Kinetic Equations, Hydrodynamic Limits, Applications

Th. Goudon

SIMPAF-INRIA Futurs & CNRS, Labo. Paul Painlevé, Lille

From Microscopic to Macroscopic

N particles	$\longmapsto$ Boltzmann's Equation
Na^2 fixed, $N \to \infty$	Statistical Description $f(t, x, v)$
Boltzmann's Equation	$\longmapsto$ Fluid Mechanics Eq.
$Kn \to 0$	Density, Velocity, Temperature...

Part of D. Hilbert's 6^{th} problem (ICM, Paris, 1900)

O. Lanford, '73

Bardos-Golse-Levermore, Lions-Masmoudi, Golse-StRaymond, 89-01...

Kinetic Equation

$$\underbrace{\partial_t f + v \cdot \nabla_x f}_{\text{transport}} + \underbrace{\nabla_v \cdot (F f)}_{\text{force field}} = \frac{1}{Kn} \underbrace{Q(f)}_{\text{collisions}}$$

- If $F = 0$ and $Q = 0$, then $f(t,x,v) = f_0(x - tv, v)$
- If $F = -\nabla_x \Phi(t,x)$ and $Q = 0$, then $f(t,x,v) = f_0\big(X(0,t,x,v), V(0,t,x,v)\big)$ where

$$\frac{\mathrm{d}}{\mathrm{d}s} X(s) = V(s), \quad \frac{\mathrm{d}}{\mathrm{d}s} V(s) = -\nabla_x \Phi(s, X(s)),$$
$$X_{|s=t} = x, \qquad V_{|s=t} = v$$

- Q= integral operator wrt v.

From Boltzmann to Fluid Equations

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{Kn} Q(f)$$

- Conservation: $m(v) = (1, v, v^2/2)$, $\displaystyle\int m(v) Q(f)\,\mathrm{d}v = 0$

  Conservation laws (local) $\partial_t \langle m f \rangle + \nabla_x \langle v\ m f \rangle = 0$

- Dissipation: $\displaystyle\int \Psi'(f) Q(f)\,\mathrm{d}v \le 0$, Ψ convex, thus

$$\partial_t \langle \Psi(f) \rangle + \nabla_x \langle v \Psi(f) \rangle \le 0$$

- Equilibrium: $Q(f_{\mathrm{eq}}) = 0$ iff

$$f_{\mathrm{eq}} = \frac{\rho(t,x)}{(2\pi T(t,x))^{3/2}} \exp\Big(- \frac{|v - u(t,x)|^2}{2T(t,x)} \Big)$$

BGK Model

$$Q(f) = M_{\rho,u,T}(v) - f$$

with

$$M_{\rho,u,T}(v) = \frac{\rho(t,x)}{(2\pi T(t,x))^{3/2}} \exp\Big(-\frac{|v-u(t,x)|^2}{2T(t,x)}\Big)$$

et

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} f \,\mathrm{d}v = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} M_{\rho,u,T} \,\mathrm{d}v = \begin{pmatrix} \rho \\ \rho u \\ \rho u^2 + 3\rho T \end{pmatrix} (t,x)$$

Hydrodynamic Limit

As $Kn \to 0$ we get $f \simeq f_{\text{eq}}$ and the conservation laws become

$$\partial_t \langle m f_{\text{eq}} \rangle + \nabla_x \langle v m f_{\text{eq}} \rangle = 0$$

$$= \partial_t \begin{pmatrix} \rho \\ \rho u \\ \rho u^2 + 3\rho T \end{pmatrix} + \nabla_x \begin{pmatrix} \rho u \\ \rho u \otimes u + \rho T \mathbb{I} \\ (\rho u^2 + 5\rho T) u \end{pmatrix} = 0$$

EULER EQUATIONS

Incompressible Models

Perturbation of a global equilibrium $f = M(1 + \epsilon^r g)$,
$M(v) = (2\pi)^{-3/2} e^{-v^2/2}$ with scaling $Kn = \epsilon$ et $\partial_t \to \epsilon \partial_t$.

$$\begin{aligned}
\epsilon \partial_t g + v \cdot \nabla_x g &= \frac{1}{\epsilon} \frac{1}{\epsilon^r M} \Big(Q(M) + \epsilon^r DQ(M)(g) \\
&\qquad + \epsilon^{2r} D^2 Q(M)(g,g) \dots \Big) \\
&= \frac{1}{\epsilon} \Big(\mathcal{L}(g) + \epsilon^r \mathcal{R}(g,g) \dots \Big)
\end{aligned}$$

leads to Incompressible Stokes $(r > 1)$ or Navier-Stokes $(r = 1)$ equations.

Rosseland Approximation in Radiative Transfer

$$\epsilon \partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon = \frac{1}{\epsilon} \sigma(\rho_\epsilon)(\rho_\epsilon - f_\epsilon),$$

$$v \in \mathbb{S}^{N-1} \qquad \rho_\epsilon = \int f_\epsilon \, \mathrm{d}v \qquad \sigma(\rho) \simeq \rho^\alpha, \quad |\alpha| < 1.$$

Hilbert Expansion: $f_\epsilon = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \dots$

- $\mathcal{O}(1/\epsilon)$ term: $\sigma(\rho^{(0)})(\rho^{(0)} - f^{(0)}) = 0$ yields $f^{(0)} = \rho^{(0)}(t, x)$,
- $\mathcal{O}(1)$ term: $\sigma(\rho^{(0)})(\rho^{(1)} - f^{(1)}) = v \cdot \nabla_x \rho^{(0)}(t, x)$ yields $f^{(1)} = -\frac{v}{\sigma(\rho^{(0)})} \nabla_x \rho^{(0)}$,
- $\mathcal{O}(\epsilon)$ term: $\partial_t f^{(0)} + v \cdot \nabla_x f^{(1)} =$ smthg with vanishing average hence

$$\partial_t \rho - \nabla_x \cdot \left(\frac{\int v \otimes v \, \mathrm{d}v}{\sigma(\rho)} \nabla_x \rho \right) = 0.$$

Mathematical Tools

• Entropy dissipation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \Psi(f_\epsilon)\,\mathrm{d}v\,\mathrm{d}x = -\frac{1}{\epsilon^2}\int \sigma(\rho^\epsilon)\big(\Psi'(f_\epsilon)-\Psi'(\rho_\epsilon)\big)\big(f_\epsilon-\rho_\epsilon\big)\,\mathrm{d}v\,\mathrm{d}x \le 0.$$

so that $f_\epsilon = \rho_\epsilon(t,x) + \epsilon g_\epsilon$ with (L^2) bounds on g_ϵ.

• One needs Strong Compactness to deal with nonlinearities.
Alternative:

⋆ Average Lemma

Bardos-Golse-Perthame-Sentis, '88 ;

⋆ Div-Curl Lemma

Murat-Tartar '78 (Homogenization & Conservation Laws) ;
Lions-Toscani '98, G.-Poupaud '01

Average lemma

If $f(x,v) \in L^2(\mathbb{R}^N \times \mathcal{V})$ and $v \cdot \nabla_x f \in L^2(\mathbb{R}^N \times \mathcal{V})$ then

$$\int_{\mathcal{V}} f\psi(v)\,\mathrm{d}v \in H^{1/2}(\mathbb{R}^N).$$

Crucial Assumption: $\forall \xi \in \mathrm{S}^{N-1}$, $\Big|\{v \in \mathcal{V} \text{ such that } v \cdot \xi = 0\}\Big| = 0.$

Div-Curl Lemma

Let $U_n = (u_n^1, \dots, u_n^N) \rightharpoonup U$, $V_n \rightharpoonup V$ in $L^2(\Omega)$ with furthermore $\mathrm{div} U_n = \sum \partial_i u_n^i$ and $\mathrm{curl} V_n = \left[\partial_j v_n^i - \partial_i v_n^j\right]_{ij}$ compact in H^{-1} then

$$U_n \cdot V_n = \sum_{i=1}^{N} u_n^i v_n^i \rightharpoonup U \cdot V \text{ in } \mathcal{D}'.$$

Crucial Assumption: $\forall \xi \in \mathrm{S}^{N-1}$, $\Big|\{v \in \mathcal{V} \text{ such that } v \cdot \xi \neq 0\}\Big| > 0.$
Thus it works for discrete velocity models $v \in \{v^1, \dots, v^M\}$, $\mathrm{d}v = \sum_{i=1}^{M} \omega_i \delta_{v=v^i}$.

How does it work ?

Moment equations

$$\begin{cases} (\rho_\epsilon,\ J_\epsilon,\ \mathbb{P}_\epsilon)(t,x) = \displaystyle\int_{\mathbb{S}^{N-1}} (1,\ v/\epsilon,\ v\otimes v) f_\epsilon \,\mathrm{d}v \\ \partial_t \rho_\epsilon + \mathrm{div}_x J_\epsilon = 0 \\ \epsilon^2 \partial_t J_\epsilon + \mathrm{Div}_x \mathbb{P}_\epsilon = -\sigma(\rho_\epsilon) J_\epsilon \end{cases}$$

But $f_\epsilon = \rho_\epsilon + \epsilon g_\epsilon$ yields $\mathbb{P}_\epsilon = \int v\otimes v \,\mathrm{d}v \rho_\epsilon + \epsilon \mathbb{R}_\epsilon$ so that

$$\mathrm{div}_{t,x}(\rho_\epsilon, J_\epsilon) \text{ and } \mathrm{curl}_{t,x}(\rho_\epsilon, 0, \ldots, 0) = \begin{pmatrix} 0 & -\nabla_x \rho_\epsilon^T \\ \nabla_x \rho_\epsilon & 0 \end{pmatrix}$$

belong to a compact set of $H^{-1}_{\mathrm{loc}}((0,T)\times\mathbb{R}^N)$.

Coupling to Homogeneization: Neutron Transport

We seek reduced models for routine computations that take into account heterogeneities of the medium

[Allaire with Bal, Capeboscq, Sieiss ; G.-Poupaud, G.-Mellet]

$$\epsilon\partial_t f + v \cdot \nabla_x f = \frac{1}{\epsilon}\Big(\int \sigma(x, x/\epsilon, v, v_\star) f(v_\star)\, \mathrm{d}v_\star - \int \sigma(x, x/\epsilon, v_\star, v)\, \mathrm{d}v_\star \; f(v) \Big)$$

Set $T = v \cdot \nabla_y - Q$, $y = x/\epsilon$ and expand $f_\epsilon = \sum \epsilon^j f^{(j)}(t, x, x/\epsilon, v)$

- $Tf^{(0)} = 0$ is solved by $\rho(t, x)M(x, y, v)$
- $Tf^{(1)} = v \cdot \nabla_x f_0 = vM \cdot \nabla_x \rho + v \cdot \nabla_x M \rho$. If $\int vM \,\mathrm{d}v\, \mathrm{d}y = 0$ then $f^{(1)} = -\chi \cdot \nabla_x \rho + \lambda\rho$ where

$$T\chi = -vM, \qquad T\lambda = v \cdot \nabla_x M$$

Eventually, we get

$$\partial_t \rho - \nabla_x \cdot (D(x) \nabla_x \rho - U(x) \rho) = 0,$$

$$D(x) = \int \int v \otimes \chi(x, y, v) \, \mathrm{d}v \, \mathrm{d}y, \qquad U(x) = \int \int v \lambda(x, y, v) \, \mathrm{d}v \, \mathrm{d}y$$

Convection term related to the space dependance of the equilibrium function.

Degond-G.-Poupaud, 2003 ;
Chalub-Markowich-Perthame-Schmeiser, 2004

Treatment of the homogeneization aspect relies on double scale technics

$$\int_\Omega u_\epsilon \psi(x, x/\epsilon) \, \mathrm{d}x \xrightarrow[\epsilon \to 0]{} \int_{(0,1)^N} \int_\Omega U \psi(x, y) \, \mathrm{d}x \, \mathrm{d}y$$

Intermediate Models

Assuming $\mathcal{V} \subset (-1,+1)$, $\int_{\mathcal{V}} \mathrm{d}v = 1$, $\int_{\mathcal{V}} v \,\mathrm{d}v = 0$, $\int_{\mathcal{V}} v^2 \,\mathrm{d}v = d > 0$, solutions of

$$\epsilon \partial_t f_\epsilon + v \partial_x f_\epsilon = \frac{1}{\epsilon}\Big(\int_{\mathcal{V}} f_\epsilon \,\mathrm{d}v - f_\epsilon\Big)$$

converge to $\rho(t,x)$ solution of

$$\partial_t \rho - d \partial^2_{xx} \rho = 0.$$

One seeks intermediate models for $0 < \epsilon \ll 1$:

$\star$ heat eq. propagates at infinite speed instead of $\mathcal{O}(1/\epsilon)$,

$\star$ $\rho - \epsilon v \partial_x \rho$ does not preserve non-negativeness, nor the flux limited condition

$$\Big| \int_{\mathcal{V}} \frac{v}{\epsilon} f_\epsilon \,\mathrm{d}v \Big| \leq \frac{1}{\epsilon} \int_{\mathcal{V}} f_\epsilon \,\mathrm{d}v.$$

Minimum Entropy Principle Closure

The Moment System

$$\begin{cases} \partial_t \rho_\epsilon + \mathrm{div}_x J_\epsilon = 0 \\ \epsilon^2 \partial_t J_\epsilon + \mathrm{Div}_x \mathbb{P}_\epsilon = -J_\epsilon \end{cases}$$

is closed by imposing [Levermore '97]

$$\mathbb{P}_\epsilon = \int_{\mathcal{V}} v^2 f_\epsilon^\star \, \mathrm{d}v$$

where $f_\epsilon^\star$ minimizes

$$\int_{\mathcal{V}} f \ln f \, \mathrm{d}v, \ \text{ with } \int_{\mathcal{V}} (1, \ v/\epsilon) f \, \mathrm{d}v = (\rho_\epsilon, \ J_\epsilon)$$

One obtains an hyperbolic system $\mathbb{P}_\epsilon = \rho_\epsilon \psi(\epsilon J_\epsilon / \rho_\epsilon)$ which is globally well-posed for small enough initial data, and consistent to the diffusion eq. as ϵ goes to 0. [Coulombel-Golse-G.'06]

Astrophysics

Diffusion regimes might lead to singularity phenomena!

Vlasov-Poisson-Fokker-Planck Equation

$$\begin{aligned}\epsilon\partial_t f + v\cdot\nabla_x f - \nabla_x\Phi\cdot\nabla_v f &= \frac{1}{\epsilon}\nabla_v\cdot(vf+\nabla_v f)\\ &= \frac{1}{\epsilon}\nabla_v\cdot(M\nabla_v(f/M))\end{aligned}$$

with $M(v) = (2\pi)^{-N/2}\, e^{-v^2/2}$ coupled to $\Delta\Phi = \rho$

For small ϵ's, we guess [Chandrasekhar, 1943] $f \simeq \rho(t,x)M(v)$ where ρ satisfies

$$\partial_t\rho - \nabla_x\cdot(\nabla_x\rho + \rho\nabla_x\Phi) = 0, \qquad \Delta\Phi = \rho$$

Smoluchowski eq (or Keller-Segel in biology)

Competition between diffusion and Ricatti's like behavior

$$\partial_t\rho - \nabla_x\Phi\cdot\nabla_x\rho = \rho\Delta_x\Phi = \rho^2$$

2D Case

Threshold phenomena: If $\int_{\mathbb{R}^2} \rho_0^2 < M_\star$ then global existence of "smooth" solutions, If $\int_{\mathbb{R}^2} \rho_0^2 > M_\star$, formation of Dirac mass in finite time.

Beckner, Carlen-Loss estimate ('92): Let $\rho \geq 0$ and $\int_{\mathbb{R}^2} \rho \,\mathrm{d}x = 1$ then

$$-4 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho(x)\rho(y) \ln(|x-y|)\mathrm{d}y\,\mathrm{d}x \leq C_* + 2\int_{\mathbb{R}^2} \rho \ln(\rho)\,\mathrm{d}x.$$

Entropy estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^4} f_\epsilon\Big(\ln(f_\epsilon)+\frac{v^2}{2}+\Phi_\epsilon\Big)\,\mathrm{d}v\,\mathrm{d}x \leq -\frac{4}{\epsilon^2}\int_{\mathbb{R}^4} \left|\nabla_v \sqrt{f_\epsilon/M}\right|^2 M\,\mathrm{d}v\,\mathrm{d}x.$$

Global convergence $\epsilon \longrightarrow 0$ in the subcritical case [G.'06].

Numerics based on DG methods [Gamba-G.-Proft]

Fluid/Particles Flows

Applications: Rocket Propulsors, Diesel Engines, Steel Production Processes...

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \mathcal{U} \cdot \nabla_v f + \nabla_v \cdot ((u - v) f) = \Delta_v f$$

Modeling discussions: Viscosity, Incompressibiliy, Coagulation-fragmentation...

Identification of Relevant Parameters and Asymptotic Regimes: e. g. Stokes settling time $\ll$ observation time leads to Hydrodynamic Limits, but the regime also depends on ρ_p/ρ_g...

Turbulent Flows

Given u with high and fast random variations

Behavior of $\lim_{\cdots} \langle f \rangle$? [G.Poupaud 04]: Diffusion-convection/Convection/Fokker-Planck

“Turbulence” for Dummies

Crucial assumption: $u(t)$ and $u(s)$ decorrelate when $|t-s| \geq 1$

Toy Model: $\dfrac{\mathrm{d}}{\mathrm{d}t} a_\epsilon(t) = i\dfrac{1}{\epsilon} u(t/\epsilon^2) a_\epsilon(t).$

Duhamel's Formula: $a_\epsilon(t) = a_\epsilon(t-\epsilon^2) + \dfrac{1}{\epsilon}\displaystyle\int_{t-\epsilon^2}^{t} iu(s/\epsilon^2) a_\epsilon(s)\,\mathrm{d}s$

$$\frac{u(t/\epsilon^2)}{\epsilon} a(t) = \underbrace{\frac{u(t/\epsilon^2)}{\epsilon} a(t-\epsilon^2)}_{\mathbb{E}(\dots)=0} + \underbrace{\frac{1}{\epsilon^2}\int_{t-\epsilon^2}^{t} iu(t/\epsilon^2)u(s/\epsilon^2)a_\epsilon(s)\,\mathrm{d}s}_{\mathcal{O}(1)}$$

$$\mathbb{E}\frac{i}{\epsilon}u(t/\epsilon^2)a(t) = -\int_0^1 \mathbb{E}\big(u(t/\epsilon^2)u(t/\epsilon^2-\tau)\big)\,\mathrm{d}\tau\ \mathbb{E}a_\epsilon(t) + \text{small terms}$$

so that the limit eq. is

$$\frac{\mathrm{d}}{\mathrm{d}t} a = -\lambda a, \qquad \lambda = \int_0^1 \mathbb{E}\big(u(0)u(-\tau)\big)\,\mathrm{d}\tau$$

Coupling with Hydrodynamics

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \mathcal{U} \cdot \nabla_v f + \nabla_v \cdot ((u - v) f) = \Delta_v f$$

Coupled to Euler or Navier-Stokes equations for n and u with force term

$$\int (v - u) f \, \mathrm{d}v$$

Hydrodynamic limit leads to two-phase flows equations.

Compactness methods (but the problem does not fit with the average lemma framework)

Relative Entropy approach: $\dfrac{\mathrm{d}}{\mathrm{d}t} H(f_\epsilon | f_{\lim}) + D(u_\epsilon - u_{\lim})$

Stability of Stationary Solutions

[G. 00, G.-Jabin-Vasseur 04, Carrillo-G '06]

Asymptotically-Induced Schemes

$$\partial_t f_\epsilon + \frac{v}{\epsilon}\partial_x f_\epsilon = \frac{1}{\epsilon^2}(\langle f_\epsilon \rangle - f_\epsilon)$$

We have $f_\epsilon = \rho_\epsilon(t,x) + \epsilon g_\epsilon$ so that

$$\underbrace{\partial_t f_\epsilon + v\partial_x g_\epsilon}_{\text{transport-like}} = \underbrace{\frac{1}{\epsilon^2}(\rho_\epsilon - f_\epsilon) - \frac{v}{\epsilon}\partial_x \rho_\epsilon = -\frac{1}{\epsilon} g_\epsilon - \frac{v}{\epsilon}\partial_x \rho_\epsilon}_{\text{Stiff sources with } \langle\cdot\rangle = 0.}$$

Splitting Approach

- Solving $\partial_t f_\epsilon + v\partial_x g_\epsilon = 0$ defines $f^{n+1/2}$, $\rho^{+1/2}$
- Solve ODEs $\partial_t f_\epsilon = \frac{1}{\epsilon^2}(\rho_\epsilon - f_\epsilon) - \frac{v}{\epsilon}\partial_x \rho_\epsilon$

  Since $\langle \mathrm{rhs} \rangle = 0$ we have $\rho^{n+1} = \rho^{n+1/2}$

and we write

$$f^{n+1} = e^{-\Delta t/\epsilon^2} f^{n+1/2} + (1 - e^{-\Delta t/\epsilon^2})\rho^{n+1/2}$$
$$g^{n+1} = e^{-\Delta t/\epsilon^2} g^{n+1/2} - (1 - e^{-\Delta t/\epsilon^2}) v\partial_x \rho^{n+1/2}$$

The scheme is Asymptotic Preserving by construction.

Fully Explicit.

Stability condition unclear ($\Delta t/\Delta x^2$...)

Cheap scheme adapted for intermediate regimes $0 \leq \epsilon \ll 1$.

Radiative Transfer Problems

$$\partial_t n + \partial_x(nu) = 0,$$
$$\partial_((nu) + \partial_x(nu^2 + p) = -S_m,$$
$$\partial_t(nE) + \partial_x((nE + p)u) = -S_e$$

coupled to

$$\epsilon\partial_t f + v\partial_x f = \frac{1}{\epsilon}Q_s + \epsilon Q_a$$
$$Q_s = \sigma_s\Big(\frac{1}{\Lambda^3}\langle\Lambda^2 f\rangle - \Lambda f\Big), \qquad Q_a = \sigma_a\Big(\frac{1}{\Lambda^3}\frac{1}{\pi}\theta^4 - \Lambda f\Big).$$

with $\Lambda = (1 - \epsilon uv)/\sqrt{1 - \epsilon^2 u^2}$ and $S_m = \frac{1}{\epsilon}\langle vQ_s\rangle + \epsilon\langle vQ_a\rangle$,
$S_e = \frac{1}{\epsilon^2}\langle Q_s\rangle + \langle Q_a\rangle$.

As $\epsilon \to 0$, f_ϵ becomes proportional to Λ^{-4}, which has a $\mathcal{O}(\epsilon)$ flux.

Non Equilibrium Diffusion Regime

Full Model:

$$\begin{cases} \partial_t n + \partial_x(nu) = 0, \\ \partial_t(nu) + \partial_x(nu^2 + p) = -\mathcal{P}\dfrac{\partial_x \rho}{3}, \\ \partial_t(nE) + \partial_x(nEu + pu) = -\mathcal{P}\dfrac{1}{3}u\partial_x\rho + \mathcal{P}\sigma_a(\rho - \theta^4), \\ \partial_t\rho - \dfrac{1}{3\sigma_s}\partial^2_{xx}\rho + \dfrac{4}{3}\partial_x(\rho u) - \dfrac{1}{3}u\partial_x\rho = \sigma_a(\theta^4 - \rho). \end{cases}$$

Simplified Model:

$$\begin{cases} \partial_t n + \partial_x(nu) = 0, \\ \partial_t(nu) + \partial_x(nu^2 + p) = 0, \\ \partial_t(nE) + \partial_x(nEu + pu) = \rho - \theta^4, \\ -\partial^2_{xx}\rho = \theta^4 - \rho, \end{cases}$$

Questions are related to the effects of the Energy Exchanges on the features of the usual Euler system:

- Smoothing effects on the shock profile [Lin-Coulombel-G. '06]

- Stability questions (of constants, of shocks profiles...)

- Asymptotic problems [G.-Lafitte '06]

- Numerical Experiments