A finite volume scheme on staggered grids for the Euler equations: unstructured meshes, stability analysis and energy conservation

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Abstract

We set up a numerical strategy for the simulation of the Euler equations, in the framework of finite volume staggered discretizations where numerical densities, energies and velocities are stored on different locations. The main difficulty relies on the treatment of the total energy, which mixes quantities stored on different grids. The proposed method is strongly inspired, on the one hand, from the kinetic framework for the definition of the numerical fluxes, and, on the other hand, from the Discrete Duality Finite Volume (DDFV) framework, which has been designed for the simulation of elliptic equations on complex meshes. We exhibit stability conditions that guaranty the positivity of the discrete densities and internal energies. Moreover, while the scheme works on the internal energy equation, we can define a discrete total energy which satisfies a *local* conservation equation. We provide a set of numerical simulations to illustrate the behaviour of the scheme.

Keywords: Euler equations. Staggered grids. Discrete Duality Finite Volume.

Contents

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1 Introduction

This work is concerned with the simulation of the Euler system of gas dynamics

$$\begin{cases} \partial_t\rho + \boldsymbol{\nabla}\cdot\big(\rho\mathbf{u}\big) = 0, \\ \partial_t\big(\rho\mathbf{u}\big) + \boldsymbol{\nabla}\cdot\big(\rho\mathbf{u}\otimes\mathbf{u}\big) + \boldsymbol{\nabla}p = 0, \\ \partial_t\big(\rho E\big) + \boldsymbol{\nabla}\cdot\big(\rho E\mathbf{u}\big) + \boldsymbol{\nabla}\cdot\big(p\mathbf{u}\big) = 0. \end{cases} \tag{1}$$

The unknowns depend on the time and space variables $(t, x) \in [0,\infty)\times\Omega$ with $\Omega\subset\mathbb{R}^2$, a regular and bounded domain. In (1), ρ, $\mathbf{u}$, E and p stand for the mass density, the velocity field, the total energy and the pressure respectively. The pressure is related to the independent unknowns $(\rho, \mathbf{u}, E)$ through an equation of state that depends of the adiabatic exponent $\gamma > 1$; in what follows we set

$$E = \frac{\|\mathbf{u}\|^2}{2} + e \text{ and } p = (\gamma-1)\rho e,$$

where e is the internal energy.

The originality of our approach is to consider staggered grids, which means that the numerical unknowns are stored at different locations of the mesh. Consequently, in contrast to the usual approaches for the Euler equations, (1) is not treated as a system for the conserved quantities $(\rho, \rho\mathbf{u}, \rho E)$, but instead each equation is considered “independently” on its own grid. Therefore, the scheme is not based on resolution of local Riemann problems, but it uses only a relevant (and simple) notion of numerical fluxes, and the upwinding principles. This viewpoint is particularly motivated by the will to address simulations of complex models for mixtures which include an additional constraint on the velocity field, or the simulation of flows in low Mach regimes. It is indeed known that standard colocalized methods might lead to severe stability constraints, or simply fail in capturing the correct solutions in these regimes, see [9, 10, 17, 18]. These issues are beyond the scope of the present paper: here, we focus on the preliminary step which consists in designing an efficient scheme for the Euler system on staggered grids.

This contribution takes place in a series of works, which started in [3] where the method is introduced for the barotropic Euler equation in one-dimension. In particular, the proposed scheme introduced numerical fluxes strongly inspired from the framework of kinetic schemes. The scheme is shown to be stable, under suitable CFL-conditions, it preserves the entropy-structure and the consistency analysis *à la* Lax-Wendroff can be performed too [2]. This approach is further developped to handle complex systems for mixture flows [4]. The method has been adapted to deal with multi-dimensional equations, when working on Cartesian grids in [16]. It also includes MUSCL strategies that make the scheme second order accurate (for smooth solutions), for both the barotropic and the full Euler system. When considering the full Euler equations, we face the difficulty that the total energy mixes up quantities, typically densities/energies vs. velocities, naturally defined on different grids. It leads to work with the internal energy equation, at the price of taking into account appropriately the kinetic energy balance, as in [20, 22]. Restricting to the barotropic case, the scheme has been adapted, still on Cartesian grids, to set up an asymptotic preserving method in the low Mach regimes [15]. In such a situation, the asymptotic regime leads to an incompressible limit. Accordingly the limit scheme correctly handles the incompressibility constraint: owing to the staggered strategy, the scheme has enforced stability/consistency properties, reminiscient of the MAC approach, in the spirit of the pioneering work of F. H. Harlow and J. E. Welch [19]. However, these versions of the scheme use strongly the Cartesian geometry of the grid. Here, we address the question of the design of the method on unstructured meshes, for the full Euler system. The main difficulty consists in finding consistent transfer procedures between the different grids, and a relevant approach for the energy equation, which cannot be treated by mere averages, due to the complex geometry of the mesh.

To this end, and motivated by the treatment of incompressibility constraints and low Mach regimes, the adaptation we are going to discuss on unstructured grids is strongly inspired by the Discrete Duality Finite Volume (DDFV) framework. The DDFV framework has been introduced in the 2000s in [11, 23] to solve the Laplace equation on general 2D meshes, including non-conformal meshes, and, more generally, to numerically deal with elliptic operators $\nabla \cdot (A\nabla \mathbf{u})$, $x \mapsto A(x)$ being a matrix valued function. In the Finite Volume approach, one has to define numerical fluxes $A\nabla \mathbf{u} \cdot \mathbf{n}$ on the interfaces of the control volumes, and finding a relevant formula that uses only unknowns stored at the center of the control volumes is not possible without severe restrictions on the mesh geometry[1]. The DDFV approach has been extended to the Stokes equations in [6, 7, 8, 25] and to the Navier-Stokes equations in [12, 13, 26, 29]. The main idea of the DDFV method is to introduce additional unknowns so that full gradients can be reconstructed, and to mimic at the discrete level the duality formula involving differential operators we are used to for continuous quantities. The Euler systems does not involve any elliptic operator, and it only involves first order derivatives. Nevertheless, the staggered scheme we propose gets its inspiration from the DDFV approach designed in [12] to solve the non homogeneous Navier-Stokes equations. We shall use ideas from [12] that consists in duplicating variables, together with a suitable treatment of the convection terms in order to restore the consistency for the equations on the primal and the dual meshes. We do not address this issue here, but we expect this approach to be well-adapted to handle low Mach regimes on unstructured meshes. Moreover, the duplication of variables can also open perspectives to reconstruct gradients and to design a second-order version of the scheme, in the spirit of [5] where a MUSCL scheme for the Euler equation is constructed based on DDFV principles.

The paper is organized as follows. In Section 2 we set up a few useful notations and make the definition of the different mesh related quantities we are going to use precise. Section 3 details the construction of the scheme. Two ingredients are crucial. First, the definition of the mass fluxes, inspired from the kinetic framework, is very specific and induces several properties which play a central role in the stability-consistency analysis. Second, suitable footbridges should be introduced in order to transfer information from a grid to another. We pay attention to make such a transfer consistently, which relies on a general statement, as discussed in [14]. We shall equally discuss the stability analysis: under a suitable constraint on the time step, the scheme preserves the positivity of the mass density and of the internal energy. In Section 4, we turn to the balance of total energy. Precisely, we are able to provide a relevant definition of the discrete total energy and we justify that its time evolution satisfies a local conservation law. Finally, Section 5 validates the scheme by a series of numerical experiments.

2 Notation: meshes, unknowns

From now on, we suppose that Ω is an open bounded polygonal domain of $\mathbb{R}^2$ and its boundary is denoted $\partial\Omega$.

Meshes. The construction uses three meshes: the primal mesh, the dual mesh and the diamond mesh; the main steps of the construction are illustrated by Fig. 1.

[1] if x_K and x_L stand for centers of adjacent control volumes, we only get an approximation of ∇v in the direction of $[x_K, x_L]$, while we need a full gradient.

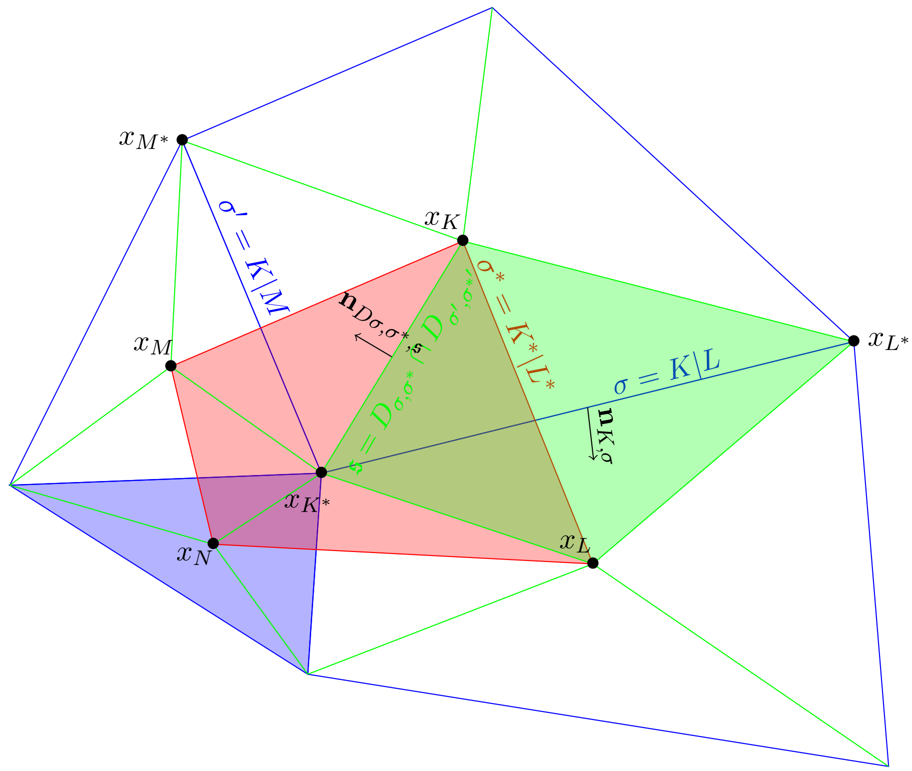
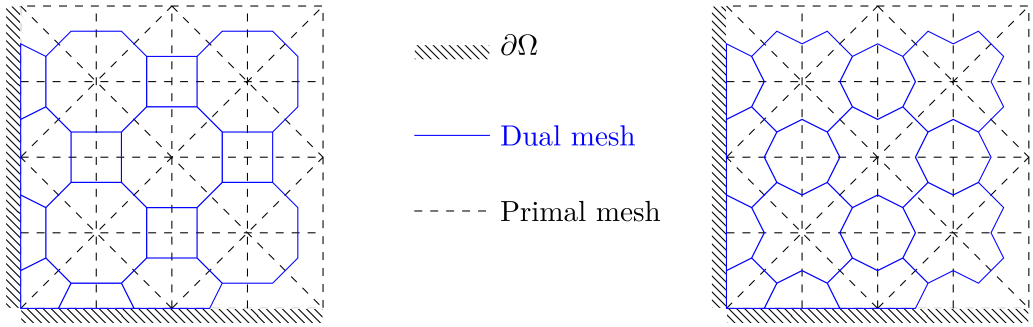


Figure 1: Meshes and associated notations.

- The primal mesh $\mathfrak{M}$ consists of disjoints, non-degenerate, convex polygons K called "primal cells". We associate to each cell K its barycenter x_K (see the blue cell in Fig. 1).
- The dual mesh $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$ is made of cells built around the vertices x_{K^*} of the primal mesh. We distinguish the interior dual mesh $\mathfrak{M}^*$, the vertices x_{K^*} of which do not belong to $\partial\Omega$, and the boundary of the dual mesh $\partial\mathfrak{M}^*$ for which $x_{K^*} \in \partial\Omega$. There are two options to construct the dual mesh, see Fig. 2:
  - ⓐ with the direct approach the interior dual mesh $\mathfrak{M}^*$ consists of cells K^*, built around the vertex $x_{K^*} \notin \partial\Omega$, by joining the centers x_K of all cells having K^* as a vertex (see the red cell in Fig. 1) The boundary dual mesh $\partial\mathfrak{M}^*$ is the set of cells K^* such that $x_{K^*} \in \partial\Omega$ and in this specific case, a dual cell is made by joining the centers of the cells that share the vertex x_{K^*} and the centers of the two boundary edges containing x_{K^*}.
  - ⓑ the barycentric mesh is obtained by joining the centers x_K to the midpoints of the edges of the primal mesh, [7, 8].

  Clearly, the barycentric dual mesh can have a much more complicated structure than the direct dual mesh, but it might have a practical interest in some circumstances.
- The diamond mesh $\mathfrak{D}$ is made of quadrilateral cells D_{σ,σ^*} obtained by joining the endpoints of the edges $\sigma = [x_{K^*}, x_{L^*}]$ of the primal mesh to the centers of the primal mesh that share this edge, which defines $\sigma^* = [x_K, x_L]$. We distinguish the diamonds of the boundary $\mathfrak{D}_{ext} = \{D_{\sigma,\sigma^*} \in \mathfrak{D}$ such that $\sigma \in \partial\Omega\}$ and $\mathfrak{D}_{int} = \mathfrak{D} \setminus \mathfrak{D}_{ext}$. In the specific case where $D_{\sigma,\sigma^*} \in \mathfrak{D}_{ext}$, the diamond cell D_{σ,σ^*} degenerates to a triangle.

In what follows, we assume



(a) Classic dual mesh

(b) Barycentric dual mesh

Figure 2: Structured triangular mesh

- either the direct dual mesh, case ⓐ, is such that all associated diamonds are convex. In such a case the diagonals $\sigma = [x_{K^*}, x_{L^*}]$, and $\sigma^* = [x_K, x_L]$, which are equally edges of the primal – resp. dual – mesh, are included in the diamonds. When referring to the direct mesh ⓐ, we always assume this convexity assumption.
- or we work with the barycentric dual mesh ⓑ. We refer the reader to Fig. 3 for an example where there are non convex diamond cells. In this situation, considering a diamond cell D_{σ,σ^*}, $\sigma = [x_{K^*}, x_{L^*}]$ is still included in the diamond, and there are two edges of the dual mesh, hereafter denoted σ_K^* and σ_L^*, which belong to this diamond, while possibly $\sigma^* = [x_K, x_L] \not\subset D_{\sigma,\sigma^*}$.

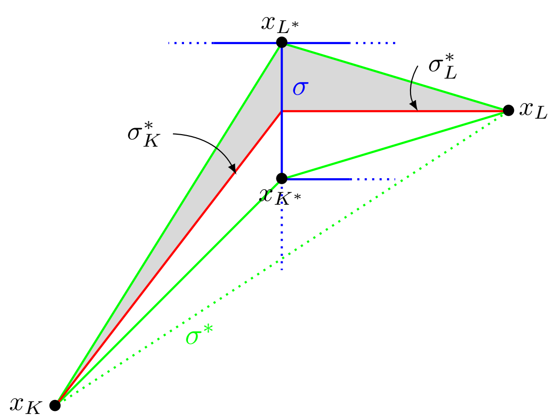


Figure 3: A case of non convex diamond cells: there are two edges of the dual mesh included in the diamond cell. The shaded area is $D_{\sigma,\sigma^*} \cap L^*$

Of course, the three meshes cover the computational domain:

$$\Omega = \bigcup_{K \in \mathfrak{M}} K = \bigcup_{K^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*} K^* = \bigcup_{D_{\sigma,\sigma^*} \in \mathfrak{D}} D_{\sigma,\sigma^*}.$$

The discretization is thus defined as a pair $(\mathfrak{T}, \mathfrak{D})$ where $\mathfrak{T} = \mathfrak{M} \cup \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ combines the primal mesh $\mathfrak{M}$ and the dual mesh $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$ and $\mathfrak{D}$ stands for the diamond mesh. Note that contrarily to standard DDFV notation, we do not introduce here the notation $\partial\mathfrak{M}$ for the set of edges of the primal mesh $\mathfrak{M}$ included in $\partial\Omega$, considered as degenerate cells.

Boundaries. For boundary conditions, we distinguish

- the zero-flux boundaries with nothing going out or going in at the interface between the domain and the outside,
- the outflow boundaries with no information coming from the outside of the domain,
- the Dirichlet boundaries where the variables are supposed to be known, equal to $(\rho_D, \mathbf{u}_D, e_D)$.

Notations. We refer the reader to Fig. 1 for the following notations.

- We denote $\sigma = K|L = [x_{K^*}, x_{L^*}]$ the edge separating two adjacent cells K and L of the primal mesh, and $\sigma^* = [x_K, x_L]$ is the segment that joins the centers of the cells K and L. For the direct mesh, under the convexity assumption, we have $\sigma^* = K^*|L^*$, the edge separating the adjacent cells K^* and L^* of the dual mesh.
- We denote $\mathfrak{s} = D_{\sigma,\sigma^*}|D_{\sigma',\sigma^{*\prime}}$ the face separating two diamond cells D_{σ,σ^*} and $D_{\sigma',\sigma^{*\prime}}$.
- For $K \in \mathfrak{M}$, we denote $\mathfrak{D}_K = \{D_{\sigma,\sigma^*} \in \mathfrak{D}, \sigma \in \partial K\}$. For $K^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$, we similarly denote $\mathfrak{D}_{K^*} = \{D_{\sigma,\sigma^*} \in \mathfrak{D}, \sigma^* \in \partial K^*\}$ if we work with the direct mesh. For the barycentric mesh, the definition becomes $\mathfrak{D}_{K^*} = \{D_{\sigma,\sigma^*} \in \mathfrak{D}, \partial K^* \cap D_{\sigma,\sigma^*} \neq \emptyset\}$.
- The area of a cell X of $\mathfrak{M}$, $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$ or $\mathfrak{D}$ is denoted $|X|$ and the length of an edge $\mathfrak{x}$ of type σ, σ^* or $\mathfrak{s}$ is denoted $|\mathfrak{x}|$.
- For a cell X of $\mathfrak{M}$, $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$ or $\mathfrak{D}$ and for $\mathfrak{x} \in \partial X$, we define a unit vector $\mathbf{n}_{X,\mathfrak{x}}$ normal to the face $\mathfrak{x}$ of the cell X and pointing outwards: $\mathbf{n}_{K,\sigma}$ (with $\sigma \in \partial K$ for $K \in \mathfrak{M}$), $\mathbf{n}_{K^*,\sigma^*_K}$ and $\mathbf{n}_{K^*,\sigma^*_L}$ (with $\sigma^*_K, \sigma^*_L \in \partial K^*$ for $K^* \in \mathfrak{M}^*$), and $\mathbf{n}_{D\sigma,\sigma^*,\mathfrak{s}}$ (with $\mathfrak{s} \in \partial D_{\sigma,\sigma^*} \cap D_{\sigma',\sigma^{*\prime}}$ for $D_{\sigma,\sigma^*} \in \mathfrak{D}$). Note that

$$\mathbf{n}_{K,\sigma} = -\mathbf{n}_{L,\sigma}, \quad \mathbf{n}_{K^*,\sigma^*_K} = -\mathbf{n}_{L^*,\sigma^*_K}, \quad \mathbf{n}_{K^*,\sigma^*_L} = -\mathbf{n}_{L^*,\sigma^*_L}, \quad \mathbf{n}_{D\sigma,\sigma^*,\mathfrak{s}} = -\mathbf{n}_{D\sigma',\sigma^{*\prime},\mathfrak{s}}.$$

- In order to analyze the preservation of the non negativity of the density and the internal energy, we need to introduce a positive number $\mathrm{reg}\,(\mathfrak{T})$ that measures the regularity of the mesh $\mathfrak{M}$:

$$\begin{aligned}\mathrm{reg}\,(\mathfrak{T}) = \sup\Bigg(&\Big\{\frac{|D_{\sigma,\sigma^*}|}{|D_{\sigma,\sigma^*} \cap X|},\ X \in \mathfrak{M} \cup \mathfrak{M}^* \cup \partial\mathfrak{M}^*, D_{\sigma,\sigma^*} \in \mathfrak{D}_X \cap \mathfrak{D}_{int}\Big\}\\ &\cup \Big\{\frac{|X|}{|D_{\sigma,\sigma^*}|},\ X \in \mathfrak{M} \cup \partial\mathfrak{M}^*, D_{\sigma,\sigma^*} \in \mathfrak{D}_X \cap \mathfrak{D}_{ext}\Big\}\Bigg) > 1.\end{aligned}$$

Unknowns.

- Density, internal energy and pressure are stored on the edges of the initial mesh: ρ_{σ,σ^*} and e_{σ,σ^*} are constant on the diamond cell $D_{\sigma,\sigma^*} \in \mathfrak{D}$ and we set $p_{\sigma,\sigma^*} = (\gamma - 1)\rho_{\sigma,\sigma^*} e_{\sigma,\sigma^*}$.
- Velocity fields are stored at both the centers and the vertices of the cell of the primal mesh: $\mathbf{u}_K$ is constant on the primal cell $K \in \mathfrak{M}$ and $\mathbf{u}_{K^*}$ is constant on the dual cell $K^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$.

Observe that, in contrast to the Cartesian framework studied in [16], we store all the components of the velocity on the centers and vertices of the primal meshes. The Cartesian case is less demanding in terms of storage since the geometry allows us to store only the horizontal or the vertical component at a given location, in the same fashion as the MAC discretization for incompressible flows.

It is finally convenient to introduce two further scalar quantities, related to the internal energy and the velocity, that are stored at the edges of the diamond mesh.

Definition 2.1. For $\mathfrak{s} = D_{\sigma,\sigma^*}|D_{\sigma',\sigma^{*\prime}}$, we denote

$$e_{\mathfrak{s}} := \frac{e_{\sigma,\sigma^*} + e_{\sigma',\sigma^{*\prime}}}{2}.$$

For $\mathfrak{s} = [x_K, x_{K^*}]$ an edge of D_{σ,σ^*}, we denote

$$u_{D\sigma,\sigma^*,\mathfrak{s}} := \frac{\mathbf{u}_K + \mathbf{u}_{K^*}}{2} \cdot \mathbf{n}_{D\sigma,\sigma^*,\mathfrak{s}}.$$

For $\sigma = [x_{K^*}, x_{L^*}]$ an edge of K such that $\sigma \subset \partial\Omega$, we denote

$$e_\sigma = \begin{cases} e_{\sigma,\sigma^*} & \text{if } \sigma \text{ is a zero-flux or an outflow boundary,} \\ e_D & \text{if } \sigma \text{ is a Dirichlet boundary,} \end{cases}$$

$$\mathbf{u}_\sigma = \begin{cases} 0 & \text{if } \sigma \text{ is a zero-flux boundary,} \\ \dfrac{\mathbf{u}_{K^*} + \mathbf{u}_{L^*}}{2} & \text{if } \sigma \text{ is an outflow boundary,} \\ \mathbf{u}_D & \text{if } \sigma \text{ is a Dirichlet boundary,} \end{cases}$$

and

$$u_\sigma := \mathbf{u}_\sigma \cdot \mathbf{n}_{K,\sigma}.$$

Note that

$$u_{D\sigma,\sigma^*,\mathfrak{s}} = -u_{D\sigma',\sigma^{*\prime},\mathfrak{s}} \quad \text{if} \quad \mathfrak{s} = D_{\sigma,\sigma^*}|D_{\sigma',\sigma^{*\prime}}.$$

Remark 2.2. This discretisation technique differs from the staggered approach developed in [20, 22]: dealing with grids made of quadrilaterals in dimension 2, in [20, 22] densities and pressures are stored at the center of the control volumes and, using ideas reminiscent to Rannacher-Turek or Crouzeix-Ravart finite element methods, velocities are stored at the center of the faces of the mesh. The corresponding scheme can be shown to conserve globally the total energy, and stability/consistency properties are further discussed in [20, 22]. While the method differs by many aspects (discretization, definition of the numerical mass and momentum fluxes), the analysis of our scheme is based on manipulations close to the proofs of [20, 22].

In what follows, we shall repeatedly use the following elementary claim.

Lemma 2.3. *Consider a triangle ABC. For a given vertex V, we denote $|v|$ the length of the edge v that does not contain V and $\mathbf{n}_V$ stands for the outward unit vector, see Fig. 4. The following equality holds:* $|a|\mathbf{n}_A + |b|\mathbf{n}_B + |c|\mathbf{n}_C = 0$.

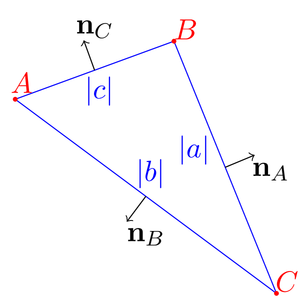


Figure 4: Triangle ABC

3 Definition of the scheme

For further purposes, we remind the reader that the sound speed of (1) is

$$c(e) = \sqrt{\gamma(\gamma-1)e},$$

which only depends on the internal energy e. This quantity naturally enters in the definition of the numerical fluxes since it is related to the speed of propagation of the information as driven by (1). Indeed, let us write (1) in the non-conservative form

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \boldsymbol{\nabla} \rho + \rho \boldsymbol{\nabla} \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + \rho^{-1} \boldsymbol{\nabla} p = 0, \\ \partial_t e + \mathbf{u} \cdot \boldsymbol{\nabla} e + \rho^{-1} p \boldsymbol{\nabla} \cdot \mathbf{u} = 0. \end{cases}$$

Denoting $U = (\rho, \mathbf{u}, e)$, it can be cast in matrix from as $\partial_t U + A(U, \boldsymbol{\nabla})U = 0$ with

$$A(U, \boldsymbol{\nabla}) = \begin{pmatrix} \mathbf{u} \cdot \boldsymbol{\nabla} & \rho \boldsymbol{\nabla} \cdot & 0 \\ \rho^{-1} \frac{\partial p}{\partial \rho}|_e \boldsymbol{\nabla} & \mathbf{u} \cdot \boldsymbol{\nabla} & \rho^{-1} \frac{\partial p}{\partial e}|_\rho \boldsymbol{\nabla} \\ 0 & \rho^{-1} p \boldsymbol{\nabla} \cdot & u \cdot \boldsymbol{\nabla} \end{pmatrix}.$$

Let $\xi \in \mathbb{R}^2$ with $\|\xi\| = 1$. Then, denoting $u = \mathbf{u} \cdot \xi$, $\lambda_-(u, c) = u - c$, u and $\lambda_+(u, c) = u + c$ are the eigenvalues of the matrix $A(U, \xi)$.

3.1 Mass conservation on the diamond cells

We start by defining the mass fluxes on the interfaces of the diamond cell. The construction of the fluxes proposed in [3] involves the following function, parametrized by $\rho \geqslant 0$, $c \geqslant 0$, $u \in \mathbb{R}$,

$$\xi \in \mathbb{R} \longmapsto \mathcal{M}_{[\rho,c,u]}(\xi) = \frac{\rho}{2c} \mathbb{1}_{|\xi - u| \leqslant c},$$

which has a compact support, precisely limited by the characteristic speeds of the Euler system. This function arises in the definition of kinetic schemes for solving the Euler system [24], and the support property plays a crucial role in the stability analysis of such schemes [31, 32]. The numerical fluxes are defined by using this function, accounting for both a direction of propagation and the characteristic speeds.

Definition 3.1. Let

$$\mathcal{F}^+(\rho, c, u) = \int_{\xi > 0} \xi \mathcal{M}_{[\rho,c,u]}(\xi) d\xi = \begin{cases} 0 & \text{if } u \leqslant -c, \\ \dfrac{\rho}{4c} \lambda_+(u, c)^2 & \text{if } |u| \leqslant c, \\ \rho u & \text{if } u \geqslant c, \end{cases} \tag{2}$$

and

$$\mathcal{F}^-(\rho, c, u) = \int_{\xi < 0} \xi \mathcal{M}_{[\rho,c,u]}(\xi) d\xi = \begin{cases} \rho u & \text{if } u \leqslant -c, \\ -\dfrac{\rho}{4c} \lambda_-(u, c)^2 & \text{if } |u| \leqslant c, \\ 0 & \text{if } u \geqslant c. \end{cases} \tag{3}$$

It is worth pointing out that, despite its “kinetic” flavor, the definition of the flux function has a very simple expression and does not need any numerical computation of integrals. The following properties are fundamental for analysing the scheme [2, 3, 4, 16].

Lemma 3.2. *The functions $\mathcal{F}^\pm$ satisfy the following properties*

- *symmetry :*

$$\mathcal{F}^-(\rho, c, u) = -\mathcal{F}^+(\rho, c, -u), \tag{4}$$

- *consistency :*
$$\mathcal{F}^+(\rho, c, u) + \mathcal{F}^-(\rho, c, u) = \rho u, \tag{5}$$
- *for any $u \in \mathbb{R}$, $\rho \geqslant 0$ and $c \geqslant 0$, we have*
$$0 \leqslant \mathcal{F}^+(\rho, c, u) \leqslant \rho[\lambda_+(c, u)]^+ \quad et \quad -\rho[\lambda_-(c, u)]^- \leqslant \mathcal{F}^-(\rho, c, u) \leqslant 0. \tag{6}$$

The scheme is next based on the upwinding principles applied to the expression

$$\mathcal{F}^\pm = \int_{\xi \lesseqgtr 0} \xi \mathcal{M}\, \mathrm{d}\xi.$$

Namely, given an interface, the mass flux associated to the positive (resp. negative) kinetic velocities ξ uses the backward (frontward) density. This definition significantly differs from the scheme introduced in [20, 21, 22] which is based instead on the material velocity only (and not on the characteristic speeds), in the spirit of AUSM schemes [28, 27]. It induces naturally some numerical diffusion which prevents the formation of oscillations when the material u velocity becomes small, see [3, Appendix B].

We thus define the mass flux $\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}}$ from the diamond cell D_{σ,σ^*} through the interface $\mathfrak{s} = D_{\sigma,\sigma^*}|D_{\sigma',\sigma^{*\prime}}$ as follows

$$\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}} = \mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}} + \mathcal{F}^-_{D\sigma,\sigma^*,\mathfrak{s}}$$

with

$$\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}} = \mathcal{F}^+(\rho_{\sigma,\sigma^*}, c(e_\mathfrak{s}), u_{D\sigma,\sigma^*,\mathfrak{s}}) \quad \text{and} \quad \mathcal{F}^-_{D\sigma,\sigma^*,\mathfrak{s}} = \mathcal{F}^-(\rho_{\sigma',\sigma^{*\prime}}, c(e_\mathfrak{s}), u_{D\sigma,\sigma^*,\mathfrak{s}}).$$

It uses the velocity and sound speed naturally given on the interface by Definition 2.1, and upwinds the density. The symmetry property (4) implies that

$$\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}} = -\mathcal{F}_{D\sigma',\sigma^{*\prime},\mathfrak{s}}$$

and thus $\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}}$ is a conservative flux. Moreover we have the following two equalities:

$$\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}} = -\mathcal{F}^-_{D\sigma',\sigma^{*\prime},\mathfrak{s}} \quad \text{and} \quad \mathcal{F}^-_{D\sigma,\sigma^*,\mathfrak{s}} = -\mathcal{F}^+_{D\sigma',\sigma^{*\prime},\mathfrak{s}}.$$

The discrete mass equation on a cell $D_{\sigma,\sigma^*} \in \mathfrak{D}_{int}$ is given by

$$\frac{\overline{\rho}_{\sigma,\sigma^*} - \rho_{\sigma,\sigma^*}}{\delta t} + \frac{1}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s} \in \partial D_{\sigma,\sigma^*}} |\mathfrak{s}| \mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}} = 0. \tag{7}$$

The definition needs to be slightly adapted at the boundary. For the diamond cell $D_{\sigma,\sigma^*} \in \mathfrak{D}_{ext}$, we have to define the outgoing mass flux $\mathcal{F}_\sigma = \mathcal{F}^+_\sigma + \mathcal{F}^-_\sigma$ through the boundary edge σ. Denoting K the primal cell whose σ is an edge, we adopt the following definition:

- $\mathcal{F}^+_\sigma = 0$ and $\mathcal{F}^-_\sigma = 0$ for Zero-flux conditions,
- $\mathcal{F}^+_\sigma = \mathcal{F}^+(\rho_{\sigma,\sigma^*}, c(e_\sigma), u_\sigma)$ and $\mathcal{F}^-_\sigma = 0$ for Outflow conditions,
- $\mathcal{F}^+_\sigma = \mathcal{F}^+(\rho_{\sigma,\sigma^*}, c(e_\sigma), u_\sigma)$ and $\mathcal{F}^-_\sigma = \mathcal{F}^-(\rho_D, c(e_\sigma), u_\sigma)$ for Dirichlet conditions.

The discrete mass equation on a cell $D_{\sigma,\sigma^*} \in \mathfrak{D}_{ext}$ is given by

$$\frac{\overline{\rho}_{\sigma,\sigma^*} - \rho_{\sigma,\sigma^*}}{\delta t} + \frac{1}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s} \in \partial D_{\sigma,\sigma^*} \setminus \partial\Omega} |\mathfrak{s}| \mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}} + \frac{|\sigma|}{|D_{\sigma,\sigma^*}|} \mathcal{F}_\sigma = 0. \tag{8}$$

3.2 Transfer lemma and mass conservation on primal and dual cells

In order to transfer the information from a grid to another, we shall make use of the following transfer lemma, extracted from [14].

Lemma 3.3. *Consider a cell* $\mathscr{C}$ *and for each edge* $\mathfrak{s}$ *of* $\mathscr{C}$*, with unit outward normal* $\mathbf{n}_{\mathscr{C},\mathfrak{s}}$*, we consider a given flux-type quantity* $\mathcal{X}_{\mathscr{C},\mathfrak{s}}$*. There exists a function* $\omega_{\mathscr{C}}$*, which is* H_{div} *such that*

$$\boldsymbol{\nabla} \cdot \omega_{\mathscr{C}} = \frac{1}{|\mathscr{C}|} \sum_{\mathfrak{s} \in \partial \mathscr{C}} |\mathfrak{s}| \mathcal{X}_{\mathscr{C},\mathfrak{s}} \tag{9}$$

and

$$\int_{\mathfrak{s}} \omega_{\mathscr{C}} \cdot \mathbf{n}_{\mathscr{C},\mathfrak{s}} = |\mathfrak{s}| \mathcal{X}_{\mathscr{C},\mathfrak{s}}. \tag{10}$$

This claim appeared in [12], in the specific case of quadrilateral and convex cells, as a key ingredient for designing convection fluxes for the incompressible Navier-Stokes equation with variable density; it has been generalized in [14], which, furthermore, provides practical procedures to compute the transfer formula. This statement will be used in different places. First, we need it in order to define the momentum fluxes to be used for updating the velocity. To this end, we apply Lemma 3.3 on the diamond cells $\mathscr{C} = D_{\sigma,\sigma^*} \in \mathfrak{D}$. This allows us to define numerical densities on the primal and dual meshes, together with numerical *conservative* mass fluxes, so that a discrete mass conservation holds on these meshes too. Second, we will work the other way around, with $\mathscr{C} = K$ or K^* in Section 4 in order to justify the local conservation of the total energy.

Let us explain how this works on the diamond cells $\mathscr{C} = D_{\sigma,\sigma^*} \in \mathfrak{D}$. We remind the reader that the edges $\mathfrak{s}$ of the diamond cell D_{σ,σ^*} are of the form $\mathfrak{s}_{XZ^*} = [x_X, x_{Z^*}]$ with $(X, Z^*) \in \{(K, K^*), (L, K^*), (L, L^*), (K, L^*)\}$, see Fig. 3 and 5, and at each edge is associated a mass flux $\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}}$.

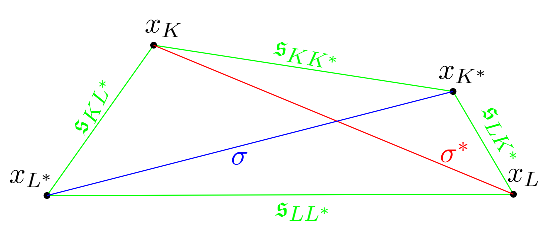


Figure 5: Diamond cell D_{σ,σ^*}.

Since the diagonal σ is an edge of the primal mesh, say for the cell K, we get a mass flux by setting

$$|\sigma| \mathcal{F}_{K,\sigma} = \int_\sigma \omega_{D_{\sigma,\sigma^*}} \cdot \mathbf{n}_{K,\sigma}. \tag{11}$$

This quantity has actually a simple expression by means of the original mass fluxes $\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}}$. Indeed, we have

$$\begin{aligned} |\sigma| \mathcal{F}_{K,\sigma} &= \int_{D_{\sigma,\sigma^*} \cap K} \boldsymbol{\nabla} \cdot \omega_{D_{\sigma,\sigma^*}} - \int_{\mathfrak{s}_{KK^*}} \omega_{D_{\sigma,\sigma^*}} \cdot \mathbf{n}_{\mathfrak{s}_{KK^*}} - \int_{\mathfrak{s}_{KL^*}} \omega_{D_{\sigma,\sigma^*}} \cdot \mathbf{n}_{\mathfrak{s}_{KL^*}} \\ &= \frac{|D_{\sigma,\sigma^*} \cap K|}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s} \in \partial D_{\sigma,\sigma^*}} |\mathfrak{s}| \mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}} - |\mathfrak{s}_{KK^*}| \mathcal{F}_{\sigma,\mathfrak{s}_{KK^*}} - |\mathfrak{s}_{KL^*}| \mathcal{F}_{\sigma,\mathfrak{s}_{KL^*}}. \end{aligned}$$

Using the obvious relation $|D| = |D \cap K| + |D \cap L|$, for the interface $\sigma = K|L$ we finally arrive at

$$\mathcal{F}_{K,\sigma} = \frac{|D_{\sigma,\sigma^*} \cap K|}{|D_{\sigma,\sigma^*}|} \sum_{\substack{\mathfrak{s} \in \partial D_{\sigma,\sigma^*} \\ \mathfrak{s} \subset L}} \frac{|\mathfrak{s}|}{|\sigma|} \mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}} - \frac{|D_{\sigma,\sigma^*} \cap L|}{|D_{\sigma,\sigma^*}|} \sum_{\substack{\mathfrak{s} \in \partial D_{\sigma,\sigma^*} \\ \mathfrak{s} \subset K}} \frac{|\mathfrak{s}|}{|\sigma|} \mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}}. \tag{12}$$

These mass fluxes will be used to obtain a mass conservation equation on the primal mesh, but they will also enter into the definition of the momentum fluxes. For this purpose, we need to split

$$\mathcal{F}_{K,\sigma} = \mathcal{F}^+_{K,\sigma} + \mathcal{F}^-_{K,\sigma}, \qquad \pm \mathcal{F}^{\pm}_{K,\sigma} \geqslant 0,$$

since we wish to apply upwinding principles. A naive attempt would consist in performing the same construction starting from the original decomposition $\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}} = \mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}} + \mathcal{F}^-_{D\sigma,\sigma^*,\mathfrak{s}}$; but, as it will be further detailed later on, there is no reason that the corresponding fluxes (11) on the interfaces of K and K^* preserve the sign property. Instead, we simply rearrange terms in (12)

$$\begin{aligned} \mathcal{F}_{K,\sigma} \quad = \quad & \underbrace{\frac{|D_{\sigma,\sigma^*} \cap K|}{|D_{\sigma,\sigma^*}|} \sum_{\substack{\mathfrak{s} \in \partial D_{\sigma,\sigma^*} \\ \mathfrak{s} \subset L}} \frac{|\mathfrak{s}|}{|\sigma|} \mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}} - \frac{|D_{\sigma,\sigma^*} \cap L|}{|D_{\sigma,\sigma^*}|} \sum_{\substack{\mathfrak{s} \in \partial D_{\sigma,\sigma^*} \\ \mathfrak{s} \subset K}} \frac{|\mathfrak{s}|}{|\sigma|} \mathcal{F}^-_{D\sigma,\sigma^*,\mathfrak{s}}}_{\geqslant 0} \\ & \underbrace{+ \frac{|D_{\sigma,\sigma^*} \cap K|}{|D_{\sigma,\sigma^*}|} \sum_{\substack{\mathfrak{s} \in \partial D_{\sigma,\sigma^*} \\ \mathfrak{s} \subset L}} \frac{|\mathfrak{s}|}{|\sigma|} \mathcal{F}^-_{D\sigma,\sigma^*,\mathfrak{s}} - \frac{|D_{\sigma,\sigma^*} \cap L|}{|D_{\sigma,\sigma^*}|} \sum_{\substack{\mathfrak{s} \in \partial D_{\sigma,\sigma^*} \\ \mathfrak{s} \subset K}} \frac{|\mathfrak{s}|}{|\sigma|} \mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}}_{\leqslant 0}. \end{aligned}$$

This is the definition we are going to use for $\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}$ and $\mathcal{F}^-_{D\sigma,\sigma^*,\mathfrak{s}}$, namely

$$\mathcal{F}^{\pm}_{K,\sigma} = \frac{|D_{\sigma,\sigma^*} \cap K|}{|D_{\sigma,\sigma^*}|} \sum_{\substack{\mathfrak{s} \in \partial D_{\sigma,\sigma^*} \\ \mathfrak{s} \subset L}} \frac{|\mathfrak{s}|}{|\sigma|} \mathcal{F}^{\pm}_{D\sigma,\sigma^*,\mathfrak{s}} - \frac{|D_{\sigma,\sigma^*} \cap L|}{|D_{\sigma,\sigma^*}|} \sum_{\substack{\mathfrak{s} \in \partial D_{\sigma,\sigma^*} \\ \mathfrak{s} \subset K}} \frac{|\mathfrak{s}|}{|\sigma|} \mathcal{F}^{\mp}_{D\sigma,\sigma^*,\mathfrak{s}}. \tag{13}$$

As far as we are dealing with the direct dual mesh ⓐ, by the convexity assumption, the diagonal σ^* is an edge of the dual mesh, and we obtain similarly the following expression for the mass fluxes on the dual cells

$$\mathcal{F}_{K^*,\sigma^*} = \mathcal{F}^+_{K^*,\sigma^*} + \mathcal{F}^-_{K^*,\sigma^*}$$

with

$$\mathcal{F}^{\pm}_{K^*,\sigma^*} = \frac{|D_{\sigma,\sigma^*} \cap K^*|}{|D_{\sigma,\sigma^*}|} \sum_{\substack{\mathfrak{s} \in \partial D_{\sigma,\sigma^*} \\ \mathfrak{s} \subset L^*}} \frac{|\mathfrak{s}|}{|\sigma^*|} \mathcal{F}^{\pm}_{D\sigma,\sigma^*,\mathfrak{s}} - \frac{|D_{\sigma,\sigma^*} \cap L^*|}{|D_{\sigma,\sigma^*}|} \sum_{\substack{\mathfrak{s} \in \partial D_{\sigma,\sigma^*} \\ \mathfrak{s} \subset K^*}} \frac{|\mathfrak{s}|}{|\sigma^*|} \mathcal{F}^{\mp}_{D\sigma,\sigma^*,\mathfrak{s}}. \tag{14}$$

What is crucial is the fact the fluxes satisfy the conservation property: with $\sigma = K|L$ and $\sigma^* = K^*|L^*$,

$$\mathcal{F}_{K,\sigma} = -\mathcal{F}_{L,\sigma} \quad \text{and} \quad \mathcal{F}_{K^*,\sigma^*} = -\mathcal{F}_{L^*,\sigma^*}.$$

More specifically we have

$$\mathcal{F}^{\pm}_{K,\sigma} = -\mathcal{F}^{\mp}_{L,\sigma} \quad \text{and} \quad \mathcal{F}^{\pm}_{K^*,\sigma^*} = -\mathcal{F}^{\mp}_{L^*,\sigma^*}. \tag{15}$$

Let us detail how this approach generalizes for the barycentric dual mesh ⓑ, the geometry of which is much more intricate. There is no restriction at all on the geometry of the cells, in defining a mass flux on an edge $\mathfrak{s}$ (of $X = K$ or $X = K^*$) contained in D_{σ,σ^*}, with the unit outward normal $\mathbf{n}_{\mathfrak{s}}$ by the formula

$$|\mathfrak{s}|\mathcal{F}_{X,\mathfrak{s}} = \int_{\mathfrak{s}} \omega_{D_{\sigma,\sigma^*}} \cdot \mathbf{n}_{\mathfrak{s}}. \tag{16}$$

This is still a conservative quantity: if $\mathfrak{s} = X|Y \subset D_{\sigma,\sigma^*}$, then $\mathcal{F}_{X,\mathfrak{s}} = -\mathcal{F}_{Y,\mathfrak{s}}$, and we adopt this definition from now on. In particular, it allows us to deal with the barycentric mesh. We refer the reader to Fig. 3 again. The domain $D_{\sigma,\sigma^*} \cap L^*$ is delimited by σ^*_K, σ^*_L (which are not necessarily on the same direction), $\mathfrak{s}_{KL^*}$, and $\mathfrak{s}_{L^*L}$. The mass fluxes are already known on $\mathfrak{s}_{KL^*}$, and $\mathfrak{s}_{L^*L}$, which are boundaries of the diamond D_{σ,σ^*}, and we wish to clarify their expression on σ^*_K, σ^*_L, the boundaries of L^*. Therefore, by definition of $\omega_{D_{\sigma,\sigma^*}}$ and the divergence theorem, we obtain

$$\begin{aligned}
\int_{D_{\sigma,\sigma^*}\cap L^*} \boldsymbol{\nabla}\cdot\omega_{D_{\sigma,\sigma^*}} &= \frac{|D_{\sigma,\sigma^*}\cap L^*|}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}} \\
&= |\sigma^*_K|\mathcal{F}_{L^*,\sigma^*_K} + |\sigma^*_L|\mathcal{F}_{L^*,\sigma^*_L} \\
&\quad + |\mathfrak{s}_{KL^*}|\mathcal{F}_{D_{\sigma,\sigma^*},\mathfrak{s}_{KL^*}} + |\mathfrak{s}_{L^*L}|\mathcal{F}_{D_{\sigma,\sigma^*},\mathfrak{s}_{L^*L}}.
\end{aligned}$$

We write

$$1 = \frac{|D_{\sigma,\sigma^*}\cap L^*|}{|D_{\sigma,\sigma^*}|} + \frac{|D_{\sigma,\sigma^*}\cap K^*|}{|D_{\sigma,\sigma^*}|}$$

so that

$$\begin{aligned}
|\sigma^*_K|\mathcal{F}_{L^*,\sigma^*_K} + |\sigma^*_L|\mathcal{F}_{L^*,\sigma^*_L} &= \frac{|D_{\sigma,\sigma^*}\cap L^*|}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}} \\
&\quad - |\mathfrak{s}_{KL^*}|\mathcal{F}_{D_{\sigma,\sigma^*},\mathfrak{s}_{KL^*}} - |\mathfrak{s}_{L^*L}|\mathcal{F}_{D_{\sigma,\sigma^*},\mathfrak{s}_{L^*L}} \\
&= \frac{|D_{\sigma,\sigma^*}\cap L^*|}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\cap K^*} |\mathfrak{s}|\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}} \\
&\quad - \frac{|D_{\sigma,\sigma^*}\cap K^*|}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\cap L^*} |\mathfrak{s}|\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}}.
\end{aligned} \tag{17}$$

We recover the same expression as for the direct mesh, and it makes sense to still denote this sum by $\mathcal{F}_{K^*,\sigma^*}$ (which thus does not need to determine explicitly $|\sigma^*_K|\mathcal{F}_{L^*,\sigma^*_K}$ and $|\sigma^*_L|\mathcal{F}_{L^*,\sigma^*_L}$: the sum can be evaluated directly form the knowledge of the mass fluxes on the interfaces of the diamond cells).

Next, we can define the splitting $\mathcal{F}_{K^*,\sigma^*} = \mathcal{F}^+_{K^*,\sigma^*} + \mathcal{F}^-_{K^*,\sigma^*}$ as we did for the primal mesh or the direct mesh, which leads to (14). It it worth detailing further this decomposition and explaining its consistency. Indeed, the auxilliary function $\omega_{D_{\sigma,\sigma^*}}$ depends on the fluxes $\mathcal{F}_{D_{\sigma,\sigma^*},\mathfrak{r}} = \mathcal{F}^+_{D_{\sigma,\sigma^*},\mathfrak{r}} + \mathcal{F}^-_{D_{\sigma,\sigma^*},\mathfrak{r}}$, and, it turns out that $\mathcal{F}_{X,\mathfrak{s}}$, defined by (16) appears as a linear combination of these fluxes; we get

$$\mathcal{F}_{X,\mathfrak{s}} = \sum_{\mathfrak{r}\in\partial D_{\sigma,\sigma^*},\ D_{\sigma,\sigma^*}\in\mathfrak{D}_X} \eta_{\mathfrak{r}}\mathcal{F}_{D_{\sigma,\sigma^*},\mathfrak{r}}.$$

The expression of the coefficients is fully detailed in [14], but we dot not need this here: we point out the the $\eta_{\mathfrak{r}}$ are real, without definite sign; this explains why a construction applying directly Lemma 3.3 to the $\mathcal{F}^{\pm}_{D_{\sigma,\sigma^*},\mathfrak{r}}$ would fail: it is not guaranteed that the combination of the

$\mathcal{F}^{\pm}_{D_{\sigma,\sigma^*},\mathfrak{r}}$ keeps the sign. Nevertheless, we can write

$$\begin{aligned}\mathcal{F}_{X,\mathfrak{s}} \quad = \quad & \underbrace{\sum_{\eta_\mathfrak{r}\geqslant 0} \eta_\mathfrak{r}\mathcal{F}^{+}_{D_{\sigma,\sigma^*},\mathfrak{r}} + \sum_{\eta_\mathfrak{r}\leqslant 0} \eta_\mathfrak{r}\mathcal{F}^{-}_{D_{\sigma,\sigma^*},\mathfrak{r}}}_{\geqslant 0} \\ & + \underbrace{\sum_{\eta_\mathfrak{r}\leqslant 0} \eta_\mathfrak{r}\mathcal{F}^{+}_{D_{\sigma,\sigma^*},\mathfrak{r}} + \sum_{\eta_\mathfrak{r}\geqslant 0} \eta_\mathfrak{r}\mathcal{F}^{-}_{D_{\sigma,\sigma^*},\mathfrak{r}}}_{\leqslant 0}.\end{aligned}$$

This defines the splitting $\mathcal{F}_{X,\mathfrak{s}} = \mathcal{F}^{+}_{X,\mathfrak{s}} + \mathcal{F}^{-}_{X,\mathfrak{s}}$. We apply this construction to define $\mathcal{F}_{K^*,\sigma^*_K} = \mathcal{F}^{+}_{K^*,\sigma^*_K} + \mathcal{F}^{-}_{K^*,\sigma^*_K}$ and $\mathcal{F}_{K^*,\sigma^*_L} = \mathcal{F}^{+}_{K^*,\sigma^*_L} + \mathcal{F}^{-}_{K^*,\sigma^*_L}$. By summing these contribution, we go back to

$$|\sigma^*|\mathcal{F}_{K^*,\sigma^*} = |\sigma^*_K|\mathcal{F}_{K^*,\sigma^*_K} + |\sigma^*_L|\mathcal{F}_{K^*,\sigma^*_L},$$

and then we obtain

$$|\sigma^*|\mathcal{F}^{\pm}_{K^*,\sigma^*} = |\sigma^*_K|\mathcal{F}^{\pm}_{K^*,\sigma^*_K} + |\sigma^*_L|\mathcal{F}^{\pm}_{K^*,\sigma^*_L}. \tag{18}$$

For interfaces σ and σ^* such that $D_{\sigma,\sigma^*} \in \mathfrak{D}_{ext}$, we naturally set

$$\mathcal{F}_{K,\sigma} = \mathcal{F}_\sigma \quad \text{and} \quad \mathcal{F}_{K^*,\sigma} = \mathcal{F}_\sigma.$$

since $\mathcal{F}_\sigma$ was previously defined as the flux outgoing through σ. Moreover, denoting $\sigma = [x_{K^*}, x_{L^*}]$, we set $\mathcal{F}_{K^*,\sigma^*} = \mathcal{F}^{+}_{K^*,\sigma^*} + \mathcal{F}^{-}_{K^*,\sigma^*}$ with

$$\begin{aligned}|\sigma^*|\mathcal{F}^{\pm}_{K^*,\sigma^*} = \frac{|D_{\sigma,\sigma^*}\cap K^*|}{|D_{\sigma,\sigma^*}|}|\mathfrak{s}_{KL^*}|\mathcal{F}^{\pm}_{\sigma,\mathfrak{s}_{KL^*}} - \frac{|D_{\sigma,\sigma^*}\cap L^*|}{|D_{\sigma,\sigma^*}|}|\mathfrak{s}_{KK^*}|\mathcal{F}^{\mp}_{\sigma,\mathfrak{s}_{KK^*}} \\ + \frac{|D_{\sigma,\sigma^*}\cap K^*|}{2|D_{\sigma,\sigma^*}|}|\sigma|\mathcal{F}^{\pm}_{\sigma} - \frac{|D_{\sigma,\sigma^*}\cap L^*|}{2|D_{\sigma,\sigma^*}|}|\sigma|\mathcal{F}^{\mp}_{\sigma}.\end{aligned}$$

With the motivation of writing a conservative equation for the momentum $\rho\mathbf{u}$, we introduce averaged densities on $\mathfrak{T}$.

Definition 3.4. The averaged density on a cell K of the primal mesh is defined by

$$\rho_K = \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K} \frac{|D_{\sigma,\sigma^*}\cap K|}{|K|}\rho_{\sigma,\sigma^*} \text{ for } K \in \mathfrak{M}$$

and on a cell K^* of the dual mesh, we set

$$\rho_{K^*} = \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}} \frac{|D_{\sigma,\sigma^*}\cap K^*|}{|K^*|}\rho_{\sigma,\sigma^*} \text{ for } K^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*.$$

With this definition at hand, considering either the primal or the dual mesh, the averaged densities ρ_K and ρ_{K^*} satisfy conservative equations, as observed in [12].

Proposition 3.5. *The averaged densities ρ_K, ρ_{K^*} satisfy the following conservative equations for any $K \in \mathfrak{M}$ and any $K^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$:*

$$|K|\frac{\overline{\rho}_K - \rho_K}{\delta t} + \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K} |\sigma|\mathcal{F}_{K,\sigma} = 0,$$

$$|K^*|\frac{\overline{\rho}_{K^*} - \rho_{K^*}}{\delta t} + \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}} |\sigma^*|\mathcal{F}_{K^*,\sigma^*} + \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}} \frac{|\sigma|}{2}\mathcal{F}_{K^*,\sigma} = 0.$$

Proof. This is a consequence of the construction of the fluxes. Indeed, when considering $|K|\frac{\overline{\rho}_K-\rho_K}{\delta t}$, we are led to compute

$$\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}}\left(\frac{|D_{\sigma,\sigma^*}\cap K|}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}}\right)$$
$$+\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{ext}}\left(\frac{|D_{\sigma,\sigma^*}\cap K|}{|D_{\sigma,\sigma^*}|}\Big[|\sigma|\mathcal{F}_\sigma+\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\setminus\partial\Omega}|\mathfrak{s}|\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}}\Big]\right)$$

By definition and going back to Lemma 3.3, we get

$$\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}}\left(\frac{|D_{\sigma,\sigma^*}\cap K|}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}}\right)$$
$$=\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}}|D_{\sigma,\sigma^*}\cap K|\boldsymbol{\nabla}\cdot\omega_{D_{\sigma,\sigma^*}}=\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}}\int_{D_{\sigma,\sigma^*}\cap K}\boldsymbol{\nabla}\cdot\omega_{D_{\sigma,\sigma^*}}.$$

By virtue of the divergence theorem, this becomes

$$\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}}\left(\sum_{\mathfrak{s}\in\partial(D\cap K)}\int_{\mathfrak{s}}\omega_{D_{\sigma,\sigma^*}}\cdot\mathbf{n}_{\mathfrak{s}}\right)$$
$$=\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}}\left(\int_\sigma\omega_{D_{\sigma,\sigma^*}}\cdot\mathbf{n}_\sigma+\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*},\ s\subset K}\int_{\mathfrak{s}}\omega_{D_{\sigma,\sigma^*}}\cdot\mathbf{n}_{\mathfrak{s}}\right)$$
$$=\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}}\left(|\sigma|\mathcal{F}_{K,\sigma}+\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*},\ \mathfrak{s}\subset K}|\mathfrak{s}|\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}}\right)$$
$$=\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}}|\sigma|\mathcal{F}_{K,\sigma}+0.$$

A similar computation holds for the external cells and the dual cells. Note that the same formula holds for both the direct and the barycentric meshes, by merging contributions as in (17). For the primal cells (and the dual cells for the direct dual mesh), the result can be checked by using the explicit formula (13) and (14), see [30]. ■

3.3 Momentum equation on the primal and dual cells

We now turn to the definition of the momentum fluxes $\mathcal{G}_{K,\sigma}$ for the primal cells and $\mathcal{G}_{K^*,\sigma^*}$ for the dual cells. We first consider the case of interfaces $\sigma\not\subset\partial\Omega$. In this case, for the primal cells we set

$$\mathcal{G}_{K,\sigma}=\mathcal{F}^+_{K,\sigma}\mathbf{u}_K+\mathcal{F}^-_{K,\sigma}\mathbf{u}_L. \tag{19}$$

Namely, for defining the momentum fluxes we use the mass fluxes $\mathcal{F}^\pm_{K,\sigma}$ given by (13). For the dual cells, a similar formula based on (14)

$$\mathcal{G}_{K^*,\sigma^*}=\mathcal{F}^+_{K^*,\sigma^*}\mathbf{u}_{K^*}+\mathcal{F}^-_{K^*,\sigma^*}\mathbf{u}_{L^*}. \tag{20}$$

applies directly when we use the direct mesh ⓐ: the convexity assumption of the diamond cells implies that σ^* is an edge for the dual cell and is included in D_{σ,σ^*}. For the barycentric dual mesh ⓑ, there are two viewpoints:

*) either we define the mass fluxes on all interfaces of the dual mesh, by using the general formula (16). Then, on these interfaces, say σ^*_K, we split as described above the flux into

positive and negative parts $\mathcal{F}_{K^*,\sigma_K^*} = \mathcal{F}^+_{K^*,\sigma_K^*} + \mathcal{F}^-_{K^*,\sigma_K^*}$, and we apply the upwinding principles accordingly by setting $\mathcal{G}_{K^*,\sigma_K^*} = \mathcal{F}^+_{K^*,\sigma_K^*}\mathbf{u}_{K^*} + \mathcal{F}^-_{K^*,\sigma_K^*}\mathbf{u}_{L^*}$. As in (17), we can merge the contributions $|\sigma_K^*|\mathcal{G}_{K^*,\sigma_K^*}$ and $|\sigma_L^*|\mathcal{G}_{K^*,\sigma_L^*}$: the sum can be cast into

$$(|\sigma_K^*|\mathcal{F}^+_{K^*,\sigma_K^*} + |\sigma_L^*|\mathcal{F}^+_{K^*,\sigma_L^*})\mathbf{u}_{K^*} + (|\sigma_K^*|\mathcal{F}^-_{K^*,\sigma_K^*} + |\sigma_L^*|\mathcal{F}^+_{K^*,\sigma_L^*})\mathbf{u}_{L^*}.$$

**) or, we use directly the averaged formula (17), which defines $\mathcal{F}^{\pm}_{K^*,\sigma^*}$ by (14) too, and then we make use of (20).

The two viewpoints coincide, owing to (18). Note that the momentum fluxes are conservative as a consequence of (15). For the boundary conditions, that is for $\sigma \subset \partial\Omega$, we define:

$$\mathcal{G}_{K,\sigma} = \mathcal{F}^+_\sigma \mathbf{u}_K + \mathcal{F}^-_\sigma \mathbf{u}_\sigma \quad \text{and} \quad \mathcal{G}_{K^*,\sigma} = \mathcal{F}^+_\sigma \mathbf{u}_{K^*} + \mathcal{F}^-_\sigma \mathbf{u}_\sigma.$$

Remark 3.6. For the fluxes $\mathcal{G}_{K,\sigma}$, the formula (19) is valid also for $\sigma \subset \partial\Omega$ if we use the convention $\mathbf{u}_L = \mathbf{u}_\sigma$ in this case.

The momentum equation also requires to introduce a discrete pressure gradient. It is obtained by mimicking the formula

$$\int_X \boldsymbol{\nabla} p = \int_{\partial X} p\mathbf{n}.$$

Definition 3.7. The discrete pressure gradient $\boldsymbol{\nabla}_{\boldsymbol{d}} p$ is defined on $\mathfrak{T}$ by

$$(\boldsymbol{\nabla}_{\boldsymbol{d}} p)_K = \frac{1}{|K|} \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K} |\sigma| p_{\sigma,\sigma^*}\mathbf{n}_{K,\sigma}, \qquad \text{for } K \in \mathfrak{M},$$

$$(\boldsymbol{\nabla}_{\boldsymbol{d}} p)_{K^*} = \frac{1}{|K^*|} \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}} |\sigma| p_{\sigma,\sigma^*}\mathbf{n}_{K^*,\sigma^*}, \qquad \text{for } K^* \in \mathfrak{M}^*,$$

$$(\boldsymbol{\nabla}_{\boldsymbol{d}} p)_{K^*} = \frac{1}{|K^*|} \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}} |\sigma^*| p_{\sigma,\sigma^*}\mathbf{n}_{K^*,\sigma^*} + \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}} \frac{|\sigma|}{2} p_{\sigma,\sigma^*}\mathbf{n}_{K,\sigma}, \qquad \text{for } K^* \in \partial\mathfrak{M}^*.$$

For the barycentric mesh, the formula for $(\boldsymbol{\nabla}_{\boldsymbol{d}} p)_{K^*}$ still makes sense, by setting

$$|\sigma^*|\mathbf{n}_{K^*,\sigma^*} = |\sigma_K^*|\mathbf{n}_{K^*,\sigma_K^*} + |\sigma_L^*|\mathbf{n}_{K^*,\sigma_L^*}, \tag{21}$$

remarking that this vector is indeed orthogonal to σ^*, by virtue of Lemma 2.3.

The discrete momentum equation is given for $K \in \mathfrak{T}$ by

$$\begin{aligned} &\frac{\overline{\rho}_K \overline{\mathbf{u}}_K - \rho_K \mathbf{u}_K}{\delta t} + \frac{1}{|K|} \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K} |\sigma|\mathcal{G}_{K,\sigma} + (\boldsymbol{\nabla}_{\boldsymbol{d}} p)_K = 0, \\ &\frac{\overline{\rho}_{K^*} \overline{\mathbf{u}}_{K^*} - \rho_{K^*} \mathbf{u}_{K^*}}{\delta t} + \frac{1}{|K^*|} \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}} |\sigma^*|\mathcal{G}_{K^*,\sigma^*} \\ &\qquad + \frac{1}{|K^*|} \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}} \Big[|\sigma^*|\mathcal{G}_{K^*,\sigma^*} + \frac{|\sigma|}{2}\mathcal{G}_{K^*,\sigma}\Big] + (\boldsymbol{\nabla}_{\boldsymbol{d}} p)_{K^*} = 0, \end{aligned} \tag{22}$$

with ρ_K and ρ_{K^*} given by Definition 3.4, fluxes defined in (19), (20) and pressure gradients in Definition 3.7.

3.4 Internal energy balance on the diamond mesh

At the continuous level, considering smooth enough functions, the internal energy equation is deduced from the kinetic energy balance which is itself obtained by multiplying the momentum equation by $\mathbf{u}$. At the discrete level, multiplying the discrete momentum equation by $\mathbf{u}_K$ or $\mathbf{u}_{K^*}$, whatever the considered mesh, introduces a remainder term that has to be taken into account to write the discrete internal energy equation [20, 22]. To this end, let us introduce kinetic fluxes $\mathcal{K}_{K,\sigma}$ from primal cells and $\mathcal{K}_{K^*,\sigma^*}$, $\mathcal{K}_{K^*,\sigma}$ from dual cells. For $\sigma \not\subset \partial\Omega$, we set

$$\mathcal{K}_{K,\sigma} = \mathcal{F}^+_{K,\sigma}\frac{\|\mathbf{u}_K\|^2}{2} + \mathcal{F}^-_{K,\sigma}\frac{\|\mathbf{u}_L\|^2}{2}, \quad \text{and} \quad \mathcal{K}_{K^*,\sigma^*} = \mathcal{F}^+_{K^*,\sigma^*}\frac{\|\mathbf{u}_{K^*}\|^2}{2} + \mathcal{F}^-_{K^*,\sigma^*}\frac{\|\mathbf{u}_{L^*}\|^2}{2}. \tag{23}$$

These fluxes are conservative ($\mathcal{K}_{K,\sigma} = -\mathcal{K}_{L,\sigma}$, and so on...) as a consequence of (15). For the boundary conditions, that is $\sigma \subset \partial\Omega$, we set

$$\mathcal{K}_{K,\sigma} = \mathcal{F}^+_{\sigma}\frac{\|\mathbf{u}_K\|^2}{2} + \mathcal{F}^-_{\sigma}\frac{\|\mathbf{u}_\sigma\|^2}{2} \quad \text{and} \quad \mathcal{K}_{K^*,\sigma} = \mathcal{F}^+_{\sigma}\frac{\|\mathbf{u}_{K^*}\|^2}{2} + \mathcal{F}^-_{\sigma}\frac{\|\mathbf{u}_\sigma\|^2}{2}.$$

Note that for the barycentric mesh, we can define the "intermediate" kinetic energy fluxes $\mathcal{F}_{K^*,\sigma^*_K}$, $\mathcal{F}_{K^*,\sigma^*_L}$ by going back to (18). It yields

$$\begin{aligned}
\mathcal{K}_{K^*,\sigma^*_K} &= \Big(\mathcal{F}^+_{K^*,\sigma^*_K}\frac{\|\mathbf{u}_{K^*}\|^2}{2} + \mathcal{F}^-_{K^*,\sigma^*_K}\frac{\|\mathbf{u}_{L^*}\|^2}{2}\Big),\\
\mathcal{K}_{K^*,\sigma^*_L} &= \Big(\mathcal{F}^+_{K^*,\sigma^*_L}\tfrac{\|\mathbf{u}_{K^*}\|^2}{2} + \mathcal{F}^-_{K^*,\sigma^*_L}\tfrac{\|\mathbf{u}_{L^*}\|^2}{2}\Big),\\
|\sigma^*|\mathcal{K}_{K^*,\sigma^*_K} &= |\sigma^*_K|\mathcal{K}_{K^*,\sigma^*_K} + |\sigma^*_L|\mathcal{K}_{K^*,\sigma^*_L}.
\end{aligned} \tag{24}$$

Remark 3.8. Note that, as for momentum fluxes (see Remark 3.6), formula (23) for fluxes $\mathcal{K}_{K,\sigma}$, is valid also for $\sigma \subset \partial\Omega$ if we use the convention $\mathbf{u}_L = \mathbf{u}_\sigma$ in this case.

Definition 3.9. For $K \in \mathfrak{M}$ we set

$$\mathcal{R}_K = \frac{\overline{\rho}_K}{2\delta t}\|\overline{\mathbf{u}}_K - \mathbf{u}_K\|^2 + \frac{1}{|K|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K}|\sigma|\mathcal{F}^-_{K,\sigma}\left(\frac{\|\overline{\mathbf{u}}_K - \mathbf{u}_K\|^2}{2} - \frac{\|\overline{\mathbf{u}}_K - \mathbf{u}_L\|^2}{2}\right),$$

with the convention that $\mathbf{u}_L = \mathbf{u}_\sigma$ when $\sigma \subset \partial\Omega$.

For $K^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ we set

$$\begin{aligned}
\mathcal{R}_{K^*} = \frac{\overline{\rho}_{K^*}}{2\delta t}\|\overline{\mathbf{u}}_{K^*} - \mathbf{u}_{K^*}\|^2 &+ \frac{1}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}}|\sigma^*|\mathcal{F}^-_{K^*,\sigma^*}\left(\frac{\|\overline{\mathbf{u}}_{K^*} - \mathbf{u}_{K^*}\|^2}{2} - \frac{\|\overline{\mathbf{u}}_{K^*} - \mathbf{u}_{L^*}\|^2}{2}\right)\\
&+ \frac{1}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}}\frac{|\sigma|}{2}\mathcal{F}^-_{\sigma}\left(\frac{\|\overline{\mathbf{u}}_{K^*} - \mathbf{u}_{K^*}\|^2}{2} - \frac{\|\overline{\mathbf{u}}_{K^*} - \mathbf{u}_{\sigma}\|^2}{2}\right).
\end{aligned}$$

Lemma 3.10. *The discrete balance of kinetic energy is given for $K \in \mathfrak{M}$ and $K^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ by*

$$\frac{\overline{\rho}_K\frac{\|\overline{\mathbf{u}}_K\|^2}{2} - \rho_K\frac{\|\mathbf{u}_K\|^2}{2}}{\delta t} + \frac{1}{|K|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K}|\sigma|\mathcal{K}_{K,\sigma} + (\boldsymbol{\nabla}_{\boldsymbol{d}}p)_K\cdot\overline{\mathbf{u}}_K = -\mathcal{R}_K. \tag{25}$$

$$\begin{aligned}
\frac{\overline{\rho}_{K^*}\frac{\|\overline{\mathbf{u}}_{K^*}\|^2}{2} - \rho_{K^*}\frac{\|\mathbf{u}_{K^*}\|^2}{2}}{\delta t} &+ \frac{1}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}}|\sigma^*|\mathcal{K}_{K^*,\sigma^*}\\
&+ \frac{1}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}}\frac{|\sigma|}{2}\mathcal{K}_{K^*,\sigma} + (\boldsymbol{\nabla}_{\boldsymbol{d}}p)_{K^*}\cdot\overline{\mathbf{u}}_{K^*} = -\mathcal{R}_{K^*}.
\end{aligned} \tag{26}$$

Proof. For $X \in \mathfrak{T}$, we multiply by $\overline{\mathbf{u}}_X$ the momentum equation (22) and use the averaged mass equation in Proposition (3.5).

a) Let $K \in \mathfrak{M}$. In what follows, we use the convention that $\mathbf{u}_L = \mathbf{u}_\sigma$ when the edge $\sigma \subset \partial\Omega$; so that the expressions (19) and (23) are valid for $\sigma \subset \partial\Omega$ too (see Remarks 3.6 and 3.8). We start by remarking that

$$\begin{aligned}\frac{\overline{\rho}_K \overline{\mathbf{u}}_K - \rho_K \mathbf{u}_K}{\delta t} \cdot \overline{\mathbf{u}}_K =& \frac{1}{\delta t}\left(\overline{\rho}_K \frac{\|\overline{\mathbf{u}}_K\|^2}{2} - \rho_K \frac{\|\mathbf{u}_K\|^2}{2} + \frac{\overline{\rho}_K}{2}\|\overline{\mathbf{u}}_K - \mathbf{u}_K\|^2\right)\\ &- \frac{\overline{\rho}_K - \rho_K}{\delta t}\left(\frac{\|\mathbf{u}_K\|^2}{2} - \mathbf{u}_K \cdot \overline{\mathbf{u}}_K\right).\end{aligned}$$

Thus using the average mass balance stated in Proposition (3.5) we get

$$\begin{aligned}\frac{\overline{\rho}_K \overline{\mathbf{u}}_K - \rho_K \mathbf{u}_K}{\delta t} \cdot \overline{\mathbf{u}}_K =& \frac{1}{\delta t}\left(\overline{\rho}_K \frac{\|\overline{\mathbf{u}}_K\|^2}{2} - \rho_K \frac{\|\mathbf{u}_K\|^2}{2} + \frac{\overline{\rho}_K}{2}\|\overline{\mathbf{u}}_K - \mathbf{u}_K\|^2\right)\\ &+ \frac{1}{|K|}\sum_{\sigma\in\partial K} |\sigma| \mathcal{F}_{K,\sigma}\left(\frac{\|\mathbf{u}_K\|^2}{2} - \mathbf{u}_K \cdot \overline{\mathbf{u}}_K\right).\end{aligned}$$

Next, using the notation $\mathcal{F}^{|\cdot|} = \mathcal{F}^+ - \mathcal{F}^-$ and bearing in mind that $\mathcal{F} = \mathcal{F}^+ + \mathcal{F}^-$, we get

$$\mathcal{G}_{K,\sigma} = \frac{\mathcal{F}^{|\cdot|}_{K,\sigma} + \mathcal{F}_{K,\sigma}}{2}\mathbf{u}_K + \frac{\mathcal{F}_{K,\sigma} - \mathcal{F}^{|\cdot|}_{K,\sigma}}{2}\mathbf{u}_L = \mathcal{F}_{K,\sigma}\frac{\mathbf{u}_K + \mathbf{u}_L}{2} + \mathcal{F}^{|\cdot|}_{K,\sigma}\frac{\mathbf{u}_K - \mathbf{u}_L}{2}.$$

Hence, the momentum equation multiplied by $\overline{\mathbf{u}}_K$ becomes

$$\frac{\overline{\rho}_K \frac{\|\overline{\mathbf{u}}_K\|^2}{2} - \rho_K \frac{\|\mathbf{u}_K\|^2}{2}}{\delta t} + \frac{\overline{\rho}_K}{2\delta t}\|\overline{\mathbf{u}}_K - \mathbf{u}_K\|^2 + (\boldsymbol{\nabla}_{\boldsymbol{d}} p)_K \cdot \overline{\mathbf{u}}_K + \frac{1}{|K|}\sum_{\sigma\in\partial K}|\sigma| B_{K,\sigma} = 0,$$

where

$$\begin{aligned}B_{K,\sigma} &= \mathcal{F}_{K,\sigma}\left(\frac{\|\mathbf{u}_K\|^2}{2} - \frac{\mathbf{u}_K - \mathbf{u}_L}{2}\cdot\overline{\mathbf{u}}_K\right) + \mathcal{F}^{|\cdot|}_{K,\sigma}\frac{\mathbf{u}_K - \mathbf{u}_L}{2}\cdot\overline{\mathbf{u}}_K\\ &= \left(\mathcal{F}^+_{K,\sigma}\frac{\|\mathbf{u}_K\|^2}{2} + \mathcal{F}^-_{K,\sigma}\frac{\|\mathbf{u}_L\|^2}{2}\right) + \mathcal{F}^-_{K,\sigma}\left(\frac{\|\mathbf{u}_K\|^2}{2} - \frac{\|\mathbf{u}_L\|^2}{2} - (\mathbf{u}_K - \mathbf{u}_L)\cdot\overline{\mathbf{u}}_K\right).\end{aligned}$$

With definition (23) we are left with

$$B_{K,\sigma} = \mathcal{K}_{K,\sigma} + \mathcal{F}^-_{K,\sigma}\left(\frac{\|\overline{\mathbf{u}}_K - \mathbf{u}_K\|^2}{2} - \frac{\|\overline{\mathbf{u}}_K - \mathbf{u}_L\|^2}{2}\right),$$

and thus we obtain (25).

b) For $K^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$, as previously, we first remark that

$$\begin{aligned}\frac{\overline{\rho}_{K^*} \overline{\mathbf{u}}_{K^*} - \rho_{K^*} \mathbf{u}_{K^*}}{\delta t} \cdot \overline{\mathbf{u}}_{K^*} =& \frac{1}{\delta t}\left(\overline{\rho}_{K^*} \frac{\|\overline{\mathbf{u}}_{K^*}\|^2}{2} - \rho_{K^*} \frac{\|\mathbf{u}_{K^*}\|^2}{2} + \frac{\overline{\rho}_{K^*}}{2}\|\overline{\mathbf{u}}_{K^*} - \mathbf{u}_{K^*}\|^2\right)\\ &- \frac{\overline{\rho}_{K^*} - \rho_{K^*}}{\delta t}\left(\frac{\|\mathbf{u}_{K^*}\|^2}{2} - \mathbf{u}_{K^*} \cdot \overline{\mathbf{u}}_{K^*}\right).\end{aligned}$$

Thus using the average mass balance stated in Proposition (3.5) we get

$$\begin{aligned}\frac{\overline{\rho}_{K^*}\overline{\mathbf{u}}_{K^*}-\rho_{K^*}\mathbf{u}_{K^*}}{\delta t}\cdot\overline{\mathbf{u}}_{K^*} =& \frac{1}{\delta t}\left(\overline{\rho}_{K^*}\frac{\|\overline{\mathbf{u}}_{K^*}\|^2}{2}-\rho_{K^*}\frac{\|\mathbf{u}_{K^*}\|^2}{2}+\frac{\overline{\rho}_{K^*}}{2}\|\overline{\mathbf{u}}_{K^*}-\mathbf{u}_{K^*}\|^2\right)\\ &+\frac{1}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}}|\sigma^*|\mathcal{F}_{K^*,\sigma^*}\left(\frac{\|\mathbf{u}_{K^*}\|^2}{2}-\mathbf{u}_{K^*}\cdot\overline{\mathbf{u}}_{K^*}\right)\\ &+\frac{1}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}}\frac{|\sigma|}{2}\mathcal{F}_{K^*,\sigma}\left(\frac{\|\mathbf{u}_{K^*}\|^2}{2}-\mathbf{u}_{K^*}\cdot\overline{\mathbf{u}}_{K^*}\right).\end{aligned}$$

The momentum equation multiplied by $\overline{\mathbf{u}}_{K^*}$ becomes

$$\begin{aligned}&\frac{\overline{\rho}_{K^*}\frac{\|\overline{\mathbf{u}}_{K^*}\|^2}{2}-\rho_{K^*}\frac{\|\mathbf{u}_{K^*}\|^2}{2}}{\delta t}+\frac{\overline{\rho}_{K^*}}{2\delta t}\|\overline{\mathbf{u}}_{K^*}-\mathbf{u}_{K^*}\|^2+(\boldsymbol{\nabla}_{\boldsymbol{d}}p)_{K^*}\cdot\overline{\mathbf{u}}_{K^*}\\ &+\frac{1}{|K^*|}\left(\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}}|\sigma^*|\mathcal{F}_{K^*,\sigma^*}+\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}}\frac{|\sigma|}{2}\mathcal{F}_{\sigma}\right)\left(\frac{\|\mathbf{u}_{K^*}\|^2}{2}-\mathbf{u}_{K^*}\cdot\overline{\mathbf{u}}_{K^*}\right)\\ &+\frac{1}{|K^*|}\left(\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}}|\sigma^*|\mathcal{G}_{K^*,\sigma^*}+\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}}\frac{|\sigma|}{2}\mathcal{G}_{K^*,\sigma}\right)\cdot\overline{\mathbf{u}}_{K^*}=0.\end{aligned}$$

We obtain (26) by remarking that, as in the first part of the proof, we have for all $D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}$

$$\begin{aligned}&\mathcal{F}_{K^*,\sigma^*}\left(\frac{\|\mathbf{u}_{K^*}\|^2}{2}-\mathbf{u}_{K^*}\cdot\overline{\mathbf{u}}_{K^*}\right)+\mathcal{G}_{K^*,\sigma^*}\cdot\overline{\mathbf{u}}_{K^*}\\ &\qquad=\mathcal{K}_{K^*,\sigma^*}+\mathcal{F}^-_{K^*,\sigma^*}\left(\frac{\|\overline{\mathbf{u}}_{K^*}-\mathbf{u}_{K^*}\|^2}{2}-\frac{\|\overline{\mathbf{u}}_{K^*}-\mathbf{u}_{L^*}\|^2}{2}\right).\end{aligned}$$

and similarly, we also get for all $D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}$ that

$$\begin{aligned}&\mathcal{F}_{\sigma}\left(\frac{\|\mathbf{u}_{K^*}\|^2}{2}-\mathbf{u}_{K^*}\cdot\overline{\mathbf{u}}_{K^*}\right)+\mathcal{G}_{K^*,\sigma}\cdot\overline{\mathbf{u}}_{K^*}\\ &\qquad=\mathcal{K}_{K^*,\sigma}+\mathcal{F}^-_{\sigma}\left(\frac{\|\overline{\mathbf{u}}_{K^*}-\mathbf{u}_{K^*}\|^2}{2}-\frac{\|\overline{\mathbf{u}}_{K^*}-\mathbf{u}_{\sigma}\|^2}{2}\right).\end{aligned}$$

■

We remind the reader that the kinetic energy equation does not have to be solved since we already know the velocity on the primal and dual meshes from the momentum equation. This computation only aims at defining the remainder terms $\mathcal{R}_K$ and $\mathcal{R}_{K^*}$ that will be used in the equation for updating the internal energy. Before this, we need to introduce a discrete divergence operator, naturally inspired from

$$\int_X \boldsymbol{\nabla}\cdot\mathbf{u}=\int_{\partial X}\mathbf{u}\cdot\mathbf{n}.$$

Definition 3.11. The discrete divergence operator on a cell $D_{\sigma,\sigma^*}\in\mathfrak{D}$ is defined as

$$(\boldsymbol{\nabla}_{\boldsymbol{d}}\cdot\mathbf{u})_{\sigma,\sigma^*}=\frac{1}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|u_{D\sigma,\sigma^*,\mathfrak{s}},$$

when $D_{\sigma,\sigma^*}\in\mathfrak{D}_{int}$, while for $D_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}$, we set

$$(\boldsymbol{\nabla}_{\boldsymbol{d}}\cdot\mathbf{u})_{\sigma,\sigma^*}=\frac{1}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\setminus\partial\Omega}|\mathfrak{s}|u_{D\sigma,\sigma^*,\mathfrak{s}}+\frac{|\sigma|}{2|D_{\sigma,\sigma^*}|}\left(\mathbf{u}_{\sigma}+\frac{\mathbf{u}_{K^*}+\mathbf{u}_{L^*}}{2}\right)\cdot\mathbf{n}_{K,\sigma}.$$

The following statement gives an equivalent formulation of the discrete divergence; it will be useful to study the stability of the scheme.

Lemma 3.12. *The discrete divergence operator on a cell $D_{\sigma,\sigma^*} \in \mathfrak{D}$ recasts as*

$$(\boldsymbol{\nabla}_{\boldsymbol{d}} \cdot \mathbf{u})_{\sigma,\sigma^*} = \frac{1}{2|D_{\sigma,\sigma^*}|} \left(|\sigma| \, (\mathbf{u}_L - \mathbf{u}_K) \cdot \mathbf{n}_{K,\sigma} + |\sigma^*| \, (\mathbf{u}_{L^*} - \mathbf{u}_{K^*}) \cdot \mathbf{n}_{K^*,\sigma^*} \right), \quad \forall D_{\sigma,\sigma^*} \in \mathfrak{D}_{int}.$$

$$(\boldsymbol{\nabla}_{\boldsymbol{d}} \cdot \mathbf{u})_{\sigma,\sigma^*} = \frac{1}{2|D_{\sigma,\sigma^*}|} \left(|\sigma| \, (\mathbf{u}_\sigma - \mathbf{u}_K) \cdot \mathbf{n}_{K,\sigma} + |\sigma^*| \, (\mathbf{u}_{L^*} - \mathbf{u}_{K^*}) \cdot \mathbf{n}_{K^*,\sigma^*} \right), \quad \forall D_{\sigma,\sigma^*} \in \mathfrak{D}_{ext}.$$

Proof. Let us first assume that $D_{\sigma,\sigma^*} \in \mathfrak{D}_{int}$. Using the definition of $u_{D\sigma,\sigma^*,\mathfrak{s}}$ and of $(\boldsymbol{\nabla}_{\boldsymbol{d}} \cdot \mathbf{u})_{\sigma,\sigma^*}$, we have

$$\sum_{\mathfrak{s} \in \partial D_{\sigma,\sigma^*}} |\mathfrak{s}| u_{D\sigma,\sigma^*,\mathfrak{s}} = \frac{\mathbf{u}_K}{2} \cdot \left(|\mathfrak{s}_{KK^*}| \mathbf{n}_{\sigma,\mathfrak{s}_{KK^*}} + |\mathfrak{s}_{KL^*}| \mathbf{n}_{\sigma,\mathfrak{s}_{KL^*}} \right) + \frac{\mathbf{u}_{K^*}}{2} \cdot \left(|\mathfrak{s}_{KK^*}| \mathbf{n}_{\sigma,\mathfrak{s}_{KK^*}} + |\mathfrak{s}_{LK^*}| \mathbf{n}_{\sigma,\mathfrak{s}_{LK^*}} \right)$$
$$+ \frac{\mathbf{u}_L}{2} \cdot \left(|\mathfrak{s}_{LK^*}| \mathbf{n}_{\sigma,\mathfrak{s}_{LK^*}} + |\mathfrak{s}_{LL^*}| \mathbf{n}_{\sigma,\mathfrak{s}_{LL^*}} \right) + \frac{\mathbf{u}_{L^*}}{2} \cdot \left(|\mathfrak{s}_{KL^*}| \mathbf{n}_{\sigma,\mathfrak{s}_{KL^*}} + |\mathfrak{s}_{LL^*}| \mathbf{n}_{\sigma,\mathfrak{s}_{LL^*}} \right).$$

By using Lemma 2.3, it follows that

$$\sum_{\mathfrak{s} \in \partial D_{\sigma,\sigma^*}} |\mathfrak{s}| u_{D\sigma,\sigma^*,\mathfrak{s}} = -\frac{\mathbf{u}_K}{2} \cdot |\sigma| \mathbf{n}_{K,\sigma} - \frac{\mathbf{u}_{K^*}}{2} \cdot |\sigma^*| \mathbf{n}_{K^*,\sigma^*} - \frac{\mathbf{u}_L}{2} \cdot |\sigma| \mathbf{n}_{L,\sigma} - \frac{\mathbf{u}_{L^*}}{2} \cdot |\sigma^*| \mathbf{n}_{L^*,\sigma^*}.$$

We conclude by using $-\mathbf{n}_{L,\sigma} = \mathbf{n}_{K,\sigma}$ and $-\mathbf{n}_{L^*,\sigma^*} = \mathbf{n}_{K^*,\sigma^*}$ (with definition (21) for $\mathbf{n}_{K^*,\sigma^*}$ on the barycentric mesh).

The proof for $D_{\sigma,\sigma^*} \in \mathfrak{D}_{ext}$ follows exactly the same lines and is left to the reader. ■

For the discretization of the internal energy equation, we define the following numerical fluxes, for all $D_{\sigma,\sigma^*} \in \mathfrak{D}_{int}$ and $\mathfrak{s} = D_{\sigma,\sigma^*} | D_{\sigma',\sigma^{*\prime}} \in \partial D_{\sigma,\sigma^*}$,

$$\mathcal{E}_{D\sigma,\sigma^*,\mathfrak{s}} = e_{\sigma,\sigma^*} \mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}} + e_{\sigma',\sigma^{*\prime}} \mathcal{F}^-_{D\sigma,\sigma^*,\mathfrak{s}}. \tag{27}$$

We observe that the fluxes $\mathcal{E}_{D\sigma,\sigma^*,\mathfrak{s}}$ are conservative by definition. For $D_{\sigma,\sigma^*} \in \mathfrak{D}_{ext}$, we have to define the outgoing flux $\mathcal{E}_\sigma$ through the primal egde $\sigma \subset \partial D_{\sigma,\sigma^*} \cap \partial\Omega$. We take

$$\mathcal{E}_\sigma = e_{\sigma,\sigma^*} \mathcal{F}^+_\sigma + e_\sigma \mathcal{F}^-_\sigma$$

Finally, we also give the definition of a remainder term $\mathcal{R}_{\sigma,\sigma^*}$ on the diamond cell, which is based on the remainder term $\mathcal{R}_K$ and $\mathcal{R}_{K^*}$ given in Definition 3.9,

$$\mathcal{R}_{\sigma,\sigma^*} = \frac{|D_{\sigma,\sigma^*} \cap K| \mathcal{R}_K + |D_{\sigma,\sigma^*} \cap L| \mathcal{R}_L + |D_{\sigma,\sigma^*} \cap K^*| \mathcal{R}_{K^*} + |D_{\sigma,\sigma^*} \cap L^*| \mathcal{R}_{L^*}}{2|D_{\sigma,\sigma^*}|}, \tag{28}$$

with the convention that $\mathcal{R}_L = 0$ if $D_{\sigma,\sigma^*} \in \mathfrak{D}_{ext}$. This definition comes from the derivation of the local conservation of the total energy that will be discussed in the forthcoming Section 4. The remainder $\mathcal{R}_{\sigma,\sigma^*}$ is defined so that it exactly balances the kinetic energy contributions that will appear when summing the internal energy equation and the kinetic energy equations.

The discrete internal energy equation is given by

$$\begin{aligned}
&\frac{\overline{\rho}_{\sigma,\sigma^*} \overline{e}_{\sigma,\sigma^*} - \rho_{\sigma,\sigma^*} e_{\sigma,\sigma^*}}{\delta t} + \frac{1}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s} \in \partial D_{\sigma,\sigma^*}} |\mathfrak{s}| \mathcal{E}_{D\sigma,\sigma^*,\mathfrak{s}} \\
&\qquad + p_{\sigma,\sigma^*} \left(\boldsymbol{\nabla}_{\boldsymbol{d}} \cdot \overline{\mathbf{u}} \right)_{\sigma,\sigma^*} = \mathcal{R}_{\sigma,\sigma^*}, \quad \forall D_{\sigma,\sigma^*} \in \mathfrak{D}_{int} \\
&\frac{\overline{\rho}_{\sigma,\sigma^*} \overline{e}_{\sigma,\sigma^*} - \rho_{\sigma,\sigma^*} e_{\sigma,\sigma^*}}{\delta t} + \frac{1}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s} \in \partial D_{\sigma,\sigma^*}} |\mathfrak{s}| \mathcal{E}_{D\sigma,\sigma^*,\mathfrak{s}} + \frac{|\sigma|}{|D_{\sigma,\sigma^*}|} \mathcal{E}_\sigma \\
&\qquad + p_{\sigma,\sigma^*} \left(\boldsymbol{\nabla}_{\boldsymbol{d}} \cdot \overline{\mathbf{u}} \right)_{\sigma,\sigma^*} = \mathcal{R}_{\sigma,\sigma^*}, \quad \forall D_{\sigma,\sigma^*} \in \mathfrak{D}_{ext}
\end{aligned} \tag{29}$$

where the flux $\mathcal{E}_{D\sigma,\sigma^*,\mathfrak{s}}$ is defined by (27), $\boldsymbol{\nabla}_{\boldsymbol{d}} \cdot \overline{\mathbf{u}}$ is given by Definition 3.11, and $\mathcal{R}_{\sigma,\sigma^*}$ by (28).

3.5 Stability analysis

We now turn to the stability analysis of the scheme. Firstly, we exhibit a CFL-condition which ensures that the numerical density remains non-negative. Secondly, in order to justify that the scheme preserves the non-negativity of the internal energy too, we exhibit a condition guarantying that the remainder terms $\mathcal{R}_K$ and $\mathcal{R}_{K^*}$, and thus $\mathcal{R}_{\sigma,\sigma^*}$, are kept positive. To this end, we shall use Lemma 3.2 and specifically property (6) of the numerical flux.

Proposition 3.13. *Let $\rho_{\sigma,\sigma^*} \geqslant 0$. We assume that the following CFL-like conditions are satisfied*

$$\begin{aligned}
&\frac{\delta t}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|[\lambda_+(c(e_\mathfrak{s}), u_{D\sigma,\sigma^*,\mathfrak{s}})]^+ \leqslant 1, && \forall D_{\sigma,\sigma^*} \in \mathfrak{D}_{int} \\
&\frac{\delta t}{|D_{\sigma,\sigma^*}|} \Bigg[\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\setminus\partial\Omega} |\mathfrak{s}|[\lambda_+(c(e_\mathfrak{s}), u_{D\sigma,\sigma^*,\mathfrak{s}})]^+ + |\sigma|[\lambda_+(c(e_\sigma), u_\sigma)]^+ \Bigg] \leqslant 1, && \forall D_{\sigma,\sigma^*} \in \mathfrak{D}_{ext}.
\end{aligned} \tag{30}$$

Then, the non negativity of the density ρ_{σ,σ^*} is preserved: $\overline{\rho}_{\sigma,\sigma^*} \geqslant 0$.

Proof. Let $D_{\sigma,\sigma^*} \in \mathfrak{D}_{int}$. We go back to the mass conservation equation (7) and we make use of Lemma (3.2) and we are thus led to

$$\begin{aligned}
\overline{\rho}_{\sigma,\sigma^*} &= \rho_{\sigma,\sigma^*} - \frac{\delta t}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}| \left(\mathcal{F}^+(\rho_{\sigma,\sigma^*}, u_{D\sigma,\sigma^*,\mathfrak{s}}) + \mathcal{F}^-(\rho_{\sigma',\sigma^{*\prime}}, u_{D\sigma,\sigma^*,\mathfrak{s}})\right) \\
&\geqslant \rho_{\sigma,\sigma^*} - \frac{\delta t}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\mathcal{F}^+(\rho_{\sigma,\sigma^*}, u_{D\sigma,\sigma^*,\mathfrak{s}}) \\
&\geqslant \rho_{\sigma,\sigma^*} - \frac{\delta t}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\rho_{\sigma,\sigma^*}[\lambda_+(c(e_\mathfrak{s}), u_{D\sigma,\sigma^*,\mathfrak{s}})]^+.
\end{aligned}$$

With $\rho_{\sigma,\sigma^*} \geqslant 0$, the right hand side of this inequality remains non negative under the CFL-like condition (30). The proof for $D_{\sigma,\sigma^*} \in \mathfrak{D}_{ext}$ follows exactly the same lines and is left to the reader. ■

Remark 3.14. In order to compare the stability condition with the CFL condition obtained in 1d and on MAC grids, see [3, 16], we remind the reader that $u_{D\sigma,\sigma^*,\mathfrak{s}} = -u_{\sigma',\mathfrak{s}}$ so that the characteristic speeds of the system satisfy:

$$[\lambda_+(c(e_\mathfrak{s}), u_{D\sigma,\sigma^*,\mathfrak{s}})]^+ = [\lambda_+(c(e_\mathfrak{s}), -u_{\sigma',\mathfrak{s}})]^+ = [-\lambda_-(c(e_\mathfrak{s}), u_{\sigma',\mathfrak{s}})]^+ = [\lambda_-(c(e_\mathfrak{s}), u_{\sigma',\mathfrak{s}})]^-.$$

It allows us to rewrite the conditions (30) in a form similar to what has been obtained on Cartesian grids. For instance, the first one can be recast as

$$\frac{\delta t}{|D_{\sigma,\sigma^*}|} \left(\sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K}} |\mathfrak{s}|[\lambda_+(c(e_\mathfrak{s}), u_{D\sigma,\sigma^*,\mathfrak{s}})]^+ + \sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset L}} |\mathfrak{s}|[\lambda_-(c(e_\mathfrak{s}), u_{\sigma',\mathfrak{s}})]^- \right) \leqslant 1.$$

We now turn to the remainder term $\mathcal{R}_K$. The fact that $\mathcal{R}_K$ remains non-negative depends on the mesh-regularity coefficient $\mathrm{reg}\,(\mathfrak{T})$, and the obtained condition is stronger than (30).

Proposition 3.15. *Let us assume that the following CFL-like conditions are satisfied*

$$\frac{\delta t}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|[\lambda_+(c(e_\mathfrak{s}), u_{D\sigma,\sigma^*,\mathfrak{s}})]^+ \leqslant \frac{1}{1+\mathrm{reg}\,(\mathfrak{T})}, \quad \forall D_{\sigma,\sigma^*} \in \mathfrak{D}_{int}, \tag{31}$$

$$\frac{\delta t}{|D_{\sigma,\sigma^*}|}\Big(\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\setminus\partial\Omega}|\mathfrak{s}|[\lambda_+(c(e_\mathfrak{s}),u_{D\sigma,\sigma^*,\mathfrak{s}})]^+ + |\sigma|[\lambda_+(c(e_\sigma),u_\sigma)]^+\Big) \leqslant \frac{1}{\mathrm{reg}\,(\mathfrak{T})}, \quad \forall D_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}, \tag{32}$$

$$\begin{aligned}\frac{\delta t}{|D_{\sigma,\sigma^*}|}\Big(\big(1+\mathrm{reg}\,(\mathfrak{T})\big)\frac{|\sigma|}{2}[\lambda_+(c(e_\sigma),u_\sigma)]^+ \\ &+ \frac{|\sigma|}{2\big(\mathrm{reg}\,(\mathfrak{T})-1\big)}\frac{\rho_D}{\rho_{\sigma,\sigma^*}}[\lambda_+(c(e_\sigma),u_D)]^- \\ &+ \mathrm{reg}\,(\mathfrak{T})\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\setminus\partial\Omega}|\mathfrak{s}|[\lambda_+(c(e_\mathfrak{s}),u_{D\sigma,\sigma^*,\mathfrak{s}})]^+\Big) \leqslant 1, \quad \forall D_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}.\end{aligned} \tag{33}$$

Then, $\mathcal{R}_K \geqslant 0\ \forall K\in\mathfrak{M}$, $\mathcal{R}_{K^*}\geqslant 0\ \forall K^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*$ *and* $\mathcal{R}_{\sigma,\sigma^*}\geqslant 0\ \forall D_{\sigma,\sigma^*}\in\mathfrak{D}$.

Proof. Let us split the proof into two parts depending on the type of cell we consider.

a) Let $K\in\mathfrak{M}$. By using the averaged mass equation in Proposition 3.5, the remainder term $\mathcal{R}_K$ in Definition 3.9 can be rewritten as

$$\mathcal{R}_K = \frac{\|\overline{\mathbf{u}}_K-\mathbf{u}_K\|^2}{2\delta t}\left(\rho_K - \frac{\delta t}{|K|}\sum_{\sigma\in\partial K}|\sigma|\mathcal{F}^+_{K,\sigma}\right) - \frac{1}{|K|}\sum_{\sigma\in\partial K}|\sigma|\mathcal{F}^-_{K,\sigma}\frac{\|\overline{\mathbf{u}}_K-\mathbf{u}_L\|^2}{2},$$

where the last contribution is non negative since $\mathcal{F}^-_{K,\sigma}\leqslant 0$. Hence, we get

$$\mathcal{R}_K \geqslant \frac{\|\overline{\mathbf{u}}_K-\mathbf{u}_K\|^2}{2\delta t}\mathcal{A}_K \text{ where } \mathcal{A}_K = \rho_K - \frac{\delta t}{|K|}\sum_{\sigma\in\partial K}|\sigma|\mathcal{F}^+_{K,\sigma}. \tag{34}$$

Having $\mathcal{A}_K\geqslant 0$ is thus enough to ensure $\mathcal{R}_K\geqslant 0$. Going back to Definition 3.4 for ρ_K and to (13) for $\mathcal{F}^+_{K,\sigma}$, we modify the expression of $\mathcal{A}_K$, and we arrive at

$$\begin{aligned}\mathcal{A}_K = &\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{ext}}\left(\frac{|D_{\sigma,\sigma^*}\cap K|}{|K|}\rho_{\sigma,\sigma^*} - \frac{\delta t}{|K|}|\sigma|\mathcal{F}^+_\sigma\right) \\ &+ \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}}\Bigg(\frac{|D_{\sigma,\sigma^*}\cap K|}{|K|}\rho_{\sigma,\sigma^*} - \frac{\delta t}{|K|}\frac{|D_{\sigma,\sigma^*}\cap K|}{|D_{\sigma,\sigma^*}|}\sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset L}}|\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}} \\ &\qquad + \frac{\delta t}{|K|}\frac{|D_{\sigma,\sigma^*}\cap L|}{|D_{\sigma,\sigma^*}|}\sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K}}|\mathfrak{s}|\mathcal{F}^-_{D\sigma,\sigma^*,\mathfrak{s}}\Bigg).\end{aligned}$$

Adding and substracting $\displaystyle\frac{|D_{\sigma,\sigma^*}\cap K|}{|D_{\sigma,\sigma^*}|}\sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K}}|\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}$ leads to:

$$\begin{aligned}\mathcal{A}_K = &\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{ext}}\left(\frac{|D_{\sigma,\sigma^*}\cap K|}{|K|}\rho_{\sigma,\sigma^*} - \frac{\delta t}{|K|}|\sigma|\mathcal{F}^+_\sigma\right) \\ &+ \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}}\frac{|D_{\sigma,\sigma^*}\cap K|}{|K|}\left(\rho_{\sigma,\sigma^*} - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}\right) \\ &+ \frac{\delta t}{|K|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}}\frac{1}{|D_{\sigma,\sigma^*}|}\sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K}}|\mathfrak{s}|\left(|D_{\sigma,\sigma^*}\cap K|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}} + |D_{\sigma,\sigma^*}\cap L|\mathcal{F}^-_{D\sigma,\sigma^*,\mathfrak{s}}\right).\end{aligned}$$

Let us start by analysing the third sum. By using the equality $|D_{\sigma,\sigma^*}\cap L| = |D_{\sigma,\sigma^*}| - |D_{\sigma,\sigma^*}\cap K|$, we have

$$\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}} \frac{1}{|D_{\sigma,\sigma^*}|} \sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K}} |\mathfrak{s}| \left(|D_{\sigma,\sigma^*}\cap K|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}} + |D_{\sigma,\sigma^*}\cap L|\mathcal{F}^-_{D\sigma,\sigma^*,\mathfrak{s}}\right)$$

$$= \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}} \frac{1}{|D_{\sigma,\sigma^*}|} \sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K}} |\mathfrak{s}| \left(|D_{\sigma,\sigma^*}\cap K|\mathcal{F}^{|\cdot|}_{D\sigma,\sigma^*,\mathfrak{s}} + |D_{\sigma,\sigma^*}|\mathcal{F}^-_{D\sigma,\sigma^*,\mathfrak{s}}\right).$$

Since the flux $\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}}$ is conservative, we get

$$\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}} \sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K}} |\mathfrak{s}|\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}} = -\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{ext}} \sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\setminus\partial\Omega} |\mathfrak{s}|\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}},$$

and thus

$$\begin{aligned}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}} \sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K}} |\mathfrak{s}|\mathcal{F}^-_{D\sigma,\sigma^*,\mathfrak{s}} \quad = \quad & -\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}} \sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K}} |\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}} \\ & -\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{ext}} \sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\setminus\partial\Omega} |\mathfrak{s}|\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}}.\end{aligned}$$

Plugging this result in the definition of $\mathcal{A}_K$ yields

$$\begin{aligned}\mathcal{A}_K = & \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{ext}} \left(\frac{|D_{\sigma,\sigma^*}\cap K|}{|K|}\rho_{\sigma,\sigma^*} - \frac{\delta t}{|K|}|\sigma|\mathcal{F}^+_\sigma - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\setminus\partial\Omega} |\mathfrak{s}|\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}}\right) \\ & + \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}} \frac{|D_{\sigma,\sigma^*}\cap K|}{|K|}\left(\rho_{\sigma,\sigma^*} - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}\right) \\ & + \frac{\delta t}{|K|}\left(\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}} \frac{|D_{\sigma,\sigma^*}\cap K|}{|D_{\sigma,\sigma^*}|}\sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K}} |\mathfrak{s}|\mathcal{F}^{|\cdot|}_{D\sigma,\sigma^*,\mathfrak{s}} - \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}} \sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K}} |\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}\right).\end{aligned}$$

We are going to establish a bound from below. To this end, we proceed as follows. Since $\mathcal{F}^{|\cdot|} \geqslant 0$ and $\mathcal{F} \leqslant \mathcal{F}^+$, we get

$$\mathcal{A}_K \geqslant \mathcal{B}^{ext}_K + \mathcal{B}^{int}_K,$$

with

$$\mathcal{B}^{ext}_K = \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{ext}} \left(\frac{|D_{\sigma,\sigma^*}\cap K|}{|K|}\rho_{\sigma,\sigma^*} - \frac{\delta t}{|K|}|\sigma|\mathcal{F}^+_\sigma - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\setminus\partial\Omega} |\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}\right),$$

and

$$\begin{aligned}\mathcal{B}^{int}_K = & \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}} \frac{|D_{\sigma,\sigma^*}\cap K|}{|K|}\left(\rho_{\sigma,\sigma^*} - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}\right) \\ & - \frac{\delta t}{|K|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}} \sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K}} |\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}.\end{aligned}$$

Substracting the non negative term $\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}} \sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset L}} |\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}$ to $\mathcal{B}_K^{int}$, we get

$$\begin{aligned}\mathcal{B}_K^{int} \geqslant & \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}} \frac{|D_{\sigma,\sigma^*}\cap K|}{|K|}\left(\rho_{\sigma,\sigma^*} - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}\right)\\ &\qquad\qquad - \frac{\delta t}{|K|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}\\ \geqslant & \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}} \frac{|D_{\sigma,\sigma^*}\cap K|}{|K|}\rho_{\sigma,\sigma^*}\left(1-\frac{\delta t}{|D_{\sigma,\sigma^*}|}\left(1+\frac{|D_{\sigma,\sigma^*}|}{|D_{\sigma,\sigma^*}\cap K|}\right)\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|\frac{\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}}{\rho_{\sigma,\sigma^*}}\right).\end{aligned}$$

By using Lemma 3.2 we end up with

$$\mathcal{B}_K^{int} \geqslant \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{int}} \frac{|D_{\sigma,\sigma^*}\cap K|}{|K|}\rho_{\sigma,\sigma^*}\left(1-\frac{\delta t}{|D_{\sigma,\sigma^*}|}\left(1+\frac{|D_{\sigma,\sigma^*}|}{|D_{\sigma,\sigma^*}\cap K|}\right)\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|[\lambda_+(c(e_\mathfrak{s}),u_{D\sigma,\sigma^*,\mathfrak{s}})]^+\right). \tag{35}$$

Therefore $\mathcal{B}_K^{int} \geqslant 0$ holds when

$$1 \geqslant \frac{\delta t}{|D_{\sigma,\sigma^*}|}\left(1+\frac{|D_{\sigma,\sigma^*}|}{|D_{\sigma,\sigma^*}\cap K|}\right)\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|[\lambda_+(c(e_\mathfrak{s}),u_{D\sigma,\sigma^*,\mathfrak{s}})]^+.$$

This inequality holds when (31) is fulfilled since $\operatorname{reg}(\mathfrak{T}) \geqslant \dfrac{|D_{\sigma,\sigma^*}|}{|D_{\sigma,\sigma^*}\cap K|}$.

b) We now turn to the study of $\mathcal{B}_K^{ext}$. Since, for all $D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{ext}$, $|D_{\sigma,\sigma^*}\cap K| = |D_{\sigma,\sigma^*}|$, we have

$$\begin{aligned}\mathcal{B}_K^{ext} = & \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{ext}} \frac{|D_{\sigma,\sigma^*}|}{|K|}\rho_{\sigma,\sigma^*}\left(1-\frac{\delta t}{|D_{\sigma,\sigma^*}|}|\sigma|\frac{\mathcal{F}^+_\sigma}{\rho_{\sigma,\sigma^*}} - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\frac{|K|}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\setminus\partial\Omega}|\mathfrak{s}|\frac{\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}}{\rho_{\sigma,\sigma^*}}\right)\\ \geqslant & \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K\cap\mathfrak{D}_{ext}} \frac{|D_{\sigma,\sigma^*}|}{|K|}\rho_{\sigma,\sigma^*}\left(1-\frac{\delta t}{|D_{\sigma,\sigma^*}|}|\sigma|[\lambda_+(c(e_\sigma),u_\sigma)]^+\right.\\ &\qquad\qquad\left. - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\operatorname{reg}(\mathfrak{T})\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\setminus\partial\Omega}|\mathfrak{s}|[\lambda_+(c(e_\mathfrak{s}),u_{D\sigma,\sigma^*,\mathfrak{s}})]^+\right).\end{aligned}$$

Therefore $\mathcal{B}_K^{ext} \geqslant 0$ holds when

$$1 \geqslant \frac{\delta t}{|D_{\sigma,\sigma^*}|}\left(|\sigma|[\lambda_+(c(e_\sigma),u_\sigma)]^+ + \operatorname{reg}(\mathfrak{T})\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\setminus\partial\Omega}|\mathfrak{s}|[\lambda_+(c(e_\mathfrak{s}),u_{D\sigma,\sigma^*,\mathfrak{s}})]^+\right)$$

This inequality holds when (32) is fulfilled since $\operatorname{reg}(\mathfrak{T}) \geqslant 1$. Thus, $\mathcal{R}_K \geqslant 0$, for any $K\in\mathfrak{M}$.

We now turn to the study of $\mathcal{R}_{K^*}$ for $K^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*$. The proof follows the same lines. By using the averaged mass equation in Proposition 3.5, the remainder term $\mathcal{R}_{K^*}$ in Definition 3.9

can be rewritten as

$$\mathcal{R}_{K^*} = \frac{\|\overline{\mathbf{u}}_{K^*} - \mathbf{u}_{K^*}\|^2}{2\delta t}\Big(\rho_{K^*} - \frac{\delta t}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}} |\sigma^*|\mathcal{F}^+_{K^*,\sigma^*} - \frac{\delta t}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}} \frac{|\sigma|}{2}\mathcal{F}^+_{\sigma}\Big)$$
$$- \frac{1}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}} |\sigma^*|\mathcal{F}^-_{K^*,\sigma^*}\frac{\|\overline{\mathbf{u}}_{K^*} - \mathbf{u}_{L^*}\|^2}{2}$$
$$- \frac{1}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}} \frac{|\sigma|}{2}\mathcal{F}^-_{\sigma}\frac{\|\overline{\mathbf{u}}_{K^*} - \mathbf{u}_{\sigma}\|^2}{2},$$

where the two last contribution are non negative since $\mathcal{F}^-_{K^*,\sigma^*} \leqslant 0$ and $\mathcal{F}^-_{\sigma} \leqslant 0$. Hence, we get

$$\mathcal{R}_{K^*} \geqslant \frac{\|\overline{\mathbf{u}}_{K^*} - \mathbf{u}_{K^*}\|^2}{2\delta t}\mathcal{A}_{K^*}, \tag{36}$$

where

$$\mathcal{A}_{K^*} = \rho_{K^*} - \frac{\delta t}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}} |\sigma^*|\mathcal{F}^+_{K^*,\sigma^*} - \frac{\delta t}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}} \frac{|\sigma|}{2}\mathcal{F}^+_{\sigma}.$$

Having $\mathcal{A}_{K^*} \geqslant 0$ is thus enough to ensure $\mathcal{R}_{K^*} \geqslant 0$. Going back to Definition 3.4 for ρ_{K^*} and to (14) for $\mathcal{F}^+_{K^*,\sigma^*}$, we modify the expression of $\mathcal{A}_{K^*}$, and we arrive at

$$\mathcal{A}_{K^*} = \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}} \Big(\frac{|D_{\sigma,\sigma^*}\cap K^*|}{|K^*|}\rho_{\sigma,\sigma^*} - \frac{\delta t}{|K^*|}\frac{|\sigma|}{2}\mathcal{F}^+_{\sigma}$$
$$- \frac{\delta t}{|K^*|}\frac{|D_{\sigma,\sigma^*}\cap K^*|}{|D_{\sigma,\sigma^*}|}|\mathfrak{s}_{KL^*}|\mathcal{F}^+_{\sigma,\mathfrak{s}_{KL^*}} + \frac{\delta t}{|K^*|}\frac{|D_{\sigma,\sigma^*}\cap L^*|}{|D_{\sigma,\sigma^*}|}|\mathfrak{s}_{KK^*}|\mathcal{F}^-_{\sigma,\mathfrak{s}_{KK^*}}$$
$$- \frac{\delta t}{|K^*|}\frac{|D_{\sigma,\sigma^*}\cap K^*|}{2|D_{\sigma,\sigma^*}|}|\sigma|\mathcal{F}^+_{\sigma} + \frac{\delta t}{|K^*|}\frac{|D_{\sigma,\sigma^*}\cap L^*|}{2|D_{\sigma,\sigma^*}|}|\sigma|\mathcal{F}^-_{\sigma}\Big)$$
$$+ \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}} \Big(\frac{|D_{\sigma,\sigma^*}\cap K^*|}{|K^*|}\rho_{\sigma,\sigma^*} - \frac{\delta t}{|K^*|}\frac{|D_{\sigma,\sigma^*}\cap K^*|}{|D_{\sigma,\sigma^*}|}\sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset L^*}} |\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}$$
$$+ \frac{\delta t}{|K^*|}\frac{|D_{\sigma,\sigma^*}\cap L^*|}{|D_{\sigma,\sigma^*}|}\sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K^*}} |\mathfrak{s}|\mathcal{F}^-_{D\sigma,\sigma^*,\mathfrak{s}}\Big).$$

Adding and substracting $\frac{|D_{\sigma,\sigma^*}\cap K^*|}{|D_{\sigma,\sigma^*}|}\sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K^*}} |\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}$ leads to:

$$\mathcal{A}_{K^*} = \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}} \frac{|D_{\sigma,\sigma^*}\cap K^*|}{|K^*|}\Big(\rho_{\sigma,\sigma^*} - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\frac{|\sigma|}{2}\Big(\big(1 + \frac{|D_{\sigma,\sigma^*}|}{|D_{\sigma,\sigma^*}\cap K^*|}\big)\mathcal{F}^+_{\sigma} - \frac{|D_{\sigma,\sigma^*}\cap L^*|}{|D_{\sigma,\sigma^*}\cap K^*|}\mathcal{F}^-_{\sigma}\Big)\Big)$$
$$+ \frac{\delta t}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}} \Big(-\frac{|D_{\sigma,\sigma^*}\cap K^*|}{|D_{\sigma,\sigma^*}|}|\mathfrak{s}_{KL^*}|\mathcal{F}^+_{\sigma,\mathfrak{s}_{KL^*}} + \frac{|D_{\sigma,\sigma^*}\cap L^*|}{|D_{\sigma,\sigma^*}|}|\mathfrak{s}_{KK^*}|\mathcal{F}^-_{\sigma,\mathfrak{s}_{KK^*}}\Big)$$
$$+ \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}} \frac{|D_{\sigma,\sigma^*}\cap K^*|}{|K^*|}\Big(\rho_{\sigma,\sigma^*} - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}\Big)$$
$$+ \frac{\delta t}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}} \frac{1}{|D_{\sigma,\sigma^*}|}\sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K^*}} |\mathfrak{s}|\Big(|D_{\sigma,\sigma^*}\cap K^*|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}} + |D_{\sigma,\sigma^*}\cap L^*|\mathcal{F}^-_{D\sigma,\sigma^*,\mathfrak{s}}\Big).$$

Let us start by analysing the fourth sum. Using equality $|D_{\sigma,\sigma^*}\cap L^*| = |D_{\sigma,\sigma^*}| - |D_{\sigma,\sigma^*}\cap K^*|$, we have

$$\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}} \frac{1}{|D_{\sigma,\sigma^*}|} \sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K^*}} |\mathfrak{s}| \left(|D_{\sigma,\sigma^*}\cap K^*|\mathcal{F}^{+}_{D\sigma,\sigma^*,\mathfrak{s}} + |D_{\sigma,\sigma^*}\cap L^*|\mathcal{F}^{-}_{D\sigma,\sigma^*,\mathfrak{s}}\right)$$

$$= \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}} \frac{1}{|D_{\sigma,\sigma^*}|} \sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K^*}} |\mathfrak{s}| \left(|D_{\sigma,\sigma^*}\cap K^*|\mathcal{F}^{|\cdot|}_{D\sigma,\sigma^*,\mathfrak{s}} + |D_{\sigma,\sigma^*}|\mathcal{F}^{-}_{D\sigma,\sigma^*,\mathfrak{s}}\right).$$

Since the flux $\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}}$ is conservative, we get

$$\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}} \sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K^*}} |\mathfrak{s}|\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}} = -\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}} \sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\setminus\partial\Omega\\ \mathfrak{s}\subset K^*}} |\mathfrak{s}|\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}},$$

and thus

$$\begin{aligned}
&\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}} \sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K^*}} |\mathfrak{s}|\mathcal{F}^{-}_{D\sigma,\sigma^*,\mathfrak{s}}\\
&= -\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}} \sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K^*}} |\mathfrak{s}|\mathcal{F}^{+}_{D\sigma,\sigma^*,\mathfrak{s}}\\
&\quad -\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}} \sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\setminus\partial\Omega\\ \mathfrak{s}\subset K^*}} |\mathfrak{s}|\mathcal{F}_{D\sigma,\sigma^*,\mathfrak{s}}.
\end{aligned}$$

Plugging this result in the definition of $\mathcal{A}_{K^*}$ yields

$$\begin{aligned}
\mathcal{A}_{K^*} &= \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}} \frac{|D_{\sigma,\sigma^*}\cap K^*|}{|K^*|}\left(\rho_{\sigma,\sigma^*} - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\frac{|\sigma|}{2}\left(\left(1+\frac{|D_{\sigma,\sigma^*}|}{|D_{\sigma,\sigma^*}\cap K^*|}\right)\mathcal{F}^{+}_{\sigma} - \frac{|D_{\sigma,\sigma^*}\cap L^*|}{|D_{\sigma,\sigma^*}\cap K^*|}\mathcal{F}^{-}_{\sigma}\right)\right)\\
&-\frac{\delta t}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}}\left(\frac{|D_{\sigma,\sigma^*}\cap K^*|}{|D_{\sigma,\sigma^*}|}|\mathfrak{s}_{KL^*}|\mathcal{F}^{+}_{\sigma,\mathfrak{s}_{KL^*}} + \frac{|D_{\sigma,\sigma^*}\cap K^*|}{|D_{\sigma,\sigma^*}|}|\mathfrak{s}_{KK^*}|\mathcal{F}^{-}_{\sigma,\mathfrak{s}_{KK^*}} + |\mathfrak{s}_{KK^*}|\mathcal{F}^{+}_{\sigma,\mathfrak{s}_{KK^*}}\right)\\
&+\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}} \frac{|D_{\sigma,\sigma^*}\cap K^*|}{|K^*|}\left(\rho_{\sigma,\sigma^*} - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|\mathcal{F}^{+}_{D\sigma,\sigma^*,\mathfrak{s}}\right)\\
&+\frac{\delta t}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}} \frac{1}{|D_{\sigma,\sigma^*}|}\sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K^*}}|\mathfrak{s}||D_{\sigma,\sigma^*}\cap K^*|\mathcal{F}^{|\cdot|}_{D\sigma,\sigma^*,\mathfrak{s}}\\
&-\frac{\delta t}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}} \sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K^*}}|\mathfrak{s}|\mathcal{F}^{+}_{D\sigma,\sigma^*,\mathfrak{s}}.
\end{aligned}$$

Since $\mathcal{F}^{|\cdot|}\geqslant 0$ and $\mathcal{F}^{-}_{\sigma,\mathfrak{s}_{KK^*}}\leqslant 0$, we have

$$\mathcal{A}_{K^*}\geqslant \mathcal{B}^{ext}_{K^*}+\mathcal{B}^{int}_{K^*},$$

with

$$\begin{aligned}
\mathcal{B}^{ext}_{K^*} &= \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}} \frac{|D_{\sigma,\sigma^*}\cap K^*|}{|K^*|}\left(\rho_{\sigma,\sigma^*} - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\frac{|\sigma|}{2}\left(\left(1+\frac{|D_{\sigma,\sigma^*}|}{|D_{\sigma,\sigma^*}\cap K^*|}\right)\mathcal{F}^{+}_{\sigma} - \frac{|D_{\sigma,\sigma^*}\cap L^*|}{|D_{\sigma,\sigma^*}\cap K^*|}\mathcal{F}^{-}_{\sigma}\right)\right)\\
&-\frac{\delta t}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}}\left(\frac{|D_{\sigma,\sigma^*}\cap K^*|}{|D_{\sigma,\sigma^*}|}|\mathfrak{s}_{KL^*}|\mathcal{F}^{+}_{\sigma,\mathfrak{s}_{KL^*}} + |\mathfrak{s}_{KK^*}|\mathcal{F}^{+}_{\sigma,\mathfrak{s}_{KK^*}}\right)
\end{aligned}$$

and

$$\mathcal{B}^{int}_{K^*} = \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}} \frac{|D_{\sigma,\sigma^*}\cap K^*|}{|K^*|}\left(\rho_{\sigma,\sigma^*} - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}\right) - \frac{\delta t}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}}\sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset K^*}} |\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}.$$

Substracting the non negative term $\frac{\delta t}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}}\sum_{\substack{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\\ \mathfrak{s}\subset L^*}} |\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}$ to $\mathcal{B}^{int}_{K^*}$, we get

$$\begin{aligned}\mathcal{B}^{int}_{K^*} &\geqslant \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}} \frac{|D_{\sigma,\sigma^*}\cap K^*|}{|K^*|}\left(\rho_{\sigma,\sigma^*} - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}\right) \\ &\qquad - \frac{\delta t}{|K^*|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}} \\ &\geqslant \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}} \frac{|D_{\sigma,\sigma^*}\cap K^*|}{|K^*|}\rho_{\sigma,\sigma^*}\left(1 - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\left(1+\frac{|D_{\sigma,\sigma^*}|}{|D_{\sigma,\sigma^*}\cap K^*|}\right)\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\frac{\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}}{\rho_{\sigma,\sigma^*}}\right).\end{aligned}$$

By using Lemma 3.2 we end up with

$$\mathcal{B}^{int}_{K^*} \geqslant \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{int}} \frac{|D_{\sigma,\sigma^*}\cap K^*|}{|K^*|}\rho_{\sigma,\sigma^*}\left(1 - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\left(1+\frac{|D_{\sigma,\sigma^*}|}{|D_{\sigma,\sigma^*}\cap K^*|}\right)\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|[\lambda_+(c(e_\mathfrak{s}),u_{D\sigma,\sigma^*,\mathfrak{s}})]^+\right).$$

Therefore $\mathcal{B}^{int}_{K^*}\geqslant 0$ holds when

$$1 \geqslant \frac{\delta t}{|D_{\sigma,\sigma^*}|}\left(1+\frac{|D_{\sigma,\sigma^*}|}{|D_{\sigma,\sigma^*}\cap K^*|}\right)\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|[\lambda_+(c(e_\mathfrak{s}),u_{D\sigma,\sigma^*,\mathfrak{s}})]^+.$$

This inequality holds when (31) is fulfilled since $\mathrm{reg}\,(\mathfrak{T}) \geqslant \dfrac{|D_{\sigma,\sigma^*}|}{|D_{\sigma,\sigma^*}\cap K^*|}$.

We now turn to the study of $\mathcal{B}^{ext}_{K^*}$. Since, for all $D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}$, $|D_{\sigma,\sigma^*}\cap K^*| = |D_{\sigma,\sigma^*}|$, we have

$$\begin{aligned}\mathcal{B}^{ext}_{K^*} &= \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}} \frac{|D_{\sigma,\sigma^*}\cap K^*|}{|K^*|}\Bigg(\rho_{\sigma,\sigma^*} - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\frac{|\sigma|}{2}\left(\left(1+\frac{|D_{\sigma,\sigma^*}|}{|D_{\sigma,\sigma^*}\cap K^*|}\right)\mathcal{F}^+_\sigma - \frac{|D_{\sigma,\sigma^*}\cap L^*|}{|D_{\sigma,\sigma^*}\cap K^*|}\mathcal{F}^-_\sigma\right) \\ &\qquad - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\left(|\mathfrak{s}_{KL^*}|\mathcal{F}^+_{\sigma,\mathfrak{s}_{KL^*}} + \frac{|D_{\sigma,\sigma^*}|}{|D_{\sigma,\sigma^*}\cap K^*|}|\mathfrak{s}_{KK^*}|\mathcal{F}^+_{\sigma,\mathfrak{s}_{KK^*}}\right)\Bigg) \\ &\geqslant \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{K^*}\cap\mathfrak{D}_{ext}} \frac{|D_{\sigma,\sigma^*}\cap K^*|}{|K^*|}\rho_{\sigma,\sigma^*}\Bigg(1 - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\frac{|\sigma|}{2}\left(\left(1+\mathrm{reg}\,(\mathfrak{T})\right)\frac{\mathcal{F}^+_\sigma}{\rho_{\sigma,\sigma^*}} - \frac{1}{\mathrm{reg}\,(\mathfrak{T})-1}\frac{\mathcal{F}^-_\sigma}{\rho_{\sigma,\sigma^*}}\right) \\ &\qquad - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\mathrm{reg}\,(\mathfrak{T})\left(|\mathfrak{s}_{KL^*}|\frac{\mathcal{F}^+_{\sigma,\mathfrak{s}_{KL^*}}}{\rho_{\sigma,\sigma^*}} + |\mathfrak{s}_{KK^*}|\frac{\mathcal{F}^+_{\sigma,\mathfrak{s}_{KK^*}}}{\rho_{\sigma,\sigma^*}}\right)\Bigg).\end{aligned} \tag{37}$$

Therefore $\mathcal{B}_K^{ext} \geqslant 0$ holds when

$$1 \geqslant \frac{\delta t}{|D_{\sigma,\sigma^*}|}\Big(\big(1+\mathrm{reg}\,(\mathfrak{T})\big)\frac{|\sigma|}{2}[\lambda_+(c(e_\sigma),u_\sigma)]^+ + \frac{|\sigma|}{2\big(\mathrm{reg}\,(\mathfrak{T})-1\big)}\frac{\rho_D}{\rho_{\sigma,\sigma^*}}[\lambda_+(c(e_\sigma),u_D)]^- + \mathrm{reg}\,(\mathfrak{T})\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\setminus\partial\Omega}|\mathfrak{s}|[\lambda_+(c(e_\mathfrak{s}),u_{D\sigma,\sigma^*,\mathfrak{s}})]^+\Big)$$

This inequality holds when (33) is fulfilled. Thus, $\mathcal{R}_{K^*} \geqslant 0$.

That $\mathcal{R}_{D_{\sigma,\sigma^*}}$ is non negative follows from the fact that $\mathcal{R}_K$, $\mathcal{R}_L$, $\mathcal{R}_{K^*}$ and $\mathcal{R}_{L^*}$ are all non negative. ∎

We are now able to exhibit the CFL-like condition that ensures the non-negativity of the internal energy e_{σ,σ^*}.

Proposition 3.16. *Let $e_{\sigma,\sigma^*} \geqslant 0$. We assume that (31), (32), (33) are fulfilled, and, moreover, that the following CFL-like conditions are satisfied on each diamond cell $D_{\sigma,\sigma^*} \in \mathfrak{D}$:*

$$\frac{1}{\gamma} \geqslant \frac{\delta t}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|\left(\frac{|\sigma^*|+|\sigma|}{\sqrt{2}|\mathfrak{s}|}c(e_{\sigma,\sigma^*})+c(e_\mathfrak{s})+[u_{D\sigma,\sigma^*,\mathfrak{s}}]^+\right),\ \forall D_{\sigma,\sigma^*}\in\mathfrak{D}_{int} \tag{38}$$

$$\frac{\gamma}{1+\mathrm{reg}\,(\mathfrak{T})} \geqslant \frac{\delta t}{|D_{\sigma,\sigma^*}|}\left(1+\frac{c(e_{\sigma,\sigma^*})|\sigma|}{\sqrt{2}}\mathrm{reg}\,(\mathfrak{T})\right),\ \forall D_{\sigma,\sigma^*}\in\mathfrak{D}_{int} \tag{39}$$

$$\frac{1}{\gamma}\geqslant\frac{\delta t}{|D_{\sigma,\sigma^*}|}\left(\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|\left(\frac{|\sigma^*|+|\sigma|}{\sqrt{2}|\mathfrak{s}|}c(e_{\sigma,\sigma^*})+c(e_\mathfrak{s})+[u_{D\sigma,\sigma^*,\mathfrak{s}}]^+\right)+|\sigma|([u_\sigma]^++c(e_\sigma))\right),\ \forall D_{\sigma,\sigma^*}\in\mathfrak{D}_{ext} \tag{40}$$

$$\frac{\gamma}{1+\mathrm{reg}\,(\mathfrak{T})}\geqslant\frac{\delta t}{|D_{\sigma,\sigma^*}|}\left(1+\frac{c(e_{\sigma,\sigma^*})|\sigma|}{\sqrt{2}}\mathrm{reg}\,(\mathfrak{T})+\frac{|\sigma|\gamma/2}{1-\mathrm{reg}\,(\mathfrak{T})^2}\frac{\rho_D}{\rho_{\sigma,\sigma^*}}[\lambda_+(c(e_\sigma),u_D)]^-\right),\ \forall D_{\sigma,\sigma^*}\in\mathfrak{D}_{ext} \tag{41}$$

Then the non negativity of the internal energy is preserved: we have $\overline{e}_{\sigma,\sigma^*} \geqslant 0$.

Proof. We start by observing that (31), (32) implies (30), so that $\overline{\rho}_{\sigma,\sigma^*} \geqslant 0$ if $\rho_{\sigma,\sigma^*} \geqslant 0$. Next, we turn to the non negativity of $\overline{e}_{\sigma,\sigma^*}$; we follow the arguments in [16].
Let us write $(\boldsymbol{\nabla_d}\cdot\overline{\mathbf{u}})_{\sigma,\sigma^*}$ as in Lemma 3.12 and then apply the Young inequality for each four terms. For $X \in \mathfrak{M}$ and $i \in \{0,1\}$ we write

$$\begin{aligned}(-1)^i p_{\sigma,\sigma^*}\overline{\mathbf{u}}_X\mathbf{n}_{K,\sigma} &= (-1)^i(\gamma-1)\left[\rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*}(\overline{\mathbf{u}}_X-\mathbf{u}_X)\cdot\mathbf{n}_{K,\sigma}+\rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*}\mathbf{u}_X\cdot\mathbf{n}_{K,\sigma}\right]\\ &\geqslant -\rho_{\sigma,\sigma^*}\left[\frac{c(e_{\sigma,\sigma^*})}{2\sqrt{2}\gamma}\|\overline{\mathbf{u}}_X-\mathbf{u}_X\|^2+(\gamma-1)e_{\sigma,\sigma^*}\left(\frac{c(e_{\sigma,\sigma^*})}{\sqrt{2}}-(-1)^i\mathbf{u}_X\mathbf{n}_{K,\sigma}\right)\right],\end{aligned}$$

so that

$$\begin{aligned}-\frac{\delta t|\sigma|}{2|D_{\sigma,\sigma^*}|}p_{\sigma,\sigma^*}(\overline{\mathbf{u}}_L-\overline{\mathbf{u}}_K)\mathbf{n}_{K,\sigma} \geqslant &-\frac{\delta t|\sigma|}{2|D_{\sigma,\sigma^*}|}\rho_{\sigma,\sigma^*}\frac{c(e_{\sigma,\sigma^*})}{2\sqrt{2}\gamma}\left(\|\overline{\mathbf{u}}_K-\mathbf{u}_K\|^2+\|\overline{\mathbf{u}}_L-\mathbf{u}_L\|^2\right)\\ &-\frac{\delta t|\sigma|}{2|D_{\sigma,\sigma^*}|}(\gamma-1)\rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*}\left(\frac{2c(e_{\sigma,\sigma^*})}{\sqrt{2}}+(\mathbf{u}_L-\mathbf{u}_K)\mathbf{n}_{K,\sigma}\right).\end{aligned}$$

For $X \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ and $i \in \{0,1\}$ we write

$$\begin{aligned}(-1)^i p_{\sigma,\sigma^*}\overline{\mathbf{u}}_X\mathbf{n}_{K^*,\sigma^*} &= (-1)^i(\gamma-1)\left[\rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*}(\overline{\mathbf{u}}_X-\mathbf{u}_X)\cdot\mathbf{n}_{K^*,\sigma^*}+\rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*}\mathbf{u}_X\cdot\mathbf{n}_{K^*,\sigma^*}\right]\\ &\geqslant -\rho_{\sigma,\sigma^*}\left[\frac{c(e_{\sigma,\sigma^*})}{2\sqrt{2}\gamma}\|\overline{\mathbf{u}}_X-\mathbf{u}_X\|^2+(\gamma-1)e_{\sigma,\sigma^*}\left(\frac{c(e_{\sigma,\sigma^*})}{\sqrt{2}}-(-1)^i\mathbf{u}_X\mathbf{n}_{K^*,\sigma^*}\right)\right],\end{aligned}$$

so that

$$-\frac{\delta t|\sigma^*|}{2|D_{\sigma,\sigma^*}|}p_{\sigma,\sigma^*}(\overline{\mathbf{u}}_{L^*}-\overline{\mathbf{u}}_{K^*})\mathbf{n}_{K^*,\sigma^*} \geqslant -\frac{\delta t|\sigma^*|}{2|D_{\sigma,\sigma^*}|}\rho_{\sigma,\sigma^*}\frac{c(e_{\sigma,\sigma^*})}{2\sqrt{2}\gamma}\left(\|\overline{\mathbf{u}}_{K^*}-\mathbf{u}_{K^*}\|^2+\|\overline{\mathbf{u}}_{L^*}-\mathbf{u}_{L^*}\|^2\right)$$
$$-\frac{\delta t|\sigma^*|}{2|D_{\sigma,\sigma^*}|}(\gamma-1)\rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*}\left(\frac{2c(e_{\sigma,\sigma^*})}{\sqrt{2}}+(\mathbf{u}_{L^*}-\mathbf{u}_{K^*})\mathbf{n}_{K^*,\sigma^*}\right).$$

Let us split the proof into two steps, depending on the localisation of D_{σ,σ^*} in $\mathfrak{D}_{int}$ or $\mathfrak{D}_{ext}$.

a) Suppose that $D_{\sigma,\sigma^*}\in\mathfrak{D}_{int}$, we get $\overline{\rho}_{\sigma,\sigma^*}\overline{e}_{\sigma,\sigma^*}\geqslant T_0+T_K+T_L+T_{K^*}+T_{L^*}$ with

$$\begin{aligned}T_0=\rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*}&-\rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*}\frac{\delta t|\sigma|}{2|D_{\sigma,\sigma^*}|}(\gamma-1)\left(\frac{2c(e_{\sigma,\sigma^*})}{\sqrt{2}}+(\mathbf{u}_L-\mathbf{u}_K)\mathbf{n}_{K,\sigma}\right)\\&-\rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*}\frac{\delta t|\sigma^*|}{2|D_{\sigma,\sigma^*}|}(\gamma-1)\left(\frac{2c(e_{\sigma,\sigma^*})}{\sqrt{2}}+(\mathbf{u}_{L^*}-\mathbf{u}_{K^*})\mathbf{n}_{K^*,\sigma^*}\right)\\&-\frac{\delta t}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|\mathcal{E}_{D\sigma,\sigma^*,\mathfrak{s}},\end{aligned}$$

and for $X\in\{K,L,K^*,L^*\}$

$$T_X=\frac{\delta t}{2|D_{\sigma,\sigma^*}|}\left(|D_{\sigma,\sigma^*}\cap X|\mathcal{R}_X-\frac{|\sigma|}{2\sqrt{2}\gamma}\rho_{\sigma,\sigma^*}c(e_{\sigma,\sigma^*})\|\overline{\mathbf{u}}_X-\mathbf{u}_X\|^2\right).$$

In order to guaranty $\overline{e}_{\sigma,\sigma^*}\geqslant 0$ it is sufficient to ensure that these five terms are non negative. Using Lemma 3.12 on $(\boldsymbol{\nabla}_{\boldsymbol{d}}\cdot\mathbf{u})_{\sigma,\sigma^*}$, T_0 becomes

$$\begin{aligned}T_0=\rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*}&\left(1-\frac{\delta t}{|D_{\sigma,\sigma^*}|}(\gamma-1)\left(\frac{|\sigma^*|+|\sigma|}{\sqrt{2}}c(e_{\sigma,\sigma^*})+\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|u_{D\sigma,\sigma^*,\mathfrak{s}}\right)\right)\\&-\frac{\delta t}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|\mathcal{E}_{D\sigma,\sigma^*,\mathfrak{s}}.\end{aligned}$$

Owing to Lemma 3.2, we have $\mathcal{E}_{D\sigma,\sigma^*,\mathfrak{s}}=e_{\sigma,\sigma^*}\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}+e_{\sigma',\sigma^{*\prime}}\mathcal{F}^-_{D\sigma',\sigma^{*\prime},\mathfrak{s}}\leqslant e_{\sigma,\sigma^*}\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}}\leqslant\rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*}[\lambda_+(c(e_{\mathfrak{s}}),u_{D\sigma,\sigma^*,\mathfrak{s}})]^+$ and this allows us to bound T_0 from below as

$$\begin{aligned}T_0\geqslant\rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*}\Bigg(1&-\frac{\delta t}{|D_{\sigma,\sigma^*}|}(\gamma-1)\left(\frac{|\sigma^*|+|\sigma|}{\sqrt{2}}c(e_{\sigma,\sigma^*})+\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|u_{D\sigma,\sigma^*,\mathfrak{s}}\right)\\&-\frac{\delta t}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|[\lambda_+(c(e_{\mathfrak{s}}),u_{D\sigma,\sigma^*,\mathfrak{s}})]^+\Bigg).\end{aligned}$$

Finally, since $u_{D\sigma,\sigma^*,\mathfrak{s}}\leqslant[u_{D\sigma,\sigma^*,\mathfrak{s}}]^+$ and $[\lambda_+(c(e_{\mathfrak{s}}),u_{D\sigma,\sigma^*,\mathfrak{s}})]^+\leqslant[u_{D\sigma,\sigma^*,\mathfrak{s}}]^++c(e_{\mathfrak{s}})$, the fact that $T_0\geqslant$ follows from

$$1\geqslant\frac{\delta t}{|D_{\sigma,\sigma^*}|}(\gamma-1)\left(\frac{|\sigma^*|+|\sigma|}{\sqrt{2}}c(e_{\sigma,\sigma^*})+\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|[u_{D\sigma,\sigma^*,\mathfrak{s}}]^+\right)+\frac{\delta t}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|([u_{D\sigma,\sigma^*,\mathfrak{s}}]^++c(e_{\mathfrak{s}})).$$

Gathering the $[u_{D\sigma,\sigma^*,\mathfrak{s}}]^+$ terms and using $\gamma>1$ lead to the assumption (38). Next, we turn to the T_X term. Owing to (34), we can write

$$T_X\geqslant\frac{\|\overline{\mathbf{u}}_X-\mathbf{u}_X\|^2}{4|D_{\sigma,\sigma^*}|}\left(|D_{\sigma,\sigma^*}\cap X|\mathcal{A}_X-\rho_{\sigma,\sigma^*}c(e_{\sigma,\sigma^*})\frac{\delta t|\sigma|}{\sqrt{2}\gamma}\right).$$

Going back to (35), we have $\mathcal{A}_X \geqslant \sum_{\sigma'\in\partial X} \frac{|D_{\sigma',\sigma^{*\prime}}\cap X|}{|X|}\rho_{\sigma',\sigma^{*\prime}} Q_{\sigma',\sigma^{*\prime}}$ where

$$Q_{\sigma',\sigma^{*\prime}} = \left(1 - \frac{\delta t}{|D_{\sigma',\sigma^{*\prime}}|}\left(1 + \frac{|D_{\sigma',\sigma^{*\prime}}|}{|D_{\sigma',\sigma^{*\prime}}\cap X|}\right)\sum_{\mathfrak{s}\in\partial D_{\sigma',\sigma^{*\prime}}} |\mathfrak{s}|[\lambda_+(c(e_{\mathfrak{s}}), u_{D\sigma',\sigma^{*\prime},\mathfrak{s}})]^+\right).$$

With (31), we get $Q_{\sigma',\sigma^{*\prime}} \geqslant 0$ for each $\sigma' \in \partial X$ so that $\mathcal{A}_X \geqslant \frac{|D_{\sigma,\sigma^*}\cap X|}{|X|}\rho_{\sigma,\sigma^*}Q_{\sigma,\sigma^*}$ and

$$T_X \geqslant \frac{|D_{\sigma,\sigma^*}\cap X|^2}{4|D_{\sigma,\sigma^*}||X|}\rho_{\sigma,\sigma^*}\|\overline{\mathbf{u}}_X - \mathbf{u}_X\|^2\tilde{Q}_{\sigma,\sigma^*},$$

where

$$\tilde{Q}_{\sigma,\sigma^*} = Q_{\sigma,\sigma^*} - c(e_{\sigma,\sigma^*})\frac{\delta t}{\sqrt{2}\gamma}\frac{|\sigma||X|}{|D_{\sigma,\sigma^*}\cap X|^2}.$$

That T_X is non negative follows from $\tilde{Q}_{\sigma,\sigma^*} \geqslant 0$ and we have

$$\begin{aligned}
\tilde{Q}_{\sigma,\sigma^*} &\geqslant Q_{\sigma,\sigma^*} - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\frac{c(e_{\sigma,\sigma^*})|\sigma|\mathrm{reg}\,(\mathfrak{T})}{\sqrt{2}\gamma}\frac{|X|}{|D_{\sigma,\sigma^*}\cap X|}\\
&\geqslant 1 - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\left((1+\mathrm{reg}\,(\mathfrak{T}))\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|[\lambda_+(c(e_{\mathfrak{s}}), u_{D\sigma,\sigma^*,\mathfrak{s}})]^+ + \frac{c(e_{\sigma,\sigma^*})|\sigma|\mathrm{reg}\,(\mathfrak{T})}{\sqrt{2}\gamma}\frac{|X|}{|D_{\sigma,\sigma^*}\cap X|}\right).
\end{aligned}$$

Substracting the non-negative term $\frac{\delta t}{|D_{\sigma,\sigma^*}|}(1+\mathrm{reg}\,(\mathfrak{T}))\frac{|\sigma^*|+|\sigma|}{\sqrt{2}}c(e_{\sigma,\sigma^*})$ on the right hand side and using (38) yield

$$\begin{aligned}
\tilde{Q}_{\sigma,\sigma^*} &\geqslant 1 - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\left(\frac{1+\mathrm{reg}\,(\mathfrak{T})}{\gamma} + \frac{c(e_{\sigma,\sigma^*})|\sigma|\mathrm{reg}\,(\mathfrak{T})}{\sqrt{2}\gamma}\frac{|X|}{|D_{\sigma,\sigma^*}\cap X|}\right)\\
&\geqslant 1 - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\frac{1+\mathrm{reg}\,(\mathfrak{T})}{\gamma}\left(1 + \frac{c(e_{\sigma,\sigma^*})|\sigma|}{\sqrt{2}}\frac{|X|}{|D_{\sigma,\sigma^*}\cap X|}\right),
\end{aligned}$$

which is thus non negative when (39) holds.

b) Suppose now that $D_{\sigma,\sigma^*} \in \mathfrak{D}_{ext}$, we get $\overline{\rho}_{\sigma,\sigma^*}\overline{e}_{\sigma,\sigma^*} \geqslant T_0 + T_K + T_L + T_{K^*} + T_{L^*}$ with

$$\begin{aligned}
T_0 = \rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*} &- \rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*}\frac{\delta t|\sigma|}{2|D_{\sigma,\sigma^*}|}(\gamma-1)\left(\frac{2c(e_{\sigma,\sigma^*})}{\sqrt{2}} + (\mathbf{u}_L - \mathbf{u}_K)\mathbf{n}_{K,\sigma}\right)\\
&- \rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*}\frac{\delta t|\sigma^*|}{2|D_{\sigma,\sigma^*}|}(\gamma-1)\left(\frac{2c(e_{\sigma,\sigma^*})}{\sqrt{2}} + (\mathbf{u}_{L^*} - \mathbf{u}_{K^*})\mathbf{n}_{K^*,\sigma^*}\right)\\
&- \frac{\delta t}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|\mathcal{E}_{D\sigma,\sigma^*,\mathfrak{s}} - \frac{\delta t}{|D_{\sigma,\sigma^*}|}|\sigma|\mathcal{E}_{\sigma},
\end{aligned}$$

and for $X \in \{K, L, K^*, L^*\}$

$$T_X = \frac{\delta t}{2|D_{\sigma,\sigma^*}|}\left(|D_{\sigma,\sigma^*}\cap X|\mathcal{R}_X - \frac{|\sigma|}{2\sqrt{2}\gamma}\rho_{\sigma,\sigma^*}c(e_{\sigma,\sigma^*})\|\overline{\mathbf{u}}_X - \mathbf{u}_X\|^2\right).$$

In order to guaranty $\overline{e}_{\sigma,\sigma^*} \geqslant 0$ it is sufficient to ensure that these five terms are non negative. Using Lemma 3.12 on $(\boldsymbol{\nabla}_{\boldsymbol{d}}\cdot\mathbf{u})_{\sigma,\sigma^*}$, T_0 becomes

$$\begin{aligned}
T_0 = \rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*}&\left(1 - \frac{\delta t}{|D_{\sigma,\sigma^*}|}(\gamma-1)\left(\frac{|\sigma^*|+|\sigma|}{\sqrt{2}}c(e_{\sigma,\sigma^*}) + \sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|u_{D\sigma,\sigma^*,\mathfrak{s}}\right)\right)\\
&- \frac{\delta t}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|\mathcal{E}_{D\sigma,\sigma^*,\mathfrak{s}} - \frac{\delta t}{|D_{\sigma,\sigma^*}|}|\sigma|\mathcal{E}_{\sigma}.
\end{aligned}$$

Equation (6) yields

$$\mathcal{E}_{D\sigma,\sigma^*,\mathfrak{s}} = e_{\sigma,\sigma^*}\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}} + e_{\sigma',\sigma^{*\prime}}\mathcal{F}^-_{D\sigma',\sigma^{*\prime},\mathfrak{s}} \leqslant e_{\sigma,\sigma^*}\mathcal{F}^+_{D\sigma,\sigma^*,\mathfrak{s}} \leqslant \rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*}[\lambda_+(c(e_\mathfrak{s}), u_{D\sigma,\sigma^*,\mathfrak{s}})]^+$$

and $\mathcal{E}_\sigma = e_{\sigma,\sigma^*}\mathcal{F}^+_\sigma + e_\sigma\mathcal{F}^-_\sigma \leqslant e_{\sigma,\sigma^*}\mathcal{F}^+_\sigma \leqslant \rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*}[\lambda_+(c(e_\sigma), u_\sigma)]^+$ and this allows us to bound T_0 from below as

$$T_0 \geqslant \rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*}\Bigg(1 - \frac{\delta t}{|D_{\sigma,\sigma^*}|}(\gamma-1)\left(\frac{|\sigma^*|+|\sigma|}{\sqrt{2}}c(e_{\sigma,\sigma^*}) + \sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|u_{D\sigma,\sigma^*,\mathfrak{s}}\right) - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|[\lambda_+(c(e_\mathfrak{s}), u_{D\sigma,\sigma^*,\mathfrak{s}})]^+ - \frac{\delta t}{|D_{\sigma,\sigma^*}|}|\sigma|[\lambda_+(c(e_\sigma), u_\sigma)]^+\Bigg)\Bigg).$$

Finally, since $u_{D\sigma,\sigma^*,\mathfrak{s}} \leqslant [u_{D\sigma,\sigma^*,\mathfrak{s}}]^+$ and $[\lambda_+(c(e_\mathfrak{s}), u_{D\sigma,\sigma^*,\mathfrak{s}})]^+ \leqslant [u_{D\sigma,\sigma^*,\mathfrak{s}}]^+ + c(e_\mathfrak{s})$, the fact that $T_0 \geqslant$ follows from

$$1 \geqslant \frac{\delta t}{|D_{\sigma,\sigma^*}|}(\gamma-1)\left(\frac{|\sigma^*|+|\sigma|}{\sqrt{2}}c(e_{\sigma,\sigma^*}) + \sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|[u_{D\sigma,\sigma^*,\mathfrak{s}}]^+\right) + \frac{\delta t}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|([u_{D\sigma,\sigma^*,\mathfrak{s}}]^+ + c(e_\mathfrak{s})) + \frac{\delta t}{|D_{\sigma,\sigma^*}|}|\sigma|([u_\sigma]^+ + c(e_\sigma)).$$

Gathering the $[u_{D\sigma,\sigma^*,\mathfrak{s}}]^+$ terms and using $\gamma > 1$ lead to the assumption (40). Next, we turn to the T_X term: if $X \in \mathfrak{M}$ we follow the same sketch as a) and we get $T_X \geqslant 0$ under assumption (39); we are thus left with the case of $X^* \in \mathfrak{M}^*$. Owing to (36), we can write

$$T_{X^*} \geqslant \frac{\|\overline{\mathbf{u}}_{X^*} - \mathbf{u}_{X^*}\|^2}{4|D_{\sigma,\sigma^*}|}\left(|D_{\sigma,\sigma^*}\cap X^*|\mathcal{A}_{X^*} - \rho_{\sigma,\sigma^*}c(e_{\sigma,\sigma^*})\frac{\delta t|\sigma|}{\sqrt{2}\gamma}\right).$$

Going back to (37), we have $\mathcal{A}_{X^*} \geqslant \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{X^*}\cap\mathfrak{D}_{ext}} \frac{|D_{\sigma',\sigma^{*\prime}}\cap X^*|}{|X^*|}\rho_{\sigma',\sigma^{*\prime}}Q_{\sigma',\sigma^{*\prime}}$ where

$$Q_{\sigma',\sigma^{*\prime}} \geqslant 1 - \frac{\delta t}{|D_{\sigma',\sigma^{*\prime}}|}\frac{|\sigma'|}{2}\Big(\big(1+\mathrm{reg}\,(\mathfrak{T})\big)\frac{\mathcal{F}^+_\sigma}{\rho_{\sigma',\sigma^{*\prime}}} - \frac{1}{1-\mathrm{reg}\,(\mathfrak{T})}\frac{\mathcal{F}^-_\sigma}{\rho_{\sigma',\sigma^{*\prime}}}\Big) - \frac{\delta t}{|D_{\sigma',\sigma^{*\prime}}|}\mathrm{reg}\,(\mathfrak{T})\left(|\mathfrak{s}_{XZ^*}|\frac{\mathcal{F}^+_{\sigma',\mathfrak{s}_{XZ^*}}}{\rho_{\sigma',\sigma^{*\prime}}} + |\mathfrak{s}_{XX^*}|\frac{\mathcal{F}^+_{\sigma',\mathfrak{s}_{XX^*}}}{\rho_{\sigma',\sigma^{*\prime}}}\right).$$

With (33), we get $Q_{\sigma',\sigma^{*\prime}} \geqslant 0$ for each $\sigma^{*'} \in \partial X^*$ so that $\mathcal{A}_{X^*} \geqslant \frac{|D_{\sigma,\sigma^*}\cap X^*|}{|X^*|}\rho_{\sigma,\sigma^*}Q_{\sigma,\sigma^*}$ and

$$T_{X^*} \geqslant \frac{|D_{\sigma,\sigma^*}\cap X^*|^2}{4|D_{\sigma,\sigma^*}||X^*|}\rho_{\sigma,\sigma^*}\|\overline{\mathbf{u}}_{X^*} - \mathbf{u}_{X^*}\|^2\tilde{Q}_{\sigma,\sigma^*},$$

where

$$\tilde{Q}_{\sigma,\sigma^*} = Q_{\sigma,\sigma^*} - c(e_{\sigma,\sigma^*})\frac{\delta t}{\sqrt{2}\gamma}\frac{|\sigma^*||X^*|}{|D_{\sigma,\sigma^*}\cap X^*|^2}.$$

That T_{X^*} is non negative follows from $\tilde{Q}_{\sigma,\sigma^*} \geqslant 0$ and we have

$$\tilde{Q}_{\sigma,\sigma^*} \geqslant Q_{\sigma,\sigma^*} - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\frac{c(e_{\sigma,\sigma^*})|\sigma^*|\mathrm{reg}\,(\mathfrak{T})}{\sqrt{2}\gamma}\frac{|X^*|}{|D_{\sigma,\sigma^*}\cap X^*|}$$

which leads to

$$\tilde{Q}_{\sigma,\sigma^*} \geqslant 1 - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\left(\begin{array}{l} \frac{|\sigma|}{2}\Big(\big(1+\mathrm{reg}\,(\mathfrak{T})\big)[\lambda_+(c(e_\sigma),u_\sigma)]^+ + \frac{1}{1-\mathrm{reg}\,(\mathfrak{T})}\frac{\rho_D}{\rho_{\sigma,\sigma^*}}[\lambda_+(c(e_\sigma),u_D)]^-\Big) \\ + \mathrm{reg}\,(\mathfrak{T}) \sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\setminus\partial\Omega} |\mathfrak{s}|[\lambda_+(c(e_\mathfrak{s}),u_{D\sigma,\sigma^*,\mathfrak{s}})]^+ \\ + \frac{c(e_{\sigma,\sigma^*})|\sigma^*|\mathrm{reg}\,(\mathfrak{T})}{\sqrt{2}\gamma}\frac{|X^*|}{|D_{\sigma,\sigma^*}\cap X^*|} \end{array}\right)$$

Substracting the non-negative term

$$\frac{\delta t}{|D_{\sigma,\sigma^*}|}\left((1+\mathrm{reg}\,(\mathfrak{T}))\left(\frac{|\sigma^*|+|\sigma|}{\sqrt{2}}c(e_{\sigma,\sigma^*}) + \frac{|\sigma|}{2}[\lambda_+(c(e_\sigma),u_\sigma)]^+\right) + \sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}\setminus\partial\Omega}|\mathfrak{s}|[\lambda_+(c(e_\mathfrak{s}),u_{D\sigma,\sigma^*,\mathfrak{s}})]^+\right)$$

on the right hand side and using (40) yield

$$\tilde{Q}_{\sigma,\sigma^*} \geqslant 1 - \frac{\delta t}{|D_{\sigma,\sigma^*}|}\left(\begin{array}{l} \frac{|\sigma|}{2}\frac{1}{1-\mathrm{reg}\,(\mathfrak{T})}\frac{\rho_D}{\rho_{\sigma,\sigma^*}}[\lambda_+(c(e_\sigma),u_D)]^- \\ + \frac{1+\mathrm{reg}\,(\mathfrak{T})}{\gamma}\left(1+\frac{c(e_{\sigma,\sigma^*})|\sigma^*|}{\sqrt{2}}\frac{|X^*|}{|D_{\sigma,\sigma^*}\cap X^*|}\right) \end{array}\right)$$

which is thus non negative when (41) holds.

■

4 Conservation of total energy

Definition 4.1. We define a kinetic energy $E^{\mathrm{kin}}_{\sigma,\sigma^*}$, stored on the cell $D_{\sigma,\sigma^*} \in \mathfrak{D}_{int}$, by the formula

$$E^{\mathrm{kin}}_{\sigma,\sigma^*} = \frac{|D_{\sigma,\sigma^*}\cap K|\frac{\rho_K\|\mathbf{u}_K\|^2}{2} + |D_{\sigma,\sigma^*}\cap L|\frac{\rho_L\|\mathbf{u}_L\|^2}{2} + |D_{\sigma,\sigma^*}\cap K^*|\frac{\rho_{K^*}\|\mathbf{u}_{K^*}\|^2}{2} + |D_{\sigma,\sigma^*}\cap L^*|\frac{\rho_{L^*}\|\mathbf{u}_{L^*}\|^2}{2}}{2|D_{\sigma,\sigma^*}|\rho_{\sigma,\sigma^*}}$$

and a total energy E_{σ,σ^*}, stored on the cell $D_{\sigma,\sigma^*} \in \mathfrak{D}_{int}$, by setting

$$E_{\sigma,\sigma^*} = e_{\sigma,\sigma^*} + E^{\mathrm{kin}}_{\sigma,\sigma^*}.$$

We wish to write a *local* conservation equation for the total energy $\rho_{\sigma,\sigma^*}E_{\sigma,\sigma^*}$. This property is in fact related to the duality relations between discrete operators, as discussed in [14], and based on Lemma 3.3.

Proposition 4.2. *The discrete total energy $\rho_{\sigma,\sigma^*}E_{\sigma,\sigma^*}$ satisfies the following conservative equation on $\mathfrak{D}_{int}$:*

$$\frac{\overline{\rho}_{\sigma,\sigma^*}\overline{E}_{\sigma,\sigma^*} - \rho_{\sigma,\sigma^*}E_{\sigma,\sigma^*}}{\delta t} + \frac{1}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|\mathcal{T}_{D\sigma,\sigma^*,\mathfrak{s}} + \frac{1}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|q_{D\sigma,\sigma^*,\mathfrak{s}} = 0,$$

where

- $\mathcal{T}_{D\sigma,\sigma^*,\mathfrak{s}}$ *is a conservative total energy flux through the interface $\mathfrak{s}$ of the diamond cell D_{σ,σ^*},*
- $\frac{1}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|q_{D\sigma,\sigma^*,\mathfrak{s}}$ *is a conservative discrete version of $\boldsymbol{\nabla}\cdot(p\overline{\mathbf{u}})$ on the diamond cell D_{σ,σ^*}.*

Proof. We identify the corresponding fluxes by using Lemma 3.3. It is helpful to bear in mind the typical shape of a diamond cell with vertices x_K, x_{K^*}, x_L, x_{L^*} as depicted in Fig. 3 and 5. Let $X \in \{K, L, K^*, L^*\}$ and multiply the kinetic energy balance equation on the cell X by $\frac{|D_{\sigma,\sigma^*} \cap X|}{2|D_{\sigma,\sigma^*}|}$. Next, add the four relations to the equation for the internal energy (29) on the cell D_{σ,σ^*}.

*) The first task is to identify some conservative fluxes $\mathcal{K}_{D\sigma,\sigma^*,\mathfrak{s}}$ such that

$$\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\mathcal{K}_{D\sigma,\sigma^*,\mathfrak{s}} = \frac{|D_{\sigma,\sigma^*} \cap K|}{2|K|} \sum_{\sigma'\in\partial K} |\sigma'|\mathcal{K}_{K,\sigma'} + \frac{|D_{\sigma,\sigma^*} \cap L|}{2|L|} \sum_{\sigma'\in\partial L} |\sigma'|\mathcal{K}_{L,\sigma'}$$
$$+ \frac{|D_{\sigma,\sigma^*} \cap K^*|}{2|K^*|} \sum_{\sigma'\in\partial K^*} |\sigma'|\mathcal{K}_{K^*,\sigma'} + \frac{|D_{\sigma,\sigma^*} \cap L^*|}{2|L^*|} \sum_{\sigma'\in\partial L^*} |\sigma'|\mathcal{K}_{L^*,\sigma'}. \quad (42)$$

Applying Lemma 3.3, on each primal cell $\mathscr{C} = K$ with $\mathcal{X}_{K,\sigma} = \mathcal{K}_{K,\sigma}$ provides a function ω_K which satisfies (9) and (10). For the dual mesh, we distinguish the construction for the direct mesh ⓐ and the barycentric mesh ⓑ:

- for ⓐ, we proceed as for for the primal mesh to define ω_{K^*} from the fluxes $\mathcal{X}_{K^*,\sigma^*} = \mathcal{K}_{K^*,\sigma^*}$ on the interfaces of K^*.
- for ⓑ, we still apply Lemma 3.3, but in order to define ω_{K^*}, we should use the two quantities $\mathcal{X}_{K^*,\sigma^*_K} = \mathcal{K}_{K^*,\sigma^*_K}$ and $\mathcal{X}_{K^*,\sigma^*_L} = \mathcal{K}_{K^*,\sigma^*_L}$, see (24), associated to the two interfaces σ^*_K and σ^*_L of K^* that belong to D_{σ,σ^*}.

We next define a conservative flux of kinetic energy for each $\mathfrak{s} = [x_K, x_{K^*}] \in \partial D_{\sigma,\sigma^*}$ as follows

$$\mathcal{K}_{D\sigma,\sigma^*,\mathfrak{s}} = \frac{1}{2|\mathfrak{s}|}\int_{\mathfrak{s}} (\omega_K + \omega_{K^*})\mathbf{n}_{D\sigma,\sigma^*,\mathfrak{s}} \text{ where } \mathfrak{s} = \mathfrak{s}_{KK^*}.$$

With a convenient reorganization of the terms, we write $\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\mathcal{K}_{D\sigma,\sigma^*,\mathfrak{s}}$ as

$$\frac{1}{2}\left(\int_{\mathfrak{s}_{KK^*}} \omega_K \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{KK^*}} + \int_{\mathfrak{s}_{KL^*}} \omega_K \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{KL^*}}\right)$$
$$+ \frac{1}{2}\left(\int_{\mathfrak{s}_{LK^*}} \omega_L \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{LK^*}} + \int_{\mathfrak{s}_{LL^*}} \omega_L \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{LL^*}}\right)$$
$$+ \frac{1}{2}\left(\int_{\mathfrak{s}_{KK^*}} \omega_{K^*} \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{KK^*}} + \int_{\mathfrak{s}_{LK^*}} \omega_{K^*} \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{LK^*}}\right)$$
$$+ \frac{1}{2}\left(\int_{\mathfrak{s}_{KL^*}} \omega_{L^*} \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{KL^*}} + \int_{\mathfrak{s}_{LL^*}} \omega_{L^*} \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{LL^*}}\right).$$

First, let us explain how we proceed with the mesh ⓐ. Since the kinetic energy fluxes $\mathcal{K}_{K,\sigma}$ and $\mathcal{K}_{K^*,\sigma^*}$ are conservative, we have

$$\mathcal{K}_{K,\sigma} + \mathcal{K}_{L,\sigma} = 0 \text{ and } \mathcal{K}_{K^*,\sigma^*} + \mathcal{K}_{L^*,\sigma^*} = 0. \quad (43)$$

Applying again (10) in Lemma 3.3, these two equalities recast as

$$\frac{1}{2}\left(\int_\sigma \omega_K \mathbf{n}_{K,\sigma} + \int_\sigma \omega_L \mathbf{n}_{L,\sigma}\right) = 0$$

and

$$\frac{1}{2}\left(\int_{\sigma^*} \omega_{K^*} \mathbf{n}\mathcal{K}_{K^*,\sigma^*} + \int_{\sigma^*} \omega_{L^*} \mathbf{n}_{L^*,\sigma^*}\right) = 0. \quad (44)$$

We add these expressions in the sum $\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\mathcal{K}_{D\sigma,\sigma^*,\mathfrak{s}}$ and we get

$$\begin{aligned}
\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\mathcal{K}_{D\sigma,\sigma^*,\mathfrak{s}} = \frac{1}{2}\left(\int_\sigma \omega_K \mathbf{n}_{K,\sigma} + \int_{\mathfrak{s}_{KK^*}} \omega_K \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{KK^*}} + \int_{\mathfrak{s}_{KL^*}} \omega_K \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{KL^*}}\right) \\
+ \frac{1}{2}\left(\int_\sigma \omega_L \mathbf{n}_{L,\sigma} + \int_{\mathfrak{s}_{LK^*}} \omega_L \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{LK^*}} + \int_{\mathfrak{s}_{LL^*}} \omega_L \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{LL^*}}\right) \\
+ \frac{1}{2}\left(\int_{\sigma^*} \omega_{K^*} \mathbf{n}_{K^*,\sigma^*} + \int_{\mathfrak{s}_{KK^*}} \omega_{K^*} \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{KK^*}} + \int_{\mathfrak{s}_{LK^*}} \omega_{K^*} \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{LK^*}}\right) \\
+ \frac{1}{2}\left(\int_{\sigma^*} \omega_{L^*} \mathbf{n}_{L^*,\sigma^*} + \int_{\mathfrak{s}_{KL^*}} \omega_{L^*} \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{KL^*}} + \int_{\mathfrak{s}_{LL^*}} \omega_{L^*} \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{LL^*}}\right).
\end{aligned}$$

Thus, by the divergence theorem, and because $\boldsymbol{\nabla}\cdot\omega_X$ is constant over the cell X, we get

$$\begin{aligned}
\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\mathcal{K}_{D\sigma,\sigma^*,\mathfrak{s}} = \frac{|D_{\sigma,\sigma^*}\cap K|}{2}\boldsymbol{\nabla}\cdot\omega_K + \frac{|D_{\sigma,\sigma^*}\cap L|}{2}\boldsymbol{\nabla}\cdot\omega_L \\
+ \frac{|D_{\sigma,\sigma^*}\cap K^*|}{2}\boldsymbol{\nabla}\cdot\omega_{K^*} + \frac{|D_{\sigma,\sigma^*}\cap L^*|}{2}\boldsymbol{\nabla}\cdot\omega_{L^*}.
\end{aligned}$$

Applying (9) in Lemma 3.3 shows that (42) is satisfied. Finally, we define a conservative flux of total energy $\mathcal{T}_{D\sigma,\sigma^*,\mathfrak{s}}$ through the interface $\mathfrak{s}$ of the diamond cell D_{σ,σ^*} by

$$\mathcal{T}_{D\sigma,\sigma^*,\mathfrak{s}} = \mathcal{K}_{D\sigma,\sigma^*,\mathfrak{s}} + \mathcal{E}_{D\sigma,\sigma^*,\mathfrak{s}}.$$

For the mesh ⓑ, we should modify (43) since we take into account the two interfaces σ_K^* and σ_L^*; on the dual cell K^* we now have

$$\mathcal{K}_{K^*,\sigma_K^*} + \mathcal{K}_{L^*,\sigma_K^*} = 0 = \mathcal{K}_{K^*,\sigma_L^*} + \mathcal{K}_{L^*,\sigma_L^*}.$$

Accordingly, (44) becomes

$$\frac{1}{2}\left(\int_{\sigma_K^*} \omega_{K^*}\mathbf{n}_{K^*,\sigma_K^*} + \int_{\sigma_K^*} \omega_{L_K^*}\mathbf{n}_{L^*,\sigma_K^*}\right) = 0 = \frac{1}{2}\left(\int_{\sigma_L^*} \omega_{K^*}\mathbf{n}_{K^*,\sigma_L^*} + \int_{\sigma_L^*} \omega_{L^*}\mathbf{n}_{L^*,\sigma_L^*}\right).$$

We are thus led to

$$\begin{aligned}
\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\mathcal{K}_{D\sigma,\sigma^*,\mathfrak{s}} = \frac{1}{2}\left(\int_\sigma \omega_K \mathbf{n}_{K,\sigma} + \int_{\mathfrak{s}_{KK^*}} \omega_K \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{KK^*}} + \int_{\mathfrak{s}_{KL^*}} \omega_K \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{KL^*}}\right) \\
+ \frac{1}{2}\left(\int_\sigma \omega_L \mathbf{n}_{L,\sigma} + \int_{\mathfrak{s}_{LK^*}} \omega_L \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{LK^*}} + \int_{\mathfrak{s}_{LL^*}} \omega_L \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{LL^*}}\right) \\
+ \frac{1}{2}\Bigg(\int_{\sigma_K^*} \omega_{K^*} \mathbf{n}_{K^*,\sigma_K^*} + \int_{\sigma_L^*} \omega_{K^*} \mathbf{n}_{K^*,\sigma_L^*} \\
+ \int_{\mathfrak{s}_{KK^*}} \omega_{K^*} \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{KK^*}} + \int_{\mathfrak{s}_{LK^*}} \omega_{K^*} \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{LK^*}}\Bigg) \\
+ \frac{1}{2}\Bigg(\int_{\sigma_K^*} \omega_{L^*} \mathbf{n}_{L^*,\sigma_K^*} + \int_{\sigma_L^*} \omega_{L^*} \mathbf{n}_{L^*,\sigma_L^*} \\
+ \int_{\mathfrak{s}_{KL^*}} \omega_{L^*} \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{KL^*}} + \int_{\mathfrak{s}_{LL^*}} \omega_{L^*} \mathbf{n}_{D_{\sigma,\sigma^*},\mathfrak{s}_{LL^*}}\Bigg).
\end{aligned}$$

>From this, we can apply the divergence theorem and conclude as for the mesh ⓐ.

**) We now turn to the pressure term. There are four terms coming from the sum of the kinetic energy equations and the discrete version of $p\boldsymbol{\nabla}\cdot\mathbf{u}$, namely

$$\begin{aligned}\frac{|D_{\sigma,\sigma^*}\cap K|}{2}\overline{\mathbf{u}}_K\cdot(\boldsymbol{\nabla}_{\boldsymbol{d}}p)_K+\frac{|D_{\sigma,\sigma^*}\cap L|}{2}\overline{\mathbf{u}}_L\cdot(\boldsymbol{\nabla}_{\boldsymbol{d}}p)_L\\ +\frac{|D_{\sigma,\sigma^*}\cap K^*|}{2}\overline{\mathbf{u}}_{K^*}\cdot(\boldsymbol{\nabla}_{\boldsymbol{d}}p)_{K^*}+\frac{|D_{\sigma,\sigma^*}\cap L^*|}{2}\overline{\mathbf{u}}_{L^*}\cdot(\boldsymbol{\nabla}_{\boldsymbol{d}}p)_{L^*}\\ +|D_{\sigma,\sigma^*}|p_{\sigma,\sigma^*}\left(\boldsymbol{\nabla}_{\boldsymbol{d}}\cdot\mathbf{u}\right)_{\sigma,\sigma^*}.\end{aligned}\tag{45}$$

We wish to rewrite this sum as $\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|q_{D\sigma,\sigma^*,\mathfrak{s}}$ with $q_{D\sigma,\sigma^*,\mathfrak{s}}$ verifying the conservation property $q_{D\sigma,\sigma^*,\mathfrak{s}}=-q_{D\sigma',\sigma^{*\prime},\mathfrak{s}}$ where $\mathfrak{s}=D_{\sigma,\sigma^*}|D_{\sigma',\sigma^{*\prime}}$.

To this end, we apply Lemma 3.3 again on each primal (resp. dual) cell $\mathscr{C}=K$ (resp. K^*) with, now, $\mathcal{X}_{K,\sigma}=p_{\sigma,\sigma^*}\overline{\mathbf{u}}_K\cdot\mathbf{n}_{K,\sigma}$ (resp. $\mathcal{X}_{K^*,\sigma^*}=p_{\sigma,\sigma^*}\overline{\mathbf{u}}_{K^*}\cdot\mathbf{n}_{K^*,\sigma^*}$ for mesh ⓐ, or $\mathcal{X}_{K^*,\sigma^*_K}=p_{\sigma,\sigma^*}\overline{\mathbf{u}}_{K^*}\cdot\mathbf{n}_{K^*,\sigma^*_K}$, $\mathcal{X}_{K^*,\sigma^*_L}=p_{\sigma,\sigma^*}\overline{\mathbf{u}}_{K^*}\cdot\mathbf{n}_{K^*,\sigma^*_L}$ for ⓑ). It provides functions ω_K (resp. ω_{K^*}) that satisfy (9) and (10).

We next define, for each $\mathfrak{s}=[x_K,x_{K^*}]\in\partial D_{\sigma,\sigma^*}$,

$$q_{D\sigma,\sigma^*,\mathfrak{s}}=\frac{1}{2|\mathfrak{s}|}\int_{\mathfrak{s}}(\omega_K+\omega_{K^*})\cdot\mathbf{n}_{D\sigma,\sigma^*,\mathfrak{s}}\text{ where }\mathfrak{s}=\mathfrak{s}_{KK^*}.$$

By construction, this quantity is conservative.

We are now going to check that the sum $\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|q_{D\sigma,\sigma^*,\mathfrak{s}}$ coincides with (45). With a convenient reorganization of the terms, we write $\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|q_{D\sigma,\sigma^*,\mathfrak{s}}$ as

$$\begin{aligned}&\frac12\left(\int_{\mathfrak{s}_{KK^*}}\omega_K\cdot\mathbf{n}_{\sigma,\mathfrak{s}_{KK^*}}+\int_{\mathfrak{s}_{KL^*}}\omega_K\cdot\mathbf{n}_{\sigma,\mathfrak{s}_{KL^*}}\right)+\frac12\left(\int_{\mathfrak{s}_{LK^*}}\omega_L\cdot\mathbf{n}_{\sigma,\mathfrak{s}_{LK^*}}+\int_{\mathfrak{s}_{LL^*}}\omega_L\cdot\mathbf{n}_{\sigma,\mathfrak{s}_{LL^*}}\right)\\&+\frac12\left(\int_{\mathfrak{s}_{KK^*}}\omega_{K^*}\cdot\mathbf{n}_{\sigma,\mathfrak{s}_{KK^*}}+\int_{\mathfrak{s}_{LK^*}}\omega_{K^*}\cdot\mathbf{n}_{\sigma,\mathfrak{s}_{LK^*}}\right)+\frac12\left(\int_{\mathfrak{s}_{KL^*}}\omega_{L^*}\cdot\mathbf{n}_{\sigma,\mathfrak{s}_{KL^*}}+\int_{\mathfrak{s}_{LL^*}}\omega_{L^*}\cdot\mathbf{n}_{\sigma,\mathfrak{s}_{LL^*}}\right).\end{aligned}$$

We only detail the computation with the mesh ⓐ. We make use again of Lemma 3.3 to write

$$\int_\sigma\omega_K\cdot\mathbf{n}_{K,\sigma}=|\sigma|p_{\sigma,\sigma^*}\overline{\mathbf{u}}_K\cdot\mathbf{n}_{K,\sigma}\text{ and }\int_{\sigma^*}\omega_{K^*}\cdot\mathbf{n}_{K^*,\sigma^*}=|\sigma^*|p_{\sigma,\sigma^*}\overline{\mathbf{u}}_{K^*}\cdot\mathbf{n}_{K^*,\sigma^*}.$$

(For the mesh ⓑ, the latter is replaced by the sum over the interfaces σ^*_K and σ^*_L.) For $X\in\{K,L\}$ we add $\frac12\int_\sigma\omega_X\cdot\mathbf{n}_{X,\sigma}+\frac12\int_{\sigma^*}\omega_{X^*}\cdot\mathbf{n}_{X^*,\sigma^*}$ and substract $\frac{|\sigma|}{2}p_{\sigma,\sigma^*}\overline{\mathbf{u}}_X\cdot\mathbf{n}_{X,\sigma}+\frac{|\sigma^*|}{2}p_{\sigma,\sigma^*}\overline{\mathbf{u}}_{X^*}\cdot\mathbf{n}_{X^*,\sigma^*}$ in the above expression and we get

$$\begin{aligned}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}}|\mathfrak{s}|q_{D\sigma,\sigma^*,\mathfrak{s}}=&\frac12\left(\int_\sigma\omega_K\cdot\mathbf{n}_{K,\sigma}+\int_{\mathfrak{s}_{KK^*}}\omega_K\cdot\mathbf{n}_{\sigma,\mathfrak{s}_{KK^*}}+\int_{\mathfrak{s}_{KL^*}}\omega_K\cdot\mathbf{n}_{\sigma,\mathfrak{s}_{KL^*}}\right)\\&+\frac12\left(\int_\sigma\omega_L\cdot\mathbf{n}_{L,\sigma}+\int_{\mathfrak{s}_{LK^*}}\omega_L\cdot\mathbf{n}_{\sigma,\mathfrak{s}_{LK^*}}+\int_{\mathfrak{s}_{LL^*}}\omega_L\cdot\mathbf{n}_{\sigma,\mathfrak{s}_{LL^*}}\right)\\&+\frac12\left(\int_{\sigma^*}\omega_{K^*}.\mathbf{n}_{K^*,\sigma^*}+\int_{\mathfrak{s}_{KK^*}}\omega_{K^*}\cdot\mathbf{n}_{\sigma,\mathfrak{s}_{KK^*}}+\int_{\mathfrak{s}_{LK^*}}\omega_{K^*}\cdot\mathbf{n}_{\sigma,\mathfrak{s}_{LK^*}}\right)\\&+\frac12\left(\int_{\sigma^*}\omega_{L^*}\cdot\mathbf{n}_{L^*,\sigma^*}+\int_{\mathfrak{s}_{KL^*}}\omega_{L^*}\cdot\mathbf{n}_{\sigma,\mathfrak{s}_{KL^*}}+\int_{\mathfrak{s}_{LL^*}}\omega_{L^*}\cdot\mathbf{n}_{\sigma,\mathfrak{s}_{LL^*}}\right)\\&-\frac{|\sigma|}{2}p_{\sigma,\sigma^*}(\overline{\mathbf{u}}_K\cdot\mathbf{n}_{K,\sigma}+\overline{\mathbf{u}}_L\cdot\mathbf{n}_{L,\sigma})-\frac{|\sigma^*|}{2}p_{\sigma,\sigma^*}(\overline{\mathbf{u}}_{K^*}\cdot\mathbf{n}_{K^*,\sigma^*}+\overline{\mathbf{u}}_{L^*}\cdot\mathbf{n}_{L^*,\sigma^*}).\end{aligned}$$

Thus, using equation (9) of Lemma 3.3 together with Definition 3.11 of the discrete divergence operator yields

$$\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}| q_{D\sigma,\sigma^*,\mathfrak{s}} = \frac{|D_{\sigma,\sigma^*}\cap K|}{2}\boldsymbol{\nabla}\cdot\omega_K + \frac{|D_{\sigma,\sigma^*}\cap L|}{2}\boldsymbol{\nabla}\cdot\omega_L$$
$$+ \frac{|D_{\sigma,\sigma^*}\cap K^*|}{2}\boldsymbol{\nabla}\cdot\omega_{K^*} + \frac{|D_{\sigma,\sigma^*}\cap L^*|}{2}\boldsymbol{\nabla}\cdot\omega_{L^*}$$
$$+ |D_{\sigma,\sigma^*}| p_{\sigma,\sigma^*} \left(\boldsymbol{\nabla}_{\boldsymbol{d}}\cdot\overline{\mathbf{u}}\right)_{\sigma,\sigma^*}.$$

Applying (10) in Lemma 3.3 shows that

$$\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}| q_{D\sigma,\sigma^*,\mathfrak{s}} = \frac{|D_{\sigma,\sigma^*}\cap K|}{2|K|}\sum_{\sigma'\in\partial K}|\sigma'|q_{K,\sigma'} + \frac{|D_{\sigma,\sigma^*}\cap L|}{2|L|}\sum_{\sigma'\in\partial L}|\sigma'|q_{L,\sigma'}$$
$$= \frac{|D_{\sigma,\sigma^*}\cap K^*|}{2|K^*|}\sum_{{\sigma^*}'\in\partial K^*}|{\sigma^*}'|q_{K^*,{\sigma^*}'} + \frac{|D_{\sigma,\sigma^*}\cap L^*|}{2|L^*|}\sum_{{\sigma^*}'\in\partial L^*}|{\sigma^*}'|q_{L^*,{\sigma^*}'}$$
$$+ |D_{\sigma,\sigma^*}| p_{\sigma,\sigma^*} \left(\boldsymbol{\nabla}_{\boldsymbol{d}}\cdot\overline{\mathbf{u}}\right)_{\sigma,\sigma^*}.$$

Finally, coming back to the definition of $q_{K,\sigma}$ and next, to Definition 3.7 of the discrete pressure gradient, we remark that for $X \in \{K, L, K^*, L^*\}$

$$\frac{1}{|X|}\sum_{\sigma'\in\partial X}|\sigma'|q_{X,\sigma'} = \frac{1}{|X|}\sum_{\sigma'\in\partial X}|\sigma'|p_{\sigma',{\sigma^*}'}\overline{\mathbf{u}}_X\cdot\mathbf{n}_{X,\sigma'} = \overline{\mathbf{u}}_X\cdot(\boldsymbol{\nabla}_{\boldsymbol{d}}p)_X.$$

We can conclude that

$$\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}| q_{D\sigma,\sigma^*,\mathfrak{s}} = \frac{|D_{\sigma,\sigma^*}\cap K|}{2}\overline{\mathbf{u}}_K\cdot(\boldsymbol{\nabla}_{\boldsymbol{d}}p)_K + \frac{|D_{\sigma,\sigma^*}\cap L|}{2}\overline{\mathbf{u}}_L\cdot(\boldsymbol{\nabla}_{\boldsymbol{d}}p)_L$$
$$+ \frac{|D_{\sigma,\sigma^*}\cap K^*|}{2}\overline{\mathbf{u}}_{K^*}\cdot(\boldsymbol{\nabla}_{\boldsymbol{d}}p)_{K^*} + \frac{|D_{\sigma,\sigma^*}\cap L^*|}{2}\overline{\mathbf{u}}_{L^*}\cdot(\boldsymbol{\nabla}_{\boldsymbol{d}}p)_{L^*}$$
$$+ |D_{\sigma,\sigma^*}| p_{\sigma,\sigma^*} \left(\boldsymbol{\nabla}_{\boldsymbol{d}}\cdot\mathbf{u}\right)_{\sigma,\sigma^*}.$$

■

5 Numerical simulations

In this Section we present some numerical test cases on unstructured grids. We compare the performance of the scheme to the MAC discretization [16]. The unstructured primal mesh is a tessellation made of triangles, provided by `GMSH`, which leads to a dual mesh which cells are polygons of any type.

5.1 Consistency analysis with a 2D manufactured solution

In order to numerically validate the scheme, we compute the solution of the 2D problem

$$\begin{cases} \partial_t\rho + \boldsymbol{\nabla}\cdot\big(\rho\mathbf{u}\big) = 0,\\ \partial_t\big(\rho\mathbf{u}\big) + \boldsymbol{\nabla}\cdot\big(\rho\mathbf{u}\otimes\mathbf{u}\big) + \boldsymbol{\nabla}p = f(t,\mathbf{x}),\\ \partial_t(\rho e) + \boldsymbol{\nabla}\cdot(\rho e\mathbf{u}) + p\boldsymbol{\nabla}\cdot\mathbf{u} = 0, \end{cases}$$

where the force field $(t, \mathbf{x}) \mapsto f(t, \mathbf{x})$ is tailored so that the smooth solution reads

$$\begin{cases} \rho^{\mathrm{ex}}(t, \mathbf{x}) = \exp\Big(-2\sqrt{(x\cos(t) + y\sin(t) + 1)^2 + (-x\sin(t) + y\cos(t) - 0.1)^2} \Big), \\ u^{\mathrm{ex}}(t, \mathbf{x}) = -y, \\ v^{\mathrm{ex}}(t, \mathbf{x}) = x, \\ e^{\mathrm{ex}}(t, \mathbf{x}) = \dfrac{\exp\Big(-3\sqrt{(x\cos(t) + y\sin(t) + 1)^2 + (-x\sin(t) + y\cos(t) + 0.1)^2} \Big)}{\rho(t, \mathbf{x})}, \end{cases}$$

where $\mathbf{x} = (x, y)$. We perform the simulations for $t \in [0, 0.2]$ with $\gamma = 1.4$ on the circle of center $(0, 0)$ and radius 2. We use a series of tessellations made of triangles, provided by `GMSH`: the characteristic length used in `GMSH` (the quantity that determines the mesh size) is divided by 2 between each mesh.

The discrete L^2 norms of the errors between the discrete and the exact solutions, for the density, the internal energy and the first component of the velocity, on the different meshes, are reported in Table 1 and Table 2:

$$\mathfrak{e}_{2,\rho} = \Bigg(\sum_{\substack{D_{\sigma,\sigma^*} \in \mathfrak{D} \\ \sigma^* = [\mathbf{x}_{K^*}, \mathbf{x}_{L^*}]}} \big|D_{\sigma,\sigma^*}\big| \Big| \rho_{D_{\sigma,\sigma^*}} - \rho^{\mathrm{ex}}\Big(\frac{\mathbf{x}_{K^*} + \mathbf{x}_{L^*}}{2}\Big) \Big|^2 \Bigg)^{\frac{1}{2}},$$

$$\mathfrak{e}_{2,e} = \Bigg(\sum_{\substack{D_{\sigma,\sigma^*} \in \mathfrak{D} \\ \sigma^* = [\mathbf{x}_{K^*}, \mathbf{x}_{L^*}]}} \big|D_{\sigma,\sigma^*}\big| \Big| e_{D_{\sigma,\sigma^*}} - e^{\mathrm{ex}}\Big(\frac{\mathbf{x}_{K^*} + \mathbf{x}_{L^*}}{2}\Big) \Big|^2 \Bigg)^{\frac{1}{2}},$$

$$\mathfrak{e}_{2,u} = \Bigg(\sum_{K \in \mathfrak{M}} |K| \Big| u_K - u^{\mathrm{ex}}\big(\mathbf{x}_K\big) \Big|^2 \Bigg)^{\frac{1}{2}}, \quad \mathfrak{e}_{2,v} = \Bigg(\sum_{K \in \mathfrak{M}} |K| \Big| v_K - v^{\mathrm{ex}}\big(\mathbf{x}_K\big) \Big|^2 \Bigg)^{\frac{1}{2}},$$

$$\mathfrak{e}_{2,u^*} = \Bigg(\sum_{K \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*} |K^*| \Big| u^*_K - u^{\mathrm{ex}}\big(\mathbf{x}_{K^*}\big) \Big|^2 \Bigg)^{\frac{1}{2}}, \quad \mathfrak{e}_{2,v^*} = \Bigg(\sum_{K \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*} |K^*| \Big| v^*_K - v^{\mathrm{ex}}\big(\mathbf{x}_{K^*}\big) \Big|^2 \Bigg)^{\frac{1}{2}}.$$

We also report the characteristic length h of the different meshes computed as follows

$$h = \max(|\sigma|, |\sigma^*|)$$

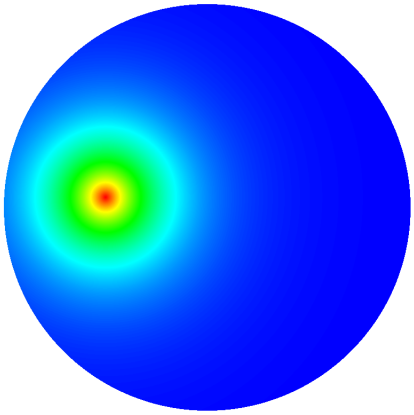
The results are almost the same for the second component. For this test case, we have set $\delta t = 10^{-4}$: the small value of the time step ensures that the stability condition is satisfied for all the considered grids. We observe as expected a first order convergence. In Fig. 6 we show the resolution of the density and the internal energy and their exact solutions, on the finest mesh.

i	$h^{(i)}$	$\mathfrak{e}_{2,\rho}^{(i)}$	$\dfrac{\log(\mathfrak{e}_{2,\rho}^{(i)}/\mathfrak{e}_{2,\rho}^{(i-1)})}{\log(h^{(i)}/h^{(i-1)})}$	$\mathfrak{e}_{2,e}^{(i)}$	$\dfrac{\log(\mathfrak{e}_{2,e}^{(i)}/\mathfrak{e}_{2,e}^{(i-1)})}{\log(h^{(i)}/h^{(i-1)})}$
1	2.62×10^{-1}	4.10×10^{-2}	$--$	1.78×10^{-1}	$--$
2	1.43×10^{-1}	2.50×10^{-2}	0.82	9.70×10^{-2}	1.00
3	7.10×10^{-2}	1.41×10^{-2}	0.82	4.91×10^{-2}	0.97
4	3.74×10^{-2}	8.08×10^{-3}	0.87	2.59×10^{-2}	1.00
5	1.96×10^{-2}	4.47×10^{-3}	0.91	1.36×10^{-2}	1.00
6	1.01×10^{-2}	2.38×10^{-3}	0.96	6.96×10^{-3}	1.01

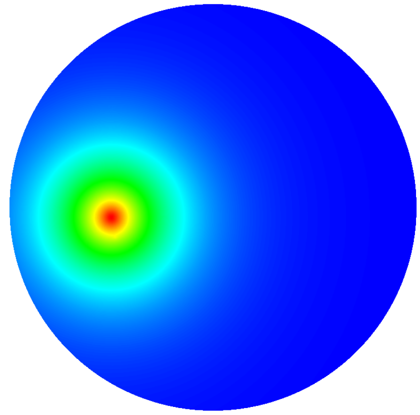
Table 1: Error in L^2-norm between approximate and exact solutions for the density and the internal energy on several meshes.

i	$h^{(i)}$	$\mathfrak{e}_{2,u}^{(i)}$	$\dfrac{\log(\mathfrak{e}_{2,u}^{(i)}/\mathfrak{e}_{2,u}^{(i-1)})}{\log(h^{(i)}/h^{(i-1)})}$	$\mathfrak{e}_{2,u^*}^{(i)}$	$\dfrac{\log(\mathfrak{e}_{2,u^*}^{(i)}/\mathfrak{e}_{2,u^*}^{(i-1)})}{\log(h^{(i)}/h^{(i-1)})}$
1	2.62×10^{-1}	1.16×10^{-1}	$--$	3.07×10^{-1}	$--$
2	1.43×10^{-1}	6.54×10^{-2}	0.95	1.74×10^{-1}	0.94
3	7.10×10^{-2}	3.66×10^{-2}	0.83	9.95×10^{-2}	0.79
4	3.74×10^{-2}	2.05×10^{-2}	0.90	5.75×10^{-2}	0.86
5	1.96×10^{-2}	1.17×10^{-2}	0.88	3.29×10^{-2}	0.86
6	1.01×10^{-2}	6.55×10^{-3}	0.87	1.90×10^{-2}	0.83

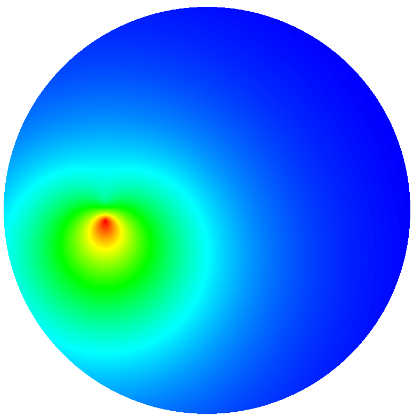
Table 2: Error in L^2-norm between approximate and exact solutions for the first component of the velocity on several meshes.



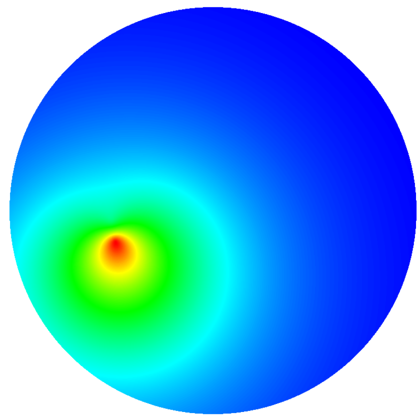
(a) Density, $t = 0$



(b) Density, $t = 0.2$



(c) Internal energy, $t = 0$



(d) Internal energy, $t = 0.2$

Figure 6: Density and internal energy, numerical solution on a mesh with 923868 primal cells.

5.2 Numerical simulations in 2D

1) We use the scheme for the simulation of the 2D Mach 3 wind tunnel with a step. The computational domain Ω is the L-shaped domain

$$\Omega = \Omega_0 \setminus \Omega_{\text{step}}, \quad \Omega_0 = [0, 3] \times [0, 1], \quad \Omega_{\text{step}} = [0.6, 3] \times [0, 0.2].$$

We perform the simulation for $t \in [0, 4]$ with $\gamma = 1.4$ and $\delta t = 10^{-4}$. The initial data reads $\rho = 1.4$, $\mathbf{u} = (3, 0)$ and $p = 1$. On the top and bottom walls, we use reflection boundary conditions which means zero flux boundary conditions. To make the flow enter through the left boundary we use a Dirichlet boundary condition, $\rho = 1.4$, $p = 1$ and $\mathbf{u} = (3, 0)$, whereas a free boundary condition is used for the right section.
In Fig. 7 (at the top), we present the result obtained at $T = 4$ with an unstructured primal mesh made of 204 254 triangles and 307 081 edges. We compare this result to the one obtained in [16] on a 960×320 cartesian grid which is reported in Fig. 7 (at the bottom).

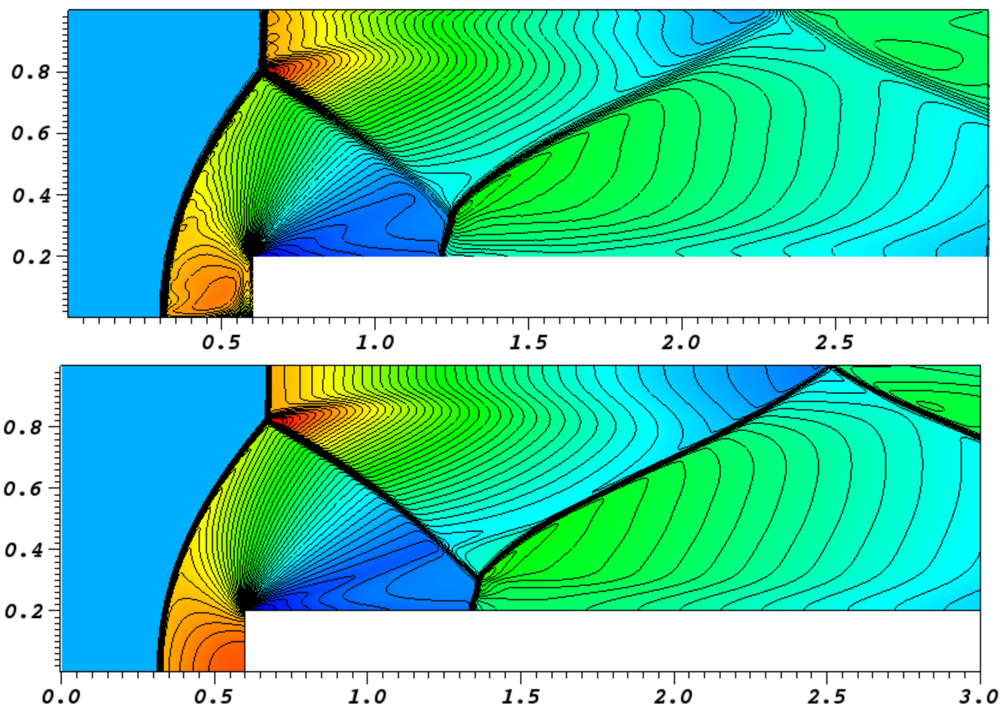


Figure 7: Test 1, simulation of the 2D Mach 3 wind tunnel with a step, density with 50 contour lines on a MAC mesh (bottom) and on an unstructured triangular grid (top)

2) The next test case is inspired from [1]: we consider the 2D simulation of three falling columns into a rectangular basin. The computational domain is the two-dimensional square $[-1,1] \times [-1,1]$. We deal with the full Euler system with $\gamma = 2$. Initially we suppose a constant initial temperature (or internal energy e) in the basin. The PDE system is endowed with zero flux boundary conditions and the following initial data

$$\begin{cases} \rho(0,\mathbf{x}) = 3 + \mathbb{1}_{(x-0.5)^2+(y-0.5)^2<(0.15)^2} + \mathbb{1}_{(x+0.5)^2+(y+0.5)^2<(0.15)^2} + 2 \cdot \mathbb{1}_{x^2+y^2<(0.2)^2}, \\ e(0,\mathbf{x}) = 1, \\ u(0,\mathbf{x}) = 0, \\ v(0,\mathbf{x}) = 0, \end{cases}$$

with $\mathbf{x} = (x,y)$. In Fig. 8 and 9, we show the density and the internal energy at time $T = 1.035$ with $\delta t = 10^{-4}$. The mesh has 43400 primal cells made of triangles and 65356 edges. The result is compared with the same simulation made on a 255×255 Cartesian grid with the MAC scheme presented in [16].

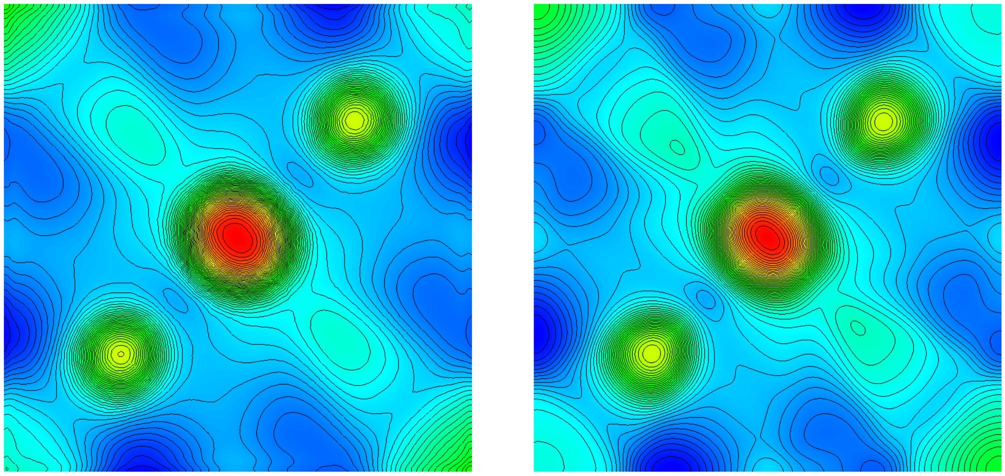


Figure 8: Test 2): Simulation of the 2D three falling columns into a rectangular basin: Density with 50 contour lines on an unstructured grid (left) and a 255×255 Cartesian mesh (right).

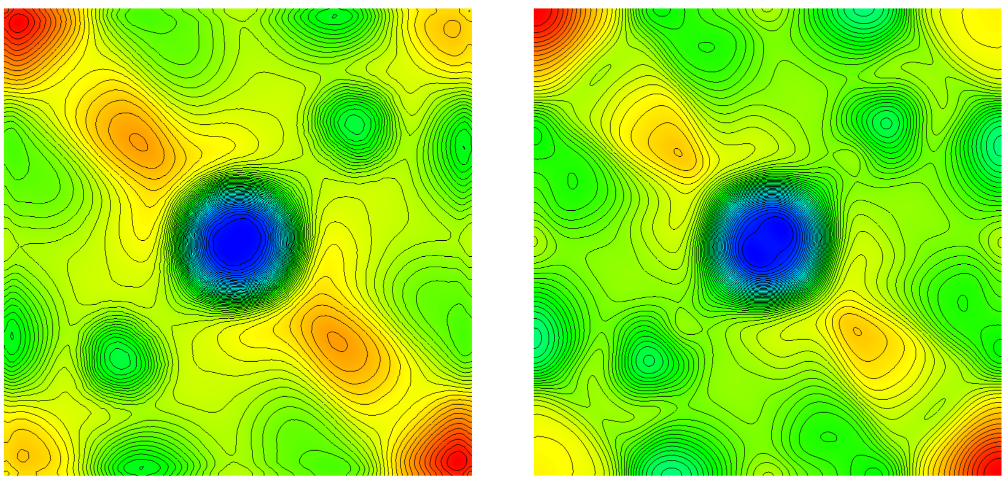


Figure 9: Test 2): Simulation of the 2D three falling columns into a rectangular basin: Internal energy with 50 contour lines on an unstructured grid (left) and on a 255×255 Cartesian mesh (right).

3) The last test case is a 2D supersonic flow in a channel with a circular arc bump. The fluid flows from left to right. Due to the geometry of the obstacle, the MAC grid proposed in [16] is not appropriate. The computational domain Ω is the rectangle $[0, 3] \times [0, 1]$ with a circular arc bump of length 1 and thickness 0.04 located at the bottom, at a distance 1 from the inlet. We perform the simulation up to the final time $T = 5$. We have set $\gamma = 1.4$ and the time step is $\delta t = 10^{-4}$. The initial data are given by $\rho = 1$, $\mathbf{u} = (1.65, 0)$ and $p = 1$, so that the initial Mach number is 1.65. On the top and bottom walls, we use zero flux boundary conditions and on the left we use Dirichlet boundary condition $\rho = 1$, $\mathbf{u} = (1.65, 0)$ and $p = 1$ whereas on the right boundary a free boundary condition is used. In Fig. 10, we show the Mach Number with 50 contour lines. The simulation

is performed on a triangular mesh with 516192 primal cells and 775313 edges. Oblique shocks are formed at the two extremities of the bump. All the shocks are well resolved. The simulation can be compared with the result presented in [33].

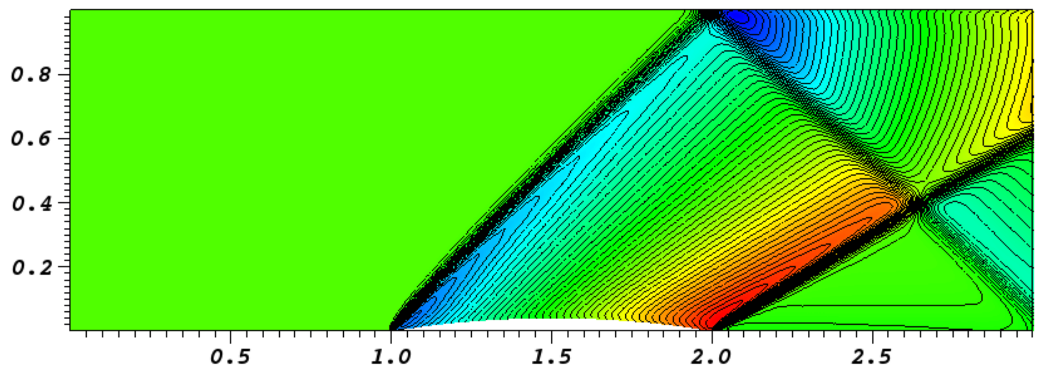


Figure 10: Test 3): Simulation of the 2D subsonic flow in a channel with a circular arc bump flows from left to right: Mach number at time $T = 5$ on an unstructured mesh.

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