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A Model of Particles Interacting with Thermal Traps

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Abstract
We consider simple models where particles interact with thermal traps: the particles move freely, except when they hit the traps where there are exchanges of mass, momentum and energy. We investigate the relaxation properties of the model, and we show that indeed the solutions relax towards a uniform Maxwellian state, entirely determined by the initial data.

Keywords Dynamical Lorentz gas · Thermal traps · Relaxation to equilibrium

Mathematics Subject Classification 82C70 · 74A25

1 Presentation of the Model

We consider a set of free particles, interacting with an array of traps with which the particles can exchange mass, momentum and energy. The particles evolve in the N-dimensional torus $\mathbb{T}^N$. We adopt the description by means of the particle distribution function f, which depends on time, space and velocity variables: given $\Omega \subset \mathbb{T}^N$ and $\mathcal{V} \subset \mathbb{R}^N$, $\iint_{\Omega\times\mathcal{V}} f(t, z, v)\,\mathrm{d}v\,\mathrm{d}x$ gives the number of particles occupying, at time $t \geq 0$, a position $x \in \Omega$, and moving with a velocity $v \in \mathcal{V}$. On $\mathbb{T}^N$, there are K non-overlapping traps; each trap is characterized by

- a center $x_k \in \mathbb{T}^N$,
- a form function $\sigma_k(x)$, which is valued in $[0, 1]$, smooth, and compactly supported in a ball $\omega_k = B(x_k, r_k)$. For instance we can set $\sigma_k(x) = \frac{1}{r_k^N}\sigma\big(\frac{x-x_k}{r_k}\big)$ with a common function $\sigma \in C_c^\infty(\mathbb{T}^N)$, $0 \leq \sigma \leq 1$, $\mathrm{supp}(\sigma) \subset B(0, 1)$, and $r_k \ll 1$ small enough to avoid overlaps. The quantity

$$\mathscr{V}_k = \int_{\mathbb{T}^N} \sigma_k(x)\,\mathrm{d}x$$

is interpreted as the *effective volume* of the kth trap.

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Moreover, the state of the kth trap is characterized by a number of trapped particles n_k, a total momentum $n_k u_k$, and a total energy $n_k E_k$. We define the temperature θ_k of the trap by setting $E_k = \frac{u_k^2 + N\theta_k}{2}$. The particles travel freely in straight line, with constant velocity, except when they hit a trap, which is a location for mass, momentum and energy exchanges: the particles are trapped with a certain rate μ, while the trap can also release free particles, with a rate λ. When a particle is re-emitted by the kth trap, its velocity v is chosen randomly according to a certain probability distribution M_k, parametrized by n_k, u_k, θ_k. To be specific, we assume that

$$\int \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} M_k \, \mathrm{d}v = n_k \begin{pmatrix} 1 \\ u_k \\ 2E_k \end{pmatrix}. \tag{1}$$

In what follows, we consider the typical example of the Maxwellian distribution

$$M_k(v) = \frac{n_k}{(2\pi\theta_k)^{N/2}} \exp\Big(- \frac{|v - u_k|^2}{2\theta_k} \Big),$$

but the approach can be adapted to consider more general emission laws. The evolution of the system is therefore driven by the BGK-like equation [4]

$$\partial_t f + v \cdot \nabla_x f = \sum_{k=1}^{K} \sigma_k \left(\lambda M_k - \mu f \right) \tag{2}$$

coupled to the differential equations

$$\partial_t n_k = \mu \iint \sigma_k f \frac{\mathrm{d}v \, \mathrm{d}x}{\mathscr{V}_k} - \lambda n_k, \tag{3a}$$

$$\partial_t (n_k u_k) = \mu \iint \sigma_k v f \frac{\mathrm{d}v \, \mathrm{d}x}{\mathscr{V}_k} - \lambda n_k u_k, \tag{3b}$$

$$\partial_t (n_k E_k) = \mu \iint \sigma_k \frac{v^2}{2} f \frac{\mathrm{d}v \, \mathrm{d}x}{\mathscr{V}_k} - \lambda n_k E_k. \tag{3c}$$

It is worth remarking that (3c) can be recast as an equation for the temperature since

$$\begin{aligned} \frac{N}{2} \partial_t (n_k \theta_k) &= \partial_t (n_k E_k) - \frac{1}{2} \partial_t (n_k u_k^2) \\ &= \partial_t (n_k E_k) - \frac{1}{2} \Big(u_k \cdot \partial_t (n_k u_k) + n_k u_k \cdot \partial_t u_k \Big) \\ &= \partial_t (n_k E_k) - u_k \cdot \partial_t (n_k u_k) + \frac{u_k^2}{2} \partial_t n_k. \end{aligned}$$

Hence, with (3a)–(3c), we get

$$\frac{N}{2} \partial_t (n_k \theta_k) = \mu \iint \frac{|v - u_k|^2}{2} f \frac{\mathrm{d}v \, \mathrm{d}x}{\mathscr{V}_k} - \lambda \frac{N}{2} n_k \theta_k \tag{4}$$

an expression which justifies that θ_k remains non negative. We wish to investigate some properties of this model, and in particular the large time behavior of the solution.

2 Motivation

This works takes its inspiration from many different roots:

- In the seminal paper [12], A. Caldeira and A. Legget suggested to describe dissipation arising on a physical system as the result of interaction mechanisms with a complex environment, that ultimately lead to the transfer of energy and its evacuation in the environment. The case in which the environmental variables are vibrational degrees of freedom has received a great deal of attention, see for instance [10, 30, 31]. This approach has been revisited in the framework of kinetic equations [18, 26–28, 38, 39].
- Certain of these models can be interpreted in terms of dynamical Lorentz gases where particles interact with an array of scatterers: when a particle hits a scatterer, it spends some time before being re-emitted from some point of the boundary of the scatterer, with a new velocity, but the re-emission law depends on the state of the scatterer, which itself is influenced by the presence of particles. Such dynamics also lead to interpretation by means of random walk. We refer the reader to [1, 15–17, 32] for thorough investigation, both theoretically and numerically, of this viewpoint. It is known that the repartition of the scatterers is crucial for the asymptotic behavior of the standard Lorentz gas, see [7, 11, 21, 24, 25, 34] and the references therein. Roughly speaking, Boltzmann-type equation can be derived asymptotically assuming a random repartition of the scatterers [7, 24] while a periodic distribution leads to intricate memory effects [11, 25, 34]. This opens the question to decide whether or not the energy exchanges with dynamical scatterers induce dissipation mechanisms, independently of the space repartition of the scatterers.
- Recently the analysis of kinetic models of particles interacting with thermal reservoirs, that means source of heat, has drawn the attention to quite complex relaxation phenomena, driven by non equilibrium steady states involving the different, but fixed, temperatures of the reservoirs [13, 14]. Beware that here the temperature of the traps is affected by the mass and energy exchanges.
- Finally, it has been proven that relaxation phenomena might occur in collisional kinetic models, even when the collisional term is degenerate: the relaxation then depends on geometrical conditions, inspired from control theory [3, 33], on the repartition of the collisional spots [5, 6, 20, 23, 29, 35]. Again, this raises the issue of the role of the space repartition of the traps.

The model (2), (3a)–(3c) is somehow related to all these situations. The traps act a dynamical environment which can be expected to drive the particle distribution towards equilibrium. However, the non homogeneous space repartition of the traps raises the issue of the effectiveness of this relaxation mechanism, and the possibility to obtain an equilibrium state.

The paper is organized as follows. In Sect. 3 we derive the fundamental conservation and dissipation properties of the model. It turns out that a H-Theorem holds, which allows us to identify the equilibrium states. Next, these estimates are used to establish the asymptotic trend to equilibrium as time becomes large in Sect. 4. This argument does not provide information about the rate of convergence. A more detailed analysis can be performed on the linearized problem, by adapting the arguments of [29] to this situation with several conserved quantities and a coupling between the kinetic equation and ODEs for the traps. This is the object of Sect. 5. The results are further illustrated by numerical simulations in Sect. 6. Finally, Sect. 7 proposes an extension of the model by taking into account the time spent by the particles within the trap.

3 Conservation, H-Theorem and Equilibrium

The system (2), (3a)–(3c) conserves the total mass, momentum and energy. Indeed, we observe that

$$\iint \begin{pmatrix} 1 \\ v \\ v^2/2 \end{pmatrix} \sigma_k\,(\lambda M_k - \mu f)\,\mathrm{d}v\,\mathrm{d}x = \lambda \mathscr{V}_k \begin{pmatrix} n_k \\ n_k u_k \\ n_k E_k \end{pmatrix} - \mu \iint \sigma_k \begin{pmatrix} 1 \\ v \\ v^2/2 \end{pmatrix} f\,\mathrm{d}v\,\mathrm{d}x$$

counterbalances the right hand side of (3a)–(3c). Next, we consider entropy dissipation. We restrict the discussion to the case where M_k is a Maxwellian, but the approach can be adapted, just changing the definition of the entropy functionals. Let us compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\iint f\ln(f)\,\mathrm{d}v\,\mathrm{d}x &= \sum_{k=1}^{K}\iint \sigma_k(\lambda M_k - \mu f)(1+\ln f)\,\mathrm{d}v\,\mathrm{d}x \\
&= (1-\ln(\mu))\sum_{k=1}^{K}\iint \sigma_k(\lambda M_k - \mu f)\,\mathrm{d}v\,\mathrm{d}x \\
&\quad -\sum_{k=1}^{K}\iint \sigma_k(\mu f - \lambda M_k)\ln\Big(\frac{\mu f}{\lambda M_k}\Big)\,\mathrm{d}v\,\mathrm{d}x \\
&\quad -\sum_{k=1}^{K}\iint \sigma_k(\mu f - \lambda M_k)\ln(\lambda M_k)\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

The last term can be recast as

$$\begin{aligned}
&-\sum_{k=1}^{K}\iint \sigma_k(\mu f - \lambda M_k)\Big(\ln\lambda + \ln n_k - \frac{N}{2}\ln(2\pi\theta_k) - \frac{|v-u_k|^2}{2\theta_k}\Big)\,\mathrm{d}v\,\mathrm{d}x \\
&= \sum_{k=1}^{K}\Big[\Big(\ln\lambda + \ln n_k - \frac{N}{2}\ln(2\pi\theta_k)\Big)\Big(\lambda n_k\mathscr{V}_k - \mu\iint \sigma_k f\,\mathrm{d}v\,\mathrm{d}x\Big) - \lambda\frac{N}{2}n_k\mathscr{V}_k\Big] \\
&\quad -\mu\sum_{k=1}^{K}\int \frac{\sigma_k}{2\theta_k}\Big(\int v^2 f\,\mathrm{d}v - 2u_k\cdot\int vf\,\mathrm{d}v + u_k^2\int f\,\mathrm{d}v\Big).
\end{aligned}$$

For the traps, we consider the evolution of the following functional

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int M_k\ln\Big(\frac{\lambda M_k}{\mu}\Big)\,\mathrm{d}v \\
&= \int M_k(1+\ln\Big(\frac{\lambda}{\mu}\Big) + \ln(M_k))\Big(\frac{\partial_t n_k}{n_k} + \Big(\frac{|v-u_k|^2}{2\theta_k^2} - \frac{N}{2\theta_k}\Big)\partial_t\theta_k + \frac{v-u_k}{\theta_k}\cdot\partial_t u_k\Big)\,\mathrm{d}v \\
&= \Big(1+\ln\Big(\frac{\lambda}{\mu}\Big)\Big)\partial_t n_k + \int\Big(\ln(n_k) - \frac{N}{2}\ln(2\pi\theta_k) - \frac{|v-u_k|^2}{2\theta_k}\Big) \\
&\quad\times M_k\Big(\Big(\frac{|v-u_k|^2}{2\theta_k^2} - \frac{N}{2\theta_k}\Big)\partial_t\theta_k + \frac{v-u_k}{\theta_k}\cdot\partial_t u_k\Big)\,\mathrm{d}v \\
&= \Big(1+\ln\Big(\frac{\lambda}{\mu}\Big) + \ln(n_k) - \frac{N}{2}\ln(2\pi\theta_k)\Big)\partial_t n_k \\
&\quad -\Big(\frac{N}{2}\partial_t n_k - \frac{n_k}{\theta_k}\Big(\Big(\frac{N}{2}\Big)^2 - \frac{N(N+2)}{4}\Big)\partial_t\theta_k\Big) \\
&= \Big(1+\ln\Big(\frac{\lambda}{\mu}\Big) + \ln(n_k) - \frac{N}{2}\ln(2\pi\theta_k)\Big)\partial_t n_k - \frac{N}{2\theta_k}\partial_t(n_k\theta_k).
\end{aligned}$$

We make use of (4) and we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int M_k \ln\Big(\frac{\lambda M_k}{\mu}\Big)\,\mathrm{d}v \\
&= \Big(1+\ln\Big(\frac{\lambda}{\mu}\Big)+\ln(n_k)-\frac{N}{2}\ln(2\pi\theta_k)\Big)\Big(\mu\iint \sigma_k f\,\frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{V}_k}-\lambda n_k\Big) \\
&\quad -\frac{1}{\theta_k}\Big(\mu\iint \sigma_k\frac{v^2}{2}f\,\frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{V}_k}-\lambda\frac{N}{2}n_k\theta_k-\lambda n_k\frac{u_k^2}{2} \\
&\quad -u_k\cdot\Big(\mu\iint \sigma_k v f\,\frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{V}_k}-\lambda n_k u_k\Big)+\frac{u_k^2}{2}\Big(\mu\iint \sigma_k f\,\frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{V}_k}-\lambda n_k\Big)\Big).
\end{aligned}$$

We conclude with the following H-theorem.

Proposition 3.1 (H-theorem) *The solutions of* (2), (3a)–(3c) *conserve total mass, total momentum and total energy:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\iint \begin{pmatrix}1\\ v\\ v^2/2\end{pmatrix} f\,\mathrm{d}v\,\mathrm{d}x+\sum_{k=1}^{K}\mathscr{V}_k\begin{pmatrix}n_k\\ n_k u_k\\ n_k E_k\end{pmatrix}\right\}=0, \tag{5}$$

and dissipate entropy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\iint f\ln f\,\mathrm{d}v\,\mathrm{d}x+\sum_{k=1}^{K}\mathscr{V}_k\int M_k\ln\Big(\frac{\lambda M_k}{\mu}\Big)\mathrm{d}v\right\}+\sum_{k=1}^{K}D_k(f,M_k)=0, \tag{6}$$

where we denote

$$D_k(f,M_k)=\iint \mu\sigma_k\Big(\frac{\lambda}{\mu}M_k-f\Big)\ln\Big(\frac{\lambda M_k}{\mu f}\Big)\,\mathrm{d}v\,\mathrm{d}x\geq 0.$$

Following [29], we can check that the dissipation term characterizes the distribution function for which the interaction terms vanish.

Lemma 3.2 (Weak coercivity) *The following properties are equivalent:*

(i) $\sum_{k=1}^{K} D_k(f,M_k)=0$ *for any* $0\leq t\leq T<\infty$,
(ii) *For all* $k\in\{1,\dots,K\}$*, we have* $f(t,x,v)=\frac{\lambda}{\mu}M_k(v)$ *on* $(0,T)\times\omega_k\times\mathbb{R}^N$.

We are led to characterize the equilibrium states as follows.

Proposition 3.3 (Unique continuation property) *There exists a unique solution of*

$$\partial_t f+v\cdot\nabla_x f=0,$$

satisfying $f(t,x,v)=\frac{\lambda}{\mu}M_k(v)$ *on* $(0,T)\times\omega_k\times\mathbb{R}^N$ *for all* $k\in\{1,\dots,K\}$*, with prescribed total mass, momentum and energy*

$$\iint \begin{pmatrix}1\\ v\\ v^2/2\end{pmatrix} f\,\mathrm{d}v\,\mathrm{d}x+\sum_{k=1}^{K}\mathscr{V}_k\int\begin{pmatrix}1\\ v\\ v^2/2\end{pmatrix}M_k\,\mathrm{d}v=\begin{pmatrix}m_0\\ p_0\\ \mathscr{E}_0\end{pmatrix},$$

which is

$$f(t,x,v)=\frac{\lambda}{\mu}\times\frac{n_e}{(2\pi\theta_e)^{N/2}}\exp\Big(-\frac{|v-u_e|^2}{2\theta_e}\Big)=\frac{\lambda}{\mu}M_e(v),$$

with

$$\begin{pmatrix} n_e \\ n_e u_e \\ (n_e u_e^2 + N n_e \theta_e)/2 \end{pmatrix} = \frac{1}{(|\mathbb{T}^N|\lambda/\mu + \mathscr{V})} \begin{pmatrix} m_0 \\ p_0 \\ \mathscr{E}_0 \end{pmatrix},$$

where we have denoted $\mathscr{V} = \sum_{k=1}^{K} \mathscr{V}_k \in (0, |\mathbb{T}^N|)$.

Proof For almost every $(x, v) \in \mathbb{T}^N \times \mathbb{R}^N$, the set $\{x - tv, t \in \mathbb{R}\}$ is dense in $\mathbb{T}^N$: the "almost everywhere infinite time geometric control condition" is satisfied, see [29, Lemma 6.4 and Proposition 3.1]. Since f is constant along the lines $x - tv$, and any trap is connected to another one by such a line, we have $f(t, x, v) = \frac{\lambda}{\mu} M_k(v) = \frac{\lambda}{\mu} M_\ell(v)$, for any $k, \ell \in \{1, \dots, K\}$. □

The main issue is to determine whether or not the solutions of (2), (3a)–(3c) converge as time becomes large to the equilibrium identified in Proposition 3.3. (Note that there exists other measure-valued equilibria, like $\delta_{x=x_*} \otimes \delta_{v=0}$ where x_* does not meet the support of the σ_k's.) This is performed in [5, 6, 29] for linear kinetic equations, with degenerate collision kernels (in [35] these results are revisited within the viewpoint of semigroup theory and by using general L^1 compactness statements). We shall adapt the approach to deal with the non linear problem (2), (3a)–(3c). (We point out that more quantitative estimates are derived in [20].) To this end, we reinterpret the entropy as a relative entropy with respect to the equilibrium state. We remind the reader that

$$H(F|G) = F\ln(F) - G\ln(G) - (1 + \ln(G))(F - G) = F\ln(F/G) - (F - G) \geq 0$$

is non negative and vanishes iff $F = G$. Therefore, we set

$$\mathscr{H} = \iint H\Big(f\Big|\frac{\lambda}{\mu} M_e\Big)\,\mathrm{d}v\,\mathrm{d}x + \sum_{k=1}^{K} \mathscr{V}_k \int H(M_k|M_e)\,\mathrm{d}v.$$

Owing to the conservation properties, we actually have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\iint f\ln f\,\mathrm{d}v\,\mathrm{d}x + \sum_{k=1}^{K} \mathscr{V}_k \int M_k \ln\Big(\frac{\lambda M_k}{\mu}\Big)\,\mathrm{d}v\right\}.$$

Indeed, $\mathscr{H}$ can be cast as

$$\begin{aligned} \mathscr{H} = &\iint f\ln(f)\,\mathrm{d}v\,\mathrm{d}x + \sum_{k=1}^{K} \mathscr{V}_k \int M_k \ln\Big(\frac{\lambda M_k}{\mu}\Big)\,\mathrm{d}v \\ &- \left(\iint f\ln\Big(\frac{\lambda M_e}{\mu}\Big)\,\mathrm{d}v\,\mathrm{d}x + \sum_{k=1}^{K} \mathscr{V}_k \int M_k \ln\Big(\frac{\lambda M_e}{\mu}\Big)\,\mathrm{d}v\right) \\ &- \left(\iint f\,\mathrm{d}v\,\mathrm{d}x + \sum_{k=1}^{K} \mathscr{V}_k \int M_k\,\mathrm{d}v - \iint \frac{\lambda M_e}{\mu}\,\mathrm{d}v\,\mathrm{d}x - \sum_{k=1}^{K} \mathscr{V}_k \int M_e\,\mathrm{d}v\right). \end{aligned}$$

By definition, the last term

$$\iint f\,\mathrm{d}v\,\mathrm{d}x + \sum_{k=1}^{K} \mathscr{V}_k n_k - n_e\Big(|\mathbb{T}^N|\frac{\lambda}{\mu} + \mathscr{V}\Big) = m_0 - n_e\Big(|\mathbb{T}^N|\frac{\lambda}{\mu} + \mathscr{V}\Big) = 0$$

vanishes, while the second term reads

$$\ln\left(\frac{\lambda n_e}{\mu(2\pi\theta_e)^{N/2}}+\frac{|u_e|^2}{2\theta_e}\right)\left(\iint f\,\mathrm{d}v\,\mathrm{d}x+\sum_{k=1}^{K}\mathscr{V}_k n_k\right)$$
$$+\frac{u_e}{\theta_e}\cdot\left(\iint vf\,\mathrm{d}v\,\mathrm{d}x+\sum_{k=1}^{K}\mathscr{V}_k n_k u_k\right)+\frac{1}{2\theta_e}\left(\iint v^2 f\,\mathrm{d}v\,\mathrm{d}x+\sum_{k=1}^{K}\mathscr{V}_k n_k E_k\right)$$

which is a sum of conserved quantities. This quantity $\mathscr{H}$ can thus be used to evaluate how far the solution is from the equilibrium. It is dissipated since $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}=-\sum_k D_k(f,M_k)\le 0$. That the solutions of (2), (3a)–(3c) converge to equilibrium follows from a contradiction-compactness argument, that will be detailed in Sect. 4.

For further purposes, we shall need another estimate, uniform on time $t\ge 0$, on higher moments of the particle distribution function. To this end, we adapt the trick presented in [8] (see also [36, Lemma 2]).

Lemma 3.4 *Let $f\in C([0,\infty);L^1(\mathbb{T}^N\times R^N))$ satisfy*

$$\partial_t f+v\cdot\nabla_x f+\sigma f=\sigma g$$

with $x\mapsto\sigma(x)\ge 0$ in L^∞, $g\ge 0$, $f\big|_{t=0}=f_0\ge 0$ and

$$\iint(1+v^2)f_0\,\mathrm{d}v\,\mathrm{d}x<\infty,\qquad \sup_{t\ge 0}\iint(1+v^2)g\,\mathrm{d}v\,\mathrm{d}x<\infty.$$

Then, there exists two constants C_0, C_1 such that for any $0\le t_1\le t_2<\infty$, we have

$$\int_{t_1}^{t_2}\iint|v|^3 f\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\le C_0+C_1(t_2-t_1)<\infty.$$

Proof We multiply the kinetic equation by the function

$$\Phi(x,v)=\sqrt{1+v^2}\,\frac{v\cdot x}{\sqrt{1+x^2}}$$

remarking that

$$v\cdot\nabla_x\Phi(x,v)=v^2\sqrt{\frac{1+v^2}{1+x^2}}\left(1-\frac{(v\cdot x)^2}{v^2(1+x^2)}\right)\ge v^2\frac{\sqrt{1+v^2}}{(1+x^2)^{3/2}}.$$

Hence, we get

$$\int_{t_1}^{t_2}\iint v\cdot\nabla_x\Phi f\,\mathrm{d}\,\mathrm{d}x\,\mathrm{d}t=\iint\Phi(f(t_2)-f(t_1))\,\mathrm{d}v\,\mathrm{d}x-\int_{t_1}^{t_2}\iint\Phi\sigma(g-f)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.$$

It follows that

$$\frac{1}{(1+\mathrm{diam}^2(\mathbb{T}^N))^{3/2}}\int_{t_1}^{t_2}\iint|v|^3 f\,\mathrm{d}\,\mathrm{d}x\,\mathrm{d}t\le\int_{t_1}^{t_2}\iint v^2\frac{\sqrt{1+v^2}}{(1+x^2)^{3/2}}f\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t$$
$$\le\iint|\Phi|(f(t_2)+f(t_1))\,\mathrm{d}v\,\mathrm{d}x+\|\sigma\|_\infty\int_{t_1}^{t_2}\iint|\Phi|(g+f)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t$$
$$\le\iint(1+v^2)(f(t_2)+f(t_1))\,\mathrm{d}v\,\mathrm{d}x+\|\sigma\|_\infty\int_{t_1}^{t_2}\iint(1+v^2)(g+f)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.$$

□

Existence-uniqueness can be addressed by using the analysis of the BGK operator in [36] and [37]. In particular, we can use the Lipschitz property of the application

$$(n,u,\theta)\longmapsto M[n,u,\theta](v)=\frac{n}{(2\pi\theta)^{N/2}}\exp\Big(-\frac{|v-u|^2}{2\theta}\Big).$$

Given (n_j,u_j,θ_j), with $j\in\{1,2\}$ satisfying

$$0<c_\star\le n_j,\theta_j\le c^\star,\qquad |u_j|,\le c^\star$$

we have

$$\int(1+v^2)|M[n_1,u_1,\theta_1](v)-M[n_2,u_2,\theta_2](v)|\,\mathrm{d}v\le C(|n_1-n_2|+|u_1-u_2|+|\theta_1-\theta_2|)$$

where C depends on $c_\star, c^\star$. Then, we consider the iteration scheme

$$\partial_t f^{(\ell+1)}+v\cdot\nabla_x f^{(\ell+1)}=\sum_{k=1}^K\sigma_k(\lambda M_k^{(\ell)}\mu-f^{(\ell+1)}),$$
$$\partial_t\begin{pmatrix} n_k^{(\ell+1)}\\ n_k^{(\ell+1)}u_k^{(\ell+1)}\\ \frac{N}{2}n_k^{(\ell+1)}\theta_k^{(\ell+1)}\end{pmatrix}+\lambda\begin{pmatrix} n_k^{(\ell+1)}\\ n_k^{(\ell+1)}u_k^{(\ell+1)}\\ \frac{N}{2}n_k^{(\ell+1)}\theta_k^{(\ell+1)}\end{pmatrix}=\mu\iint\sigma_k\begin{pmatrix}1\\ v\\ |v-u_k^{(\ell+1)}|^2/2\end{pmatrix}f^{(\ell+1)}\frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{V}_k}$$

with initial data

$$f^{(\ell+1)}=f_{\mathrm{Init}},\quad n_k^{(\ell+1)}=n_{\mathrm{Init},k},\quad u_k^{(\ell+1)}=u_{\mathrm{Init},k},\quad \theta_k^{(\ell+1)}=\theta_{\mathrm{Init},k}.$$

We assume $f_{\mathrm{Init}}\ge 0$, $n^\star\ge n_{\mathrm{Init},k}\ge n_\star>0$, $|u_{\mathrm{Init},k}|\le u^\star$, and $\theta^\star\ge\theta_{\mathrm{Init},k}\ge\theta_\star>0$. Therefore, we have $f^{(\ell+1)}\ge 0$ and for any $0<T<\infty$, we can find $c_\star, c^\star>0$ (depending on T) such that

$$0<c_\star\le n_k^{(\ell+1)},\theta_k^{(\ell+1)}\le c^\star,\qquad |u_k^{(\ell+1)}|\le c^\star,\qquad \iint(1+v^2)f^{(\ell+1)}\,\mathrm{d}v\,\mathrm{d}x\le c^\star.$$

Moreover, $|n_k^{(\ell+1)}-n_k^{(\ell)}|+|u_k^{(\ell+1)}-u_k^{(\ell)}|+|\theta_k^{(\ell+1)}-\theta_k^{(\ell)}|$ can be dominated by $\|\sigma_k\|_\infty\int_0^t\iint(1+v^2)|f^{(\ell+1)}-f^{(\ell)}|\,\mathrm{d}v\,\mathrm{d}x$. Therefore, the scheme is defined by a contraction on the functional space $C^0([0,T_0];L^1(\mathbb{T}^N\times\mathbb{R}^N,(1+v^2)\,\mathrm{d}v\,\mathrm{d}x))$, for $0<T_0<T<\infty$ small enough; Picard's theorem applies and provides the existence-uniqueness of the solution on $[0,T_0]$. This solution can be extended on the entire interval $[0,T]$ owing to the conservation laws.

Proposition 3.5 *Let $f_{\mathrm{Init}}\ge 0$ such that $\iint(1+v^2)f_{\mathrm{Init}}\,\mathrm{d}v\,\mathrm{d}x<\infty$ and for $k\in\{1,\dots,K\}$, positive $n_{\mathrm{Init},k},\theta_{\mathrm{Init},k}$, and $u_{\mathrm{Init},k}\in\mathbb{R}^N$. Then there exists a unique solution of the associated Cauchy problem (2), (3a)–(3c), with $f\in C^0([0,T];L^1(\mathbb{T}^N\times\mathbb{R}^N))$, $f\ge 0$, $n_k,u_k,\theta_k\in C^0([0,T])$, with $n_k,\theta_k>0$.*

The solution satisfies the mass, momentum and energy conservation, as well as the entropy dissipation, reproducing the arguments in [36]. Note that the local in time estimates of [37] equally applies.

4 Convergence to Equilibrium

We turn to the discussion of the convergence towards equilibrium and we are going to justify the following statement.

Theorem 4.1 *Let the initial data* $f_{\text{Init}}, n_{k,\text{Init}}, u_{k,\text{Init}}, \theta_{k,\text{Init}}$ *for* (2)*,* (3a)–(3c) *satisfy*

$$\iint (1+v^2+|\ln(f_{\text{Init}})|) f_{\text{Init}}\,\mathrm{d}v\,\mathrm{d}x + \sum_{k=1}^{K} \mathscr{V}_k \int (1+v^2+|\ln(M_{k,\text{Init}})|) M_{k,\text{Init}}\,\mathrm{d}v < \infty.$$

Then the associated solution of the Cauchy problem satisfies

$$\lim_{t\to\infty} \Bigg(\iint |f - M_e|(t,x,v)\,\mathrm{d}v\,\mathrm{d}x + \sum_{k=1}^{K} \mathscr{V}_k \big(|n_k - n_e| + |n_k u_k - n_e u_e| + |n_k E_k - n_e E_e|\big)(t) \Bigg) = 0.$$

The arguments in [5, 6] rely on semi-group theory, which are well-adapted to handle linear problems. Instead, we adapt the strategy introduced in [29], based on a contradiction argument (it is likely that the alternative approach presented in [20] can provide a more detailed information).

Preparation step. We suppose that we can find initial data $(f_{\text{Init}}, n_{\text{Init},k}, u_{\text{Init},k}, \theta_{\text{Init},k})$, $\epsilon > 0$ and an increasing sequence $(t_\nu)_{\nu\in\mathbb{N}}$ such that the solution associated to this data satisfies

$$\|f(t_\nu) - M_e\|_{L^1} + \sum_{k=1}^{K} \mathscr{V}_k (|n_k(t_\nu) - n_e| + |n_k u_k(t_\nu) - n_e u_e| + |n_k E_k(t_\nu) - n_e E_e|) \geq \epsilon. \tag{7}$$

Note that the left hand side can be bounded from above by a constant determined by the initial data and the conservation laws of the equation. We set $g_\nu(t,x,v) = f(t+t_\nu, x, v)$, $n_\nu(t) = n(t+t_\nu)$, $u_\nu(t) = u(t+t_\nu)$, $\theta_\nu(t) = \theta(t+t_\nu)$ which still define a solution of (2), (3a)–(3c), with the data $(g_\nu, n_\nu, u_\nu, \theta_\nu)\big|_{t=0} = (f, n, u, \theta)(t_\nu)$. Since $t \mapsto \mathscr{H}(t)$ is non increasing and non negative, it admits a limit as $t\to\infty$

$$\mathscr{H}_\infty = \inf\{\mathscr{H}(t),\ t \geq 0\} \in [0, \mathscr{H}(0)].$$

(Note that $\mathscr{H}_\infty = 0$ would contradict (7).) Accordingly the functional

$$\mathscr{H}_\nu(t) = \iint H\Big(g_\nu \Big| \frac{\lambda}{\mu} M_e\Big)(t,x,v)\,\mathrm{d}v\,\mathrm{d}x + \sum_{k=1}^{K} \mathscr{V}_k \int H(M_{\nu,k}, |M_e)(t,v)\,\mathrm{d}v$$

also tends to this limit as $\nu \to \infty$, independently of $t \geq 0$. We have

$$\begin{aligned} \mathscr{H}_\nu(t) - \mathscr{H}_\nu(0) + \sum_{k=1}^{K} \int_{t_\nu}^{t_\nu+t} D_k(f, M_k)(s)\,\mathrm{d}s = 0 \\ = \mathscr{H}_\nu(t) - \mathscr{H}_\nu(0) + \sum_{k=1}^{K} \int_0^t D_k(g_\nu, M_{\nu,k})(s)\,\mathrm{d}s, \end{aligned}$$

for any $0 < t < \infty$. It implies that

$$\lim_{\nu\to\infty} \sum_{k=1}^{K} \int_0^t D_k(g_\nu, M_{\nu,k})(t)\,\mathrm{d}t = 0. \tag{8}$$

Compactness step. Owing to conservation properties and entropy dissipation, we know that

$$\iint (1 + v^2 + \ln(g_\nu))g_\nu\,\mathrm{d}v\,\mathrm{d}x + \sum_{k=1}^{K} \mathscr{V}_k \int M_{\nu,k}\ln(M_{\nu,k})\,\mathrm{d}v + n_{\nu,k} + \frac{n_{\nu,k}}{2}(u_{\nu,k}^2 + N\theta_{\nu,k})$$

is bounded uniformly with respect to $t \geq 0$, and $\nu \in \mathbb{N}$. By standard arguments based on Dunford-Pettis' theorem, we deduce that g_ν is weakly compact in $L^1((0,T)\times\mathbb{T}^N\times\mathbb{R}^N)$. A similar conclusion applies to $M_{\nu,k}$ for any $k \in \{1, ..., K\}$. With Lemma 3.4, we also get

$$\int_0^T \iint |v|^3 g_\nu\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t = \int_{t_\nu}^{t_\nu+T} \iint |v|^3 f\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \leq C_0 + C_1 T. \tag{9}$$

Moreover, going back to the differential equations (3a)–(3c) satisfied by $n_{\nu,k}$, $n_{\nu,k}u_{\nu,k}$, and $n_{\nu,k}E_{\nu,k}$ we see that their time derivative are uniformly bounded too. According to the Arzela-Ascoli theorem, the sequences $\big(n_{\nu,k}, n_{\nu,k}u_{\nu,k}, n_{\nu,k}E_{\nu,k}\big)_{\nu\in\mathbb{N}}$ are therefore compact on $C^0([0,T])$ for any $0 < T < \infty$.

Identification of the limit step. Up to subsequences, we infer that

$$\begin{aligned}
&g_\nu \rightharpoonup g \quad \text{weakly in } L^1((0,T)\times\mathbb{T}^N\times\mathbb{R}^N),\\
&M_{\nu,k} \rightharpoonup Z_k \quad \text{weakly in } L^1((0,T)\times\mathbb{R}^N),\\
&n_{\nu,k}, n_{\nu,k}u_{\nu,k}, n_{\nu,k}E_{\nu,k} \to n_k,\ p_k = n_k u_k,\ Q_k = n_k(u_k^2 + N\theta_k)/2 \quad \text{uniformly on } [0,T].
\end{aligned}$$

Owing to (9), we have

$$\int_0^T \iint \varphi g_\nu\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \xrightarrow[\nu\to\infty]{} \int_0^T \iint \varphi g\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t,$$

for any locally bounded trial function $\varphi : (0,T)\times\mathbb{T}^N\times\mathbb{R}^N \to \mathbb{R}$ such that $\frac{\varphi(t,x,v)}{|v|^3} \to 0$ as $|v| \to \infty$. Let $\mathscr{N}_k = \{t \in (0,T),\ n_k(t) = 0\}$. We have

$$u_{\nu,k}, \theta_{\nu,k} \xrightarrow[\nu\to\infty]{} u_k, \theta_k \quad \text{a.e. on } \complement\mathscr{N}_k$$

(the complement set to $\mathscr{N}_k$ in $(0,T)$). Moreover, since high order moments of $M_{\nu,k}$ are bounded with respect to ν, we can pass to the limit in

$$\begin{aligned}
\lim_{\nu\to\infty} \int_0^T \int |v - u_{\nu,k}|^2 M_{\nu,k}\zeta(t)\,\mathrm{d}v\,\mathrm{d}t &= \lim_{\nu\to\infty} \int_0^T N n_{\nu,k}\theta_{\nu,k}\zeta(t)\,\mathrm{d}t\\
&= \int_0^T \int |v - u_k|^2 Z_k\zeta(t)\,\mathrm{d}v\,\mathrm{d}t\\
&= \int_0^T N n_k\theta_k\zeta(t)\,\mathrm{d}t,
\end{aligned}$$

for any function $\zeta \in L^\infty((0,T))$. It implies that $\theta_k > 0$ a.e. on $\complement\mathscr{N}_k$. Accordingly, we get

$$M_{\nu,k} \rightharpoonup Z_k = M_k \quad \text{a. e.} \complement\mathscr{N}_k \times \mathbb{R}^N.$$

Since

$$n_{\nu,k} = \int M_{\nu,k}\,\mathrm{d}v \xrightarrow[\nu\to\infty]{} n_k = \int Z_k\,\mathrm{d}v,$$

pointwise on $[0, T]$, $M_{\nu,k}$ converges to 0 a. e. on $\mathcal{N}_k \times \mathbb{R}^d$. Hence, we conclude that

$$M_{\nu,k} \rightharpoonup Z_k = M_k \quad \text{a. e. and strongly in} L^1((0,T)\times\mathbb{R}^N).$$

That the convergence holds strongly is a consequence of the combination of the weak-L^1 and a. e. convergence. We deal with the entropy dissipation by reproducing the arguments in [36, Proof of Theorem 1, Forth step], using the convexity of $(a, b) \mapsto (a-b)\ln(a/b)$; we obtain

$$\liminf_{\nu\to\infty} \sum_{k=1}^{K} \int_0^T D_k(g_\nu, M_{\nu,k})(t)\,\mathrm{d}t \geq \sum_{k=1}^{K} \int_0^T D_k(g, M_k)(t)\,\mathrm{d}t \geq 0.$$

Therefore, by using (8), we deduce that

$$\sum_{k=1}^{K} \int_0^T D_k(g, M_k)(t)\,\mathrm{d}t = 0.$$

Lemma 3.2 tells us that $g(t,x,v) = \frac{\lambda}{\mu} M_k(t,v)$ on $(0,T)\times\omega_k\times\mathbb{R}^N$. Moreover letting ν tend to 0 in the evolution equation satisfied by g_ν, we obtain

$$\partial_t g + v\cdot\nabla_x g = 0,$$

while the conserved quantities also pass to the limit (using (9) again) so that

$$\begin{aligned}
&\iint \begin{pmatrix} 1 \\ v \\ v^2/2 \end{pmatrix} g_\nu\,\mathrm{d}v\,\mathrm{d}x + \sum_{k=1}^{K} \mathscr{V}_k \begin{pmatrix} n_{\nu,k} \\ n_{\nu,k}u_{\nu,k} \\ n_{\nu,k}E_{\nu,k} \end{pmatrix} \\
&= \iint \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} f_{\mathrm{Init}}\,\mathrm{d}v\,\mathrm{d}x + \sum_{k=1}^{K} \mathscr{V}_k \begin{pmatrix} n_{\mathrm{Init},k} \\ n_{\mathrm{Init},k}u_{\mathrm{Init},k} \\ n_{\mathrm{Init},k}(u^2_{\mathrm{Init},k} + N\theta_{\mathrm{Init},k}) \end{pmatrix} \\
&= \iint \begin{pmatrix} 1 \\ v \\ v^2/2 \end{pmatrix} g\,\mathrm{d}v\,\mathrm{d}x + \sum_{k=1}^{K} \mathscr{V}_k \begin{pmatrix} n_k \\ n_k u_k \\ n_k E_k \end{pmatrix}.
\end{aligned}$$

Applying Proposition 3.3, we conclude that $g = \frac{\lambda}{\mu} M_e$, $M_k = M_e$, and $n_k = n_e$, $u_k = u_e$, $\theta_k = \theta_e$.

Now, let $m_\nu = |g_\nu - \frac{\lambda}{\mu} M_e|$. It is bounded in $L^\infty(0,\infty, L^1(\mathbb{T}^N\times\mathbb{R}^N))$. Hence, we can suppose that

$$m_\nu \overset{\star}{\rightharpoonup} m \;\; \text{weaky-}\star\text{in} L^\infty_w(0,T;\mathcal{M}^1(\mathbb{T}^N\times\mathbb{R}^N))$$

(the set of weakly-$\star$ measurable functions on $[0, T]$, with values in the space of bounded measures; it identifies with the dual of $L^1(0,T; C^0_0(\mathbb{T}^N\times\mathbb{R}^N))$, see [22, Chap. 8, Sect. 18]). It satisfies

$$\begin{aligned}
\partial_t m_\nu + v\cdot\nabla_x m_\nu &= \sum_{k=1}^{K} \sigma_k(\lambda M_{\nu,k} - \mu g_\nu)\mathrm{sgn}\Big(g_\nu - \frac{\lambda}{\mu} M_e\Big) \\
&= \sum_{k=1}^{K} \sigma_k\Big(\lambda(M_{\nu,k} - M_e)\mathrm{sgn}\Big(g_\nu - \frac{\lambda}{\mu} M_e\Big) - \mu m_\nu\Big).
\end{aligned}$$

We also infer that $\{t \mapsto \iint m_\nu \varphi \,\mathrm{d}v\,\mathrm{d}x,\ \nu \in \mathbb{N}\}$ is relatively compact in $C^0([0,T])$ for any $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R}^N)$, and, in fact, m_ν is compact in $C^0([0,T]; \mathcal{M}^1(\mathbb{T}^N \times \mathbb{R}^N)\text{-weak-}\star)$. Since $M_{\nu,k}$ converges to M_e in $L^1((0,T)\times\mathbb{R}^N)$, as $\nu \to 0$, we get

$$\partial_t m + v \cdot \nabla_x m = -\sum_{k=1}^{K} \mu\sigma_k m. \tag{10}$$

It yields

$$m(t,x,v) = m(0, x - tv, v) \exp\left(-\int_0^t \sum_{k=1}^{K} \mu\sigma_k(x - sv)\,\mathrm{d}s \right).$$

Moreover, we have

$$m_\nu = \mathrm{sgn}\Big(g_\nu - \frac{\lambda}{\mu} M_e\Big)\Big(\Big(g_\nu - \frac{\lambda}{\mu} M_{\nu,k}\Big) + \frac{\lambda}{\mu}(M_{\nu,k} - M_e)\Big).$$

On the one hand, we know that $\|M_{\nu,k} - M_e\|_{L^1}$ tends to 0 as $\nu \to \infty$. On the other hand, owing to the elementary inequality

$$|\sqrt{b} - \sqrt{a}|^2 = \left| \int_a^b \frac{\mathrm{d}s}{2\sqrt{s}} \right|^2 \le \frac{1}{4}(b-a)\ln(b/a)$$

we can use the entropy dissipation to obtain

$$\begin{aligned}
&\sum_{k=1}^{K} \int_0^T \iint \sigma_k \Big| g_\nu - \frac{\lambda}{\mu} M_{\nu,k} \Big| \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&\le \left(\sum_{k=1}^{K} \int_0^T \iint \sigma_k \Big| \sqrt{\frac{\lambda}{\mu} M_{\nu,k}} + \sqrt{g_\nu} \Big|^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \\
&\times \left(\sum_{k=1}^{K} \int_0^T \iint \sigma_k \Big| \sqrt{\frac{\lambda}{\mu} M_{\nu,k}} - \sqrt{g_\nu} \Big|^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \\
&\le \left(\sum_{k=1}^{K} \|\sigma_k\|_\infty \int_0^T \iint \Big(\frac{\lambda}{\mu} M_{\nu,k} + g_\nu \Big) \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \left(\sum_{k=1}^{K} \int_0^T D_k(g_\nu, M_{\nu,k})\,\mathrm{d}t \right)^{1/2} \\
&\le \left(T(1 + \lambda/\mu) m_0 \sup_{k \in \{1,\dots,K\}} \|\sigma_k\|_\infty \right)^{1/2} \left(\sum_{k=1}^{K} \int_0^T D_k(g_\nu, M_{\nu,k})\,\mathrm{d}t \right)^{1/2},
\end{aligned}$$

which tends to 0 by virtue of (8). Therefore, we actually have, for any $k \in \{1, \dots, K\}$,

$$\sigma_k m = 0.$$

Hence m vanishes on $(0,T) \times \omega_k \times \mathbb{R}^N$, and (10) becomes $\partial_t m + v \cdot \nabla_x m = 0$. We deduce that $m = 0$, by the unique continuation property. The initial data for the transport equation is determined by the weak-$\star$ limit of $m_\nu(0) = |g_\nu(0) - M_e|$ in $\mathcal{M}^1(\mathbb{T}^N \times \mathbb{R}^N)$. However, the uniform estimate on the energy tells us that the convergence holds tightly and, in particular

$$\iint m_\nu(0)\,\mathrm{d}v\,\mathrm{d}x \xrightarrow[\nu\to\infty]{} \iint m(0)\,\mathrm{d}v\,\mathrm{d}x.$$

Since $m = 0$, this limit necessarily vanishes. Since $n_{\nu,k}$, $n_{\nu,k}u_{\nu,k}$ and $n_{\nu,k}E_{\nu,k}$ converge uniformly on $[0,T]$ to the equilibrium values, we are led to a contradiction with (7). $\square$

5 Linearized Problem: Rate of Convergence to Equilibrium

It is interesting to consider the linearized problem obtained by considering fluctuations about the equilibrium state since linearity allows us to obtain more detailed information on the asymptotic trend to equilibrium. Taking derivatives of the Maxwellian $M(v) = \frac{n}{(2\pi\theta)^{N/2}} e^{-|v-u|^2/(2\theta)}$ with respect to the parameters $\mathbf{m} = (n, u, \theta)$, we get

$$\nabla_{\mathbf{m}} M(v) = \begin{pmatrix} \frac{1}{n} \\ \frac{v-u}{\theta} \\ \frac{|v-u|^2}{2\theta^2} - \frac{N}{2\theta} \end{pmatrix} M(v).$$

Let us denote

$$\begin{aligned} \tilde{\mathbf{m}} = \begin{pmatrix} \tilde{n} \\ \tilde{u} \\ \tilde{\theta} \end{pmatrix} \longmapsto \tilde{L}(v) &= \nabla_{\mathbf{m}} M_e(v) \cdot \tilde{\mathbf{m}} = \nabla_{\mathbf{m}} M_e(v) \cdot \begin{pmatrix} \tilde{n} \\ \tilde{u} \\ \tilde{\theta} \end{pmatrix} \\ &= \left(\frac{\tilde{n}}{n_e} + \frac{v-u_e}{\theta_e} \cdot \tilde{u} + (|v-u_e|^2 - N\theta_e) \frac{\tilde{\theta}}{2\theta_e^2} \right) M_e(v). \end{aligned}$$

We observe that

$$\int \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} \tilde{L} \, \mathrm{d}v = \begin{pmatrix} \tilde{n} \\ n_e \tilde{u} + \tilde{n} u_e \\ N\tilde{n}\theta_e + N n_e \tilde{\theta} + \tilde{n} u_e^2 + 2 n_e u_e \tilde{u} \end{pmatrix}.$$

The linearized problem can be written

$$\begin{aligned} \partial_t \tilde{f} + v \cdot \nabla_x \tilde{f} &= \sum_{k=1}^{K} \sigma_k (\lambda \tilde{L}_k - \mu \tilde{f}), \\ \partial_t \tilde{n}_k &= \mu \iint \sigma_k \tilde{f} \frac{\mathrm{d}v \, \mathrm{d}x}{\mathscr{V}_k} - \lambda \tilde{n}_k = \iint \sigma_k (\mu \tilde{f} - \lambda \tilde{L}_k) \frac{\mathrm{d}v \, \mathrm{d}x}{\mathscr{V}_k}, \\ n_e \partial_t \tilde{u}_k &= \iint \sigma_k (v - u_e)(\mu \tilde{f} - \lambda \tilde{L}_k) \frac{\mathrm{d}v \, \mathrm{d}x}{\mathscr{V}_k}, \\ \frac{N}{2} n_e \partial_t \tilde{\theta}_k &= \mu \frac{1}{2} \iint \sigma_k (|v-u_e|^2 - N\theta_e) \tilde{f} \frac{\mathrm{d}v \, \mathrm{d}x}{\mathscr{V}_k} - \lambda \frac{N}{2} n_e \tilde{\theta}_k \\ &= \frac{1}{2} \iint \sigma_k (|v-u_e|^2 - N\theta_e)(\mu \tilde{f} - \lambda \tilde{L}_k) \frac{\mathrm{d}v \, \mathrm{d}x}{\mathscr{V}_k}, \end{aligned} \tag{11}$$

with $\tilde{L}_k = \nabla_{\mathbf{m}} M_e \cdot \tilde{\mathbf{m}}_k$ associated to $\tilde{\mathbf{m}}_k = (\tilde{n}_k, \tilde{u}_k, \tilde{\theta}_k)$. The system conserves mass, momentum and energy in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\{ \iint \begin{pmatrix} 1 \\ v - u_e \\ |v-u_e|^2 - N\theta_e \end{pmatrix} \tilde{f} \, \mathrm{d}v \, \mathrm{d}x + \sum_{k=1}^{K} \mathscr{V}_k \begin{pmatrix} \tilde{n}_k \\ n_e \tilde{u}_k \\ N n_e \tilde{\theta}_k \end{pmatrix} \right\} = 0.$$

The dissipation property follows from the following observation: on the one hand

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \iint \frac{\tilde{f}^2}{M_e} \, \mathrm{d}v \, \mathrm{d}x = -\sum_{k=1}^{K} \iint \sigma_k (\mu \tilde{f} - \lambda \tilde{L}_k) \frac{\tilde{f}}{M_e} \, \mathrm{d}v \, \mathrm{d}x$$

and, on the other hand

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{V}_k\left(\frac{\tilde{n}_k^2}{n_e}+\frac{n_e}{\theta_e}\tilde{u}_k^2+\frac{Nn_e}{2\theta_e^2}\tilde{\theta}_k^2\right)\\
&\quad=\iint\Big(\frac{\tilde{n}_k}{n_e}+\frac{v-u_e}{\theta_e}\cdot\tilde{u}_k+\big(|v-u_e|^2-N\theta_e\big)\frac{\tilde{\theta}_k}{2\theta_e^2}\Big)\sigma_k(\mu\tilde{f}-\lambda\tilde{L}_k)\,\mathrm{d}v\,\mathrm{d}x\\
&\quad=\iint\sigma_k\frac{\tilde{L}_k}{M_e}(\mu\tilde{f}-\lambda\tilde{L}_k)\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}
$$

Therefore, we arrive at the following H-theorem

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\iint\frac{\tilde{f}^2}{M_e}\,\mathrm{d}v\,\mathrm{d}x+\sum_{k=1}^{K}\frac{\lambda}{\mu}\mathscr{V}_k\left(\frac{\tilde{n}_k^2}{n_e}+\frac{n_e\tilde{u}_k^2}{\theta_e}+\frac{Nn_e\tilde{\theta}_k^2}{2\theta_e^2}\right)\right\}\\
&\quad=-\mu\sum_{k=1}^{K}\iint\sigma_k\frac{|\tilde{f}-(\lambda/\mu)\tilde{L}_k|^2}{M_e}\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}
\tag{12}
$$

For any $\bar{\mathbf{m}}=(\bar{n},\bar{u},\bar{\theta})\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}$,

$$
\tilde{f}(v)=\frac{\lambda}{\mu}\bar{\mathbf{m}}\cdot\nabla_{\mathbf{m}}M_e(v)=\frac{\lambda}{\mu}\begin{pmatrix}\bar{n}\\ \bar{u}\\ \bar{\theta}\end{pmatrix}\cdot\nabla_{\mathbf{m}}M_e(v),\qquad n_k=\bar{n},\ u_k=\bar{u},\ \theta_k=\bar{\theta}
$$

is an equilibrium solution of the linearized equation. Given the initial data with

$$
\iint\begin{pmatrix}1\\ v-u_e\\ |v-u_e|^2-N\theta_e\end{pmatrix}\tilde{f}_{\mathrm{Init}}\,\mathrm{d}v\,\mathrm{d}x+\sum_{k=1}^{K}\mathscr{V}_k\begin{pmatrix}\tilde{n}_{\mathrm{Init},k}\\ n_e\tilde{u}_{\mathrm{Init},k}\\ Nn_e\tilde{\theta}_{\mathrm{Init},k}\end{pmatrix}=\begin{pmatrix}n_0\\ n_eu_0\\ Nn_e\theta_0\end{pmatrix},
$$

we select the equilibrium that fulfils the conservation law, namely $\bar{\mathbf{m}}$ is defined by the relation

$$
\begin{aligned}
&\iint\begin{pmatrix}1\\ v-u_e\\ |v-u_e|^2-N\theta_e\end{pmatrix}\frac{\lambda}{\mu}\bar{\mathbf{m}}\cdot\nabla_{\mathbf{m}}M_e\,\mathrm{d}v\,\mathrm{d}x+\sum_{k=1}^{K}\mathscr{V}_k\begin{pmatrix}\bar{n}\\ n_e\bar{u}\\ Nn_e\bar{\theta}\end{pmatrix}\\
&\quad=\Big(\frac{\lambda}{\mu}|\mathbb{T}^N|+\mathscr{V}\Big)\begin{pmatrix}\bar{n}\\ n_e\bar{u}\\ Nn_e\bar{\theta}\end{pmatrix}=\begin{pmatrix}n_0\\ n_eu_0\\ Nn_e\theta_0.\end{pmatrix}
\end{aligned}
$$

Let us set

$$
\mathscr{H}=\iint\Big|\tilde{f}-\frac{\lambda}{\mu}\bar{\mathbf{m}}\cdot\nabla_{\mathbf{m}}M_e\Big|^2\frac{\mathrm{d}v\,\mathrm{d}x}{2M_e}+\frac{1}{2}\sum_{k=1}^{K}\frac{\lambda}{\mu}\mathscr{V}_kA_e(\tilde{\mathbf{m}}_k-\bar{\mathbf{m}})\cdot(\tilde{\mathbf{m}}_k-\bar{\mathbf{m}})
$$

where $\tilde{\mathbf{m}}_k=(\tilde{n}_k,\tilde{u}_k,\tilde{\theta}_k)$ and A_e stands for the following diagonal matrix

$$
A_e=\begin{pmatrix}1/n_e & 0 & 0\\ 0 & n_e/\theta_e & 0\\ 0 & 0 & Nn_e/(2\theta_e^2)\end{pmatrix}.
$$

Note that $A_e\tilde{\mathbf{m}}_k \cdot \tilde{\mathbf{m}}_k = \frac{\tilde{n}_k^2}{n_e} + \frac{n_e\tilde{u}_k^2}{\theta_e} + \frac{Nn_e\tilde{\theta}_k^2}{2\theta_e^2}$. By expanding this expression, we obtain

$$\begin{aligned}
\mathscr{H} &= \frac{1}{2}\iint \frac{\tilde{f}^2}{M_e}\,\mathrm{d}v\,\mathrm{d}x + \sum_{k=1}^{K}\frac{\lambda}{\mu}\mathscr{V}_k\left(\frac{\tilde{n}_k^2}{n_e} + \frac{n_e\tilde{u}_k^2}{\theta_e} + \frac{Nn_e\tilde{\theta}_k^2}{2\theta_e^2}\right)\\
&\quad \underbrace{+\frac{1}{2}\iint \frac{\lambda^2}{\mu^2}|\bar{\mathbf{m}}\cdot\nabla_{\mathbf{m}}M_e|^2\frac{\mathrm{d}v\,\mathrm{d}x}{M_e} + \frac{1}{2}\sum_{k=1}^{K}\frac{\lambda}{\mu}\mathscr{V}_kA_e\bar{\mathbf{m}}\cdot\bar{\mathbf{m}}}_{\text{independent of time}}\\
&\quad +\frac{\lambda}{\mu}\left(\iint \tilde{f}\bar{\mathbf{m}}\frac{\nabla_{\mathbf{m}}M_e}{M_e}\,\mathrm{d}v\,\mathrm{d}x + \sum_{k=1}^{K}\mathscr{V}_kA_e\bar{\mathbf{m}}\cdot\tilde{\mathbf{m}}_k\right).
\end{aligned}$$

The last two terms combine as

$$\begin{aligned}
&\frac{\bar{n}}{n_e}\left(\iint \tilde{f}\,\mathrm{d}v\,\mathrm{d}x + \sum_{k=1}^{K}\mathscr{V}_k\tilde{n}_k\right)\\
&\quad +\frac{\bar{u}}{\theta_e}\cdot\left(\iint (v-u_e)\tilde{f}\,\mathrm{d}v\,\mathrm{d}x + n_e\sum_{k=1}^{K}\mathscr{V}_k\tilde{u}_k\right)\\
&\quad +\frac{\bar{\theta}}{2\theta_e^2}\left(\iint \left(|v-u_e|^2 - N\theta_e\right)\tilde{f}\,\mathrm{d}v\,\mathrm{d}x + Nn_e\sum_{k=1}^{K}\mathscr{V}_k\tilde{\theta}_k\right),
\end{aligned}$$

a sum of conserved quantities. Hence, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H} + \sum_{k=1}^{K}D_k(\tilde{f},\tilde{L}_k) = 0,$$

where

$$D_k(\tilde{f},\tilde{L}_k) = \mu\iint \sigma_k|\tilde{f} - (\lambda/\mu)\tilde{L}_k|^2\frac{\mathrm{d}v\,\mathrm{d}x}{M_e}$$

It is thus natural to work with initial data with "finite entropy", which means here and below that $\mathscr{H}(0)$ is finite. Like for the nonlinear problem, the functional $\mathscr{H}$ can be used to assess how far the solution of the Cauchy problem is from the equilibrium state. Accordingly, the same arguments can be repeated to justify the convergence towards equilibrium as time becomes large. It turns out that establishing whether the convergence holds exponentially fast is equivalent to proving an estimate on the dissipation rate. This viewpoint is inspired from control theory [33]. Precisely, adapting [29, Lemma 11.1], the exponential decay can be characterized as follows.

Lemma 5.1 *The following assertions are equivalent:*

(i) *We can find constant $C, \gamma > 0$ such that, for any initial data with finite entropy, $\mathscr{H}(t) \le Ce^{-\gamma t}\mathscr{H}(0)$,*
(ii) *There exists $M, T > 0$ such that, for any initial data with finite entropy,*

$$\int_0^T \sum_{k=1}^{K} D_k\,\mathrm{d}t \ge M\mathscr{H}(0).$$

Proof This claim applies for both the nonlinear and the linear problems; it only relies on the dissipative and time-invariance properties of the problem.

Assume (i) and pick $T_0 > 0$ such that $Ce^{-\gamma T_0} < 1/2$. We infer that

$$\int_0^{T_0} \sum_{k=1}^{K} D_k \,\mathrm{d}t = \mathscr{H}(0) - \mathscr{H}(T_0) \geq (1 - Ce^{-\gamma T_0})\mathscr{H}(0) \geq \frac{\mathscr{H}(0)}{2},$$

which means that (ii) holds with $T = T_0$ and $M = \frac{1}{2}$.

Assuming (ii) yields

$$\mathscr{H}(T) = \mathscr{H}(0) - \int_0^{T} \sum_{k=1}^{K} D_k \,\mathrm{d}t \leq (1 - M)\mathscr{H}(0).$$

This relation already implies $M \leq 1$ since $\mathscr{H}$ takes non negative values. If $M = 1$, then $\mathscr{H}(T) = 0 \geq \mathscr{H}(t)$ for $t \geq T$ and (i) holds for any $C, \gamma > 0$. If $0 < M < 1$, we observe that the solution evaluated at $(t + \tau)$ satisfies the same equation with the solution evaluated at τ as Cauchy data. Accordingly, we get $\mathscr{H}(2T) \leq (1 - M)\mathscr{H}(T) \leq (1 - M)^2\mathscr{H}(0)$, and, by recursion, $\mathscr{H}(nT) \leq (1 - M)^n\mathscr{H}(0)$ for any $n \in \mathbb{N}$. Set $\gamma = -\frac{\ln(1-M)}{T} > 0$ and $C = \frac{1}{1-M} = e^{\gamma T} > 0$. It provides the asserted estimate: for $nT \leq t < (n + 1)T$, we have

$$\mathscr{H}(t) \leq \mathscr{H}(nT) \leq (1 - 1/M)^n\mathscr{H}(0) \leq e^{-\gamma nT}\mathscr{H}(0) \leq Ce^{-\gamma t}\mathscr{H}(0).$$

□

The linearity can be further exploited to investigate conditions guaranteeing the exponential rate of convergence towards equilibrium. To this end, we can indeed adapt the techniques developed in [5, 6] and [29]. This approach introduces geometrical constraints, inspired from control theory [33]. Hence, let us introduce the following Lebeau constant

$$C^- = \sup_{T>0} \inf_{(x,v)\in\mathbb{T}^N\times\mathbb{R}^N} \frac{1}{T}\int_0^{T} \sum_{k=1}^{K} \sigma_k(x + tv)\,\mathrm{d}t$$

We wish to establish the following analog to [29, Theorem 2.3].

Theorem 5.2 *The following assertions are equivalent:*

(i) $C^- > 0$,

(ii) *There exists* $c, \gamma > 0$ *such that for any* $\tilde{f}_{\mathrm{Init}} \in L^2_{M_e^{-1}}$ *and* $\tilde{n}_{\mathrm{Init}}, \tilde{u}_{\mathrm{Init}}, \tilde{\theta}_{\mathrm{Init}} \in (\mathbb{R} \times \mathbb{R}^N \times \mathbb{R})^K$, *we have* $\mathscr{H}(t) \leq ce^{-\gamma t}\mathscr{H}(0)$.

In fact, for our purposes this result has a quite negative interpretation: as pointed out in [29], as far as $\mathrm{supp}(\sum_{k=1}^{K}\sigma_k) \neq \mathbb{T}^d$, which naturally holds, $C^- = 0$ and initial data can be found such that the convergence to equilibrium holds at a rate which is non exponential.

Proof Let us start by assuming $C^- = 0$. We are going to show that the control inequality of Lemma 5.1 does not hold. It means that, for any $T, \epsilon > 0$, we can find initial data verifying $\mathscr{H}(0) = 1$ and the associated solution of the Cauchy problem satisfies $\int_0^T \sum_{k=1}^K D_k \,\mathrm{d}t \leq \epsilon$. That $C^- = 0$ implies that, for any $T, \epsilon > 0$, we can find $(x_0, v_0) \in \mathbb{T}^N \times \mathbb{R}^N$ such that $\int_0^T \sum_{k=1}^K \sigma_k(x_0 + tv_0)\,\mathrm{d}t \leq \frac{\epsilon M_e(v_0)}{3}$. Let $\zeta : [0,\infty) \to [-1, +1]$ be in $C_c^\infty([0,\infty))$ and such that

$$\zeta(r) = 1 \text{ for } 0 \leq r \leq 1, \zeta(r) = 0 \text{ for } r \geq 2, \text{ and } \int_0^\infty \zeta(r)r^{N-1}\,\mathrm{d}r = 0.$$

For $\nu \gg 1$, we set

$$\tilde{f}_{\mathrm{Init},\nu}(x,v) = \frac{\zeta(\nu|x-x_0|)\zeta(\nu|v-v_0|)}{C_\nu}, \qquad \tilde{\mathbf{m}}_{\mathrm{Init},\nu} = 0,$$

with C_ν such that $\mathscr{H}(0) = 1$, that is

$$\begin{aligned} C_\nu^2 &= \iint \left|\zeta(\nu|x-x_0|)\zeta(\nu|v-v_0|)\right|^2 \frac{\mathrm{d}v\,\mathrm{d}x}{M_e(v)}. \\ &= \iint \left|\zeta(|y|)\zeta(|w|)\right|^2 \frac{\mathrm{d}w\,\mathrm{d}y}{\nu^{2N} M_e(w/\nu + v_0)} \\ &= \iint_{|w|\le 2,\ |y|\le 2} \left|\zeta(|y|)\zeta(|w|)\right|^2 \frac{\mathrm{d}w\,\mathrm{d}y}{\nu^{2N} M_e(w/\nu + v_0)}. \end{aligned}$$

On the integration domain, we have $|w| \le 2$ and

$$\begin{aligned} \frac{(2\pi\theta_e)^{N/2}}{n_e} \le \frac{1}{M_e(w/\nu + v_0)} &= \frac{(2\pi\theta_e)^{N/2}}{n_e} \exp\Big(\frac{|w/\nu - (u_e - v_0)|^2}{2\theta_e}\Big) \\ &\le \frac{(2\pi\theta_e)^{N/2} e^8}{n_e} \exp\Big(\frac{|u_e - v_0|^2}{\theta_e}\Big) \end{aligned}$$

holds for such w's. We deduce that we can find constants $c_*, c^* > 0$ such that

$$\frac{c_*}{\nu^{2N}} \le C_\nu^2 \le \frac{c^*}{\nu^{2N}}.$$

Next, this allows us to make use of the Lebesgue theorem to find

$$\lim_{\nu\to\infty} \nu^{2N} C_\nu^2 = \underbrace{\frac{|\mathbb{S}^{n-1}|^2}{M_e(v_0)} \left(\int_0^\infty |\zeta(r)|^2 r^{N-1}\,\mathrm{d}r\right)^2}_{:=\alpha>0}.$$

We end this preparation step by claiming that $\tilde{f}_{\mathrm{Init},\nu}$ converges weakly to 0 in $L^2(\mathbb{T}^N \times \mathbb{R}^N; \frac{\mathrm{d}v\,\mathrm{d}x}{M_e})$. Since $\tilde{f}_{\mathrm{Init},\nu}$ has its norm equal to 1 in this space, it suffices to justify that

$$I_\nu = \iint \tilde{f}_{\mathrm{Init},\nu}\varphi \frac{\mathrm{d}v\,\mathrm{d}x}{M_e} \xrightarrow[\nu\to\infty]{} 0$$

for any $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R}^N)$. Still by changing variables, we get

$$I_\nu = \iint \zeta(|y|)\zeta(|w|)\, \frac{\varphi(y/n + x_0, w/n + v_0)}{M_e(w/n + v_0)}\, \frac{\mathrm{d}w\,\mathrm{d}y}{\nu^{2N} C_\nu}.$$

As $\nu \to \infty$, the integrand converges pointwise to $\zeta(|y|)\zeta(|w|)\frac{\varphi(x_0,v_0)}{\alpha M_e(v_0)}$ and it is dominated by $|\zeta(|y|)\zeta(|w|)|\|\varphi\|_{L^\infty}\frac{(2\pi\theta_e)^{N/2}e^8}{n_e} e^{|u_e - v_0|^2/\theta_e}$ which is integrable over $\mathbb{T}^N \times \mathbb{R}^N$. The Lebesgue theorem yields

$$\lim_{\nu\to\infty} \frac{I_\nu}{C_\nu} = \frac{\varphi(x_0,v_0)}{\alpha M_e(v_0)} \left(|\mathbb{S}^{N-1}| \int_0^\infty \zeta(r) r^{N-1}\,\mathrm{d}r\right)^2 = 0,$$

by construction of the function ζ. Similar manipulations show that $|\tilde{f}_{\mathrm{Init},\nu}|^2$ converges to $M_e(v_0)\delta(x = x_0) \otimes \delta(v = v_0)$.

Let us denote $(\tilde{f}_\nu, \tilde{\mathbf{m}}_\nu)$ the associated solution of (11). By the H-theorem, the sequence $(\tilde{f}_\nu)_{\nu>0}$ is bounded in $L^\infty(0,\infty; L^2(\mathbb{T}^N \times \mathbb{R}^N; \frac{\mathrm{d}v\,\mathrm{d}x}{M_e}))$, and $\big(\tilde{\mathbf{m}}_{k,\nu}\big)_{\nu>0}$ are bounded in $L^\infty([0,\infty))$. Going back to the differential equations in (11), we see that the time derivative of $\big(\tilde{\mathbf{m}}_{k,\nu}\big)_{\nu>0}$ are bounded in $L^\infty([0,\infty))$ too. Therefore, for any $0<T<\infty$, $\big(\tilde{\mathbf{m}}_{k,\nu}\big)_{\nu>0}$ are compact in $C^0([0,T])$, and $(\tilde{f}_\nu)_{\nu>0}$ is compact in $C^0([0,T];L^2_{1/M_e}(\mathbb{T}^N\times\mathbb{R}^N)-\text{weak})$. We can thus assume that $(\tilde{f}_\nu,\tilde{\mathbf{m}}_\nu)$ converges to $(\tilde{f},\tilde{\mathbf{m}})$. Letting ν go to ∞, we infer that $(\tilde{f},\tilde{\mathbf{m}})$ satisfies (11), thus associated to the initial data 0. In turn, by uniqueness of the solution of the Cauchy problem $\tilde{f}=0$ and $\tilde{\mathbf{m}}=0$. We turn to consider $g_\nu=\tilde{f}_\nu^2$, which satisfies

$$(\partial_t+v\cdot\nabla_x)g_\nu=\sum_{k=1}^K 2\sigma_k(\lambda\tilde{L}_{\nu,k}\tilde{f}_\nu-\mu g_\nu). \tag{13}$$

Integrating along characteristics, we obtain

$$\begin{aligned} &g_\nu(t,x,v)\\ &\quad=\exp\left(-2\mu\int_0^t\sum_{k=1}^K\sigma_k(x-(t-s)v)\,\mathrm{d}s\right)|f_{\mathrm{Init},\nu}|^2(x-tv,v)\\ &\qquad+\int_0^t\sum_{k=1}^K\exp\left(-2\mu\int_s^t\sum_{\ell=1}^K\sigma_\ell(x-(t-\tau)v)\,\mathrm{d}\tau\right)2\lambda\sigma_k\tilde{L}_{\nu,k}\tilde{f}_\nu(s,x-(t-s)v,v)\,\mathrm{d}s. \end{aligned} \tag{14}$$

We wish to show that $\int_0^T\sum_{k=1}^K D_k(\tilde{f}_\nu,\tilde{L}_{\nu,k})\,\mathrm{d}t$ can be made small as $\nu\to\infty$. We are thus led to consider

$$\begin{aligned} &\int_0^T\sum_{k=1}^K D_k(\tilde{f}_\nu,\tilde{L}_{\nu,k})\,\mathrm{d}t\\ &\quad=\mu\sum_{k=1}^K\int_0^T\iint\sigma_k|\tilde{f}_\nu-(\lambda/\mu)\tilde{L}_{\nu,k}|^2\frac{\mathrm{d}v\,\mathrm{d}x}{M_e}\,\mathrm{d}t\\ &\quad=\mu\sum_{k=1}^K\int_0^T\iint\sigma_k g_\nu\frac{\mathrm{d}v\,\mathrm{d}x}{M_e}\,\mathrm{d}t-2\mu\sum_{k=1}^K\int_0^T\iint(\lambda/\mu)\tilde{f}_\nu\tilde{L}_{\nu,k}\frac{\mathrm{d}v\,\mathrm{d}x}{M_e}\,\mathrm{d}t\\ &\qquad+\mu\sum_{k=1}^K\int_0^T\iint\sigma_k(\lambda/\mu)^2|\tilde{L}_{\nu,k}|^2\frac{\mathrm{d}v\,\mathrm{d}x}{M_e}\,\mathrm{d}t. \end{aligned}$$

Since $\tilde{\mathbf{m}}_k$ converges uniformly over $[0,T]$ to 0, $\tilde{L}_{\nu,k}=\tilde{\mathbf{m}}_k\cdot\nabla_{\mathbf{m}}M_e$ converges strongly in $L^2_{1/M_e}((0,T)\times\mathbb{T}^N\times\mathbb{R}^N)$ to 0, and the last two terms tend to 0 as $\nu\to\infty$. We are only left to study

$$\mathcal{J}_\nu:=\sum_{k=1}^K\int_0^T\iint\sigma_k g_\nu\frac{\mathrm{d}v\,\mathrm{d}x}{M_e}\,\mathrm{d}t.$$

According to (14), it can be split into two parts $\mathcal{J}_\nu = \mathcal{K}_\nu + \mathcal{L}_\nu$, the former involving the initial data, the latter the product $\tilde{L}_{\nu,k}\tilde{f}_\nu$. We obtain

$$\begin{aligned}
0 \le \mathcal{K}_\nu &\le \sum_{k=1}^{K} \int_0^T \iint \sigma_k(x)|f_{\mathrm{Init},\nu}|^2(x-tv,v)\frac{\mathrm{d}v\,\mathrm{d}x}{M_e}\,\mathrm{d}t \\
&\le \sum_{k=1}^{K} \int_0^T \iint \sigma_k(y+tv)|f_{\mathrm{Init},\nu}|^2(y,v)\frac{\mathrm{d}v\,\mathrm{d}y}{M_e}\,\mathrm{d}t \\
&\le \iint \left(\int_0^T \sum_{k=1}^{K} \sigma_k(y+tv)\,\mathrm{d}t\right)|f_{\mathrm{Init},\nu}|^2(y,v)\frac{\mathrm{d}v\,\mathrm{d}y}{M_e}.
\end{aligned}$$

As $\nu \to \infty$ the right hand side tends to

$$\frac{1}{M_e(v_0)}\left(\int_0^T \sum_{k=1}^{K} \sigma_k(x_0+tv_0)\,\mathrm{d}t\right) \le \frac{\epsilon}{3}.$$

Hence, we can exhibit $N_0 > 0$ such that for any $\nu > N_0$, we have

$$0 \le \mathcal{K}_\nu \le \frac{2\epsilon}{3}.$$

We turn to

$$\begin{aligned}
0 \le |\mathcal{L}_\nu| &\le \sum_{k=1}^{K} \int_0^T \iint 2\lambda\sigma_k(x,v)\left(\int_0^t \sum_{\ell=1}^{K} \sigma_\ell\left|\tilde{L}_{\nu,\ell}\tilde{f}_\nu\right|(s,x-(t-s)v,v)\,\mathrm{d}s\right)\frac{\mathrm{d}v\,\mathrm{d}x}{M_e}\,\mathrm{d}t \\
&\le 2\lambda KT \int_0^T \iint \sum_{\ell=1}^{K} \sigma_\ell\left|\tilde{L}_{\nu,\ell}\tilde{f}_\nu\right|(s,x-(t-s)v,v)\frac{\mathrm{d}v\,\mathrm{d}x}{M_e}\,\mathrm{d}s \\
&\le 2\lambda KT \sum_{\ell=1}^{K} \int_0^T \iint \sigma_\ell\left|\tilde{L}_{\nu,\ell}\tilde{f}_\nu\right|(s,y,v)\frac{\mathrm{d}v\,\mathrm{d}y}{M_e}\,\mathrm{d}s \\
&\le 2\lambda KT \sum_{\ell=1}^{K} \int_0^T \left(\iint |\tilde{L}_{\nu,\ell}|^2\frac{\mathrm{d}v\,\mathrm{d}y}{M_e}\right)^{1/2}\left(\iint |\tilde{f}_\nu|^2(s,y,v)\frac{\mathrm{d}v\,\mathrm{d}y}{M_e}\right)^{1/2}\,\mathrm{d}s \\
&\le 2\lambda KT \left(\iint |\tilde{f}_{\mathrm{Init},\nu}|^2(y,v)\frac{\mathrm{d}v\,\mathrm{d}y}{M_e}\right)^{1/2} \\
&\quad \times \sum_{\ell=1}^{K} \int_0^T \left(\iint |\tilde{\mathbf{m}}_{\nu,\ell}\cdot\nabla_{\mathbf{m}}M_e|^2\frac{\mathrm{d}v\,\mathrm{d}y}{M_e}\right)^{1/2}\,\mathrm{d}s,
\end{aligned}$$

where we have used the H-theorem. Since

$$\iint |\tilde{\mathbf{m}}_{\nu,\ell}\cdot\nabla_{\mathbf{m}}M_e|^2\frac{\mathrm{d}v\,\mathrm{d}y}{M_e} \le |\tilde{\mathbf{m}}_{\nu,\ell}||\mathbb{T}^N|\int |\nabla_{\mathbf{m}}M_e|^2\frac{\mathrm{d}v}{M_e}$$

where $\tilde{\mathbf{m}}_{\nu,\ell}$ converges uniformly to 0 over $[0,T]$, we conclude that $0 \le |\mathcal{L}_\nu| \le \frac{\epsilon}{3}$ holds for $\nu \ge N_0$ provided N_0 is large enough. We deduce that $0 \le \mathcal{J}_\nu \le \epsilon$ when $\nu \ge N_0$ and the control condition is denied.

Conversely, let us now assume $C^- > 0$. (As said above, this assumption, which holds for instance if the σ_k's vanish only on a finite number of points, is not very interesting for the problem we are concerned with. However, we will illustrate in Sect. 6 a slightly different situation where it becomes relevant.) Taking advantage of the linearity, by using the conservation properties, we can replace

$$\tilde f \text{by} \tilde f - \frac{\lambda}{\mu}\bar{\mathbf{m}}\cdot\nabla M_e \text{and} \begin{pmatrix}\tilde n_k\\ \tilde u_k\\ \tilde\theta_k\end{pmatrix} \text{ by } \begin{pmatrix}\tilde n_k\\ \tilde u_k\\ \tilde\theta_k\end{pmatrix} - \bar{\mathbf{m}}$$

so that the solution of (11) is required to lie in

$$\mathscr{B}_0 = \Big\{ (\tilde f, (\tilde n_k, \tilde u_k, \tilde\theta_k)_{k\in\{1,\dots,K\}}) \in L^2_{M_e^{-1}} \times (\mathbb{R}\times\mathbb{R}^N\times\mathbb{R})^K,$$
$$\iint \tilde f \frac{\nabla M_e}{M_e}\,\mathrm{d}v\,\mathrm{d}x + \sum_{k=1}^K \mathscr{V}_k \begin{pmatrix}\tilde n_k/n_e\\ n_e\tilde u_k/\theta_e\\ n_e\tilde\theta_k/\theta_e^2\end{pmatrix} = 0\Big\}.$$

We argue by contradiction, assuming that, for any $\nu\in\mathbb{N}\setminus\{0\}$, we can find an initial data in $\mathscr{B}_0$ with

$$\|\tilde f^{(\nu)}_{\mathrm{Init}}\|^2_{L^2_{1/M_e}} + \sum_{k=1}^K \mathscr{V}_k A_e \tilde{\mathbf{m}}^{(\nu)}_{\mathrm{Init},k}\cdot\tilde{\mathbf{m}}^{(\nu)}_{\mathrm{Init},k} = 1$$

such that the associated solution to (11) satisfies

$$\int_0^\nu \sum_{k=1}^K D_k(\tilde f^{(\nu)}, \tilde{\mathbf{m}}^{(\nu)}_k)\,\mathrm{d}t \le \frac{1}{\nu}.$$

$>$ From the H-theorem

$$\|\tilde f^{(\nu)}\|^2(t)_{L^2_{1/M_e}} + \sum_{k=1}^K \mathscr{V}_k A_e \tilde{\mathbf{m}}^{(\nu)}_k(t)\cdot\tilde{\mathbf{m}}^{(\nu)}_k(t) + \int_0^t \sum_{k=1}^K D_k(\tilde f^{(\nu)}, \tilde{\mathbf{m}}^{(\nu)}_k)\,\mathrm{d}s$$
$$= \|\tilde f^{(\nu)}_{\mathrm{Init}}\|^2_{L^2_{1/M_e}} + \sum_{k=1}^K \mathscr{V}_k A_e \tilde{\mathbf{m}}^{(\nu)}_{\mathrm{Init},k}\cdot\tilde{\mathbf{m}}^{(\nu)}_{\mathrm{Init},k} = 1$$

we know that $\tilde f^{(\nu)}$ is bounded in $L^\infty(0,\infty; L^2_{1/M_e}(\mathbb{T}^N]\times\mathbb{R}^N))$ and the $\tilde{\mathbf{m}}^{(\nu)}_k$ are bounded in $L^\infty([0,\infty))$. Let $0<T<\infty$; we infer

$$1 \ge \|\tilde f^{(\nu)}\|(T)^2_{L^2_{1/M_e}} + \sum_{k=1}^K \mathscr{V}_k A_e \tilde{\mathbf{m}}^{(\nu)}_k(T)\cdot\tilde{\mathbf{m}}^{(\nu)}_k(T) \ge 1 - \frac{1}{\nu} \tag{15}$$

provided $\nu > T$.

Reproducing arguments detailed above, we can assume that $\tilde f^{(\nu)}$ converges to $\tilde f$ in $L^\infty_w(0,\infty; L^2_{1/M_e}(\mathbb{T}^N\times\mathbb{R}^N))$ and, for any $0<T<\infty$, in $C^\infty([0,T]; L^2_{1/M_e}(\mathbb{T}^N\times\mathbb{R}^N) - \text{weak})$ while $\tilde{\mathbf{m}}^{(\nu)}_k$ converges to $\tilde{\mathbf{m}}_k$ in $C^0([0,T])$. Since $(x,v)\mapsto \mathbf{1}(x)v^2$ lies in $L^2_{M_e}(\mathbb{T}^N\times\mathbb{R}^N)$, we deduce that the moments of $\tilde f^{(\nu)}$ also pass to the limit, uniformly over $[0,T]$, and, since at any time $(\tilde f^{(\nu)}, (\tilde{\mathbf{m}}^{(\nu)}_k(t))_{k\in\{1,\dots,K\}}) \in \mathscr{B}_0$, $(\tilde f(t), (\tilde{\mathbf{m}}_k(t))_{k\in\{1,\dots,K\}})$ belongs to $\mathscr{B}_0$. We have $\lim_{\nu\to\infty}\sum_{k=1}^K\int_0^T D_k(\tilde f^{(\nu)}, \tilde{\mathbf{m}}^{(\nu)}_k)\,\mathrm{d}t = 0$. By convexity, it follows that

$$\sum_{k=1}^K \int_0^T D_k(\tilde f, \tilde{\mathbf{m}}_k)\,\mathrm{d}t = 0.$$

It implies that $\sigma_k(\mu \tilde{f} - \lambda \tilde{\mathbf{m}}_k \cdot \nabla_{\mathbf{m}} M_e) = 0$, for any $k \in \{1, ..., K\}$. Hence letting ν go to ∞ in the system (11) satisfied by $(\tilde{f}_\nu, (\tilde{\mathbf{m}}_k)_{k\in\{1,...,K\}})$, we obtain that the limit satisfies

$$\begin{aligned}
&(\partial_t + v \cdot \nabla_x)\tilde{f} = 0,\\
&(\tilde{f}(t), (\tilde{\mathbf{m}}_k(t))_{k\in\{1,...,K\}}) \in \mathscr{B}_0,\\
&\tilde{f} = \tilde{\mathbf{m}}_k \cdot \nabla_{\mathbf{m}} M_e \text{ on}(0, T) \times \omega \times \mathbb{R}^N \text{for all} k \in \{1, ..., K\}.
\end{aligned}$$

The unique continuation principle (direct adaptation of Proposition 3.3 to the linearized problem) then implies that $\tilde{f} = 0$ and $\tilde{\mathbf{m}} = 0$, the unique equilibrium solution in $\mathscr{B}_0$.

Next, we set $g^{(\nu)} = |\tilde{f}^{(\nu)}|^2$, which satisfies (13). Accordingly, formula (14) applies, which can be used to further estimate $\int g^{(\nu)} \frac{\mathrm{d}v\,\mathrm{d}x}{M_e} = \||\tilde{f}^{(\nu)}|^2\|^2_{L^2_{1/M_e}}$. By definition, we can pick $0 < T_0 < \infty$ such that

$$\int_0^t \sum_{k=1}^K \sigma_k(x + sv)\,\mathrm{d}s \geq t\frac{C^-}{2},$$

holds for any $t \geq T_0$, and any $(x, v) \in \mathbb{T}^N \times \mathbb{R}^N$. It follows that, for $t \geq T_0$,

$$\begin{aligned}
0 &\leq \iint \exp\left(-2\mu \int_0^t \sum_{k=1}^K \sigma_k(x - (t-s)v)\,\mathrm{d}s\right) |f^{(\nu)}_{\text{Init}}|^2(x - tv, v)\frac{\mathrm{d}v\,\mathrm{d}x}{M_e(v)}\\
&= \iint \exp\left(-2\mu \int_0^t \sum_{k=1}^K \sigma_k(y + sv)\,\mathrm{d}s\right) |f^{(\nu)}_{\text{Init}}|^2(y, v)\frac{\mathrm{d}v\,\mathrm{d}y}{M_e(v)}\\
&\leq \iint e^{-\mu C^- t}|f^{(\nu)}_{\text{Init}}|^2(y, v)\frac{\mathrm{d}v\,\mathrm{d}y}{M_e(v)} \leq e^{-\mu C^- t}\|f^{(\nu)}_{\text{Init}}\|_{L^2_{1/M_e}} \leq e^{-\mu t C^-}.
\end{aligned}$$

Next, we estimate

$$\begin{aligned}
&\iint \Big| \int_0^t \sum_{k=1}^K \exp\left(-2\mu \int_s^t \sum_{\ell=1}^K \sigma_\ell(x - (t-\tau)v)\,\mathrm{d}\tau\right)\\
&\quad\times 2\lambda\sigma_k \tilde{L}^{(\nu)}_k \tilde{f}^{(\nu)}(s, x - (t-s)v, v)\,\mathrm{d}s\Big| \frac{\mathrm{d}v\,\mathrm{d}x}{M_e}\\
&\quad\leq \iint \int_0^t \sum_{k=1}^K 2\lambda\sigma_k \big|\tilde{L}^{(\nu)}_k \tilde{f}^{(\nu)}\big|(s, x - (t-s)v, v)\,\mathrm{d}s\frac{\mathrm{d}v\,\mathrm{d}x}{M_e}\\
&\quad\leq 2\sum_{k=1}^K \int_0^t \left(\iint |\tilde{\mathbf{m}}^{(\nu)}_k|^2 |\nabla_{\mathbf{m}} M_e|^2 \frac{\mathrm{d}v\,\mathrm{d}x}{M_e}\right)^{1/2} \left(\iint |\tilde{f}^{(\nu)}|^2 \frac{\mathrm{d}v\,\mathrm{d}x}{M_e}\right)^{1/2} \mathrm{d}s\\
&\quad\leq 2\|\nabla_{\mathbf{m}} M_e\|_{L^2_{1/M_e}} \times t \sup_{0\leq s\leq t} \sum_{k=1}^K |\tilde{\mathbf{m}}^{(\nu)}_k|(s).
\end{aligned}$$

We start by picking $T > T_0$ such that $e^{-\mu t C^-} \leq \frac{1}{4}$. Next, bearing in mind that $\tilde{\mathbf{m}}^{(\nu)}_k$ converges to 0 uniformly on $[0, T]$, there exists $N \in \mathbb{N}$ such that for any $\nu \geq N$, we have $2\|\nabla_{\mathbf{m}} M_e\|_{L^2_{1/M_e}} \times T \sup_{0\leq s\leq T} \sum_{k=1}^K |\tilde{\mathbf{m}}^{(\nu)}_k| \leq \frac{1}{4}$. We can further choose N such that $\sum_{k=1}^K \mathscr{V}_k A_e \tilde{\mathbf{m}}^{(\nu)}_k(T) \cdot \tilde{\mathbf{m}}^{(\nu)}_k(T) \leq \frac{1}{8}$. It follows that $\|f^{(\nu)}\|(T)^2_{L^2_{1/M_e}} \leq \frac{1}{2}$ for $\nu \geq N$, which contradicts (15) (when $\nu \geq 8$). □

6 Numerical Simulations

We check on numerical grounds the properties of the model. It is worthwhile to compare the behavior of solutions of the system (2), (3a)–(3c) with the standard BGK system

$$\partial_t f + v \cdot \nabla_x f = \lambda \sum_{k=1}^{K} \sigma_k \times (M_f - f) \tag{16}$$

where

$$M_f(t, x, v) = \frac{n_f(t, x)}{(2\pi\theta_f(t, x))^{N/2}} \exp\Big(- \frac{|v - u_f(t, x)|^2}{2\theta_f(t, x)}\Big),$$

is the Maxwellian with macroscopic parameters determined by moments of the distribution f

$$\begin{pmatrix} n_f \\ n_f u_f \\ N n_f \theta_f \end{pmatrix} = \int \begin{pmatrix} 1 \\ v \\ |v - u_f|^2 \end{pmatrix} f \, \mathrm{d}v = \int \begin{pmatrix} 1 \\ v \\ |v - u_f|^2 \end{pmatrix} M_f \, \mathrm{d}v.$$

As we shall see, despite a formal analogy, the behaviour of (2), (3a)–(3c) and (16) can be significantly different. We work in the one dimensional framework with periodic boundary conditions. This situation is certainly specific since for any $v \neq 0$ the trajectory $x - tv$ joins two scatterers. The domain is the slab $(-L, +L)$, with a trap located in $\omega = [a, b] \in (-L, +L)$. As form function σ, we simply use the characteristic function of the trap. We use a very basic numerical scheme; we refer the reader to [19] for a detailed numerical investigation of relaxation for degenerate linear Boltzmann equations in the two-dimensional framework. In particular, [19] brings out effects of the existence of free trajectories on the relaxation rate, which is not exponential when such trajectories do exist. Here, counterexamples to the exponential decay can be constructed, except in the exceptional configuration where $\overline{\bigcup_{k=1}^{K} \omega_k} = \mathbb{T}^N$, see also [6] and [29, Corollary 8.1, Theorem 8.2]; in the one dimensional framework, we proceed as follows. We consider a segment $I_0 = [x_0 - h, x_0 + h]$ which does not intersect the trap:

$$\min(x_0 + h - a, x_0 - h - b - L) = \delta > 0.$$

We observe that

$$\mathrm{meas}\big(\{(x, v) \in I_0 \times [-V_M, +V_M],\ \tau(x, v) > t\}\big) \geq \frac{C}{1 + t}, \tag{17}$$

where $\tau(x, v)$ is the exit time: $\tau(x, v) = \inf\{s \geq 0,\ x + sv \notin \omega\}$. (It can be interpreted as the time necessary to reach a trap when starting from a free region.) Indeed, the trap cannot be reached for times lesser than δ/V_M. When $t \geq \delta/V_M$, the admissible set in the phase plane has a trapezoidal shape, determined by the slope $1/t$, see Fig. 1. Construction of counterexamples to the exponential decay to equilibrium use this estimate and rely on estimates on the time of first collision, as pointed out in [25] for standard Lorentz gas problem and in [5, 6].

For the simulations, we set $L = 1.5$, and a single trap in centred at the position $x = 0$. We perform all simulations with $\mu = 1$, and we choose several values for λ. We also make the size of the trap vary to discuss how it influences the relaxation to equilibrium. The initial data for the trap is given by

$$n^{\mathrm{Init}} = 0.001, \quad u^{\mathrm{Init}} = 0, \quad \theta^{\mathrm{Init}} = 1.$$

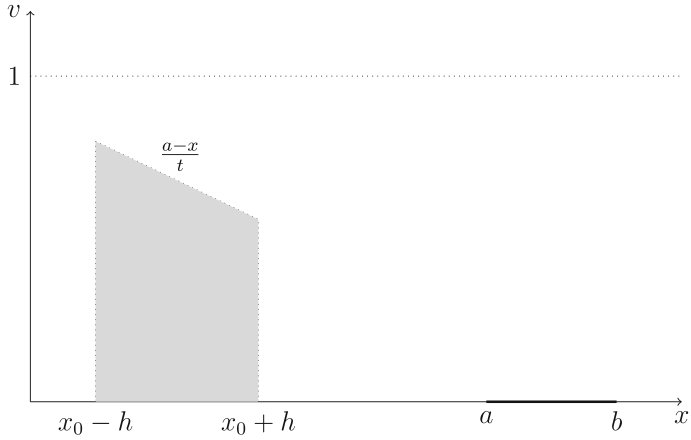


Fig. 1 Admissible domain defined by $\big(\{(x, v) \in I_0 \times [0, +1],\ \tau(x, v) > t\}\big)$

For the particles, we work either with

$$f^{\mathrm{Init}}(x, v) = \Big(\exp\Big(-\frac{|x-1/2|^2}{.2}\Big) + \exp\Big(-\frac{|x+1/5|^2}{.1}\Big)\Big)$$
$$\times\Big(\Big(\exp\Big(-|v-2|^2\Big) + \exp\Big(-\frac{|v+5/2|^2}{.1}\Big)\Big),$$

which significantly meets the trap, or with

$$f^{\mathrm{Init}}(x, v) = \Big(\exp\Big(-\frac{|x-1/2|^2}{.2}\Big) + \exp\Big(-\frac{|x+1/5|^2}{.1}\Big)\Big)$$
$$\times \mathbf{1}_{[L/5+.4, L/5+.6]}(x) \times \exp\Big(-\frac{|v-.01|^2}{.05}\Big),$$

which is peaked far from the trap, with relatively small velocities. >From the data, the equilibrium states for the trap or the BGK models are determined based on the conservation laws of the equations.

The results confirm the computations made above: the total mass, total momentum, and total energy are conserved, see Fig. 2 for typical results, the entropy decays, see Fig. 3, the concentration of trapped particles tends to the same constant, the particles distribution tends to a Maxwellian profile with macroscopic parameters completely determined by the initial state, see Fig. 4. On the latter figure, we observe that the profile of the solution is indeed close to the expected equilibrium state, with discrepancies far from the trap and for the small velocities, corresponding to particles having poor interactions with the trap. This is made more evident on Fig. 5 which show at several times the profile of the space average of the particle distribution function: it tends to the expected equilibrium solution, except a sensitive gap for small velocities (the presence of the gap is not contradictory with convergence in L^2 norm towards equilibrium).

The role of the geometry and the parameters is further illustrated in Figs. 6 and 7. In Fig. 6 we make the size of the trap vary. The smaller the trap, the slower the convergence. The convergence is significantly slower with the peaked data. The exponential decay is restored when the trap fully fills the domain, whatever the shape of the initial data. In Fig. 7, we make λ vary. The BGK model is directly impacted by changing the value of λ, which has a different

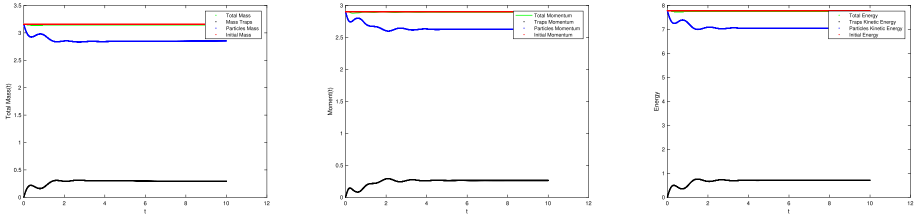


Fig. 2 Trap $= [-L/5, L/5]$. Evolution of the mass, momentum and energy

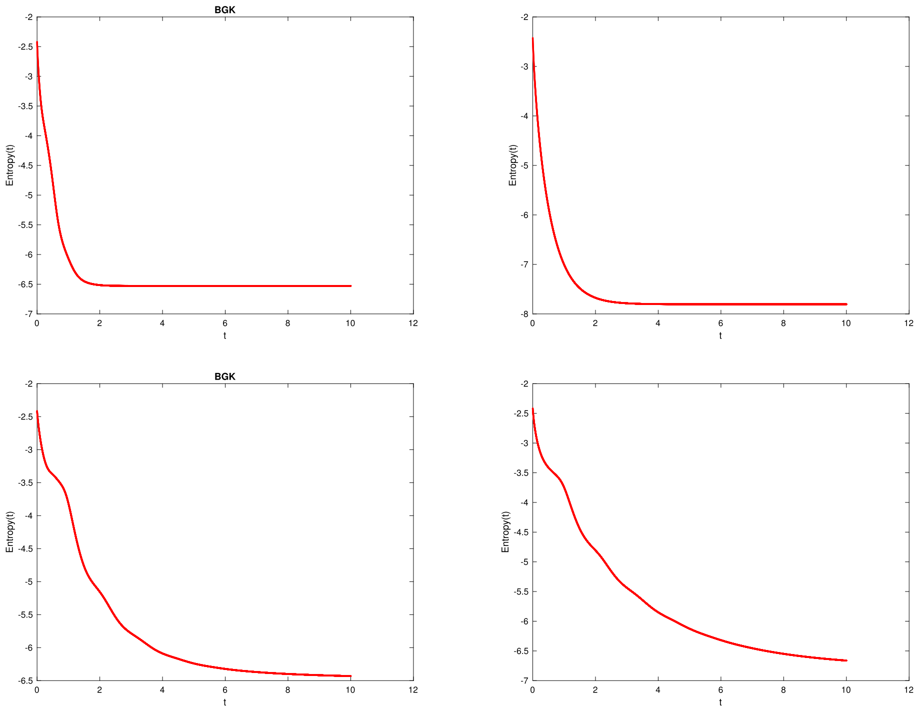


Fig. 3 Evolution of the entropy for the BGK (left) and the trap (right) model with a trap filling the domain (top) or a trap that occupies a portion of the domain (bottom)

interpretation for the trap model. This set of simulations again shows that the convergence is significantly slower for the peaked data.

In order to further illustrate the role of the parameters and of the geometry, we consider the simplest problem where there are only mass exchanges and where particles have velocities $\pm V_M$. Namely, this two-velocity trap model reads

$$\begin{aligned} \partial_t f_\pm \pm V_M \partial_x f_\pm &= \sigma\Big(\lambda \frac{n}{2} - \mu f_\pm\Big), \\ \partial_t n &= \int_{-L}^{L} \frac{\sigma}{\mathscr{V}}(\lambda n - \mu f_+ - \mu f_-)\,\mathrm{d}x, \end{aligned} \tag{18}$$

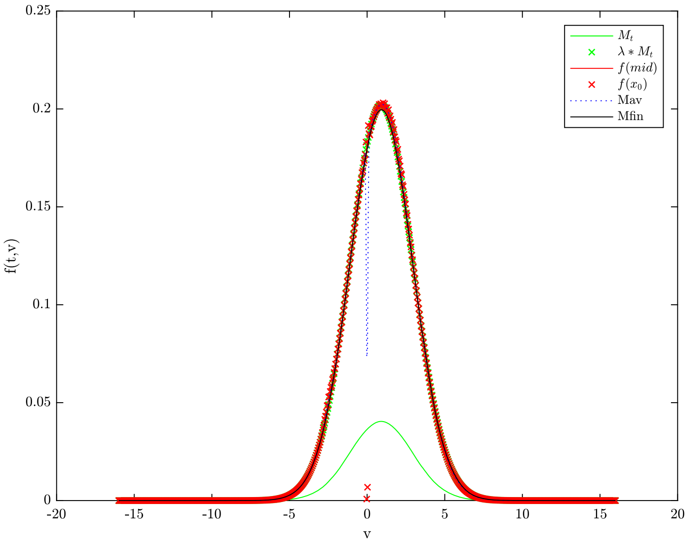


Fig. 4 Trap $= [-L/5, L/5]$. Solution at the final time evaluated at $x_0 = -1.4$, $x_{mid} = 0$, and space average of the solution (Mav)

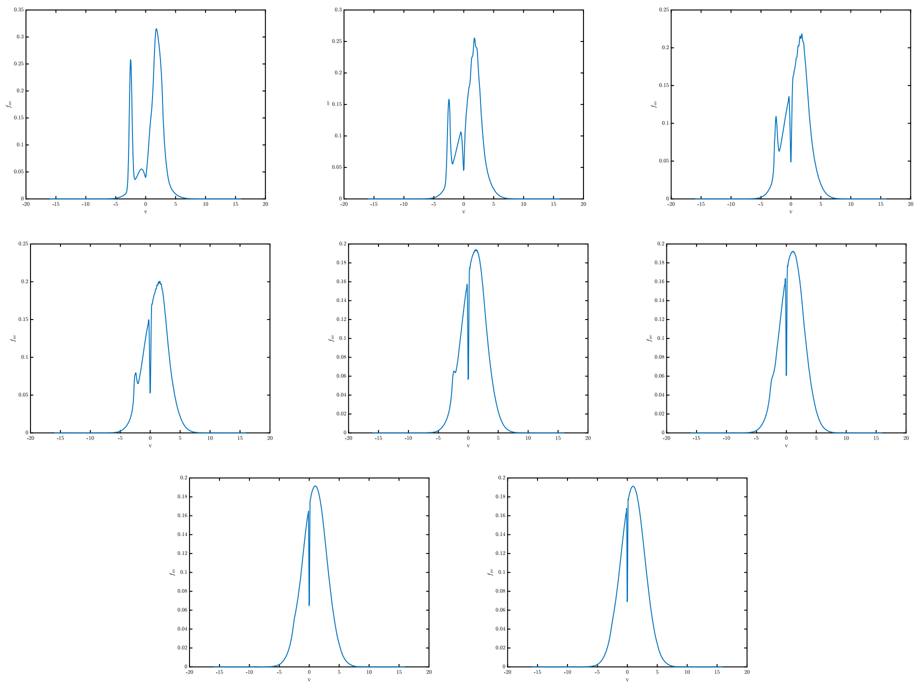


Fig. 5 Space average of the solution at time $T = T_{fin} * k/8$, $k \in \{1, ..., 8\}$, $T_{fin} = 25$

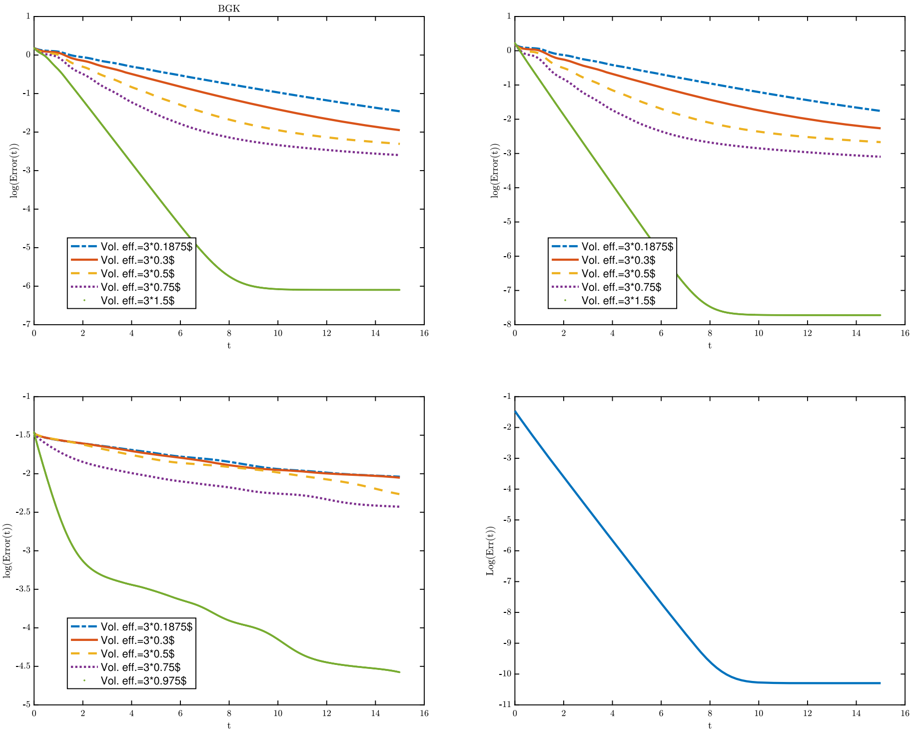


Fig. 6 Evolution of the error, for several sizes of the trap: BGK model (top left), trap model (top right). Evolution of the error for the trap model with a peaked data, for several sizes of the trap (bottom left) and for a trap filling the domain (bottom right)

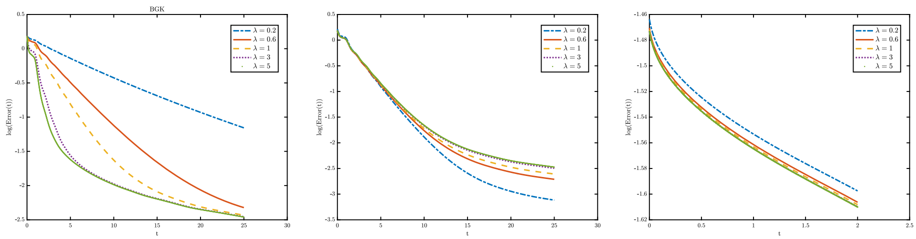


Fig. 7 Evolution of the error, for several values of λ. The trap occupies the subdomain $(-L/5, L/5)$. BGK model (left), trap model (middle), trap model with a peaked data (right)

with $\mathscr{V} = \int_{-L}^{L} \sigma(x)\,\mathrm{d}x$. We infer that the asymptotic behavior is described by the equilibrium distribution

$$f_{\pm}^{e} = \frac{\lambda n^{e}}{2\mu}, \qquad n^{e} = \frac{M_0}{2L\lambda/\mu + \mathscr{V}},$$

where M_0 is the initial mass, conserved by the equation

$$M_0 = \int_{-L}^{L} f_{+}\,\mathrm{d}x + \int_{-L}^{L} f_{-}\,\mathrm{d}x + \mathscr{V} n.$$

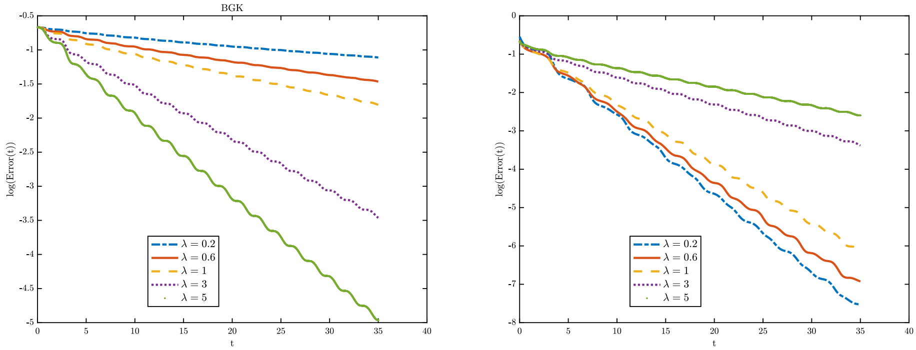


Fig. 8 Two-velocity linear models: evolution of the L^2 error to the equilibrium for the BGK model (left) and the trap model (right), for several λ

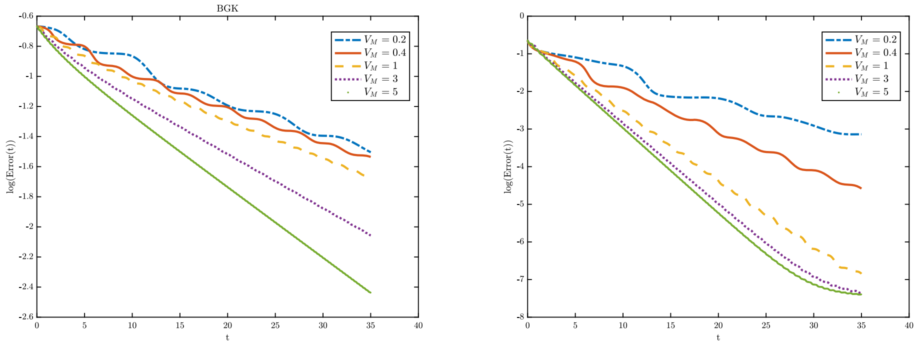
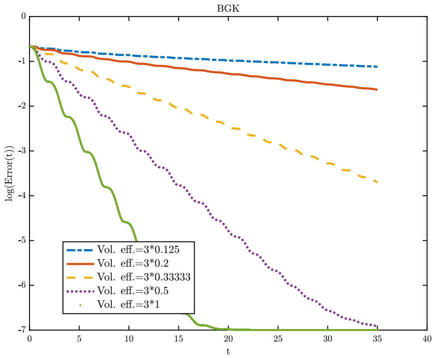


Fig. 9 Two-velocity linear models: evolution of the L^2 error to the equilibrium for the BGK model (left) and the trap model (right), for several V_M

The BGK counterpart of this model reads

$$\partial_t f_\pm \pm V_M \partial_x f_\pm = \lambda\sigma\Big(\frac{f_+ + f_-}{2} - f_\pm\Big). \tag{19}$$

Due to the fact that the velocities have modulus $V_M > 0$, a particle at distance d from the obstacle will reach it in a time $\frac{d}{V_M}$. In turn, there is no obstruction to the exponential rate of convergence towards equilibrium and indeed we see that now $C^- > 0$. We refer the reader to [5] for the analysis of this situation for the BGK model, see also [35]. Figures 8, 9 and 10 illustrate the behavior of the error to the equilibrium when we make one of the parameter vary: with $V_M = 0.8$, a size of trap $\frac{2L}{5}$ and λ varies (Fig. 8), with $\lambda = 0.8$, a size of trap $\frac{2L}{5}$ and V_M varies (Fig. 9) and with $V_M = 0.8$, $\lambda = 0.8$ and the size of the trap varies (Fig. 10). In all cases, and for both models, we observe an exponential rate of convergence. As expected, the larger the speed V_M or the size of the trap, the faster the convergence. For the BGK model, increasing λ makes also the convergence faster; this is the opposite for the trap model (but λ does not have exactly the same meaning).



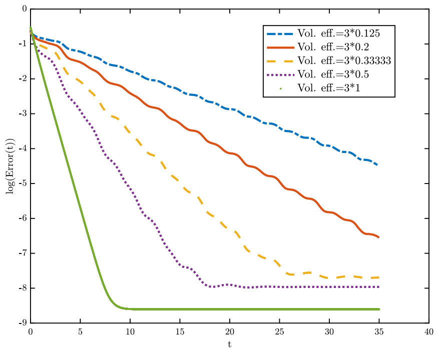


Fig. 10 Two-velocity linear models: evolution of the L^2 error to the equilibrium for the BGK model (left) and the trap model (right), for several size of the trap

7 A Model with Memory Based on Mixture Modeling

We consider the same geometrical setting, but we describe differently the dynamics of the traps. The model we propose is inspired from the derivation of BGK-type equations, see [4], for describing gas mixtures [2, 9]. The particles trapped in the kth scatterer are described by $\ell_k(t, a, v)$ the distribution of particles that entered the kth trap with velocity v and that have spent an "age" $a \geq 0$ in this trap. Particles can be re-emitted by the traps, according to an emission law characterized by the rate $\lambda(a)$ of emission of trapped particles with age a. The function $a \mapsto \lambda(a)$ is non negative; we also suppose that

$$\int_0^\infty \lambda(a)\,\mathrm{d}a = +\infty.$$

Let us set

$$\Lambda(a) = \int_0^a \lambda(\alpha)\,\mathrm{d}\alpha \quad \text{and} \quad p(a) = \lambda(a)e^{-\Lambda(a)}$$

which are non negative and satisfy

$$\Lambda(0) = 0, \qquad \int_0^\infty p(a)\,\mathrm{d}a = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}a}\Big[e^{-\Lambda(a)}\Big]\,\mathrm{d}a = 1.$$

Therefore, p can be interpreted as the probability density of the re-emission time. The definition of the interaction between particles and traps requires to introduce the following macroscopic quantities associated to f and ℓ_k

$$\begin{pmatrix} n \\ nu \\ nu^2 + Nn\theta \end{pmatrix}(t, x) = \int \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} f(t, x, v)\,\mathrm{d}v,$$

and, similarly,

$$\begin{pmatrix} n_k \\ n_k u_k \\ n_k u_k^2 + N n_k \theta_k \end{pmatrix}(t) = \int_0^\infty \int \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} \ell_k(t, a, v)\lambda(a)\,\mathrm{d}v\,\mathrm{d}a.$$

(Beware of the weight λ in the latter definition.) We can interpret n, u (resp. n_k, u_k) as the macroscopic density and bulk velocity of free particles (resp. of potentially re-emitted particles in the kth trap). We observe that

$$Nn\theta(t,x) = \int |v - u(t,x)|^2 f(t,x,v)\,\mathrm{d}v,$$

and

$$Nn_k\theta_k(t,x) = \int_0^\infty \int |v - u_k(t)|^2 \ell_k(t,a,v)\lambda(a)\,\mathrm{d}v\,\mathrm{d}a,$$

which shows that θ and θ_k are non negative; they can be interpreted as temperatures. We now combine these quantities as follows

$$\begin{aligned}
&\bar{n}_k = n + n_k,\\
&\bar{n}_k\bar{u}_k = nu + n_k u_k,\\
&\bar{n}_k\bar{u}_k^2 + N\bar{n}_k\bar{\theta}_k = nu^2 + n_k u_k^2 + N(n\theta + n_k\theta_k).
\end{aligned}$$

It might seem weird to mix up quantities which are space-dependent and defined on the whole space, with quantities that depend only on t and are attached to a given trap, but, as we shall see below, these quantities will be considered on the trap k only, where they are certainly meaningful. The physical meaning of $\bar{\theta}_k$ as a temperature makes sense, owing to the following claim.

Lemma 7.1 *The quantity $\bar{\theta}_k$ is non negative.*

Proof We expand the following integrals, which are obviously non negative

$$\begin{aligned}
&\int |v - \bar{u}_k|^2 f\,\mathrm{d}v + \int_0^\infty \int |v - \bar{u}_k|^2 \ell_k\lambda\,\mathrm{d}v\,\mathrm{d}a\\
&= \int v^2 f\,\mathrm{d}v + \int_0^\infty \int v^2 \ell_k\lambda\,\mathrm{d}v\,\mathrm{d}a\\
&\quad -2\bar{u}_k \cdot \left(\int vf\,\mathrm{d}v + \int_0^\infty \int v\ell_k\lambda\,\mathrm{d}v\,\mathrm{d}a\right) + \bar{u}_k^2\left(\int f\,\mathrm{d}v + \int_0^\infty \int \ell_k\lambda\,\mathrm{d}v\,\mathrm{d}a\right)\\
&= nu^2 + Nn\theta + n_k u_k^2 + Nn_k\theta_k - 2\bar{u}_k \cdot (nu + n_k u_k) + \bar{u}_k^2(n + n_k)\\
&= \bar{n}\bar{u}_k^2 + N\bar{n}_k\bar{\theta}_k - \bar{n}\bar{u}_k^2 = N\bar{n}_k\bar{\theta}_k \geq 0.
\end{aligned}$$

□

Let

$$\bar{M}_k(t,x,v) = (2\pi\bar{\theta}_k(t,x))^{-N/2} \exp\left(-\frac{|v - \bar{u}_k(t,x)|^2}{2\bar{\theta}_k(t,x)}\right).$$

The evolution of the particle distribution function is driven by the following BGK-like equation

$$\partial_t f + v \cdot \nabla_x f = \sum_k \sigma_k(n_k\bar{M}_k - f). \tag{20}$$

Trapped particles are subjected to aging, and emission

$$\partial_t \ell_k + \partial_a \ell_k = -\lambda\ell_k, \tag{21}$$

completed by

$$\ell_k\big|_{a=0} = \int \frac{\sigma_k}{\mathscr{V}_k} n\bar{M}_k\,\mathrm{d}x. \tag{22}$$

Equation (21) is a transport equation in age space, with a loss term that corresponds to the re-emission of trapped particles. The boundary condition (22) defines the trapped particles with age 0: their are given by the density of free particles that belong to the kth trap, but these trapped particles are instantaneously thermalized to the average state defined by $\bar{u}_k$, $\bar{\theta}_k$. Similarly, in (20) particles are re-emitted with the density n_k, in a state defined by the Maxwellian $\bar{M}_k$. As long as particles remain within the trap, their dynamics is simply disregarded : the model does not pay attention to the evolution of their position and velocity. The system (20)–(22) is endowed by initial condtions

$$f\Big|_{t=0} = f_{\mathrm{Init}}, \qquad \ell_k\Big|_{t=0} = \ell_{k,\mathrm{Init}}. \tag{23}$$

7.1 Conservation Properties

We shall see that the model conserves mass, momentum and energy.

Proposition 7.2 *We have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\iint \begin{pmatrix}1\\ v\\ v^2\end{pmatrix} f\,\mathrm{d}v\,\mathrm{d}x + \sum_k \mathscr{V}_k \int_0^\infty \int \begin{pmatrix}1\\ v\\ v^2\end{pmatrix} \ell_k\,\mathrm{d}v\,\mathrm{d}a\right\} = 0.$$

Proof We start by considering the equations for the macroscopic densities, integrating (20) over the variable v,

$$\partial_t n + \nabla_x \cdot (nu) = \sum_k \sigma_k (n_k - n).$$

In the trap, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\infty \int \ell_k\,\mathrm{d}v\,\mathrm{d}a\right) = \int \frac{\sigma_k}{\mathscr{V}_k}(n - n_k)\,\mathrm{d}x.$$

This is obtained by integrating (21) over the variables v and a, and using (22). We conclude with the mass conservation property

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\iint f\,\mathrm{d}v\,\mathrm{d}x + \sum_k \mathscr{V}_k \int_0^\infty \int \ell_k\,\mathrm{d}v\,\mathrm{d}a\right\} = 0.$$

A similar computation yields the momentum conservation

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\iint vf\,\mathrm{d}v\,\mathrm{d}x + \sum_k \mathscr{V}_k \int_0^\infty \int v\ell_k\,\mathrm{d}v\,\mathrm{d}a\right\}\\
&\quad= \sum_k \int \sigma_k (n_k\bar{u}_k - nu)\,\mathrm{d}x + \sum_k \int \sigma_k n\bar{u}_k\,\mathrm{d}x - \sum_k \mathscr{V}_k n_k u_k\\
&\quad= \sum_k \int \sigma_k \big((n_k + n)\bar{u}_k - (nu + n_k u_k)\big)\,\mathrm{d}x = 0.
\end{aligned}$$

Finally, we turn to the evolution of the kinetic energy

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}&\left\{\iint v^2 f \,\mathrm{d}v\,\mathrm{d}x + \sum_k \mathscr{V}_k \int_0^\infty \int v^2 \ell_k \,\mathrm{d}v\,\mathrm{d}a\right\} \\
&= \sum_k \int \sigma_k \big(n_k(\bar{u}_k^2 + N\bar{\theta}_k) - n(u^2 + N\theta)\big)\,\mathrm{d}x \\
&\quad + \sum_k \int \sigma_k n(\bar{u}_k^2 + N\bar{\theta}_k)\,\mathrm{d}x - \sum_k \mathscr{V}_k \int_0^\infty \int v^2 \ell_k \lambda \,\mathrm{d}v\,\mathrm{d}a \\
&= \sum_k \int \sigma_k (n + n_k)(\bar{u}_k^2 + N\bar{\theta}_k)\,\mathrm{d}x - \sum_k \int \sigma_k \big(n(u^2 + N\theta) + n_k(u_k^2 + N\theta_k)\big)\,\mathrm{d}x \\
&= 0.
\end{aligned}
$$

□

7.2 Dissipation Properties

We are going to identify an entropy functional which is dissipated by the equation.

Proposition 7.3 *We have*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left\{\iint f \ln(f)\,\mathrm{d}v\,\mathrm{d}x + \sum_k \mathscr{V}_k \int_0^\infty \int \ell_k(\Lambda + \ln(\ell_k))\,\mathrm{d}v\,\mathrm{d}a\right\} + \mathcal{D} \le 0,
$$

where $\mathcal{D} \ge 0$ vanishes iff $f = n_k \bar{M}_k$ and $\ell_k = e^{-\Lambda} n \bar{M}_k$ on the sets ω_k.

Proof Let us start by computing

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\iint f \ln f \,\mathrm{d}v\,\mathrm{d}x &= \sum_k \int \sigma_k (n_k - n)\,\mathrm{d}x \\
&\quad + \sum_k \iint \sigma_k (n_k \bar{M}_k - f) \ln\Big(\frac{f}{n_k \bar{M}_k}\Big)\,\mathrm{d}v\,\mathrm{d}x \\
&\quad + \underbrace{\sum_k \iint \sigma_k (n_k \bar{M}_k - f) \ln(n_k \bar{M}_k)\,\mathrm{d}v\,\mathrm{d}x}_{R_1}\,.
\end{aligned}
$$

By expanding

$$
\ln(n_k \bar{M}_k) = \ln(n_k) - \frac{N}{2}\ln(2\pi\bar{\theta}_k) - \frac{|v - \bar{u}_k|^2}{2\bar{\theta}_k},
$$

we can rewrite the last term as follows

$$
\begin{aligned}
R_1 &= \sum_k \int \sigma_k (n_k - n)\big(\ln(n_k) - \ln(2\pi\bar{\theta}_k)\big)\,\mathrm{d}x \\
&\quad - \sum_k \int \frac{\sigma_k}{2\bar{\theta}_k}\big(n_k N\bar{\theta}_k - nu^2 - Nn\theta + 2\bar{u}_k \cdot nu - n\bar{u}_k^2\big)\,\mathrm{d}x.
\end{aligned}
$$

Next, taking into account (22), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_k \mathscr{V}_k \int_0^\infty \int \ell_k \ln \ell_k \,\mathrm{d}v &= \sum_k \int \Big(\int \sigma_k n\bar{M}_k \,\mathrm{d}x \ln\Big(\int \frac{\sigma_k}{\mathscr{V}_k} n\bar{M}_k \,\mathrm{d}x\Big)\Big)\,\mathrm{d}v \\ &\quad - \sum_k \mathscr{V}_k \int_0^\infty \int \ell_k \,\mathrm{d}v\,\mathrm{d}a - \sum_k \mathscr{V}_k \int_0^\infty \int \lambda \ell_k \ln(\ell_k)\,\mathrm{d}v\,\mathrm{d}a.\end{aligned}$$

The last term recast as

$$\begin{aligned}&-\sum_k \int_0^\infty \iint \sigma_k \lambda e^{-\Lambda} \frac{\ell_k}{e^{-\Lambda}}\Big[\ln\Big(\frac{\ell_k}{e^{-\Lambda}}\Big) - \Lambda\Big]\,\mathrm{d}v\,\mathrm{d}a\,\mathrm{d}x \\ &\quad = \sum_k \int_0^\infty \iint \sigma_k \lambda e^{-\Lambda}\Big(n\bar{M}_k \ln(n\bar{M}_k) - \frac{\ell_k}{e^{-\Lambda}}\ln\Big(\frac{\ell_k}{e^{-\Lambda}}\Big)\Big)\,\mathrm{d}v\,\mathrm{d}a\,\mathrm{d}x \\ &\qquad - \sum_k \iint \sigma_k n\bar{M}_k \ln(n\bar{M}_k)\,\mathrm{d}v\,\mathrm{d}x + \sum_k \mathscr{V}_k \int_0^\infty \int \lambda\Lambda \ell_k \,\mathrm{d}v\,\mathrm{d}a.\end{aligned}$$

Finally, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_k \mathscr{V}_k \int_0^\infty \int \Lambda \ell_k \,\mathrm{d}v &= \sum_k \int_0^\infty \int \sigma_k \lambda (1-\Lambda)\ell_k \,\mathrm{d}v\,\mathrm{d}x \\ &= \sum_k \mathscr{V}_k n_k - \sum_k \mathscr{V}_k \int_0^\infty \int \lambda\Lambda \ell_k \,\mathrm{d}v.\end{aligned}$$

Now, we are going to use Jensen's inequality: since $\int \frac{\sigma_k}{\mathscr{V}_k}\,\mathrm{d}x = 1$, we have

$$\int \Big(\int \frac{\sigma_k}{\mathscr{V}_k} n\bar{M}_k \,\mathrm{d}x \ln\Big(\int \frac{\sigma_k}{\mathscr{V}_k} n\bar{M}_k \,\mathrm{d}x\Big)\Big)\,\mathrm{d}v \le \iint \frac{\sigma_k}{\mathscr{V}_k} n\bar{M}_k \ln(n\bar{M}_k)\,\mathrm{d}v\,\mathrm{d}x.$$

Therefore, we are led to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_k \mathscr{V}_k \int_0^\infty \int \ell_k(\Lambda + \ln \ell_k)\,\mathrm{d}v\,\mathrm{d}a \\ &\quad \le \sum_k \int_0^\infty \iint \sigma_k \lambda e^{-\Lambda}\Big(n\bar{M}_k \ln(n\bar{M}_k) - \frac{\ell_k}{e^{-\Lambda}}\ln\Big(\frac{\ell_k}{e^{-\Lambda}}\Big)\Big)\,\mathrm{d}v\,\mathrm{d}x.\end{aligned} \tag{24}$$

We wish to make a relative entropy appear, namely we set

$$H(F|G) = \sum_k \int_0^\infty \iint \sigma_k \lambda e^{-\Lambda}\Big(F\ln(F) - G\ln(G) - (1+\ln(G))(F-G)\Big)\,\mathrm{d}v\,\mathrm{d}x.$$

Clearly, we have $H(F|G) \ge 0$, and $H(F|G)$ vanishes iff $F = G$. The right hand side of (24) can be recast as

$$\begin{aligned}&-H\Big(\frac{\ell_k}{e^{-\Lambda}}|n\bar{M}_k\Big) - \sum_k \int_0^\infty \iint \sigma_k \lambda e^{-\Lambda}\Big(\frac{\ell_k}{e^{-\Lambda}} - n\bar{M}_k\Big)\,\mathrm{d}v\,\mathrm{d}x \\ &\quad - \sum_k \int_0^\infty \iint \sigma_k \lambda e^{-\Lambda}\ln(n\bar{M}_k)\Big(\frac{\ell_k}{e^{-\Lambda}} - n\bar{M}_k\Big)\,\mathrm{d}v\,\mathrm{d}x \\ &= -H\Big(\frac{\ell_k}{e^{-\Lambda}}|n\bar{M}_k\Big) - \sum_k \int \sigma_k(n_k - n) + R_2\end{aligned}$$

where

$$
\begin{aligned}
R_2 &= -\sum_k \int_0^\infty \iint \sigma_k \lambda e^{-\Lambda}\Big(\frac{\ell_k}{e^{-\Lambda}} - n\bar{M}_k\Big)\Big(\ln(n) - \frac{|v-\bar{u}_k|^2}{2\bar{\theta}_k} - \frac{N}{2}\ln(2\pi\bar{\theta}_k)\Big)\,\mathrm{d}v\,\mathrm{d}x \\
&= -\sum_k \int \sigma_k(n_k - n)\Big(\ln(n) - \frac{N}{2}\ln(2\pi\bar{\theta}_k)\Big)\,\mathrm{d}x \\
&\quad + \sum_k \int \sigma_k \frac{n_k}{2\bar{\theta}_k}(u_k^2 + \bar{u}_k^2 - 2\bar{u}_k\cdot u_k + N\bar{\theta}_k)\,\mathrm{d}x - \sum_k \int \sigma_k n \frac{N\bar{\theta}_k}{2\bar{\theta}_k}\,\mathrm{d}x.
\end{aligned}
$$

We observe that $R_1 + R_2 = 0$. We conclude that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\sum_k \mathscr{V}_k \int_0^\infty \int \ell_k(\Lambda + \ln \ell_k)\,\mathrm{d}v\,\mathrm{d}a + \iint f\ln(f)\,\mathrm{d}v\,\mathrm{d}x\Big\} \\
&\quad \le -H\Big(\frac{\ell_k}{e^{-\Lambda}}|n\bar{M}_k\Big) - \sum_k \iint \sigma_k(f - n_k\bar{M}_k)\ln\Big(\frac{f}{n_k\bar{M}_k}\Big)\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad + \sum_k \int \sigma_k(n_k - n)\ln(n_k - n)\,\mathrm{d}x.
\end{aligned}
$$

Let us denote by $(-\mathcal{D})$ the right hand side. The two first terms are non positive, the third term is non negative. However, we can use the Jensen inequality again. Let F, G be given non negative functions defined on $\mathbb{R}^D$. We denote $\langle G\rangle = \int G\,\mathrm{d}y$. We write

$$
(\langle F\rangle - \langle G\rangle)\ln\Big(\frac{\langle F\rangle}{\langle G\rangle}\Big) = \langle G\rangle\Big(\int \frac{F}{G}\frac{G}{\langle G\rangle}\,\mathrm{d}y - 1\Big)\ln\Big(\int \frac{F}{G}\frac{G}{\langle G\rangle}\,\mathrm{d}y\Big) = \langle G\rangle\Phi\Big(\int \frac{F}{G}\,\mathrm{d}\mu\Big),
$$

with the strictly convex function $\Phi(s) = (s-1)\ln(s)$ and the probability measure $\mathrm{d}\mu(y) = \frac{G\,\mathrm{d}y}{\langle G\rangle}$. We get

$$
(\langle F\rangle - \langle G\rangle)\ln\Big(\frac{\langle F\rangle}{\langle G\rangle}\Big) \le \langle G\rangle \int \Phi\Big(\frac{F}{G}\Big)\,\mathrm{d}\mu = \int (F - G)\ln\Big(\frac{F}{G}\Big)\,\mathrm{d}y.
$$

Moreover equality holds iff F is proportional to G. It follows that $\mathcal{D} \ge 0$. Eventually, we observe that $\mathcal{D}$ vanishes when the following two relations hold for every k and a. e (x, a, v): $\frac{\ell_k}{e^{-\Lambda}} = n\bar{M}_k$ and $\frac{f}{n_k\bar{M}_k} = c_k$, with c_k a fixed constant in $\mathbb{R}$. Combining these conditions, we get $n_k = n$ and $c_k = 1$. Therefore, $\mathcal{D}$ vanishes iff a. e. on $(0, T)\times(0,\infty)\times\omega_k\times\mathbb{R}^N$

$$
\ell_k = e^{-\Lambda} n\bar{M}_k = e^{-\Lambda} f.
$$

It follows that f satisfies the free transport equation $\partial_t f + v\cdot\nabla_x f = 0$. The unique continuation principle holds: we deduce that the macroscopic quantities in the traps are independent of k, and the distributions ℓ_k are all equal to the same Maxwellian $\bar{n}\bar{M}e^{-\Lambda}$. □

Remark 7.4 Similar computations can be performed by assuming that the re-emission rate λ depend on k. We find a similar conclusion, the only difference if the fact that in this case the distributions ℓ_k depend on k through the factor $e^{-\Lambda_k}$: $\ell_k = e^{-\Lambda_k} n\bar{M}_k = e^{-\Lambda_k} f$.

Remark 7.5 Some simplification occur in the specific case where $\lambda(a) = \lambda > 0$ is constant. In other words the re-emission time follows the exponential law with parameter λ. In such a case, we can simply work with the quantity

$$
\bar{\ell}_k(t, v) = \int_0^\infty \ell_k(t, a, v)\,\mathrm{d}a.
$$

The system indeed becomes

$$(\partial_t f + v\partial_x f)(t,x,v) = \sum_k \sigma_k(x) \left(\lambda \bar{M}_k(t,x,v) \int_0^\infty \int \overline{\ell}_k(t,v_\star)\,\mathrm{d}v_\star - f(t,x,v) \right),$$

$$\partial_t \overline{\ell}_k(t,v) = \int \frac{\sigma_k}{\mathscr{V}_k}(x) f(t,x,v)\,\mathrm{d}x - \lambda \overline{\ell}_k(t,v). \tag{25}$$

In this specific case, we can in fact disregard the microscopic equation for the trapped particles. Indeed, we simply have

$$\partial_t n_k = \frac{\lambda}{\mathscr{V}_k} \int \sigma_k (n - n_k)\,\mathrm{d}x,$$

$$\partial_t (n_k u_k) = \frac{\lambda}{\mathscr{V}_k} \int \sigma_k (nu - n_k u_k)\,\mathrm{d}x,$$

$$\partial_t (n_k u_k^2 + N n_k \theta_k) = \frac{\lambda}{\mathscr{V}_k} \int \sigma_k (nu^2 + Nn\theta - n_k u_k^2 - N n_k \theta_k)\,\mathrm{d}x.$$

Together with the definition of $\bar{M}_k$ and the evolution equation for f, it constitutes a closed system of equations, which coincides with (3a)–(3c) with $\mu = \lambda$.

7.3 Stationary Solutions

Let us detail the expression of the stationary solutions. In order to keep a finite total mass, these manipulations are meaningful when working on a bounded domain, like the torus. On the one hand, with (20), we get $f = n_k \bar{M}_k$ (which thus does not depend on k), which implies $n = n_k$ (which thus naturally does not depend on x neither), $nu = n_k \bar{u}_k$, $nu^2 + Nn\theta = n_k \bar{u}_k^2 + N n_k \bar{\theta}_k$, and thus $u = \bar{u}_k$, $\theta = \bar{\theta}_k$ (which thus do not depend on k). On the other hand, (21) becomes $\partial_a \ell_k = -\lambda \ell_k$, and it yields, together with (22), $\ell_k(a,v) = e^{-\Lambda(a)} \int \sigma_k n \bar{M}_k\,\mathrm{d}x$. Accordingly, by using space homogeneity, we are led to $n_k u_k = n u_k = \int \sigma_k n \bar{u}_k\,\mathrm{d}x = nu$, $n_k u_k^2 + N n_k \theta_k = n(u_k^2 + N\theta_k) = \int \sigma_k n (\bar{u}_k^2 + N\bar{\theta}_k)\,\mathrm{d}x = n(u^2 + N\theta)$. We thus find a space homogeneous solution, parametrized by constants macroscopic quantities (n, u, θ):

$$f(v) = \frac{n}{(2\pi\theta)^{N/2}} \exp\left(-\frac{|v-u|^2}{2\theta} \right), \qquad \ell_k(a,v) = e^{-\Lambda(a)} f(v).$$

The value of the parameters is prescribed by the conservation relations. To be more specific, let us denote by $\mu \in (0,\infty)$ the volume of the domain, which is supposed to contain K traps with total effective volume $\mathscr{V}$. We set

$$C = \mu + \mathscr{V} \langle e^{-\Lambda} \rangle.$$

Then, with

$$\begin{pmatrix} n_0 \\ n_0 u_0 \\ n_0(u_0^2 + N\theta_0) \end{pmatrix} = \iint \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} f\,\mathrm{d}v\,\mathrm{d}x + \sum_{k=1}^K \mathscr{V}_k \int_0^\infty \int \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} \ell_k\,\mathrm{d}v\,\mathrm{d}a$$

that define initial mass, velocity and temperature, we arrive at

$$Cn = n_0, \quad Cnu = n_0 u_0, \quad Cn(u^2 + N\theta) = n_0(u_0^2 + N\theta_0),$$

and thus

$$Cn = n_0, \quad u = u_0, \quad \theta = \theta_0.$$

As seen above, this equilibrium state makes the entropy dissipation vanish, too. Thus it appears as a natural candidate to attract the solutions for large times.

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Data Availability Data related to the simulations presented in this study are included in this article. Further information are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The author has no relevant financial or non-financial interests to disclose. The author has no competing interests to declare that are relevant to the content of this article.

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