*Journal of Statistical Physics, Vol. 108, Nos. 1/2, July 2002 (© 2002)*

# Some Remarks on Large-Time Asymptotic of the Lifshitz–Slyozov Equations

**Jean-François Collet,[1] Thierry Goudon,[1] and Alexis Vasseur[1]**

*Received October 1, 2001; accepted March 6, 2002*

We obtain an entropy functional for the Lifshitz–Slyozov system. This can be used to investigate the time asymptotics of the system. In particular, we describe situations in which the monomers concentration either tend to 0 or saturate as time becomes large. The latter situation can be excluded under assumptions on the support of the initial data.

**KEY WORDS:** Lifshitz–Slyozov equation; phase transition; large-time behavior; entropy dissipation method.



## 1. INTRODUCTION

This paper is devoted to the asymptotic behavior of solution $(c, f)$ of the following system

$$\begin{cases} \partial_t f + \partial_x((a(x)\,c(t) - b(x))\,f) = 0 \qquad \text{in } \mathbb{R}_t^+ \times \mathbb{R}_x^+, \\ c(t) + \int_0^\infty x f(t, x)\,\mathrm{d}x = \rho > 0, \\ f_{|t=0} = f_0 \qquad \text{in } \mathbb{R}_x^+, \qquad c_{|t=0} = c_0 \end{cases} \tag{1}$$

as time becomes large. This equation has been introduced by Lifshitz and Slyozov[(1)] as a model for phase transition phenomena. Details on (1) can also be found in the more recent paper of Sagalovich and Slyozov[(2)] and in the classical book of Lifshitz and Pitaevski.[(3)] It is intended to model the evolution of aggregates interacting in a solution with free particles or monomers. The aggregates are characterized by their size $x \geqslant 0$. In this

[1] Labo. J. A. Dieudonné, UMR 6621, Université Nice-Sophia Antipolis, Parc Valrose, F-06108 Nice cedex 02, France; e-mail: goudon@math.unice.fr



0022-4715/02/0700-0341/0 © 2002 Plenum Publishing Corporation

modeling the size of the aggregates is infinitely large compared to the size of the free particles. Therefore, $f(t, x)$ is the density of clusters having size $x$ at time $t$. Evolution of the solution is governed by addition to or removal from clusters of monomers, whose concentration at time $t$ is $c(t)$. The (given) coefficients $a, b$ are interpreted as rates at which these reactions occur. For instance, in the original paper,[1] it is proposed

$$a(x) = 3x^{1/3}, \qquad b(x) = 3 \tag{2}$$

by assuming the growth process is governed by diffusion from the solution. The crucial assumption on the coefficients is related to the existence and uniqueness, at each time $t$, of a critical point $x_c(t)$ where the Lagrangian growth rate

$$V(t, x) = a(x)\, c(t) - b(x) \tag{3}$$

changes sign. Indeed, evolution of a grain having size $x$ depends on the ratio between the monomers concentration $c(t)$ arround it and the equilibrium concentration $C_{\text{eq}}(x)$ at its surface: as $c(t) > C_{\text{eq}}(x)$, the monomers are caught by the cluster, whereas for $c(t) < C_{\text{eq}}(x)$, the cluster should loss monomers. However, $C_{\text{eq}}(x)$ is a decreasing function of the size (given for instance by the Gibbs–Thomson law $C_{\text{eq}}(x) = \alpha \exp(\beta / x^{1/3}) \sim \alpha(1 + \beta / x^{1/3})$ where $\alpha, \beta > 0$). As a consequence, subcritical grains shrink (for $0 \leqslant x < x_c(t)$, the growth rate of particle $V(t, x)$ is negative) while supercritical ones grow (for $x > x_c(t)$, the rate is positive). This phenomenon is known as the Ostwald ripening.

Therefore, we will consider rates $a$ and $b$ satisfying (at least)

$$\begin{cases} a(x) > 0, \quad b(x) > 0 \qquad \text{for} \quad x > 0, \\ q(x) = \dfrac{b(x)}{a(x)} \ \text{is continuous and strictly decreasing from} \\ \qquad\qquad (0, +\infty) \ \text{to} \ (q(\infty) = 0, q(0)) \end{cases} \tag{4}$$

where $[0, \rho]$ is strictly contained in $[0, q(0))$ (we can have $q(0) = +\infty$ as in (2)). Then, the following definition of the critical size associated to a density $c$ makes sense

$$x_c = q^{-1}(c)$$

Also note that for $x = 0$ the growth rate is naturally non positive

$$a(0)\, \rho - b(0) \leqslant 0 \tag{5}$$

On a mathematical viewpoint, it explains why no boundary condition is required for (1): the characteristics associated to $V(t, x)$ are always outgoing (see (8) later). Let us mention that one often deals with a slightly different model which assumes that the monomers concentration is yet very small. In turn, the mass conservation reduces to the constraint $\int_0^\infty x f(t, x)\,\mathrm{d}x = \rho$ on the first moment. Then, the growth rate is given by (3) with

$$c(t) = \int_0^\infty b(x)\, f(t, x)\,\mathrm{d}x \times \left(\int_0^\infty a(x)\, f(t, x)\,\mathrm{d}x\right)^{-1} \tag{6}$$

For the coefficients (2) this quantity is nothing but the mean radius of the aggregates.

Results on existence-uniqueness of solutions of the Lifshitz–Slyozov model have been obtained recently by using different approaches. Niethammer and Pego[4] deal with the modified model (6) and the original one (1) with compactly supported probability measure as initial data. Collet and Goudon[5] study the case of regular kinetic coefficients $a, b$ for integrable or measure-valued data. Laurençot[6] extends the well-posedness theory for integrable data to less regular coefficients, including (2). In refs. 7 and 8 a model which takes into account encounters between particles is discussed, and refs. 9 and 10 introduce and deal with a Fokker–Planck version of (1) involving a diffusive term. The connection with the Becker–Döring discrete model is investigated by Penrose[11] and Collet *et al.*,[12] through a large-size and large-time asymptotics. A review of these recent results and more references can be found in ref. 13. Finally, some explicit formulae for the solutions are given in ref. 5, which show non trivial asymptotic behavior, even when considering simple coefficients without critical point.

However, a rigourous and complete study of the asymptotic behavior of the solution is not yet available, even if a lot of progresses have been made recently by Niethammer and Pego.[14,15] Lifshitz–Slyozov argue on physical grounds that:

– the monomers concentration $c(t)$ tends to 0 monotically as time increases,

– precisely, for the coefficients (2), $c(t)\, t^{1/3}$ tends to a universal constant $K_{\mathrm{LS}}$,

– and the solution $f(t, x)$ is described by a universal asymptotic profile $M_{\mathrm{LS}}$, independently on the shape of the initial data.

These conclusions have given rise to a very controversial debate, both on physical or mathematical viewpoint. We mention among others the arguments of Brown,[16] Meerson and Sasorov,[17] Sagalovich and Slyozov.[2] Nice examples, on a simplified version of (6), given by Carr and Penrose[18] and the results of Niethammer and Pego, both for (6),[14] and (1),[15] indicate that the behavior conjectured by Lifshitz–Slyozov cannot hold without some conditions on the initial state. The conclusion of the debate can be summarized as follows:

– we can exhibit a familly of possible asymptotic profiles $M_K$, parametrized by $K \in [K_{\mathrm{LS}}, \infty)$,

– the value of $K$ corresponds to the possible limit of $c(t)\, t^{1/3}$,

– and it also measures the regularity of the profile $M_K$ at the end of its support.

– The asymptotic state of the solution is highly unstable and depends heavily on the initial shape. In particular, for compactly supported initial data, convergence to the profile $M_{\mathrm{LS}}$ cannot hold if the initial data vanishes like $(x-x_0)^\alpha$ at the end of its support. One conjectures that the solution tends to the profile which has the same behavior at the tip of the support (the value of $K$ being explicitely related to the exponent $\alpha$).

A large part of these statements have been proved in refs. 14 and 15 to which we refer for details. We also mention the recent numerical study of ref. 19 which may shed some light on these questions.

Then, the discussion of ref. 1 contains certain mathematical difficulties, in particular concerning the behavior with time of the monomer concentration. This note is a first attempt at dealing with this problem. We shall describe some relevant situations in which $c$ tends either to 0 or $\rho$, the limit depending on the initial data. The interest of the proof is that it is based on the construction of an entropy functional suitabily adapted for the system (1): the equation dissipates some combination of $c$ and pseudo-moment of $f$. For the physical case (2), a version of this functional is given by the following nice formula

$$H(t) = \int_0^\infty x^{2/3} f(t, x)\, \mathrm{d}x + \frac{c^2(t)}{3}$$

Since $x$ represents the volume of the grains, the first term is the total surface of the grains.

Section 2 is devoted to some preliminaries and notations. In Section 3, we obtain the entropy functional, and we discuss time asymptotics in Section 4.

## 2. PRELIMINARIES

In this section, we recall some of the material from ref. 5 which will be used in the sequel. First, let us set up some notations. When the coefficients $a$, $b$ are regular enough, namely

$$a, b \text{ are } C^1([0, +\infty)) \text{ with bounded derivative} \tag{7}$$

we are allowed to define the characteristic curves $X(t; s, x)$ associated to $V$ as follows

$$\frac{\partial}{\partial t} X(t; s, x) = V(t, X(t; s, x)), \qquad X(s; s, x) = x \tag{8}$$

We also recall the well-known formulae

$$\begin{cases} \dfrac{\partial}{\partial x} X(t; s, x) = J(t; s, x) = \exp\left(\displaystyle\int_s^t \partial_x V(\sigma, X(\sigma; s, x))\, \mathrm{d}\sigma\right), \\ \dfrac{\partial}{\partial s} X(t; s, x) = -V(s, x)\, J(t; s, x) \end{cases} \tag{9}$$

We will use the fact that $J(t; s, x)$ is the jacobian of the change of variable $y = X(t; s, x)$. Forgetting the coupling between $c$ and $f$, we define a solution to the transport equation of (1) via the following mild formulation

$$f(t, x) = f_0(X(0; t, x))\, J(0; t, x) \tag{10}$$

where $X$ is the characteristic curve associated to a continuous function $c$ by (8). Of course, the definition implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty \varphi(x)\, f(t, x)\, \mathrm{d}x = \int_0^\infty \varphi'(x)\, V(t, x)\, f(t, x)\, \mathrm{d}x$$

holds for any $\varphi \in C_c^\infty((0, \infty))$, at least in the $\mathscr{D}'((0, T))$ sense. Then, existence-uniqueness of solution of (1) can be obtained through a fixed point strategy on $c$, see ref. 5. Results of ref. 5 can be extended for integrable initial data to less regular coefficients, allowing blow up of the derivative at $x = 0$ as for (2). We refer for precise assumptions on $a$, $b$ to ref. 6, keeping in mind (4) together with the following property

$$0 \leqslant a(x), \qquad b(x) \leqslant A(1 + x) \tag{11}$$

(which is clearly satisfied for (7), or (2)). We summarize the existence-uniqueness result as follows.

**Theorem 1 (refs. 5 and 6).** Let the initial data $f_0$ be nonnegative and satisfy

$$\int_0^\infty f_0(x)\,\mathrm{d}x < \infty, \qquad \int_0^\infty x f_0(x)\,\mathrm{d}x \leqslant \rho$$

Then the system (1) has a unique solution $(c, f) \in C^0(\mathbb{R}^+) \times C^0(\mathbb{R}^+, w - L^1(\mathbb{R}^+))$. The concentration $c(t)$ is strictly positive for all time $t$. The first moment $M_1(t)$ of $f$ and $c(t)$ are $C^1$ functions of time, and their time derivative satisfy

$$\frac{\mathrm{d}c}{\mathrm{d}t}(t) = -\frac{\mathrm{d}M_1(t)}{\mathrm{d}t} = -\int_0^\infty V(t, x)\, f(t, x)\,\mathrm{d}x \tag{12}$$

Furthermore, the zeroth order moment

$$M_0(t) = \int_0^\infty f(t, x)\,\mathrm{d}x$$

is a decreasing function of time.

**Remark 1.** The notation $f \in C^0(\mathbb{R}^+, w - L^1(\mathbb{R}^+))$ means that, for any $\varphi \in L^\infty(\mathbb{R}^+)$, the function $t \mapsto \int_0^\infty f(t, x)\, \varphi(x)\,\mathrm{d}x$ is continuous on $\mathbb{R}^+$.

**Remark 2.** Let $R \geqslant 0$. Then, considering regular coefficients, the following relation holds for all $t \geqslant 0$

$$\int_R^\infty f(t, x)\,\mathrm{d}x = \int_{X(0;\, t,\, R)}^\infty f_0(y)\,dy \tag{13}$$

Therefore, for $R = 0$, monotonicity of $M_0$ appears as a consequence of (5). Indeed, by (9), $t \mapsto X(0; t, 0)$ is a non decreasing function of time.

Our analysis requires crucially some bounds on the solution, uniform with respect to time. Hence, let us discuss some immediate consequences of Theorem 1. The mass conservation yields immediately estimates on $M_1(t)$ and $c(t)$, since $M_1(t) + c(t) = \rho$, with $c$ and $f$ non negative. Furthermore,

$M_0(t)$ is dominated by its initial value. Next, (11) and (12) give a Lipshitz estimate on $c$

$$\left|\frac{\mathrm{d}c}{\mathrm{d}t}(t)\right| \leqslant A \int_0^\infty (1+x)\, f(t, x)\, \mathrm{d}x$$

Let us summarize these facts as follows.

**Corollary 1.** Let $(c, f)$ be the solution obtained in Theorem 1. Then, there exists a constant $C$, depending on $f_0$ and $\rho$, such that the following estimates hold for any time

$$\begin{cases} \int_0^\infty f(t, x)\, \mathrm{d}x \leqslant \int_0^\infty f_0(x)\, \mathrm{d}x \leqslant C, & \int_0^\infty x f(t, x)\, \mathrm{d}x \leqslant \rho \leqslant C, \\ 0 \leqslant c(t) \leqslant \rho, & \left|\frac{dc}{dt}(t)\right| \leqslant C \end{cases} \tag{14}$$

## 3. ENTROPY FUNCTIONAL

Our analysis relies on the existence of an "entropy-like" functional for the Lifshitz–Slyozov equation.

**Proposition 1.** Let $k\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a $C^1$ strictly increasing and concave function satisfying $k(0) = 0$. Set

$$\begin{cases} K(c) = \int_0^c k'(q^{-1}(s))\, ds & \text{for} \quad c \in [0, \rho], \\ H(t) = \int_0^\infty k(x)\, f(t, x)\, \mathrm{d}x + K(c(t)). \end{cases}$$

Then, $t \longmapsto H(t)$ is non increasing. Precisely, the entropy dissipation is given by (in $\mathscr{D}'(0, T)$)

$$\frac{\mathrm{d}}{\mathrm{d}t} H(t) = \int_0^\infty [k'(x) - k'(x_{c(t)})]\, V(t, x)\, f(t, x)\, \mathrm{d}x \leqslant 0$$

Notice that, by definition (4), $q^{-1}$ is decreasing and $q^{-1}(s) \geqslant q^{-1}(\rho) > 0$ for $0 \leqslant s \leqslant \rho$. Since $k'$ is non-increasing, $k'(q^{-1}(s)) \leqslant k'(q^{-1}(\rho)) < +\infty$ so $K(c)$ is well-defined. Since $k(0) = 0$ and $k$ is concave, we have

$$\lim_{x \to +\infty} \sup \frac{k(x)}{x} = C < +\infty$$

Thus, by (14), $H(t)$ is well-defined. If we choose $k(x) = x$, then we recover that

$$\frac{d}{dt}\left(\int_0^\infty x f(t,x)\,\mathrm{d}x + c(t)\right) \leqslant 0$$

(but this is actually $=0$ by the mass conservation relation !).

***Proof.*** In view of our assumptions, $k$ is $C^1$ with bounded derivative $(0 \leqslant k'(x) \leqslant k'(0) < \infty)$. Let $\phi \in C_c^\infty(\mathbb{R}^+)$ satisfying $0 \leqslant \phi(x) \leqslant 1$, $\phi(x) = 1$ for $x \in [0, 1]$, $\phi(x) = 0$ for $x \geqslant 2$ and $|\phi'(x)| \leqslant C$. Let us introduce an approximation $k_\varepsilon(x) = k(x)\,\phi(\varepsilon x)$, smooth with compact support and such that $0 \leqslant k_\varepsilon(x) \leqslant k(x) \leqslant Cx$, $k_\varepsilon \to k$ pointwise. We also check that $k'_\varepsilon$ converges pointwise to $k'(x)$ with the domination $|k'_\varepsilon(x)| \leqslant k'(x) + C$. Since $k_\varepsilon(0) = 0$, integration by parts yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty k_\varepsilon(x)\,f(t,x)\,\mathrm{d}x &= \int_0^\infty k'_\varepsilon(x)\,V(t,x)\,f(t,x)\,\mathrm{d}x \\ &= \int_0^\infty [k'_\varepsilon(x) - k'_\varepsilon(x_{c(t)})]\,V(t,x)\,f(t,x)\,\mathrm{d}x \\ &\quad + k'_\varepsilon(x_{c(t)})\int_0^\infty V(t,x)\,f(t,x)\,\mathrm{d}x\end{aligned}$$

By (14), $c$ is Lipschitz and the last term can be rewritten as

$$k'_\varepsilon(x_{c(t)})\left(-\frac{\mathrm{d}}{\mathrm{d}t}c(t)\right) = -\frac{\mathrm{d}}{\mathrm{d}t}K_\varepsilon(c(t))$$

with

$$K_\varepsilon(c) = \int_0^c k'_\varepsilon(q^{-1}(s))\,ds$$

By using the estimates on $k_\varepsilon$ pointed out above combined to (14), we can apply the Lebesgue theorem to obtain

$$\lim_{\varepsilon\to 0}\left(\int_0^\infty k_\varepsilon(x)\,f(t,x)\,\mathrm{d}x + K_\varepsilon(c(t))\right) = \int_0^\infty k(x)\,f(t,x)\,\mathrm{d}x + K(c(t))$$

Next, since $k'$ is bounded, we have $|k'(x)\, V(t,x)| \leqslant C\, |V(t,x)| \leqslant CA(1+x)$, by (11). Hence, letting $\varepsilon$ tend to 0 and using the Lebesgue theorem again, we are led to

$$\frac{\mathrm{d}}{\mathrm{d}t} H(t) = \int_0^\infty [k'(x) - k'(x_{c(t)})]\, V(t,x)\, f(t,x)\, \mathrm{d}x$$

in $\mathscr{D}'((0,T))$. Since $k'$ is non-increasing, this quantity is non-positive. ∎

Finally, we can also treat cases with functions $k$ having unbounded derivative (i.e., $k'(x)$ can blow up as $x \to 0$). To this end, we consider an approximation $k_n$ satisfying $|k'_n(x)| \leqslant C_n$, $k_n(x) \to k(x)$, $k'_n(x) \to k'(x)$ and the other requirements of the statement. We have, for any test function $\zeta \in C_c^\infty((0,T))$,

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t} H_n, \zeta \right\rangle_{\mathscr{D}', \mathscr{D}((0,T))} + \int_0^\infty \int_0^\infty [k'_n(x_{c(t)}) - k'_n(x)]\, V(t,x)\, f(t,x)\, \zeta(t)\, \mathrm{d}x\, \mathrm{d}t = 0$$

By the Lebesgue theorem, we check that $H_n(t) \to H(t)$. Hence, in the dissipation term we use Fatou's lemma in order to conclude that, for any non negative test function $\zeta$

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t} H, \zeta \right\rangle_{\mathscr{D}', \mathscr{D}((0,T))} + \int_0^\infty \int_0^\infty [k'(x_{c(t)}) - k'(x)]\, V(t,x)\, f(t,x)\, \zeta(t)\, \mathrm{d}x\, \mathrm{d}t \leqslant 0$$

In particular, the derivative of $H$ is non-positive (as a distribution in time).

We can apply the reasoning to the coefficients (2); taking $k(x) = x^{2/3}$ leads to

$$H(t) = \int_0^\infty x^{2/3} f(t,x)\, \mathrm{d}x + \frac{c^2}{3}$$

which involves the surface of the agglomerates.

## 4. ASYMPTOTIC BEHAVIOR

Let $t_n$ be a sequence of time increasing to $\infty$ and consider the sequences obtained by shifting the solution

$$f_n(t,x) = f(t+t_n, x), \qquad c_n(t) = c(t+t_n)$$

Note that $(c_n, f_n)$ satisfy

$$\begin{cases} \partial_t f_n(t,x) + \partial_x(V_n(t,x)\, f_n(t,x)) = 0, \\ V_n(t,x) = a(x)\, c_n(t) - b(x), \\ c_n(t) + \int_0^\infty x f_n(t,x)\, \mathrm{d}x = \rho \end{cases} \tag{15}$$

We shall exploit the bounds of Proposition 1 to deduce compactness properties for $(c_n, f_n)$.

Let $C_0(\mathbb{R}^+)$ be the space of continuous functions on $\mathbb{R}^+$ vanishing at infinity (i.e., $\varphi \in C_0(\mathbb{R}^+)$ is continuous and such that for any $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \mathbb{R}^+$ with $|\varphi(x)| \leqslant \varepsilon$ when $x \in \mathbb{R}^+ \setminus K_\varepsilon$). We equip $C_0(\mathbb{R}^+)$ with the norm

$$\|\varphi\|_\infty = \sup_{x \in \mathbb{R}^+} |\varphi(x)|$$

Let $\mathscr{M}^1(\mathbb{R}^+)$ be the space of bounded measures on $\mathbb{R}^+$, endowed with the norm

$$\|\mu\|_{\mathscr{M}^1} = \int_{\mathbb{R}^+} \mathrm{d}|\mu|$$

It identifies with the dual of $C_0(\mathbb{R}^+)$ through the relation

$$\varphi \longmapsto \int_{\mathbb{R}^+} \varphi\, \mu(\mathrm{d}x)$$

According to Proposition 1, $c_n$ is bounded in $C^0(\mathbb{R}^+)$ with $\frac{\mathrm{d}c_n}{\mathrm{d}t}$ bounded. Therefore, by Arzela–Ascoli's theorem, we can suppose that, for a subsequence,

$$c_n(t) \to c_\infty(t) \quad \text{uniformly in } \; C^0([0,T])$$

Next, $f_n$ is a nonnegative sequence bounded in $L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^+))$ and $V_n f_n$ is also bounded in $L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^+))$, by using (11) together with (14). Therefore, the time derivative $\partial_t f_n = -\partial_x(V_n f_n)$ is bounded in $L^\infty(\mathbb{R}^+; W^{-1,1}(\mathbb{R}^+))$. By combining the Arzela–Ascoli theorem, the separability of $C_0(\mathbb{R}^+)$ and the diagonal argument, we can find a subsequence and a measure valued function $f_\infty \colon \mathbb{R}^+ \to \mathscr{M}^1(\mathbb{R}^+)$ verifying

$$\lim_{n\to\infty} \int_0^\infty \varphi(x)\, f_n(t,x)\, \mathrm{d}x = \int_0^\infty \varphi(x)\, f_\infty(t, \mathrm{d}x)$$

in $C^0([0,T])$ for any test function $\varphi \in C_0(\mathbb{R}^+)$. This means that $f_n$ converges to $f_\infty$ in $C^0([0,T]; \mathscr{M}^1(\mathbb{R}^+)\text{-}weak\text{-}*)$. Actually, we will use that this last statement implies

$$\lim_{n\to\infty} \int_0^\infty \int_0^\infty \zeta(t)\, \varphi(x)\, f_n(t,x)\, \mathrm{d}t\, \mathrm{d}x = \int_0^\infty \int_0^\infty \zeta(t)\, \varphi(x)\, f_\infty(t, \mathrm{d}x)\, \mathrm{d}t \tag{16}$$

for any $\zeta \in C_c^0(\mathbb{R}^+)$ and $\varphi \in C_0(\mathbb{R}^+)$.

Next, with the estimate on the first moment of the $f_n$'s in (14), we can check that for almost all $t \in \mathbb{R}^+$, the non negative function $x$ is integrable for the measure $f_\infty(t, \mathrm{d}x)$ and

$$\sup_{t\in\mathbb{R}^+} \int_0^\infty x f_\infty(t, \mathrm{d}x) \leqslant C$$

(consider functions $\varphi$ with compact support approaching $x$ pointwise...). As a consequence, we can enlarge the space of available test functions in (16); precisely, we shall use (16) with $\zeta \in C_c^0(\mathbb{R}^+)$ and continuous functions $\varphi$ such that

$$\lim_{x\to\infty} \frac{\varphi(x)}{x} = 0$$

Having disposed of these preliminaries, we aim at describing the possible limit point of $(c_n, f_n)$.

From now on, let us strenghten the condition (11) by assuming

$$\lim_{x\to\infty} a(x)/x, \qquad b(x)/x = 0 \tag{17}$$

This condition is fulfilled for (7), or (2). For the entropy functional, we assume that the strictly increasing and concave function $k$ verifies

$$\lim_{x\to\infty} \frac{k(x)}{x} = 0, \qquad \lim_{x\to\infty} k'(x)\, \frac{a(x)+b(x)}{x} = 0 \tag{18}$$

Note that, by (17), the second requirement holds since blow up of $k'$ may occur only at $x = 0$.

**Proposition 2.** The limit $c_\infty$ does not depend on time and

$$f_\infty(t,x) = M_{0,\infty}\, \delta_{x = x_{c_\infty}}$$

where $x_{c_\infty} = q^{-1}(c_\infty)$ and $M_{0,\infty} = \inf\{M_0(t),\ t \geqslant 0\}$. In particular, if $c_\infty = 0$ then $M_{0,\infty} = 0$.

The drawback of this result is the fact that the limit $c_\infty$ depends on the sequence $t_n$ and is not uniquely defined. However, if the initial data has an infinite support, a little bit more can be said on the behavior of $c(t)$.

**Corollary 2.** If the support of the initial data $f_0$ is infinite, then, we have

$$\lim_{t\to\infty} c(t) = 0 \qquad \lim_{t\to+\infty} \|f(t,\cdot)\|_{L^1} = 0$$

Next, in restricted but physically relevant situations we are able to provide information on the asymptotic behavior of the solution. In particular, the monomers concentration $c(t)$ is proven to go to 0 for a wide class of initial data. A part of this result will use the characteristics framework which supposes, to be absolutely rigorous, the regularity (7). Maybe the assumptions on $a$ and $b$ can be slightly weakened, but we prefer to do not insist on these purely technical difficulties which do not belong to the scope of the paper.

**Theorem 2.** Assume that $b(x) = b > 0$ and $a$ is non decreasing. We assume that $\Phi(f_0) \in L^1(\mathbb{R}^+)$ for some convex function $\Phi$: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying

$$\Phi(0) = 0, \qquad \lim_{s\to+\infty} \frac{\Phi(s)}{s} = +\infty$$

Then, as $t$ goes to $\infty$, one has $\|f(t,\cdot)\|_{L^1} \to 0$ (i.e., the total number of clusters $M_0(t)$ goes to 0) while the monomers concentration either tends to 0 or $\rho$. In particular, if there exists $\delta > 0$ such that $\operatorname{supp}(f_0) \cap [x_{c_0} + \delta, \infty[\ \neq \emptyset$, then $c(t) \to 0$.

**Remark 3.** Assumptions in this theorem are not as restrictive as they seem. Indeed, the assumptions on the coefficients are satisfied by (2). On the other hand, $s^p$, $p > 1$ or $s[\ln(s)]_+$ are examples of suitable functions $\Phi$ (sometimes referred to as Nagumo's functions). Actually, in view of the De La Vallée criterion (see ref. 20), there is no restriction at all on the initial data: for any $f_0 \in L^1(\mathbb{R}^+)$ we can find a suitable function $\Phi$, which fulfills all the conditions of Theorem 2.

**Remark 4.** If the initial data $f_0$ has an unbounded support, as certainly considered in ref. 1, then $c(t)$ tends to 0. If $\operatorname{supp}(f_0)$ belongs to

$[0, x_{c_0}]$, the two behaviors are certainly possible, depending on the initial repartition of the mass. Note finally that we are not able to prove the monotonicity of $c$, neither to obtain a sharp rate of convergence.

*Proof of Proposition 2.* The proof falls into two steps. The former uses the dissipation of the entropy, the latter uses the decrease of the zeroth order moment.

*Step 1: Entropy.* Since $t \longmapsto H(t) \geqslant 0$ is non increasing, it has a limit

$$H_\infty = \inf\{H(t), t \geqslant 0\} \geqslant 0$$

Therefore $H_n(t) = H(t+t_n) \to H_\infty$ as $n \to \infty$, the limit being independent on $t$. On the other hand, by using the uniform convergence of $c_n$, (18) and (16), we can pass to the limit in each term of $H_n$, at least in the $\mathscr{D}'((0, T))$ sense. We get

$$H_\infty = \int_0^\infty k(x)\, f_\infty(t, \mathrm{d}x) + K(c_\infty(t))$$

Similarly, the time derivative also passes to the limit (in $\mathscr{D}'((0, T))$). Let us set

$$w_n(t, x) = [k'(x_{c_n(t)}) - k'(x)]\, V_n(t, x) \geqslant 0,$$
$$w_\infty(t, x) = [k'(x_{c_\infty(t)}) - k'(x)]\, V_\infty(t, x) \geqslant 0$$

Consider first the case with $k'$ bounded. Then, one has

$$\frac{\mathrm{d}}{\mathrm{d}t} H_\infty = 0 = -\lim_{n \to \infty} \int_0^\infty w_n(t, x)\, f_n(t, x)\, \mathrm{d}x = -\int_0^\infty w_\infty(t, x)\, f_\infty(t, \mathrm{d}x)$$

in $\mathscr{D}'((0, T))$. Indeed, $w_n(t, x)$ depends on $n$ only through $c_n$. Then, we check that $w_n$ converges to $w_\infty(t, x)$, uniformly on $[0, T] \times [0, R]$ for any $0 < R < \infty$ while (17) and (18) provide a control on the large $x$'s. We deduce that the product with $f_n$ passes to the limit. Precisely, we write

$$\left| \int_0^\infty \int_0^\infty \zeta(t)\, w_n(t, x)\, f_n(t, x)\, \mathrm{d}x\, \mathrm{d}t - \int_0^\infty \int_0^\infty \zeta(t)\, w_\infty(t, x)\, f_\infty(t, \mathrm{d}x)\, dt \right|$$
$$\leqslant \int_0^\infty \int_0^\infty |\zeta(t)|\, |w_n(t, x)(x) - w_\infty(t, x)| f_n(t, x)\, \mathrm{d}x\, \mathrm{d}t$$
$$+ \left| \int_0^\infty \int_0^\infty \zeta(t)\, w_\infty(t, x)\, f_n(t, x)\, \mathrm{d}x\, \mathrm{d}t - \int_0^\infty \int_0^\infty \zeta(t)\, w_\infty(t, x)\, f_\infty(t, \mathrm{d}x)\, dt \right|$$

The first term is made small by using the uniform convergence of $w_n$ and the estimate on the moments of $f_n$. The second term is dealt with by using (16).

One deduces that for almost all time $t \geqslant 0$, the integral $\int w_\infty(t,x) \times f_\infty(t,\mathrm{d}x)$ vanishes. We can interpret $w_\infty(t,x)\, f_\infty(t,\mathrm{d}x)$ as a measure on $\mathbb{R}^+$ (it is a non negative distribution), hence this implies that $w_\infty(t,x)\, f_\infty(t,\mathrm{d}x)$ vanishes in $\mathscr{M}^1(\mathbb{R}^+)$ for almost all time. But $w_\infty(t,x)>0$ for $x \neq x_{c(t)}$, so that $\mathrm{supp}(f_\infty(t,\mathrm{d}x)) \subset \{x_{c_\infty(t)}\}$. Using again that $f_\infty(t,\cdot)$ is a zeroth order distribution, we get

$$f_\infty = d(t)\, \delta_{x = x_{c_\infty(t)}}$$

with the convention $d(t)=0$ if $c_\infty(t)=0$ (it means that the critical size is infinite).

A similar conclusion holds when $k'$ is unbounded. Indeed, we have seen that

$$\left\langle \frac{dH_n}{dt}, \zeta \right\rangle_{\mathscr{D}',\mathscr{D}} + \int_0^\infty \int_0^\infty w_n(t,x)\, f_n(t,x)\, \zeta(t)\, \mathrm{d}x\, \mathrm{d}t \leqslant 0$$

holds for any non negative test function $\zeta \in C_c^\infty(\mathbb{R}^+)$. Then, letting $n$ go to $\infty$, we verify that

$$\left\langle \frac{dH_\infty}{dt}, \zeta \right\rangle_{\mathscr{D}',\mathscr{D}} + \int_0^\infty \int_0^\infty w_\infty(t,x)\, \zeta(t)\, f_\infty(t,\mathrm{d}x)\, \mathrm{d}t \leqslant 0$$

However, $\frac{dH_\infty}{dt}=0$, therefore one has for any non negative $\zeta \in C_c^\infty(\mathbb{R}^+)$

$$\int_0^\infty \zeta(t) \left( \int_0^\infty w_\infty(t,x)\, f_\infty(t,\mathrm{d}x) \right) \mathrm{d}t \leqslant 0$$

while $w_\infty(t,x)\, f_\infty(t,\mathrm{d}x)$ is a non negative distribution. It follows that, as in the bounded case, $w_\infty(t,x)\, f_\infty(t,\mathrm{d}x)$ vanishes for almost all time.

We end this step by letting $n \to \infty$ in (15); it gives

$$\partial_t f_\infty + \partial_x (V_\infty(t,x)\, f_\infty) = 0 = \partial_t f_\infty \tag{19}$$

in the sense of distributions on $\mathbb{R}^+ \times \mathbb{R}^+$ (we treat the product $V_n f_n$ as $w_n f_n$ before, with (17), (16)). The second equality uses the fact that $V_\infty(t,x)$ vanishes on the support of $f_\infty$.

*Step 2: Zeroth Order Moment.* Since $t \longmapsto M_0(t) \geqslant 0$ is non increasing, it has a limit $M_{0,\infty} \geqslant 0$. We are thus led to

$$\lim_{n \to \infty} \int_0^\infty f_n(t, x)\, \mathrm{d}x = M_{0,\infty} = \int_0^\infty f_\infty(t, \mathrm{d}x) = d(t)$$

which proves that $d(t)$ does not depend on time. Hence, (19) becomes

$$M_{0,\infty} \partial_t (\delta_{x = x_{c_\infty(t)}}) = 0$$

This actually means that $c_\infty$ does not depend on time and completes the proof of Proposition 2. ∎

**Remark 5.** We have

$$\lim_{n \to \infty} H(t + t_n) = H_\infty = M_{0,\infty} k(x_\infty) + K(c_\infty)$$

However, it is not obvious at all that this relation determines uniquely the limit $c_\infty$ which may depend on the subsequence $t_n$.

*Proof of Corollary 2.* We split the proof into two steps. In the first step we show the result up to a subsequence and in the second step we show the convergence for the entire family.

*Step 1: Convergence Up to a Subsequence.* Suppose there exists $R > 0$ such that for all $t > 0$, $x_{c(t)} = q^{-1}(c(t)) < R$. Then $V(t, x) \geqslant 0$ for $x \geqslant R$. Consider a non decreasing function $\phi$ such that $\phi(x) \geqslant 0$, $\phi'$ is bounded, $\mathrm{supp}(\phi) \subset \,]\, R', +\infty[$ and $\phi(x) = 1$ for $x \geqslant 2R'$ with $R' > R$. Then for all $t > 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty \phi(x)\, f_n(t, x)\, \mathrm{d}x = \int_0^\infty \phi'(x)\, V(t, x)\, f_n(t, x)\, \mathrm{d}x \geqslant 0$$

It follows that

$$\int_{R'}^\infty f_n(t, x)\, \mathrm{d}x \geqslant \int_0^\infty \phi(x)\, f_n(t, x)\, \mathrm{d}x \geqslant \int_{2R'}^\infty f_0(x)\, \mathrm{d}x$$

holds. Since $f_0$ has a infinite support, the right hand side is larger than some $\eta > 0$. This would contradict the convergence $f_n \rightharpoonup M_{0,\infty} \delta_{x = x_\infty}$ with $x_\infty < R'$ which implies

$$\lim_{n \to \infty} \int_{R'}^\infty f_n(t, x)\, \mathrm{d}x = 0$$

One deduces the existence of a subsequence $t_n$ such that $x_{c(t_n)}$ goes to infinity.

*Step 2: Convergence for the Entire Familly.* Thanks to Proposition 2, Step 1 implies $M_{0,\infty}=0$ so

$$\lim_{t\to\infty} \|f(t,\cdot)\|_{L^1}=0$$

Since for every $R>0$:

$$\int_0^\infty k(x)\, f(t,x)\,\mathrm{d}x=\int_0^R k(x)\, f(t,x)\,\mathrm{d}x+\int_R^\infty \frac{k(x)}{x}\, xf(t,x)\,\mathrm{d}x$$
$$\leqslant k(R)\int_0^\infty f(t,x)\,\mathrm{d}x+\frac{k(R)}{R}\int_R^\infty xf(t,x)\,\mathrm{d}x$$

we have for every $R>0$:

$$\limsup_{t\to\infty}\int_0^\infty k(x)\, f(t,x)\,\mathrm{d}x\leqslant\frac{k(R)}{R}\,\rho$$

Thus, by using (18), one deduces that $\int_0^\infty k(x)\, f(t,x)\,\mathrm{d}x$ converges to 0 when $t$ tends to infinity. Consequently,

$$H_\infty=\lim_{t\to\infty} K(c(t))=\lim_{n\to\infty} K(c(t_n))=0$$

We conclude that the function $c(t)$ converges to 0 when $t$ goes to infinity. ∎

*Proof of Theorem 2.* We split the proof into 3 steps.

*Step 1: Limit Point of $f_n$.* We first show that the assumptions of Theorem 2 imply that the limit point of $f_n$ reduces to 0 (and thus does not depend on the sequence $t_n$). To this end, we renormalize the equation in the spirit of ref. 21 (this idea has been used also in ref. 6). Using that $V(t,x)$ lies in $W^{1,\infty}_{\mathrm{loc}}((0,\infty))$, we multiply (15) by $\Phi'(f_n)$ and we find (in $\mathscr{D}'((0,\infty))$)

$$\partial_t\Phi(f_n)+\partial_x[V(t,x)\,\Phi(f_n)]=[\Phi(f_n)-\Phi'(f_n)\,f_n]\,\partial_x V(t,x)$$

But $\Phi$ is convex and $\Phi(0)=0$ so that

$$\Phi(f_n)\leqslant\Phi'(f_n)\,f_n$$

and

$$\partial_x V(t, x) = c(t)\, a'(x) - b'(x) = c(t)\, a'(x) \geqslant 0$$

Hence, integrating yields the estimate (up to a classical approximation argument)

$$\int_0^\infty \Phi(f_n(t, x))\, dx \leqslant \int_0^\infty \Phi(f_0(x))\, dx$$

Therefore

$$(1+x)\, f_n + \Phi(f_n) \text{ is bounded in } L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^+))$$

It is a standard fact, related to the Dunford–Pettis criterion, see ref. 22, that it implies the relative weak compactness of $f_n$ in $L^1((0, T) \times \mathbb{R}^+)$. In turn, $f_n$ cannot converge to a Dirac mass $M_{0,\infty} \delta_{x=x_\infty} \dots$ except if $M_{0,\infty} = 0$!

*Step 2: Convergence of the Monomers Concentration.* Since the limit point of $f_n$ is 0, independently on the sequence $t_n$, one deduces that

$$f(t, \cdot) \rightharpoonup 0 \quad \text{in } L^1(\mathbb{R}+)$$

and, obviously, $M_0(t) \to 0$ as well as

$$\lim_{t \to 0} \int_0^\infty k(x)\, f(t, x)\, dx = 0$$

However, $H(t) \to H_\infty$, and it follows that

$$H_\infty = \lim_{t \to \infty} K(c(t))$$

Besides, $K$ is strictly monotonous, so that $c(t)$ has a limit $\alpha \in [0, \rho]$.

*Step 3: Support Property.* From now on we assume the regularity of the coefficients (7). This step starts with some interesting remarks on the support of the solution which can be easily deduced from (10). Let us set

$$x_M = \sup\{x \geqslant 0, x \in \operatorname{supp}(f_0)\}$$

If $x_M = +\infty$ the support of the solution is also infinite, while if $x_M$ is finite, we have $\operatorname{supp}(f(t, \cdot)) \subset [0, X(t; 0, x_M)]$. Furthermore, the critical point cannot reach the endpoint of the support in finite time, if it starts from its left, as shown in the following lemma which will be proved later on.

**Lemma 1.** If $x_{c_0} = q^{-1}(c_0) < x_M$, then, for all $t > 0$, one has $x_{c(t)} < X(t; 0, x_M)$.

Assume $\alpha > 0$. We will show that in this case $\alpha = \rho$ and the support of the solution should remain in a compact set independent of time. Indeed, $x_{c(t)} = q^{-1}(c(t))$ is bounded, say by $R > 0$. Therefore, (13) and (9) imply that $M_{0,R}: t \longmapsto \int_R^\infty f(t, x)\, \mathrm{d}x$ is non decreasing. However, it is dominated by $M_0(t)$ which decreases to 0. We conclude that $\mathrm{supp}(f(t, \cdot)) \subset [0, R]$ for $t \geqslant 0$. In turn, we are led to

$$\lim_{t \to \infty} M_1(t) = \lim_{t \to \infty} \int_0^R x f(t, x)\, \mathrm{d}x = 0, \qquad \lim_{t \to \infty} c(t) = \rho$$

In particular if $\mathrm{supp}(f_0)$ is unbounded, $\mathrm{supp}(f(t, \cdot))$ is never bounded, which thus yields $\lim_{t \to \infty} c(t) = 0$.

Now, consider the case $\mathrm{supp}(f_0) \cap [x_{c_0} + \delta, \infty[ \neq \emptyset$, for some $\delta > 0$ with $x_M = \sup\{x \in \mathrm{supp}(f_0)\} < \infty$. Suppose that $X(t; 0, x_M)$ is bounded. Then by Lemma 1, it increases to a finite limit *R*, so that the solution $f(t, x)$ is supported in $[0, R]$. Then, the previous step shows that $\alpha = \rho$. However, passing to the limit in (8), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} X(+\infty; 0, x_M) = a(R)(\rho - q(R)) = 0$$

which contradicts the inequalities $R > x_M > x_{c_0} = q^{-1}(c_0) > q^{-1}(\rho)$. We conclude that the support of $f$ cannot be bounded independently of time; in turn this gives $\alpha = \lim_{t \to \infty} c(t) = 0$. ▮

*Proof of Lemma 1.* At $t = 0$ the characteristic curve $X(t; 0, x_M)$ issued from $x_M$ is strictly increasing. Assume there exists $t_* > 0$ such that $x_{c(t_*)} = X(t_*; 0, x_M)$ and $x_{c(t)} < X(t; 0, x_M)$ for $t < t_*$. Then (12) says that

$$\frac{\mathrm{d}c}{\mathrm{d}t}(t_*) = -\int_0^{x_{c(t_*)}} V(t_*, x)\, f(t_*, x)\, \mathrm{d}x > 0$$

Indeed, $V(t_*, x) < 0$ a.e. in the domain of integration, while $f$ cannot be identically 0 (otherwise we would have $c(t_*) = \rho = q(X(t_*; 0, x_M))$ by the mass conservation which is incompatible with $X(t_*; 0, x_M) > x_M > x_{c_0} = q^{-1}(c_0)$). Hence $\frac{\mathrm{d}x_{c(t_*)}}{\mathrm{d}t} < 0$, which leads to a contradiction. ▮

## REFERENCES

1. I. M. Lifshitz and V. V. Slyozov, The kinetics of precipitation from supersaturated solid solutions, *J. Phys. Chem. Solids* **19**:35–50 (1961).

2. V. V. Sagalovich and V. V. Slyozov, Diffusive decomposition of solid solutions, *Sov. Phys. Usp.* **30**:23–44 (1987).
3. I. M. Lifshitz and L. Pitaevski, *Cinétique Physique, Coll. Physique Théorique*, Landau–Lifshitz (Mir, 1990).
4. B. Niethammer and R. Pego, On the initial-value problem in the Lifshitz–Slyozov–Wagner theory of Ostwald ripening, *SIAM J. Math. Anal.* **31***:467–485 (2000)*.
5. J. F. Collet and T. Goudon, On solutions of the Lifshitz–Slyozov model, *Nonlinearity* **13**:1239–1262 (2000).
6. P. Laurençot, Weak solutions to the Lifshitz–Slyozov–Wagner equation, *Indiana Univ. Math. J.* **50**:1319–1346 (2001).
7. J. F. Collet and T. Goudon, Lifshitz–Slyozov equations: The model with encounters, *Transp. Theory Stat. Phys.* **28**:545–573 (1999).
8. P. Laurençot, The Lifshitz–Slyozov equation with encounters, *Math. Models Methods Appl. Sci.* **11**:731–748 (2001).
9. J. F. Collet and S. Hariz, A modified version of the Lifshitz–Slyozov model, *Appl. Math. Letters* **12**:81–85 (1999).
10. S. Hariz, U*ne version modifiée du modèle de Lifshitz–Slyozov: existence, unicité de la solution, simulation numérique*, Thesis, Université de Nice-Sophia Antipolis, Juin 1999.
11. O. Penrose, The Becker–Döring equations at large times and their connection with the LSW theory of coarsening, *J. Stat. Phys.* **89**:305–320 (1997).
12. J. F. Collet, T. Goudon, F. Poupaud, and A. Vasseur, The Becker–Döring system and its Lifshitz–Slyozov limit, to appear in SIAM *J. Appl. Math.*
13. J. F. Collet, T. Goudon, S. Hariz, F. Poupaud, and A. Vasseur, Some recent results on the kinetic theory of phase transitions, to appear in *IMA Proceedings*.
14. B. Niethammer and R. Pego, Non-self-similar behavior in the LSW theory of Ostwald ripening, *J. Stat. Phys.* **95**:867–902 (1999).
15. B. Niethammer and R. Pego, The LSW model for domain coarsening: asymptotic behavior for conserved total mass, *J. Stat. Phys.* **104**:1113–1144 (2001).
16. L. Brown, A new examination of classical coarsening theory, *Acta Metall.* **37**:71–77 (1989).
17. B. Meerson and P. Sasorov, Domain stability, competition, growth, and selection in globally constrained bistable systems, *Phys. Rev. E* **53**:3491–3494 (1996).
18. J. Carr and O. Penrose, Asymptotic behavior of solutions to a simplified Lifshitz–Slyozov equation, *Physica D* **124**:166–176 (1998).
19. J. A. Carrillo and T. Goudon, A numerical study on large-time asymptotics of the Lifshitz–Slyozov system, Rapport INRIA n. 4287 (2001).
20. C. Dellacherie and P. A. Meyer, *Probabilités et potentiel*, Chapitres I–IV. Publications de l'Institut de Mathématique de l'Université de Strasbourg (Hermann, 1975).
21. R. Di Perna and P. L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, *Invent. Math.* **98**:511–547 (1989).
22. R. Edwards, Functional analysis, *Theory and Applications* (Dover, 1994).


File: KAPP/822-joss/108_1-2 373236(19p) - Page : 19/19 - Op: SD - Time: 10:18 - Date: 04:04:2002