

Asymptotics Problems for Wave-Particles Interactions; Quantum and Classical Models

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Quantum Model: Unknown=**Density Matrix** $\rho(t; n, m)$, $n, m \in \mathbb{N}$,

$$i\hbar \partial_t \rho + [H_q, \rho] = \frac{1}{\tau} Q_q(\rho), \quad \text{Bloch equation}$$

where $[H_q, \rho](n, m) = \sum_{k \in \mathbb{N}} (H_q(n, k)\rho(k, m) - \rho(n, k)H_q(k, m))$,

Classical Model: Unknown=**Distribution Function** $f(t; x, p) \geq 0$,
 $x, p \in \mathbb{R}^N$

$$\partial_t f + \{H_c, f\} = \frac{1}{\tau} Q_c(f),$$

where $\{H_c, f\} = \nabla_p H_c \cdot \nabla_x f - \nabla_x H_c \cdot \nabla_p f$.

Scaling Issues

$$\text{We split } H = \begin{cases} H_{0c}(x, p) + \epsilon V_c(t, t/\theta; x, p), \\ H_{0q}(n, m) + \epsilon V_q(t, t/\theta; n, m), \end{cases}$$

H_0 = time independent Hamiltonian corresponding to the **confining potential** of the atomic nucleus,

ϵV = perturbations by external electro-magnetic waves

Scaling Assumptions:

$$\frac{TH_{0c}}{LP} \simeq \frac{1}{\epsilon^2} \gg 1, \quad \frac{TH_{0q}}{\hbar} \simeq \frac{1}{\epsilon^2} \gg 1, \quad \frac{T}{\theta} \simeq \frac{1}{\epsilon^2}, \quad \frac{T}{\tau} \simeq \frac{1}{\epsilon^2}.$$

Cutting Edge Questions

$$i\partial_t \rho + \frac{1}{\epsilon^2} [H_{0q} + \epsilon V_q(t/\epsilon^2), \rho] = \frac{1}{\epsilon^2} Q_q(\rho),$$

$$\partial_t f + \frac{1}{\epsilon^2} \underbrace{\{H_{0c} + \epsilon V_c(t/\epsilon^2), f\}} = \underbrace{\frac{1}{\epsilon^2} Q_c(f)},$$

Free Hamiltonian H_0

Damping

+ Small Oscillating Perturbation

or Relaxation term

- Derivation of a New Classical Model (Relaxation operator)
- Relaxation and Homogenization all together
- Random vs. Deterministic modelling

Asymptotic Behavior

- Quantum Model: $\rho_\epsilon(t; n, m) \rightarrow \rho(t; n, n) \delta(n, m)$ verifying the **Einstein Rate Equation**

$$\partial_t \rho(t; n, n) = \sum_k A(t; n, k) (\rho(t; k, k) - \rho(t; n, n))$$

- Classical Model: $f_\epsilon(t; x, p) \rightarrow F(t; H_0(x, p))$ verifying the **Diffusion Eq.** wrt energy

$$\partial_t (h_0(E) F(t; E)) - \partial_E (d(E) h_0(E) \partial_E F(t; E)) = 0$$

$h_0(E) F(t; E) = \#$ of particles with energy E ,

$d(E) h_0(E) \partial_E F(t; E) =$ Flux through the energy surface Σ_E .

- Both limit eq. are **Irreversible Equations** (decay of L^2 norms)

Role of the Damping Term, Deterministic vs Random Perturbations

Two Difficulties:

- Homogeneization Limit (Fast Oscillations of V)
- Relaxation Limit that pushes towards $\text{Ker}([H_0, \cdot] - Q_q)$ (resp. $\text{Ker}(\{H_0, \cdot\} - Q_c) \sim$ “hydrodynamical limit”

Two Frameworks:

- (Quasi-)Periodic Oscillations and Damping is crucial
- Random Perturbation where Damping term can vanish

Background

Homogenization of Convection(-Diffusion) Equations

$$\partial_t u_\epsilon + \operatorname{div}_x (a_\epsilon(t, x) u_\epsilon) = \eta \Delta_x u_\epsilon$$

Deterministic Framework : $a_\epsilon = \frac{1}{\epsilon} a(t, t/\epsilon^2, x, x/\epsilon)$ periodic, without div-free assumption (invariant scaling for NS) [G-Poupaud 04]

$\eta > 0$ is crucial

Limit equation $\partial_t u - \operatorname{div}_x ((\eta + D) \nabla_x u) = 0$ with $(\eta + D) \xi \cdot \xi > 0$ (but D has no sign! [Avellaneda-Vergassola 97])

$\eta = 0$ is much more involved [Amirat-Hamdache-Ziani 89, E 90, Hou-Xin 92]

Role of the Damping: Cell Problems

Let $\Omega \in \mathbb{R}^D$ with **rationally independent components**. Consider

$$\lambda R + \Omega \cdot \nabla_{\vartheta} R = H,$$

with $(0, 1)^N = \Theta$ -periodic boundary condition and $\lambda \geq 0$.

- If $\lambda = 0$ then $\int_{\Theta} H \, d\vartheta = 0$ is a *necessary* condition.
- If $\lambda = 0$ and $H = 0$ then $\Omega \cdot \xi \widehat{R}(\xi) = 0$ and **R is constant**.
- But the Fredholm alternative *does not* apply to $\Omega \cdot \nabla_{\vartheta}$ due to **Small Divisors** problems: set $\widehat{R}(\xi) = -i\widehat{H}(\xi)/\Omega \cdot \xi$,

$$\text{If } |\Omega \cdot \xi| \geq C/|\xi|^\gamma \text{ then } \|R\|_{L^2} \leq C\|H\|_{H^\gamma}.$$

- With **$\lambda > 0$** we get

$$R(\vartheta) = \int_0^{+\infty} e^{-\lambda\sigma} H(\vartheta - \Omega\sigma) \, d\sigma \in L^2.$$

Example: Shear Flow

$$\partial_t u_\epsilon + \partial_x (a(y/\epsilon)u_\epsilon) = 0$$

Suppose that $a(\vartheta) = a_1 \chi_{[0,\delta]}(\vartheta) + a_2 \chi_{(\delta,1]}(\vartheta)$ and periodic

$u_\epsilon(t, x) \rightarrow u(t, x) = \int_{(0,1)} U(t, x, \vartheta) d\vartheta$ with

$$\partial_t U + a(\vartheta)\partial_x U = 0, \quad U(t, x, \vartheta) = V(x - ta(\vartheta)).$$

The limit is described by a Damped Wave Equation

$$\partial_{tt}^2 u + (a_1 + a_2)\partial_{tx}^2 u - a_1 a_2 \partial_{xx}^2 u = 0.$$

Random Oscillating Field

But considering $\partial_t u_\epsilon + \operatorname{div}_x (a_\epsilon u_\epsilon) = 0$ with a random velocity we are led to a diffusion equation, D depends on the correlation of a [Kubo 63, Khasminskii 66, Kesten-Papanicolaou 79, G.-Poupaud 06...]

How does randomness induce irreversibility ?

Toy Model: $\frac{d}{dt}u_\epsilon(t) = i\frac{1}{\epsilon}a(t/\epsilon^2)u_\epsilon(t)$ with a random, $\mathbb{E}a = 0$

Crucial assumption: $a(t)$ and $a(s)$ decorrelate when $|t - s| \geq 1$

Duhamel's Formula: $u_\epsilon(t) = u_\epsilon(t - \epsilon^2) + \frac{1}{\epsilon} \int_{t-\epsilon^2}^t ia(s/\epsilon^2)u_\epsilon(s) ds$

$$\frac{a(t/\epsilon^2)}{\epsilon}u_\epsilon(t) = \underbrace{\frac{a(t/\epsilon^2)}{\epsilon}u_\epsilon(t - \epsilon^2)}_{\mathbb{E}(\dots) = 0} + \underbrace{\frac{1}{\epsilon^2} \int_{t-\epsilon^2}^t ia(t/\epsilon^2)a(s/\epsilon^2)u_\epsilon(s) ds}_{\mathcal{O}(1)}$$

$$\mathbb{E} \frac{i}{\epsilon} a(t/\epsilon^2)u(t) = - \int_0^1 \mathbb{E}(a(t/\epsilon^2)a(t/\epsilon^2 - \tau)) d\tau \mathbb{E}u_\epsilon(t) + \text{small terms}$$

so that the limit eq. is

$$\frac{d}{dt}u = -\lambda u, \quad \lambda = \int_0^1 \mathbb{E}(a(0)a(-\tau)) d\tau > 0.$$

Other Examples

- N Particles to Boltzmann: [Lanford 73]
- Lorentz Gas: [Galavotti, Sinai-Bunimovich-Chernov 91, Bardos-Dumas-Golse 97]
- Kinetic (or Schrödinger) eq. with highly oscillating potentials: [Poupaud-Vasseur 03, Loeper-Vasseur 04, Frénod-Hamdache 96]
- Particles subject to friction forces [G. Poupaud 04]...

On the Bloch eq., using a different toolbox:

[Bidégaray-Castella-Degond 04 and
Bidégaray-Castella-Dumas-Gisclon 04]

The Quantum Model

$$\left\{ \begin{array}{l} \gamma(n, m) \geq 0, \quad \gamma(n, n) = 0, \quad \gamma(n, m) = \gamma(m, n), \\ \omega(n, m) = -\omega(m, n) \in \mathbb{R}, \\ V_\epsilon(t; n, k) = V(t, t/\epsilon^2; n, k), \quad V(n, k) = \overline{V(k, n)} + \text{bounds} \end{array} \right.$$

$$Q(\rho)(n, m) = i\gamma(n, m) (\rho(n, m)\delta(n, m) - \rho(n, m))$$

$$[H_0, \rho] = -\omega(n, m)\rho(n, m)$$

Set $Z(n, m) = \gamma(n, m) + i\omega(n, m)$:

$Z(n, m) = 0$ for $n = m$, and $\overline{Z(n, m)} = Z(m, n)$.

$$\begin{aligned} \partial_t \rho(t; n, m) + \frac{1}{\epsilon^2} Z(n, m) \rho(n, m) &= \frac{1}{\epsilon} \Theta_\epsilon[\rho](t; n, m) \\ &= -\frac{i}{\epsilon} \sum_{k \in \mathbb{N}} \left[V_\epsilon(n, k) \rho(t; k, m) - V_\epsilon(k, m) \rho(t; n, k) \right]. \end{aligned}$$

Θ_ϵ is a (uniformly) bounded operator on ℓ^2

The problem is well-posed in $C^0([0, \infty); \ell^2) +$ (uniform) estimates.

Eq. for the populations:

$$\partial_t \rho(t; n, n) = -\frac{i}{\epsilon} \sum_{k \in \mathbb{N}} \left[V_\epsilon(n, k) \rho(t; k, n) - V_\epsilon(k, n) \rho(t; n, k) \right].$$

depends only on the coherences

Eq. for the quantum coherences: ($n \neq m$)

$$\begin{aligned} \partial_t \rho(t; n, m) = & -\frac{1}{\epsilon^2} Z(n, m) \rho(n, m) \\ & -\frac{i}{\epsilon} \sum_{k \in \mathbb{N}} \left[V_\epsilon(n, k) \rho(t; k, m) - V_\epsilon(k, m) \rho(t; n, k) \right]. \end{aligned}$$

Quantum Coherences are **Damped** when $\text{Re}Z(n, m) = \gamma(n, m) \neq 0$,

Hence the solution **relaxes** to $\rho(t; n, m) \delta(n, m)$.

The Two-level Model

“1”=Ground state, “2”=Excited state

$$\frac{d}{dt}\rho_{11}^\epsilon = -\frac{2}{\epsilon}\text{Im}(V_{12}^\epsilon\rho_{21}^\epsilon), \quad \frac{d}{dt}\rho_{21}^\epsilon = -\frac{1}{\epsilon^2}(i\omega+\gamma)\rho_{21}^\epsilon + \frac{i}{\epsilon}V_{21}^\epsilon(2\rho_{11}^\epsilon-1).$$

Perturbation $V_{12}^\epsilon(t) = \exp(i(\Delta + \omega)t/\epsilon^2)$.

Set $\rho_{21}^\epsilon(0) = 0$, then as $\epsilon \rightarrow 0$

$$\frac{d}{dt}\rho_{11} = \frac{2\gamma}{\gamma^2 + \Delta^2}(\rho_{22} - \rho_{11}), \quad \frac{d}{dt}(\rho_{11} + \rho_{22}) = 0$$

($\Delta = 0$ as well.)

However, when $\gamma = 0$, we have

$$\rho_{11}^\epsilon(t) = \frac{4}{4 + \Delta^2/\epsilon^2}(\rho_{11}(0) - 1/2) \cos(\sqrt{4 + \Delta^2/\epsilon^2} t/\epsilon) + \frac{1}{4 + \Delta^2/\epsilon^2} \left(\frac{\Delta^2}{\epsilon^2} \rho_{11}(0) + 2 \right),$$

Rabi's oscillations

Quantum Model: (Quasi-)Periodic oscillations

$V_\epsilon(t; n, m) = \mathcal{V}(t, \Omega t/\epsilon^2; n, m)$ with $\Omega \in \mathbb{R}^d \setminus \{0\}$, rationally independent components and $\gamma(n, m) \geq \gamma_* > 0$ for $n \neq m$.

Multiscale Ansatz: $\rho_\epsilon(t; n, m) = \sum_j \epsilon^j \rho^{(j)}(t, \Omega t/\epsilon^2; n, m)$.

$\partial_t \rightarrow \partial_t + \frac{1}{\epsilon^2} \Omega \cdot \nabla_{\vartheta}$ yields

$$1/\epsilon^2 \text{ terms: } \Omega \cdot \nabla_{\vartheta} \rho^{(0)} + Z(n, m) \rho^{(0)} = 0,$$

$$1/\epsilon^1 \text{ terms: } \Omega \cdot \nabla_{\vartheta} \rho^{(1)} + Z(n, m) \rho^{(1)} = \Theta(t, \vartheta) [\rho^{(0)}],$$

$$1/\epsilon^0 \text{ terms: } \Omega \cdot \nabla_{\vartheta} \rho^{(2)} + Z(n, m) \rho^{(2)} = -\partial_t \rho^{(0)} + \Theta(t, \vartheta) [\rho^{(1)}], \dots$$

$$\rho^{(0)}(t, \vartheta; n, n) = \rho^{(0)}(t; n, n), \quad \rho^{(0)}(t, \vartheta; n, m) = 0 \quad \text{if } n \neq m,$$

$$\rho^{(1)}(t, \vartheta; n, m) = i\chi(t, \vartheta; n, m) (\rho^{(0)}(t; m, m) - \rho^{(0)}(t; n, n)),$$

$$\chi(t, \vartheta; n, m) = - \int_0^{+\infty} e^{-Z(n, m)\sigma} \mathcal{V}(t, \vartheta - \Omega\sigma; n, m) d\sigma, \quad n \neq m$$

Statement for the Quantum Model: (Quasi-)Periodic oscillations

Theorem. ρ_ϵ converges to $\rho(t; n, n)\delta(n, m)$ weakly in $L^2(\mathbb{R}^+; \ell^2)$; the diagonal part $\rho_\epsilon(t; n, n)$ converges to $\rho(t; n, n)$ in $C^0([0, T]; \ell^2 - \text{weak})$ and the limit satisfies the Einstein rate equation

$$\partial_t \rho(t; n, n) = \sum_{k \in \mathbb{N}} A(t; n, k) (\rho(t; k, k) - \rho(t; n, n)),$$

$$\rho(0; n, n) = \lim_{\epsilon \rightarrow 0} \rho_\epsilon^0(n, n) \quad \text{weakly in } \ell^2,$$

$$A(t; n, k) = 2\text{Re}(Z(n, k)) \int_{\Theta} |\chi(t, \vartheta; n, k)|^2 d\vartheta > 0, \text{ for } n \neq k.$$

Method of proof

- Uniform estimates
- Use Double-scale convergence [Nguetseng 89, Allaire 92] (to be adapted to the quasi-periodic framework):

Let u_ϵ be a bounded sequence in $L^2(\mathbb{R})$. Then, there exists a subsequence and $U \in L^2_{\#}(\mathbb{R} \times \Theta)$ such that for any trial function

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} u_\epsilon(t) \psi(t, \Omega t / \epsilon^2) dt = \int_{\mathbb{R}} \int_{\Theta} U(t, \vartheta) \psi(t, \vartheta) d\vartheta dt.$$

- Multiply the equation by “oscillating test functions” [Evans 89-92, Tartar 86-89]

Quantum Model: Random Framework

$V_\epsilon(t; n, m) = \mathcal{V}(t/\epsilon^2; n, m)$ with \mathcal{V} bounded random variable such that

- i) $\mathbb{E}(\mathcal{V}(\tau; n, m)) = 0,$
- ii) $E(\mathcal{V}(\tau; k, l) \mathcal{V}(\sigma; m, n)) = \mathcal{R}(\tau - \sigma; k, l, m, n)$
- iii) If $|\tau - \sigma| \geq \mathcal{T}$ then $\mathcal{V}(\tau)$ and $\mathcal{V}(\sigma)$ are independent.

Theorem. Let ρ_ϵ^0 be **deterministic**. Suppose

$Z(n, m) = \gamma(n, m) + i\omega(n, m)$ **vanishes iff $n = m$** . Then, $\mathbb{E}\rho_\epsilon(t; n, m)$ converges weakly to $\rho(t; n, n)\delta(n, m)$ and $\mathbb{E}\rho_\epsilon(t; n, n)$ converges in $C^0([0, T]; \ell^2 - weak)$ to $\rho \in L^\infty(\mathbb{R}^+; \ell^2)$ sol. of the Einstein eq. with

$$A(n, k) = 2\text{Re} \int_0^{\mathcal{T}} \mathcal{R}(\tau; n, k, k, n) e^{-Z(k, n)\tau} d\tau.$$

Comments

- If all $\gamma(n, m) = 0$ it means that $\omega(n, m) = H_0(m, m) - H_0(n, n) \neq 0$ where $H_0(n, n)$ = eigenvalues of a differential operator: **non-degeneracy assumption**.

Need relaxation for energy levels corresponding to multidimensional eigenspaces.

- Alternative: Same statement with possibly vanishing damping coefficients

$$\gamma_\epsilon(n, m) \geq \underline{\gamma}_\epsilon > 0, \quad \gamma_\epsilon(n, m) \xrightarrow{\epsilon \rightarrow 0} \gamma(n, m) \geq 0, \quad \frac{\epsilon^2}{\underline{\gamma}_\epsilon} \xrightarrow{\epsilon \rightarrow 0} 0.$$

Trick: use the “Entropy Estimate”

$$\rho_\epsilon(t; n, m) = \rho_\epsilon(t; n, n)\delta(n, m) + \frac{\epsilon}{\sqrt{\underline{\gamma}_\epsilon}} r_\epsilon(t; n, m)$$

with $r_\epsilon(t; n, m)$ bounded.

A New Classical Model

Goal: Mimic the quantum relaxation operator

Populations \simeq # of particles on a energy shell $\Sigma_E = \{H_0(x, p) = E\}$.

Hypothesis (think of $H_0(x, p) = x^2 + p^2$)

• $H_0 \in C^\infty(\mathbb{R}^{2D})$, $\lim_{|(x,p)| \rightarrow \infty} H_0(x, p) = +\infty$ (Confining).

• For a.a. $E \in \mathbb{R}$, Σ_E is a smooth orientable $(2D - 1)$ submanifold

of \mathbb{R}^{2D} . We set $\delta(H_0(x, p) - E) := \frac{d\Sigma_E(x, p)}{|\nabla_{x,p} H_0(x, p)|}$ (Liouville's

measure) and suppose $h_0(E) := \int_{\Sigma_E} \delta(H_0(x, p) - E) < +\infty$.

Define $Pf(x, p) = \frac{1}{h_0(E)} \int_{\Sigma_E} f(y, q) \delta(H_0(y, q) - E) \Big|_{E=H_0(x,p)}$

Classical Model

Fundamental properties follow from the coarea formula

$$\int_{\mathbb{R}^{2D}} f(x, p) \, dp \, dx = \int_{\mathbb{R}} \left(\int_{\Sigma_E} f(x, p) \delta(H_0(x, p) - E) \right) \, dE$$

which yields $\begin{cases} P \text{ is a bounded operator on } L^r, & P(Pf) = Pf, \\ \int Pf \, dp \, dx = \int f \, dp \, dx, & P\{H_0, f\} = 0 \end{cases}$

$$\underbrace{\partial_t f_\epsilon + \frac{1}{\epsilon^2} \{H_0, f_\epsilon\}}_{\text{Transport along}} + \underbrace{\frac{1}{\epsilon} \{V_\epsilon, f_\epsilon\}}_{\text{Fast Varying}} = \underbrace{\frac{\gamma}{\epsilon^2} (Pf_\epsilon - f_\epsilon)}_{\text{Resonant Interaction}}$$

Transport along Fast Varying Resonant Interaction

$X_\epsilon(t), P_\epsilon(t)$ Perturbation

where $\frac{d}{dt}(X_\epsilon, P_\epsilon) = \frac{1}{\epsilon^2} (\nabla_p H_0, -\nabla_x H_0)(X_\epsilon, P_\epsilon)$

“H-Theorem”: $\|f_\epsilon(t)\|_{L^2(\mathbb{R}^{2N})}^2 + \frac{\gamma}{\epsilon^2} \|f_\epsilon - Pf_\epsilon\|_{L^2((0, \infty) \times \mathbb{R}^{2N})}^2 \leq \|f_0\|_{L^2(\mathbb{R}^{2N})}^2$

Write $f_\epsilon = Pf_\epsilon + \epsilon g_\epsilon$ where

$$\partial_t Pf_\epsilon = -P\{V_\epsilon, g_\epsilon\}$$

$$\partial_t g_\epsilon = -\frac{\gamma}{\epsilon^2}g_\epsilon - \frac{1}{\epsilon^2}\{H_0, g_\epsilon\} - \frac{1}{\epsilon^2}\{V_\epsilon, Pf_\epsilon\} - \frac{1}{\epsilon}(I - P)\{V_\epsilon, g_\epsilon\}$$

The remainder is **damped** for $\gamma > 0$

and the solution **relaxes** to $Pf(t, H_0(x, p))$.

Quasi-periodic Framework

Theorem. Suppose $V_\epsilon(t; x, p) = V(t, \Omega t/\epsilon^2; x, p)$ with Ω having rationally indep. components Then, $f_\epsilon = P f_\epsilon + \epsilon g_\epsilon$ where g_ϵ is bounded in $L^2((0, T) \times \mathbb{R}^{2D})$ and, up to a subsequence, $P f_\epsilon(t; x, p)$ converges to $F(t; H_0(x, p))$ in $C^0([0, T]; L^2(\mathbb{R}^{2D}) - weak)$, with

$$\partial_t(h_0 F) = \partial_E(h_0 d \partial_E F)$$

$$d(t; E) = \Pi \left(\int_{\Theta} \{ \mathcal{V}, H_0 \} \chi \, d\vartheta \right) (E) \geq 0,$$

$$\chi(t, \vartheta; x, p) = - \int_0^\infty e^{-\gamma s} \{ \mathcal{V}, H_0 \} (t, \vartheta - \Omega s; \mathcal{X}(-s; x, p), \mathcal{P}(-s; x, p)) \, ds,$$

Crucial Assumptions

- Damping $\gamma > 0$
- **Stability Property** (or increase γ)

$$\sup_{|(x,p)| \leq R} \left| \nabla_{x,p} (\mathcal{X}(t; x, p), \mathcal{P}(t; x, p)) \right| \leq C_R (1 + |t|)^{q_R}$$

A Simple Example

$$H_0(x, p) = \frac{x^2 + p^2}{2}, \quad V(t/\epsilon^2, x) = x \cos(\omega t/\epsilon^2)$$

$$\Pi f(E) = \frac{1}{2\pi} \int_0^{2\pi} f\left(\sqrt{2E} \cos(\sigma), \sqrt{2E} \sin(\sigma)\right) d\sigma$$

$$d(E) = \frac{\pi E}{2} \left(\frac{\gamma}{(\omega + 1)^2 + \gamma^2} + \frac{\gamma}{(\omega - 1)^2 + \gamma^2} \right)$$

If $\omega = \pm 1$, the coefficient blows up as $\gamma \rightarrow 0$: resonance phenomena

Random Framework

Theorem. Let $\gamma = 0$. Suppose that $\{H_0, f\} = 0$ iff $f(y) = F(H_0(y))$. Then, $\mathbb{E}P f_\epsilon \rightarrow F(t; H_0(y))$ in $C^0([0, T]; L^2(\mathbb{R}^{2D}) - weak)$, sol. of a diffusion eq. with

$$d(E) = \Pi \left(\int_0^T \mathcal{R}(\tau; \mathcal{Y}(\tau; y), y) : J\nabla H_0(\mathcal{Y}(\tau; y)) \otimes J\nabla H_0(y) d\tau \right) (E).$$

- Is it interesting ? Not so much!

$H_0(x, p) = (x^2 + p^2)/2$, works in 1D but fails for $D \geq 2$ (since $\{H_0, x \wedge p\} = 0$). Related to ergodicity of H_0 [Knauf 87, Donnay-Liverani 91]

- But we can deal with **vanishing** damping coefficients

$$\gamma_\epsilon \rightarrow \gamma_0 \geq 0, \quad \gamma_\epsilon > 0, \quad \epsilon^2/\gamma_\epsilon \xrightarrow{\epsilon \rightarrow 0} 0$$

- Relaxed Assumptions on H_0 : bounded derivatives of order 2, 3.

Comments

- More general oscillating potentials can be dealt with **KBM**= Krylov-Bogolioubov-Mitropolski type (long time average assumption)

Difficulty: the action of P on Sobolev spaces is unclear.

- Solutions of $\{H_0, f\} = 0: H_0, I_1, \dots, I_K$, then defines P to be a projection onto given involution quantities. This would lead a $K + 1$ -dimensional diffusion equation.
- Establish relation between the quantum and the classical models through Semi-Classical Limit.