly lipschitzian, i.e. there exists $L > 0$ such that $\sup_{x\in \mathbb{R}^N, y\in Y} |\Sigma(y,x,\lambda) - \Sigma(y,x,\mu)| \leq L \; |\lambda - \mu|$. Then, the problem (11) has a unique solution ρ belonging to $C^0([0,T]; L^2(\mathbb{R}^N)) \cap L^2(0,T; H^1(\mathbb{R}^N))$.*

Proof. Let ρ_1 and ρ_2 be solutions of (11). We set $w = \rho_2 - \rho_1$ and we get

$$\left\{ \begin{array}{l} \partial_t w - \nabla_x \cdot (Q(\rho_2)\nabla_x w) = \nabla_x \cdot ((Q(\rho_2) - Q(\rho_1))\nabla_x \rho_1) \\ w_{|t=0} = \rho_{2,0} - \rho_{1,0} = w_0. \end{array} \right. \tag{55}$$

The proof works by using the L^1 norm. To this end, we introduce the following approximation of the sign function

$$Y_\eta(s) = \frac{s}{\sqrt{s^2+\eta}}.$$

We remark that Y_η is C^1 with $Y'_\eta(s) = \dfrac{\eta}{(s^2+\eta)^{3/2}}$ so that by Stampacchia's theorem $Y_\eta(w)$ belongs to $L^2(0,T;H^1(\mathbb{R}^N))$ and can be used as test function in (55). We also note that $Z_\eta(s) = \int_0^s Y_\eta(\tau)\; d\tau$ approaches $|s|$ as η goes to 0, with $0 \leq Z_\eta(s) \leq |s|$. Multiplying (55) by $Y_\eta(w)$ and integrating lead to

$$\begin{array}{l} \displaystyle\int_{\mathbb{R}^N} Z_\eta(w(t))\; dx + \int_0^t \int_{\mathbb{R}^N} Q(\rho_2)\nabla_x w \cdot \nabla_x w \; Y'_\eta(w)\; ds\; dx \\ \displaystyle = \int_{\mathbb{R}^N} Z_\eta(w_0)\; dx + \int_0^t \int_{\mathbb{R}^N} (Q(\rho_2) - Q(\rho_1))\nabla_x \rho_1 \cdot \nabla_x w \; Y'_\eta(w)\; ds\; dx. \end{array} \tag{56}$$

Since $Q(\rho_2)\nabla_x w\cdot\nabla_x w = D(\rho_2)\nabla_x w\cdot\nabla_x w \geq 0$, we can drop the second integral in the left hand side of (56); estimating the right hand side gives

$$\begin{aligned}
&\int_{I\!R^N} Z_\eta(w(t))\ dx \leq \int_{I\!R^N} |w_0|\ dx \\
&+ \int_0^t \int_{I\!R^N} |Q(\rho_2) - Q(\rho_1)|\ |\nabla_x \rho_1|\ |\nabla_x w|\ Y_\eta'(w)\ ds\ dx \\
&\leq \int_{I\!R^N} |w_0|\ dx \\
&+ \int_0^t \int_{I\!R^N} k\|\Sigma(x,\cdot,\rho_2) - \Sigma(x,\cdot,\rho_1)\|_{L^\infty(Y)}\ |\nabla_x \rho_1|\ |\nabla_x w|\ Y_\eta'(w)\ ds\ dx \\
&\leq \int_{I\!R^N} |w_0|\ dx + kL\int_0^t \int_{I\!R^N} |\nabla_x \rho_1|\ |\nabla_x w| \frac{|w|\eta}{(w^2+\eta)^{3/2}}\ ds\ dx
\end{aligned} \tag{57}$$

by using (54) and the Lipschitz property of Σ. Then, we remark that $\frac{|w|\eta}{(w^2+\eta)^{3/2}} = F(\frac{|w|}{\sqrt{\eta}})$, with $F(z) = \frac{z}{(1+z^2)^{3/2}}$, tends to 0 a.e. and is uniformly bounded while $|\nabla_x \rho_1|\ |\nabla_x w|$ obviously belongs to $L^1(0,T\times I\!R^N)$ as product of L^2 functions. Hence, it suffices to apply Lebesgue's theorem to deduce that

$$\int_{I\!R^N} |w(t)|\ dx \leq \int_{I\!R^N} |w_0|\ dx$$

holds as $\eta \to 0$, which clearly gives the uniqueness statement. □

The remainder of this Section is devoted to the proof of Theorems 3-4.

3.1. Proof of Theorem 3

Theorem 3 is classical for continuous velocity model. In this case, it relies on Fredholm alternative for compact operators; the compactness being closely related to averaging lemma (see [17], [25] and for an application close to our [4]). However, this argument clearly fails when dealing with discrete velocities model. This motivates the following proof.

Uniqueness

For any function $f \in H_\#$ we have $\int_Y\int_V v\cdot\nabla_y f\ f\ dv\ dy = 0$ by periodicity. Therefore if $g=0$, multiplying (47) by f and integrating give

$$\begin{aligned}
\int_Y\int_V \sigma(y)(f-\rho_f)\ f\ dv\ dy &= \int_Y\int_V \sigma(y)(f-\rho_f)^2\ dv\ dy = 0 \\
&\geq \underline{\sigma}\ \|f-\rho_f\|^2_{L^2(Y\times V)}
\end{aligned}$$

where $\rho_f = \int_V f\ dv$. Therefore $f = \rho_f(y)$ and (47) reduces to $v \cdot \nabla_y \rho_f(y) = 0$ *a.e.* The geometric condition $\theta = \int_V v \otimes v\ dv > 0$ implies that the orthogonal set of V is $\{0\}$. Therefore $\nabla_y \rho_f = 0$ and $f = \rho_f$ is a constant. $\square$

<u>*Existence*</u>

By integrating with respect to v and y (47), we immediatly obtain that $\int_Y \int_V g\ dv\ dy = 0$ is a necessary condition for having a solution. We prove now that it is also sufficient.

For $\lambda > 0$ and any $g \in L^2(Y \times V)$ we obtain by an explicit integration along characteristics a solution $f \in H_\#$ of the problem

$$\lambda f + v.\nabla_y f + \sigma(y) f = g.$$

We denote this solution $f = \mathcal{F}(g)$. Consider the application $f \longmapsto \mathcal{T}(f) = \mathcal{F}(g + \sigma\ \rho_f)$. We claim that for $\lambda > 0$, $\mathcal{T}$ is a contraction. Indeed, we set $\delta = \mathcal{T}(f) - \mathcal{T}(f')$ then, we have

$$\lambda \delta + v.\nabla_y \delta + \sigma(y)\delta = \sigma(\rho - \rho')$$

where $\rho = \rho_f$ and $\rho' = \rho_{f'}$. Mutiplying by δ and integrating give

$$\begin{aligned}
&\int_Y \int_V (\lambda + \sigma(y))\ \delta^2\ dy\ dv = \int_Y \int_V \sigma(y)(\rho - \rho')\delta\ dy\ dv \\
&\leq \left(\int_Y \int_V (\lambda + \sigma(y))\ \delta^2\ dy\ dv\right)^{1/2} \left(\int_Y \int_V \frac{\sigma(y)^2}{\lambda + \sigma(y)} (\rho - \rho')^2\ dy\ dv\right)^{1/2}
\end{aligned}$$

and thus we get

$$\begin{aligned}
\int_Y \int_V (\lambda + \sigma(y))\ \delta^2\ dy\ dv &\leq \int_Y \int_V \frac{\sigma(y)^2}{\lambda + \sigma(y)} (f - f')^2\ dy\ dv \\
&\leq \frac{\overline{\sigma}^2}{(\lambda + \overline{\sigma})^2} \ \int_Y \int_V (\lambda + \sigma(y))(f - f')^2\ dy\ dv,
\end{aligned}$$

by maximizing the function $s \longmapsto s/(\lambda + s)$. This proves that $\mathcal{T}$ is a contraction in L^2 for the corresponding norm. The fixed point gives a solution $f_\lambda \in H_\#$ of

$$\lambda f_\lambda + v \cdot \nabla_y f_\lambda(y, v) + \sigma(y) \left(f_\lambda(y, v) - \int_V f_\lambda(y, w)\ dw\right) = g(y, v), \qquad (58)$$
$$y \in Y,\ v \in V.$$

We wish to prove the existence of a solution of (47) by passing to the limit $\lambda \longrightarrow 0$ in (58). This becomes possible, at least for a subsequence, if we are able to show that $\|f_\lambda\|_{L^2(Y\times V)}$ remains bounded. If this is not the case there exists a subsequence $\lambda_k \to 0$ with $N_k = \|f_{\lambda_k}\|_{L^2(Y\times V)} \to \infty$. We can moreover assume that $F_k = f_{\lambda_k}/N_k$ has a weak limit F in $L^2(Y \times V)$. We denote $\rho_k = \int_V F_k \, dv$. We have

$$\lambda_k F_k + v \cdot \nabla_y F_k + \sigma(y)\,(F_k - \rho_k) = g/N_k \to 0 \; in \; L^2(Y \times V). \tag{59}$$

As usual, multiplying by F_k and integrating yield the strong convergence of $F_k - \rho_k$ towards 0 in $L^2(Y \times V)$. Therefore, as $k \to \infty$ in $F_k = \rho_k + (F_k - \rho_k)$, one deduces that $F = F(y)$ depends only on y and passing to the limit in (59) gives $v \cdot \nabla_y F(y) = 0$. Hence, F is a constant. (As in the uniqueness proof we use that the orthogonal set of V is $\{0\}$). We use now the necessary condition $\int_Y \int_V g(y,v)\, dv\, dy = 0$ to obtain by integration of (59) $\int_Y \int_V F_k(y,v)\, dv\, dy = 0$. The limit gives finally $F = 0$.

The contradiction will follow from the strong compactness of the sequence F_k. Indeed, we multiply (59) by v to obtain

$$\begin{aligned}\int_V v \otimes v \; dv \; \nabla_y \rho_k = \; & - \; div_y \left(\int_V v \otimes v \; (F_k - \rho_k) \; dv \right) \\ & + \; \int_V (-\lambda_k F_k - \sigma(y)(F_k - \rho_k) + g/N_k) \; v \; dv.\end{aligned}$$

In the right hand side, the first term tends to 0 (strongly) in $H^{-1}(Y)$ and the second is bounded in $L^2(Y)$. Therefore since $\theta = \int_V v \otimes v \; dv$ is positive definite it follows that $(\rho_k)_{k\geq 0}$ is compact in $L^2(Y)$. However, we have seen that $F_k - \rho_k$ converges strongly to 0 so that $F_k = \rho_k + (F_k - \rho_k)$ tends to $F = 0$ strongly. This contradicts the fact that $\|F_k\|_{L^2(Y\times V)} = 1$.

Finally f_λ is bounded in $L^2(Y \times V)$ and its weak limit gives the desired solution of (47). □

Remark 6 *The compactness of the sequence has been obtained without the average compactness Theorem of [17]. Therefore we only need the geometric condition $\theta > 0$ which is by no way sufficient to obtain average compactness (think to discrete velocity model). This has to be compared with the work of*

P.L. Lions and G. Toscani [22] and with the use of compensated compactness in the other Sections.

3.2. Proof of Theorem 4

Let us introduce the vector $\rho_\sigma = \int_V \chi_\sigma \, dv$ where χ_σ is the solution of (53). If we take the tensor product of the equation with χ_σ, symmetrize and integrate, we obtain :

$$\begin{aligned} &\frac{1}{2}\int_Y \int_V ((\chi_\sigma - \rho_\sigma) \otimes \chi_\sigma + \chi_\sigma \otimes (\chi_\sigma - \rho_\sigma))\sigma \; dy \; dv \\ &= \frac{1}{2}\int_Y \int_V v \otimes \chi_\sigma + \chi_\sigma \otimes v \; dy \; dv = D_\sigma \\ &= \int_Y \int_V (\chi_\sigma - \rho_\sigma) \otimes (\chi_\sigma - \rho_\sigma)\sigma \; dy \; dv \end{aligned} \tag{60}$$

which proves that D_σ is non negative. Remark also that

$$\begin{aligned} 0 \; &\leq \; D_\sigma = \frac{1}{2}\int_Y \int_V v \otimes (\chi_\sigma - \rho_\sigma) + (\chi_\sigma - \rho_\sigma) \otimes v \; dy \; dv \\ &\leq \; \left(\int_V |v|^2 dv\right)^{1/2} \|\chi_\sigma - \rho_\sigma\|_{L^2(Y\times V)} I = tr(\theta)^{1/2} \|\chi_\sigma - \rho_\sigma\|_{L^2(Y\times V)} I \end{aligned} \tag{61}$$

where I is the identity matrix. Therefore, using (60) we get

$$\underline{\sigma} \; \|\chi_\sigma - \rho_\sigma\|^2_{L^2(Y\times V)} \leq tr(D_\sigma) \leq N \; tr(\theta)^{1/2} \|\chi_\sigma - \rho_\sigma\|_{L^2(Y\times V)} \tag{62}$$

which yields

$$\|\chi_\sigma - \rho_\sigma\|_{L^2(Y\times V)} \leq \frac{N \; tr(\theta)^{1/2}}{\underline{\sigma}} \tag{63}$$

and

$$D_\sigma \leq \frac{N \; tr(\theta)}{\underline{\sigma}} I \tag{64}$$

by plugging (63) into (61). We now look for a bound from below. This time, we take the tensor product of the equation (53) by v and integrate. We obtain

$$\int_Y \int_V \sigma \; (\chi_\sigma - \rho_\sigma) \otimes v \; dy \; dv = \int_Y \int_V v \otimes v \; dy \; dv = \theta.$$

Let $\xi \in I\!R^N$. By using the Cauchy Schwartz inequality, we have

$$\begin{aligned} &\theta\xi \cdot \xi = \int_Y \int_V \sigma \; (\chi_\sigma - \rho_\sigma) \cdot \xi \; v \cdot \xi \; dy \; dv \\ &\leq \left(\int_Y \int_V \sigma \; ((\chi_\sigma - \rho_\sigma) \cdot \xi)^2 \; dy \; dv\right)^{1/2} (\overline{\sigma})^{1/2} \; \left(\int_Y \int_V (v \cdot \xi)^2 \; dy \; dv\right)^{1/2}. \end{aligned}$$

By the last equality in (60), we can rewrite the right hand side as $(D_\sigma\xi\cdot\xi)^{1/2}\ (\overline{\sigma})^{1/2}\ (\theta\xi\cdot\xi)^{1/2}$, so that we finally get

$$\theta\xi\cdot\xi \le \overline{\sigma}\ D_\sigma\xi\cdot\xi$$

which is the desired inequality.

It remains to prove that Q_σ and D_σ are Lipschitz. We set $\delta = \chi_\sigma - \chi_{\sigma'}$. We have

$$\begin{aligned} v\cdot\nabla_y\delta + \sigma(\delta-\rho_\delta) &= (\sigma-\sigma')(\chi_{\sigma'}-\rho_{\sigma'}) \\ &\qquad with \quad \rho_\delta = \int_V \delta\ dv. \end{aligned}$$

We take the scalar product with δ and integrate to obtain

$$\begin{aligned} \int_Y\int_V |\delta-\rho_\delta|^2\sigma\ dv\ dy &= \int_Y\int_V (\sigma-\sigma')(\chi_{\sigma'}-\rho_{\sigma'})\cdot\delta\ dv\ dy \\ &= \int_Y\int_V (\sigma-\sigma')(\chi_{\sigma'}-\rho_{\sigma'})\cdot(\delta-\rho_\delta)\ dv\ dy \\ &\le \|\sigma-\sigma'\|_{L^\infty(Y)}\,\|\chi_{\sigma'}-\rho_{\sigma'}\|_{L^2(Y\times V)}\,\|\delta-\rho_\delta\|_{L^2(Y\times V)} \end{aligned}$$

Using (63) (with σ') yields

$$\underline{\sigma}\|\delta-\rho_\delta\|_{L^2(Y\times V)} \le \|\sigma-\sigma'\|_{L^\infty(Y)}\frac{N\ tr(\theta)^{1/2}}{\underline{\sigma}}.$$

But we have

$$\begin{aligned} |D_\sigma - D_{\sigma'}| \le |Q_\sigma - Q_{\sigma'}| &\le \|\delta-\rho_\delta\|_{L^2(Y\times V)}tr(\theta)^{1/2} \\ &\le \frac{N\ tr(\theta)}{\underline{\sigma}^2}\ \|\sigma-\sigma'\|_{L^\infty(Y)} \end{aligned}$$

which concludes the proof. □

4. PROOF OF THEOREM 2

4.1. Estimates

We begin by deriving uniform estimates satisfied by the solution u_ε of (1). Since the initial data satisfies

$$0 \le u_0^\varepsilon \le \overline{u}, \tag{65}$$

it follows that the following estimates

$$\left\{\begin{array}{l} 0 \le u_\varepsilon(t,x,v) \le \overline{u}, \\ 0 \le \rho_\varepsilon(t,x) \le \overline{u} \end{array}\right. \tag{66}$$

hold. Then, by multiplying (1) by u_ε we are led to

$$\begin{array}{l} 1/2 \displaystyle\int_{I\!R^N \times V} u_\varepsilon^2(t) dx dv + \int_0^t \int_{I\!R^N \times V} \sigma_\varepsilon \frac{(\rho_\varepsilon - u_\varepsilon)^2(s)}{\varepsilon^2}\, dv dx ds \\ = 1/2 \displaystyle\int_{I\!R^N \times V} (u_0^\varepsilon)^2 \; dx dv. \end{array} \tag{67}$$

We denote $g_\varepsilon = \dfrac{u_\varepsilon - \rho_\varepsilon}{\varepsilon}$. One deduces that

$$\left\{\begin{array}{l} u_\varepsilon \text{ is bounded in } L^\infty(I\!R^+; L^2(I\!R^N \times V)), \\ g_\varepsilon \text{ is bounded in } L^2(I\!R^+ \times I\!R^N \times V). \end{array}\right. \tag{68}$$

It follows that

$$|j_\varepsilon| \le \left(\int_V v^2 dv\right)^{1/2} \left(\int_V |g_\varepsilon|^2 dv\right)^{1/2} \text{ is bounded in } L^2(I\!R^+ \times I\!R^N). \tag{69}$$

In (69), $j_\varepsilon(t,x)$ still stands for the current $\displaystyle\int_V \frac{v}{\varepsilon} u_\varepsilon(t,x,v) dv$. Equations for the zeroth and first moment in (1) read as follows

$$\left\{\begin{array}{l} \partial_t \rho_\varepsilon + div_x j_\varepsilon = 0, \\ \\ \partial_t j_\varepsilon + \dfrac{1}{\varepsilon^2} div_x \left(\displaystyle\int_V v \otimes v u_\varepsilon dv\right) = \dfrac{1}{\varepsilon^2} \displaystyle\int_V \sigma_\varepsilon v g_\varepsilon dv. \end{array}\right. \tag{70}$$

We have

$$\begin{array}{rcl} \displaystyle\int_V v \otimes v u_\varepsilon \; dv & = & \displaystyle\int_V v \otimes v(\rho_\varepsilon + \varepsilon g_\varepsilon) \; dv \\ & = & \theta \rho_\varepsilon + \varepsilon \kappa_\varepsilon \end{array}$$

where it is clear that the $N \times N$ matrix κ_ε is bounded in $(L^2(I\!R^+ \times I\!R^N))^{N\times N}$. Therefore, the second equation in (70) can be rewritten as follows

$$\theta \cdot \nabla_x(\rho_\varepsilon) = \sigma_\varepsilon \int_V v g_\varepsilon \; dv - \varepsilon div_x \kappa_\varepsilon - \varepsilon^2 \partial_t j_\varepsilon. \tag{71}$$

The first term in the right hand side of (71) is bounded in $L^2(I\!R^+ \times I\!R^N)$ while the two other terms tend to 0 in $H^{-1}(I\!R^+ \times I\!R^N)$ as ε goes to 0. It follows that $\nabla_x \rho_\varepsilon$ lies in a compact set in $H^{-1}_{loc}((0,T) \times I\!R^N)$. Remark that Aubin's lemma does not apply because of the time derivative in the last term in (71).

Then, we use a compensated compactness argument since $div_{t,x}(\rho_\varepsilon, j_\varepsilon) = 0$ and $curl_{t,x}(\rho_\varepsilon, 0, ..., 0)$ belong to compact sets in $H^{-1}_{loc}((0,T) \times I\!R^N)$; this leads to strong convergence for ρ_ε. Therefore, up to a subsequence, we may assume that the following convergences hold

$$\left\{ \begin{array}{l} \rho_\varepsilon \to \rho \; strongly \; in \; L^p_{loc}(I\!R^+ \times I\!R^N), \; 1 \leq p < \infty, \\ u_\varepsilon = \rho_\varepsilon + \varepsilon g_\varepsilon \to \rho \; strongly \; in \; L^p_{loc}(I\!R^+ \times I\!R^N), \; 1 \leq p < \infty, \\ g_\varepsilon \rightharpoonup g \; weakly \; in \; L^2(I\!R^+ \times I\!R^N \times V), \\ j_\varepsilon \rightharpoonup j \; weakly \; in \; L^2(I\!R^+ \times I\!R^N), \\ \nu_\varepsilon = \sigma_\varepsilon j_\varepsilon \rightharpoonup \nu \; weakly \; in \; L^2(I\!R^+ \times I\!R^N). \end{array} \right. \tag{72}$$

Then, letting ε go to 0 in (70) and (71) leads to the following limit equations

$$\left\{ \begin{array}{l} \partial_t \rho + div_x j = 0, \\ \theta \cdot \nabla_x \rho = \nu. \end{array} \right. \tag{73}$$

which shows that the limit ρ belongs to $L^2(0,T; H^1(I\!R^N))$. It remains to determine the equation relating ρ, j and ν.

4.2. Double scale limit

Our proof relies on double scale convergence arguments. Let ψ_ε be a bounded sequence in $L^2((0,T) \times I\!R^N \times V)$. According to [26], [2], there exists a subsequence, still labelled ψ_ε, which "double-scale converges" to Ψ in the sense that

$$\begin{array}{l} \displaystyle\lim_{\varepsilon \to 0} \int_{(0,T) \times I\!R^N \times V} \psi_\varepsilon(t,x,v) \; \phi(t,x,\frac{x}{\varepsilon},v) \; dvdxdt \\ \displaystyle= \int_{(0,T) \times I\!R^N \times V} \int_Y \Psi \; \phi(t,x,y,v) \; dy \; dvdxdt \end{array} \tag{74}$$

holds. In this definition, dy is the normalized Lebesgue measure on the cell Y and ϕ is a smooth and periodic function with respect to the $y-$ variable. Precisely, (74) makes sense as the test function ϕ lies in the set

$$\begin{array}{l} \mathcal{V} = \{\phi \in C^0([0,T]_t \times I\!R^N_x \times I\!R^N_y; L^2(V_v)), \\ \; \phi \text{ is } Y \text{ periodic w.r.t. the variable } y \\ \; \text{and } \phi \text{ has a compact support wrt the variable } x\}. \end{array}$$

We refer to [2] for a detailled discussion on the "admissible" test functions ϕ. In the sequel, we will also use the following set of admissible test functions

$$\begin{array}{l} \mathcal{D}_\sharp = \{\phi \in C^\infty((0,T)_t \times I\!R^N_x \times I\!R^N_y; L^\infty(V_v)), \\ \; \phi \text{ is } Y \text{ periodic w.r.t. the variable } y \\ \; \text{and there exists a compact } K \subset (0,T)_t \times I\!R^N_x \text{ such that} \\ supp(\phi) \subset K \times I\!R^N_y \times V\}. \end{array}$$

We recall the following properties of the double scale limit

• if $\psi_\varepsilon \to \psi$ strongly in $L^2(0,T) \times I\!R^N \times V)$ then $\psi(t,x,v) = \Psi(t,x,y,v)$ (the limit does not depend on the y variable).

• if $\psi_\varepsilon \rightharpoonup \psi$ weakly in $L^2(0,T) \times I\!R^N \times V)$ then one has $\psi(t,x,v) = \int_Y \Psi(t,x,y,v)dy$.

Having disposed of these preliminaries, we multiply (1) by a test function $\varepsilon\phi_\varepsilon(t,x,v) = \varepsilon\phi(t,x,\frac{x}{\varepsilon},v)$, ϕ belonging to $\mathcal{D}_\sharp$. In the following computations, $(\nabla_x\phi)_\varepsilon(t,x,v)$, $(\nabla_y\phi)_\varepsilon(t,x,v)$,... are equal to $\nabla_x\phi(t,x,\frac{x}{\varepsilon},v)$, $\nabla_y\phi(t,x,\frac{x}{\varepsilon},v)$,... and in the sequel $\langle\cdot\rangle$ stands for integration with respect to the variable v. We obtain

$$\begin{array}{ll}
- & \frac{1}{\varepsilon}\int_{(0,T)\times I\!R^N\times V} u_\varepsilon v\cdot(\nabla_y\phi)_\varepsilon dxdvds \\
- & \int_{(0,T)\times I\!R^N\times V} u_\varepsilon v\cdot(\nabla_x\phi)_\varepsilon dxdvds + \int_{(0,T)\times I\!R^N\times V} \sigma_\varepsilon g_\varepsilon\phi_\varepsilon dxdvds \\
- & \varepsilon\int_{(0,T)\times I\!R^N\times V} u_\varepsilon(\partial_t\phi)_\varepsilon dxdvds = 0 \\
= & -\frac{1}{\varepsilon}\int_{(0,T)\times I\!R^N} \rho_\varepsilon\int_V v\cdot(\nabla_y\phi)_\varepsilon dvdxds \\
- & \int_{(0,T)\times I\!R^N\times V} g_\varepsilon v\cdot(\nabla_y\phi)_\varepsilon dxdvds \\
- & \int_{(0,T)\times I\!R^N\times V} \rho_\varepsilon v\cdot(\nabla_x\phi)_\varepsilon dxdvds + \int_{(0,T)\times I\!R^N\times V} \sigma_\varepsilon g_\varepsilon\phi_\varepsilon dxdvds \\
- & \varepsilon\int_{(0,T)\times I\!R^N\times V} u_\varepsilon(\partial_t\phi)_\varepsilon dxdvds \\
- & \varepsilon\int_{(0,T)\times I\!R^N\times V} g_\varepsilon v\cdot(\nabla_x\phi)_\varepsilon dxdvds = 0.
\end{array} \tag{75}$$

To make disappear the first term (in $1/\varepsilon$), we require that

$$div_y\left(\int_V v\phi\ dv\right) = 0. \tag{76}$$

We denote by $E_\sharp$ the set of functions $\phi\in\mathcal{D}_\sharp$ which satisfy (76). For such a test function, by using the fact that $\langle g_\varepsilon\rangle = 0$, we are led to

$$\begin{array}{l}
\int_{(0,T)\times I\!R^N\times V} g_\varepsilon(-v\cdot(\nabla_y\phi)_\varepsilon + \sigma_\varepsilon(\phi_\varepsilon - \langle\phi_\varepsilon\rangle))\ dxdvds \\
= \int_{(0,T)\times I\!R^N\times V} \rho_\varepsilon v\cdot(\nabla_x\phi)_\varepsilon\ dxdvds \\
+\varepsilon\int_{(0,T)\times I\!R^N\times V} (u_\varepsilon(\partial_t\phi)_\varepsilon + g_\varepsilon v\cdot(\nabla_x\phi)_\varepsilon)\ dxdvds.
\end{array} \tag{77}$$

We introduce the double scale limit G and H of g_ε and $\sigma_\varepsilon g_\varepsilon$, respectively. Remark that the double scale limit of ρ_ε is ρ since it convergences strongly in $L^2_{loc}([0,T] \times I\!R^N)$. Letting $\varepsilon \to 0$ in (77) gives the following relation

$$\begin{aligned} &\int_{(0,T)\times I\!R^N \times V} \int_Y (-Gv\cdot\nabla_y\phi + H(\phi - \langle\phi\rangle) - \rho v\cdot\nabla_x\phi)\ dydxdvds = 0 \\ &= \int_{(0,T)\times I\!R^N \times V} \int_Y (v\cdot\nabla_y G + H - \langle H\rangle + v\cdot\nabla_x\rho)\phi\ dydxdvds \end{aligned} \tag{78}$$

which holds for all ϕ in the set $E_\sharp$, the second integral being understood in the sense of distributions. Therefore, we are left with two tasks: first, we need a characterization of the orthogonal set of $E_\sharp$, and second, we want to determine H in terms of G. The answer to the first question is given by the following claim.

Proposition 3 *Let T be in the dual $\mathcal{D}'_\sharp$ of $\mathcal{D}_\sharp$. Then, T belongs to the orthogonal set of $E_\sharp$ iff $T = v\cdot\nabla_y S$, where $S \in \mathcal{D}'_\sharp$ does not depend on v.*

Obviously, $T = v.\nabla_y Q$ with $Q = Q(t,x,y)$ is orthogonal to $E_\sharp$. The proof of the converse is quite classical in the case of continuous velocities. We only give the keypoint which allows to deal with discrete cases. For $n \in Z\!\!\!Z^N$, let $\hat{T}(t,n,x,v)$ be the Fourier coefficient (wrt y) of T belonging to the orthogonal set of $E_\sharp$. We can check that $\hat{T}(t,0,x,v) = 0$ and there exists Q, which does not depend on v, such that for $n \neq 0$, $\hat{T}(t,n,x,v) = n\cdot v\ \hat{Q}(t,n,x)$. Multiplying by v and integrating give

$$\hat{Q}(t,n,x)\ \theta.n = \int_V \hat{T}(t,n,x,v)\ v\ dv$$

and thus, it follows that

$$\hat{Q}(t,n,x) = \frac{n}{|n|^2}\cdot\theta^{-1}\int_V \hat{T}(t,n,x,v)\ v\ dv$$

holds for all $n \neq 0$ (Remark the use of the invertibility of the matrix θ). Therefore, $Q = \sum_{n\neq 0} \hat{Q}(t,n,x)\frac{1}{2i\pi}e^{2i\pi y\cdot n}$ converges in distribution sense. We finally obtain $T = v.\nabla_y Q$.

We search now for the relation between H and G. As a consequence of Proposition 3, we have

$$v \cdot \nabla_y G + H - \langle H \rangle + v \cdot \nabla_x \rho = v \cdot \nabla_y S \textit{ in } \mathcal{D}'_\sharp. \tag{79}$$

Multiplying this equation by v and integrating lead to

$$\nabla_y S = \theta^{-1}(div_y \left(\int_V v \otimes v \; G \, dv \right) + \int_V vH \, dv) + \nabla_x \rho. \tag{80}$$

Without loss of generality, we can assume that $\int_Y S \, dy = 0$ (replacing if necessary S by $S - \int_Y S \, dy$ in (79)). Furthermore, $\int_V v \otimes v \; G \, dv$, $\int_V vH \, dv$ and $\nabla_x \rho$ belong to $L^2((0,T) \times I\!R^N \times Y)$ so that the right hand side in (80) appears as the sum of functions and 1^{st} order derivatives of functions in $L^2((0,T) \times I\!R^N \times Y)$. Thus, (80) determines S as a function of $L^2((0,T) \times I\!R^N \times Y)$ (use for instance the Fourier serie associated to S...). Furthermore, (79) implies that $v \cdot \nabla_y (G - S) \in L^2((0,T) \times I\!R^N \times Y \times V)$, therefore we get

$$(G - S) \in L^2((0,T) \times I\!R^N; H_\sharp). \tag{81}$$

In order to determine H, we establish

Lemma 3 *The sequence $\sigma_\varepsilon(t,x) = \Sigma(\frac{x}{\varepsilon}, x, \rho_\varepsilon(t,x))$ double scale converges to $\Sigma(y, x, \rho(t,x))$ strongly in the sense of Theorem 5 of the Annex. As a consequence*

$$\sigma_\varepsilon \; g_\varepsilon \rightharpoonup H = G \; \Sigma(y, x, \rho(t,x)) \textit{ double scale.}$$

We postpone to the Annex the proof of this result as well as the precise definition of the "strong double scale convergence". This notion allows to pass to the limit in products and non linear terms.

Now, (79) reads as follows

$$v \cdot \nabla_y (G - S) + \Sigma(G - \langle G \rangle) + v \cdot \nabla_x \rho = 0.$$

Since S does not depend on v (cf. Proposition 3), we have $S = \langle S \rangle$ so that $G - S$ satisfies

$$\left\{ \begin{array}{l} (G - S) \in L^2((0,T) \times I\!R^N; H_\sharp), \\ v \cdot \nabla_y (G - S) + \Sigma((G - S) - \langle (G - S) \rangle) = -v \cdot \nabla_x \rho. \end{array} \right. \tag{82}$$

It suffices to apply Theorem 3 to deduce that

$$G - S = -\chi \cdot \nabla_x \rho + T \tag{83}$$

where $T = T(t,x) \in L^2((0,T) \times I\!R^N)$ and χ is defined by (48). We remark that $\langle vS \rangle = \langle vT \rangle = \langle v \rangle = 0$, then G is not uniquely defined by (83), but the averaged quantity

$$\int_V v \; G \, dv = - \int_V v \otimes \chi \, dv \cdot \nabla_x \rho$$

is. Then, we are now able to express the limit current as follows

$$j(t,x) = \int_V v \; g \, dv = \int_{V \times Y} v \; G \, dv \; dy = -Q(x, \rho(t,x)) \cdot \nabla_x \rho, \tag{84}$$

where Q is the matrix defined by (50). Then, plugging (84) in (73) leads to the equation satisfied by ρ.

It remains to check that the Cauchy data is ρ_0, the weak limit of $\rho_0^\varepsilon = \int u_0^\varepsilon \, dv$. In view of (70) and (69), the sequence $\partial_t \rho_\varepsilon$ is bounded in $L^2(0,T; H^{-1}(I\!R^N))$ while it is also known that ρ_ε is bounded in $L^\infty(0,T; L^2(I\!R^N))$, by (68). Thus, one deduces that ρ_ε is relatively compact in $C^0([0,T]; H^{-1}_{loc}(I\!R^N))$ (see for instance J. Simon [30]). But it is assumed that $\rho_\varepsilon(0) = \rho_0^\varepsilon$ converges weakly $*$ in $L^\infty(I\!R^N)$ to ρ_0, therefore $\rho(0) = \rho_0$. $\square$

5. ANNEX : STRONG DOUBLE SCALE CONVERGENCE

This Annex is devoted to double scale convergence of sequence of functions with value in Hilbert spaces and to the definition of the strong double scale convergence.

Let H be a separable Hilbert space and Ω be a bounded open set of $I\!R^N$. We define $\mathcal{D}_\#(\Omega \times I\!R^N; H)$ as to be the set of infinitely differentiable functions with value in H which are Y-periodic in the second variable and have a compact support in Ω. In the same way $\mathcal{C}_\#(\overline{\Omega} \times I\!R^N; H)$ stands for the space of continuous functions with value in H which are Y-periodic in the second variable.

We say that a sequence $u_\varepsilon \in \mathcal{D}'(\Omega; H)$ of distributions double scale converges towards $U \in \mathcal{D}'_\#(\Omega \times I\!R^N; H)$ if for any test function $\varphi \in \mathcal{D}_\#(\Omega \times I\!R^N; H)$ we have

$$\int_\Omega u_\varepsilon(x)\cdot\varphi(x,\frac{x}{\varepsilon})\,dx \to \int_\Omega\int_Y U(x,y)\cdot\varphi(x,y)\,dx\,dy \tag{85}$$

as $\varepsilon\to 0$, where $\cdot$ denotes the scalar product in H (the integral has to be taken as a duality product). We note $u_\varepsilon \rightharpoonup U$ *d.s.*

Of course arguing as in [2], cooordinates by coordinates, we have that for any bounded sequence u_ε of $L^2(\Omega;H)$ there is a subsequence which double scale converges to some $U\in L^2(\Omega\times Y;H)$. In that case (85) holds also for $\varphi\in\mathcal{C}_\#(\overline{\Omega}\times I\!R^N;H)$.

We define now what is a strong double scale convergence. This notion already appears in the work of G. Allaire [2] (see in particular Theorem 1.8 in [2]). We skip the proof which is in the spirit of G. Allaire's work (we refer also to the pedagogical expository in [19]) but we stress that the strong double scale convergence allows to pass to the limit in non linear double scale sequences (Proposition 4).

Theorem 5 *Consider $(u_\varepsilon)_{\varepsilon>0}$ a bounded sequence of $L^2(\Omega;H)$, then the following assertions are equivalent :*

(i) $\quad inf_{\varphi\in\mathcal{C}_\#}\, limsup_{\varepsilon\to 0}\,\|u_\varepsilon(x)-\varphi(x,\frac{x}{\varepsilon})\|_{L^2(\Omega;H)}=0.$

(ii) $\quad u_\varepsilon\rightharpoonup U$ *d.s and for any bounded sequence* $(v_\varepsilon)_{\varepsilon>0}$ *of* $L^2(\Omega;H)$ *such that* $v_\varepsilon\rightharpoonup V$ *d.s we have* $u_\varepsilon\cdot v_\varepsilon\rightharpoonup U\cdot V$ *d.s (as real functions).*

(iii) $\quad u_\varepsilon\rightharpoonup U$ *d.s and* $\|u_\varepsilon\|_{L^2(\Omega;H)}\to\|U\|_{L^2(\Omega\times Y;H)}$.

In this case we say that $u_\varepsilon\rightharpoonup U$ *d.s. strong.*

The terminology is justified by the fact that the double scale strong convergence allows to pass to the limit in products (see (ii)) but also in more complicated non linear terms as shown in the next Proposition.

Proposition 4 *Let (u_ε) be a bounded sequence of $L^2(\Omega;H)$ such that $u_\varepsilon\rightharpoonup U$ d.s. strong. Let $\Phi=\Phi(y,x,u)\in L^2_{loc}(I\!R^N_y;\mathcal{C}(\overline{\Omega}\times H;H))$, periodic in the y variable and (uniformly) k-Lipschitz with respect to $u\in H$. Then*

$$\Phi(\frac{x}{\varepsilon},x,u_\varepsilon(x))\rightharpoonup\Phi(y,x,U(x,y))\ d.s.\ strong.$$

Proof.

Let us first introduce the space $\mathcal{V}$ of test functions on $\Omega\times I\!R^N$, Y-periodic in the second variable and which belong to $L^2(Y;\mathcal{C}(\overline{\Omega};H))$. We claim that in the characterization (i) of Theorem 5, $\mathcal{C}_\#(\overline{\Omega}\times I\!\!R^N;H)$ can be replaced by $\mathcal{V}$. Indeed for any $\phi\in\mathcal{V}$ by regularizing with respect to y we obtain for any $\eta>0$ a function $\phi_\eta\in\mathcal{C}_\#(\overline{\Omega}\times I\!\!R^N;H)$ such that

$$\int_Y \left(\|\phi(.,y)-\phi_\eta(.,y)\|_{L^\infty(\Omega;H)}\right)^2\ dy\le\eta.$$

It follows that

$$\begin{aligned}
&\int_\Omega\left(\|\phi_\eta(x,\frac{x}{\varepsilon})-\phi(x,\frac{x}{\varepsilon})\|_H\right)^2 dx\\
&\quad\le\int_\Omega\left(\|\phi(.,\frac{x}{\varepsilon})-\phi_\eta(.,\frac{x}{\varepsilon})\|_{L^\infty(\Omega;H)}\right)^2 dx\\
&\quad\le\varepsilon^N\int_{\Omega/\varepsilon}\left(\|\phi(.,y)-\phi_\eta(.,y)\|_{L^\infty(\Omega;H)}\right)^2 dy\\
&\le\varepsilon^N\sum_{m\in Z\!\!\!Z^N;\ (m+Y)\bigcap\Omega/\varepsilon\neq\emptyset}\int_Y\left(\|\phi(.,y+m)-\phi_\eta(.,y+m)\|_{L^\infty(\Omega;H)}\right)^2 dy\\
&\quad\le\varepsilon^N\#\{m\in Z\!\!\!Z^N;\ (m+Y)\bigcap\Omega/\varepsilon\neq\emptyset\}\eta\le C\eta.
\end{aligned}$$

It is moreover now easy to check that $\phi(x,x/\varepsilon)$ d.s. converges towards $\phi(x,y)$ for such a $\phi\in\mathcal{V}$.

Now, since u_ε converges d.s. strong, there exists $U_\eta\in\mathcal{C}_\#(\overline{\Omega}\times I\!\!R^N;H)$ which satisfies

$$\limsup_{\varepsilon\to0}\ \|u_\varepsilon(x)-U_\eta(x,\frac{x}{\varepsilon})\|_{L^2(\Omega;H)}\le\eta.$$

This easily implies that $\|U-U_\eta\|_{L^2(\Omega\times Y;H)}\le\eta$. We have $\Phi(y,x,U_\eta(x,y))\in\mathcal{V}$ and by using the Lipschitz property of Φ

$$\|\Phi(\frac{x}{\varepsilon},x,u_\varepsilon(x))-\Phi(\frac{x}{\varepsilon},x,U_\eta(x,\frac{x}{\varepsilon}))\|_{L^2(\Omega;H)}\le k\|u_\varepsilon(x)-U_\eta(x,\frac{x}{\varepsilon})\|_{L^2(\Omega;H)}\quad(86)$$

which proves by taking the $\limsup_{\varepsilon\to0}$ that $(\Phi(\frac{x}{\varepsilon},x,u_\varepsilon(x))_{\varepsilon>0}$ satisfies the criterium (i) and converges d.s. strong. Let Ψ be this limit. Letting $\varepsilon\to0$ in (86) leads to

$$\|\Psi-\Phi(y,x,U_\eta(x,y))\|_{L^2(\Omega\times Y;H)}\le k\ \eta.$$

Since $U_\eta\to U$ in $L^2(\Omega\times Y;H)$ and with the help of the Lipschitz property of Φ, we check that $\Phi(y,x,U_\eta(x,y))\to\Phi(y,x,U(x,y))$ as $\eta\to0$ in $L^2(\Omega\times Y;H)$

which conclude s the proof. $\square$

We are now ready to prove Lemma 3. It follows from the assumption of equicontinuity that there is a sequence $\Sigma_k = \Sigma_k(y,x,\lambda)$ which satisfies

$$\begin{cases} \sup_{x\in I\!R^N, y\in Y} |\Sigma_k(y,x,\lambda)-\Sigma_k(y,x,\mu)| \le k|\lambda-\mu|, \\ |\Sigma(y,x,\lambda)-\Sigma_k(y,x,\lambda)| \le \alpha(k) \to 0. \end{cases}$$

We work with the space $H = L^2((0,T)\times V)$. Moreover, since we used in Section 4 test functions $\phi \in \mathcal{D}_\sharp$ having compact support in x (see the definition of the set $\mathcal{D}_\sharp$ at the beginning of subsection 4.2), it suffices to deal with $\Omega = B(0,R)$. Finally, we consider ρ_ε as a sequence of functions in $L^2(\Omega; H)$ (even if ρ_ε does not depend on v) and the functions Σ_k as functions on $\Omega \times I\!R^N \times H$ with value in H having the $k-$Lipschitz property. Since ρ_ε converges strongly, and consequentely d.s. strong, to ρ in $L^2(\Omega; H)$, Proposition 4 implies that

$$\Sigma_k(\frac{x}{\varepsilon}, x, \rho_\varepsilon(t,x)) \rightharpoonup \Sigma_k(y,x,\rho(t,x)) \ d.s.\ strong.$$

Furthermore, one has

$$\begin{aligned} &\|\Sigma_k(\frac{x}{\varepsilon}, x, \rho_\varepsilon(t,x)) - \Sigma(\frac{x}{\varepsilon}, x, \rho_\varepsilon(t,x))\|_{L^2(\Omega;H)} \\ &\le \alpha(k)(T\ meas(\Omega)\ meas(V))^{1/2}, \end{aligned}$$

uniformly in ε. Hence, we get

$$\Sigma(\frac{x}{\varepsilon}, x, \rho_\varepsilon(t,x)) \rightharpoonup \Sigma(y,x,\rho(t,x)) \ d.s.\ strong.$$

Let $\phi \in \mathcal{D}_\sharp$ (as defined in subsection 4.2). It is easy to check that $\phi(t,x,\frac{x}{\varepsilon},v)\ g_\varepsilon$ $\rightharpoonup \phi\ G\ d.s.$ (since $\phi(t,x,\frac{x}{\varepsilon},v)$ converges ds strong and using Theorem 5 (ii)), still considering this function with values in $H = L^2((0,T)\times V)$. We thus have (Theorem 5 (ii))

$$\sigma_\varepsilon\ \phi(t,x,\frac{x}{\varepsilon},v)\ g_\varepsilon \rightharpoonup \Sigma(y,x,\rho(t,x))\ \phi\ G\ d.s.$$

In particular we get

$$\begin{aligned} &\int_0^T \int_\Omega \int_V \sigma_\varepsilon\, \phi(t,x,\frac{x}{\varepsilon},v)\ g_\varepsilon\ dt\ dx\ dv \to \\ &\int_0^T \int_\Omega \int_Y \int_V \Sigma(x,y,\rho(t,x))\ G(t,x,y,v)\phi(t,x,y,v)\ dt\ dx\ dy\ dv \end{aligned}$$

which is the desired result. □

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