HOMOGENIZATION OF TRANSPORT EQUATIONS; WEAK MEAN FIELD APPROXIMATION

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Abstract. We are interested in the behavior with respect to the small parameter ϵ of solutions ρ^ϵ of the conservative advection-diffusion equation $\partial_t \rho^\epsilon + \nabla_x \cdot (\rho^\epsilon u^\epsilon) = \eta \Delta_x \rho^\epsilon$, driven by a large velocity field: $|u^\epsilon| = \mathcal{O}(1/\epsilon)$ which oscillates periodically with respect to time and space variables. The novelty compared to previous works is that we deal with the periodic case in its full generality. In particular, the cell equation which allows to compute effective coefficients is parabolic and not elliptic. We also derive estimates on the homogenized solution via entropy methods.

Key words. Homogenization. Advection-diffusion equation. Entropy dissipation

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1. Introduction.

We consider the evolution of a scalar physical field, $\rho(t,x)$, submitted to convection by a velocity field $u(t,x)$ and molecular diffusion with a diffusion coefficient $\eta > 0$. We are led to the classical equation

$$\partial_t \rho + \mathrm{div}_x(\rho u) = \eta \Delta_x \rho. \tag{1.1}$$

For instance, ρ can be the mass density or the temperature in a fluid. We can also interpret (1.1) as the Fokker-Planck equation for the probability density $\rho(t,x)$ of particles whose trajectories obey the differential Langevin equation

$$\mathrm{d}X(t) = u(t, X(t))\,\mathrm{d}t + \sqrt{2\eta}\,\mathrm{d}W(t), \tag{1.2}$$

W being a Brownian motion. We are interested in the effects of microscopic dynamics at large scale: the given velocity field u is an oscillating quantity with a very fast period. The fluctuations of u can be either deterministic or random. Without being more precise for the time being, we suppose that the variation of u depends on a characteristic scale ϵ, small compared to the large (macroscopic) scale of observation of the scalar field ρ. Therefore we are interested in deriving the bulk properties of ρ from the microscopic behavior of u, a homogenization problem.

While the problem (1.1) is linear, determination of the evolution of the average quantity, $\langle\rho\rangle$, requires the knowledge of the correlation $\langle\rho u\rangle$. Indeed averaging (1.1) yields

$$\partial_t\langle\rho\rangle = \mathrm{div}_x\langle\rho u\rangle + \eta\Delta_x\langle\rho\rangle.$$

The "turbulent moment closure problem" consists of looking for a relation between this correlation function and the average quantity $\langle\rho\rangle$. Classically, a simple linear relation is postulated

$$\langle\rho u\rangle = \langle u\rangle\ \langle\rho\rangle + \alpha\langle\rho\rangle + \beta\nabla_x\langle\rho\rangle$$

involving effective tensors α, β. It yields the following average (effective) equation

$$\partial_t\langle\rho\rangle = \mathrm{div}_x\big((\langle u\rangle + \alpha)\langle\rho\rangle\big) + \mathrm{div}_x\Big((\eta+\beta)\nabla_x\langle\rho\rangle\Big)$$

with a modification of both convection and diffusion terms by the homogenization process. The aim of this work is to determine the effective coefficients α, β and to prove rigorously (in the mathematical sense) the convergence of average concentrations $\langle\rho\rangle$ towards solutions of the above effective equation.

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The question of homogenizing transport equations with highly oscillating coefficients has motivated a lot of works. Such a problem is related to the propagation of oscillations in fluid dynamics equations, see Di Perna-Majda [12]. Clearly limit processes depend strongly on the scaling used in the equation; some examples are provided in Mc Laughlin-Papanicolaou-Pironneau [29]. We also mention the very complete and deep presentation of these problems in the recent review paper of Kramer-Majda [24]. We also refer to the classical book of Bensoussan-Lions-Papanicolaou [6], or more recently Jikov-Kozlov-Oleinik [22], for a presentation of homogenization problems and of the classical mathematical methods used to solve it.

The modeling of the fast oscillating (turbulent) velocity field u can be done in two different ways. The first one consists of considering deterministic fields which are periodic with respect to some fast variables. The second one corresponds to random fields. In this paper we focus on the former case (concerning the latter, general situations are treated by Kesten-Papanicolaou [23]; the interested reader will find details and more references for instance in [24]). Let us give some of the known results in this field.

The periodic case can induce resonant phenomena which make the problem considerably more difficult when the molecular diffusivity η vanishes. When $\eta = 0$, the problem to determine the limit of the solutions of the transport equation when the periodicity vanishes is a challenging open question. It has been attacked only by considering simplified geometry of the characteristics associated to u. The first studies are concerned essentially with shear flows: oscillations of the velocity field hold in a direction transverse to the variable of derivation of the scalar field. The approach initiated by Tartar, [31, 32], has permitted to enlight memory effects induced by the homogenization procedure as in the papers of Mascarenhas [28], Amirat-Hamdache-Ziani [2, 3], Hamdache [19, 20]... These effects can also be interpreted as an increasing of the order of the equation. The second simplified geometry situation which has been studied is the case of divergence free field in 2 space dimensions. The field is then a curl of a potential and the characteristics are level sets of this potential. This situation has been investigated in the works by Brenier [7], Hou-Xin [21], E [13]. It cannot be generalized in higher dimension. Actually a common mistake is to assume a Fredholm alternative for transport equations with periodic conditions which does not hold in general.

When $\eta \neq 0$, the positive diffusivity changes completely the asymptotic regime. The case which has been the most studied is when the velocity field is divergence free. The divergence free case is much easier from the mathematical point of view. Indeed we have in this situation L^2 and H^1 estimates which are uniform with respect to u thanks to the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |\rho(t)|^2\,\mathrm{d}x + 2\eta\int |\nabla\rho(t)|^2\,\mathrm{d}x = 0.$$

The compactness properties induced by this relation help a lot to mathematically justify the homogenized limit (see for instance [6]). When dealing with divergence free velocity fields, the original diffusion is enhanced by the homogenization process. On the contrary, compressible velocity fields can give rise to a diffusion coefficient which can be depleted in the limit. From the physical point of view this role of the compressibility has been pointed out recently by Avellaneda-Vergassola [5]. It does not seem to exist mathematical results in this case and the present work is an attempt to fill this gap.

In this work the homogenization of advection-diffusion equations ($\eta \neq 0$) with periodic oscillations situations is investigated. The most general situation is considered. The velocity field depends on fast variables in time and space but depends also on the slow variables. The equation is written in conservative form. The scaling assumption leads to large fluctuations of the velocity field (order $1/\epsilon$, with ϵ the scale of space fluctuations of the velocity). It corresponds to the invariant scaling of the Navier-Stokes equation. Therefore this scaling is particularly relevant for the study of turbulent flows. In such a case, we cannot obtain compactness properties on the solutions ρ, and the only available estimate is in L^1. We use a method of oscillating test functions (in the spirit of Tartar [11, 31] or Evans [14, 15]).

Under the assumption that a certain mean value of the velocity field (the ballistic velocity) vanishes, we prove the convergence of the solutions to the solutions of a drift-diffusion equation. According to [5], the effective diffusion coefficient is positive, but it can be depleted compared to the original one. The concentration is factorized by a bulk concentration times a given fast oscillating function. We also obtain uniform estimates on the bulk component which allows to obtain the uniqueness of the limit.

We also give partial answers to the problem when the ballistic velocity does not vanish. Actually when the velocity does not depend on the macroscopic variables we give a complete description of the homogenized limit. Up to our knowledge this result is new even from a physical view point.

The two main mathematical difficulties in this work are the following:
- The first one is, when considering oscillating in time velocities, that the cell equation is parabolic and not elliptic as usual. However considering periodic boundary conditions, the corresponding operators are proved to satisfy the Fredholm alternative.
- The second one relies on the lack of regularity of the sequence of solutions. The only avaible estimate is in L^1. This is not sufficient to guarantee the uniqueness of the limit. To overcome this difficulty we obtain uniform estimates on the bulk component of the solutions by using an entropy method. It allows to obtain enough regularity on the homogenized limit to justify the uniqueness of this limit.

2. The periodic homogenization problem. In this Section we consider a passive scalar field

$$\rho^\epsilon : (t,x) \in [0,\infty) \times \mathbb{R}^N \longmapsto \rho^\epsilon(t,x) \in \mathbb{R}^+.$$

It represents a physical quantity whose evolution is driven by a fluid flow. The velocity field of the fluid is

$$u^\epsilon : (t,x) \in [0,\infty) \times \mathbb{R}^N \longmapsto u^\epsilon(t,x) \in \mathbb{R}^N$$

which is parametrized by $\epsilon > 0$. The parameter ϵ represents the length scale of oscillations of the velocity field u^ϵ. The scalar quantity is advected by the fluid and is also possibly subject to a diffusion process with a diffusion coefficient $\eta \geq 0$. Therefore, the scalar field ρ^ϵ is solution of the following advection-diffusion equation

$$\partial_t \rho^\epsilon(t,x) + \operatorname{div}_x\big(\rho^\epsilon u^\epsilon\big)(t,x) = \eta\ \Delta_x \rho^\epsilon(t,x), \quad \forall (t,x) \in (0,\infty) \times \mathbb{R}^N \tag{2.1}$$

$$\rho^\epsilon(0,x) = \rho_I^\epsilon(x), \quad \forall x \in \mathbb{R}^N. \tag{2.2}$$

The scalar field $\rho_I^\epsilon : \mathbb{R}^N \to \mathbb{R}^+$ is a given initial data.

We are interested in situations where the velocity field can be a model for turbulence. In order to obtain in the limit $\epsilon \to 0$ an effective turbulent diffusion coefficient due to the velocity field, the strength of $|u^\epsilon|$ should be large of order $1/\epsilon$. On the other hand, the velocity field gives rise to an effective drift term $\overline{u}$ which is the mean over fluctuations of u^ϵ. Therefore, the average value of u^ϵ should remain of order 1. Let us make more precise these considerations. We get u^ϵ of the form

$$u^\epsilon(t,x) = \frac{1}{\epsilon}\left(u^0(t,x;\frac{t}{\epsilon^2},\frac{x}{\epsilon}) + \epsilon\, u^1(t,x;\frac{t}{\epsilon^2},\frac{x}{\epsilon})\right). \tag{2.3}$$

The velocity fields

$$u^{0,1} : (t,x;\tau,y) \in [0,\infty) \times \mathbb{R}^N \times \mathbb{R}^{N+1} \longmapsto u^{0,1}(t,x;\tau,y) \in \mathbb{R}^N$$

are periodic with respect to the fast variables $\tau,\ y$:

$$\begin{aligned} &\forall (t,x;\tau,y) \in [0,\infty) \times \mathbb{R}^N \times \mathbb{R}^{N+1},\ \forall n = (n_0,n') \in \mathbb{Z} \times \mathbb{Z}^N, \\ &\qquad u^{0,1}(t,x;\tau+n_0,y+n') = u^{0,1}(t,x;\tau,y). \end{aligned} \tag{2.4}$$

Finally, setting $Y = (0,1)^{N+1}$, the situation is very different if u^0 has a vanishing or not vanishing mean value

$$\forall (t,x) \in [0,\infty) \times \mathbb{R}^N, \quad \int_Y u^0(t,x;\tau,y)\,\mathrm{d}\mu(\tau,y), \tag{2.5}$$

for a measure $\mathrm{d}\mu(\tau,y)$ which will be precised later on. This mean value is the so called ballistic velocity. Hence, in the vanishing ballistic velocity case, the quantity $\overline{u}(t,x) = \int_Y u^1(t,x;\tau,y)\,\mathrm{d}\mu(\tau,y)$ corresponds to the mean value of the velocity field u^ϵ.

REMARK 1. *The scaling we use in the fast variables : $u \to u^\epsilon = \frac{1}{\epsilon} u(\frac{t}{\epsilon^2}, \frac{x}{\epsilon})$, is the invariant scaling of the incompressible Navier-Stokes equation. (If u is a solution of the Navier Stokes equation, u^ϵ is also a solution of the same equation.) It shows the relevance of this scaling for studying turbulence in fluids.*

REMARK 2. *Actually, we are working with dimensionless equations, obtained from the original variables (t',x') by setting $t = t'/T$, $x = x'/L$, where T and L are characteristic values of time and length, respectively, under which the evolution of ρ is studied. We point out that the choice of T and L is free. Let us denote by ℓ the typical length of microscopic variations of the velocity field and by τ_0 its typical time scale of variation. The typical value for the mean velocity field is denoted by U_m and for the fluctuations by U_{fl}. We define the parameter ϵ as $\epsilon = \frac{U_m}{U_{fl}}$. It is natural to choose the scale T and L such that $U_m = \frac{L}{T}$. If we want that the fluctuations of the velocity field give rise to a diffusion process, U_{fl} should be scaled as a parabolic scaling. Then L and T are completely determined by the relation $U_{fl} = \frac{L^2}{T}\,\frac{1}{\ell}$. Consequently, we have $\epsilon = \frac{\ell}{L}$.*

Up to now there is no physical assumption. The regime we are studying in this work corresponds to the assumption $U_{fl} \gg U_m$, or in other words $\epsilon \ll 1$. It is also natural to assume that the path of a particle with velocity U_{fl} during the microscopic time period τ_0 is of order ℓ. It leads to

$$U_{fl}\,\tau_0 = \frac{L^2}{\ell}\,\frac{\tau_0}{T} = \ell, \quad \frac{\tau_0}{T} = \frac{\ell^2}{L^2} = \epsilon^2.$$

We have also to assume that the molecular diffusivity acts at the macroscopic scale L and T. If η' stands for the diffusion coefficient in the physical variables, it means that η' has the same order as $\frac{L^2}{T} = U_{fl}\ell$ while $\eta = \eta' \frac{T}{L^2}$. This physical situation corresponds to the so-called weak mean field approximation.

REMARK 3. *The case of a purely periodic and divergence free velocity field with null average*

$$u^\epsilon(x) = \frac{1}{\epsilon}\, u(x/\epsilon), \qquad \int_Y u(y)\,\mathrm{d}(y) = 0, \qquad \mathrm{div}_y(u) = 0$$

actually recasts in a more classical question of homogenization of parabolic equation. Indeed, in such a case, we can associate to u a skew-symmetric matrix B by solving $\Delta_y B = \nabla_y u$. Using Fourier expansion yields

$$B_{ij}(y) = \sum_{n\neq 0} e^{2i\pi n\cdot y}\,\frac{i}{|n|^2}\Big(n_i \widehat{u_j}(n) - n_j \widehat{u_i}(n)\Big).$$

We have $\mathrm{Div}_y(B) = u(y)$ so that

$$\frac{1}{\epsilon}\, u(x/\epsilon)\cdot\nabla_x\rho + \eta\Delta\rho = \nabla_x\cdot((\eta + B(x/\epsilon))\nabla_x\rho).$$

This remark is used for instance in [4]. Difficulties arise when $\int_Y u\,\mathrm{d}(y) \neq 0$. This kind of problem arises for instance in neutron transport theory; we refer to CAPDEBOSCQ *[8, 9] for a treatment of the spectral problem.*

3. Formal asymptotics. As usual, see [6], we try to guess the result with a formal double scale ansatz

$$\rho^\epsilon(t,x) = R^0(t,x;t/\epsilon^2,x/\epsilon) + \epsilon R^1(t,x;t/\epsilon^2,x/\epsilon) + \epsilon^2 R^2(t,x;t/\epsilon^2,x/\epsilon) + ...$$

where the $R^i(t,x;\tau,y)$'s are $Y-$periodic. The action of the operator

$$\mathcal{T}_\epsilon = \partial_t(\cdot) + \mathrm{div}_x\big(u^\epsilon\ \cdot\big) - \eta\Delta_x(\cdot)$$

on functions of the form $r^\epsilon(x) = R(t,x;t/\epsilon^2,x/\epsilon)$ reads as follows

$$\mathcal{T}_\epsilon(r^\epsilon)(t,x;\frac{t}{\epsilon^2},\frac{x}{\epsilon}) = \epsilon^{-2}\mathcal{T}_0(R)(t,x;\frac{t}{\epsilon^2},\frac{x}{\epsilon}) + \epsilon^{-1}\mathcal{T}_1(R)(t,x;\frac{t}{\epsilon^2},\frac{x}{\epsilon}) + \mathcal{T}_2(R)(t,x;\frac{t}{\epsilon^2},\frac{x}{\epsilon})$$

with

$$\begin{cases} \mathcal{T}_0(R)(t,x;\tau,y) = \Big(\partial_\tau R + \mathrm{div}_y\big(u^0 R\big) - \eta\Delta_y R\Big)(t,x;\tau,y), \\ \mathcal{T}_1(R)(t,x;\tau,y) = \Big(\mathrm{div}_y\big(u^1 R\big) + \mathrm{div}_x\big(u^0 R\big) - 2\eta\nabla_x\cdot\nabla_y R\Big)(t,x;\tau,y), \\ \mathcal{T}_2(R)(t,x;\tau,y) = \Big(\partial_t R + \mathrm{div}_x\big(u^1 R\big) - \eta\Delta_x R\Big)(t,x;\tau,y). \end{cases}$$

Plugging the double scale ansatz in the equation (2.1) and identifying the terms with the same power of ϵ yield

$$\begin{array}{ll} \epsilon^{-2}\text{term:} & \mathcal{T}_0 R^0 = 0, \\ \epsilon^{-1}\text{term:} & \mathcal{T}_0 R^1 = -\mathcal{T}_1 R^0, \\ \epsilon^{-0}\text{term:} & \mathcal{T}_0 R^2 = -\mathcal{T}_2 R^0 - \mathcal{T}_1 R^1 \end{array}$$

These relations can be generically written $\mathcal{T}_0 R^p = S^p$ where S^p depends only on R^0, ..., R^{p-1}. We also remark that the operator $\mathcal{T}_0$ is a differential operator in the variables τ, y, parametrized by t and x. Thus, we are dealing with cell equations with variable $(\tau,y)\in Y$ and periodic boundary conditions. At least formally, we aim at solving recursively these cell equations.

However, by integrating over Y, we realize that $\int_Y S\,\mathrm{d}(\tau,y) = 0$ is a necessary condition for the cell problem $\mathcal{T}_0 R = S$ to have a solution. We also note that the kernel of the adjoint operator $\mathcal{T}_0^\star = -\partial_\tau - u^0\cdot\nabla_y - \eta\Delta_y$ contains the constants. Actually it can be shown that $\mathcal{T}_0$ is a Fredholm operator of index 0. Hence, the condition $\int_Y S\,\mathrm{d}(\tau,y) = 0$ is also sufficient and we will solve the cell equations by means of the Fredholm alternative. At this point a difficulty should be pointed out. When $\eta = 0$ the cell problem is no more of Fredholm type. The ergodic property that the null space of $\mathcal{T}_0^\star$ is spanned by the constants (with respect to y) is not sufficient to guaranty that the problem $\mathcal{T}_0 R = S$ has a solution under the condition $\int_Y S\,\mathrm{d}(\tau,y) = 0$. These facts will be detailed in the next subsection concerning rigorous proofs.

Then, let us assume the following facts:

(A) For $(t,x)\in[0,\infty)\times\mathbb{R}^N$ fixed, 0 is a simple eigenvalue of $\mathcal{T}_0$ and the nullspace is spanned by a normalized function

$$\mathrm{Ker}(\mathcal{T}_0) = \mathrm{Span}\{\Theta\}, \qquad \int_Y \Theta\ \mathrm{d}(\tau,y) = 1.$$

(B) The cell problem $\mathcal{T}_0 R = S$ has a unique solution (up to elements of $\mathrm{Ker}(\mathcal{T}_0)$), under the necessary and sufficient condition $\int_Y S\,\mathrm{d}(\tau,y) = 0$.

From the ϵ^{-2} equation and (A), we infer that the leading term $R^0\in\mathrm{Ker}(\mathcal{T}_0)$ reads

$$R^0(t,x;\tau,y) = \rho(t,x)\ \Theta(t,x;\tau,y).$$

Next, the ϵ^{-1} equation becomes

$$\mathcal{T}_0 R^1 = -\mathcal{T}_1(R^0) = -\Big(\mathrm{div}_x(u^0\ \Theta) + \mathrm{div}_y(u^1\ \Theta) - 2\eta\nabla_x\cdot\nabla_y\Theta\Big)\rho$$
$$-\Big(\Theta\ u^0 - 2\eta\nabla_y\Theta\Big)\cdot\nabla_x\rho.$$

Let us define $\chi(t,x;\tau,y) = (\chi_1,...,\chi_N) \in \mathbb{R}^N$ as the solution (with zero mean) of the auxiliary cell problem

$$\mathcal{T}_0\chi_j = \Theta\ {u^0}_j - 2\eta\partial_{y_j}\Theta \tag{3.1}$$

and $\kappa(t,x;\tau,y)$ the solution of

$$\mathcal{T}_0\kappa = -\mathrm{div}_x(u^0\ \Theta) - \mathrm{div}_y(u^1\ \Theta) + 2\eta\nabla_x\cdot\nabla_y\Theta.$$

Remark that, in view of (B), these problems can be solved under the condition

$$\int_Y u^0\ \Theta\,\mathrm{d}(\tau,y) = 0. \tag{3.2}$$

Therefore, the measure mentioned in (2.5) is nothing but $\mathrm{d}\mu(\tau,y) = \Theta\,\mathrm{d}(\tau,y)$. This condition is referred to as the "vanishing ballistic velocity condition". From now on we assume this relation holds true. Thus, the ϵ^{-1} equation is solved by

$$R^1(t,x;\tau,y) = -\chi(t,x;\tau,y)\cdot\nabla_x\rho(t,x) + \kappa(t,x;\tau,y)\rho(t,x) + S^1(t,x)\Theta(t,x;\tau,y)$$

where S^1 is an arbitrary function of (t,x). We will see that the choice of S^1 is irrelevant to obtain the equation on R^0. As usual in two-scale asymptotics, the determination of S^1 is only necessary to obtain the equation for R^1. Now, let us look at the ϵ^0 term which provides the equation satisfied by ρ. The solvability condition yields

$$\int_Y \mathcal{T}_2 R^0\,\mathrm{d}(\tau,y) + \int_Y \mathcal{T}_1 R^1\,\mathrm{d}(\tau,y) = 0.$$

Besides, we have

$$\int_Y \mathcal{T}_2 R^0\,\mathrm{d}(\tau,y) = \partial_t\rho(t,x) + \mathrm{div}_x\Big(\int_Y u^1\ \Theta(t,x;\tau,y)\,\mathrm{d}(\tau,y)\ \rho(t,x)\Big) - \Delta_x\rho(t,x),$$
$$\int_Y \mathcal{T}_1 R^1\,\mathrm{d}(\tau,y) = \mathrm{div}_x\left(\int_Y u^0 R^1\,\mathrm{d}(\tau,y)\right)$$
$$= -\ \mathrm{div}_x\left(\int_Y u^0\otimes\chi(t,x;\tau,y)\,\mathrm{d}(\tau,y)\ \cdot\nabla_x\rho(t,x)\right)$$
$$+\ \mathrm{div}_x\left(\int_Y u^0\kappa(t,x;\tau,y)\,\mathrm{d}(\tau,y)\,\rho(t,x)\right).$$

Let us define the turbulent diffusion matrix D by

$$D(t,x) = \int_Y u^0(t,x;\tau,y)\otimes\chi(t,x;\tau,y)\,\mathrm{d}(\tau,y) \tag{3.3}$$

and the effective drift term by

$$v(t,x) = \int_Y \big(u^1\ \Theta(t,x;\tau,y) + u^0\ \kappa(t,x;\tau,y)\big)\,\mathrm{d}(\tau,y). \tag{3.4}$$

We are finally led to the following effective equation for $\rho(t,x) = \lim_{\epsilon\to 0} \rho^\epsilon(t,x)$

$$\partial_t \rho(t,x) + \mathrm{div}_x\big(\rho\ v\big)(t,x) - \mathrm{div}_x\left(\left(D + \eta I_N\right) \cdot \nabla_x \rho\right)(t,x) = 0.$$

The matrix I_N is the $N \times N$ identity matrix. Therefore, the homogenization procedure induces an effective diffusion which is the sum of the original diffusion ηI with the matrix D.

REMARK 4. *We might wonder that the effects of the turbulent velocity field is to increase the molecular diffusivity and the (symmetric part of the) matrix D is non negative. This is not true in general. Actually, considering potential flow, we will see that D is negative. This fact has been already noticed by* AVELLANEDA-VERGASSOLA *[5]. However, we will check that $\eta I_N + D$ remains non negative.*

Also the above asymptotic is based on the vanishing ballistic velocity assumption (3.2). The problem of what happens when (3.2) does not hold is addressed in Subsection 4.5.

4. Rigorous results.

4.1. Functional preliminaries.
First, let us introduce some functional spaces.

Let $\Omega \subset \mathbb{R}^D$, and $p,\ q \in \mathbb{N}$. The set $C^p_\#(\Omega \times \mathbb{R}^{N+1})$ is the set of p-times continuously differentiable functions on $\Omega \times \mathbb{R}^{N+1}$ which are $Y-$periodic with respect to the last variable. Similarly, $C^p_{b,\#}(\Omega \times \mathbb{R}^{N+1})$ and $C^p_{c,\#}(\Omega \times \mathbb{R}^{N+1})$ are the subspaces of functions having bounded derivatives up to order p, and being supported in $K \times \mathbb{R}^{N+1}$ for some compact subset K of Ω, respectively. We also define anisotropic spaces $C^{[p;q,\alpha]}_\#([0,\infty) \times \mathbb{R}^N \times \mathbb{R}^{N+1})$ to be the spaces of continuous functions $u = u(t,x,\tau,y)$ such that

- $(\partial_t)^r(\partial_x)^s u$ exist and are continuous for $2r + s \le p$ where $(\partial_x)^s$ denotes any partial derivative of order s with respect to $x \in \mathbb{R}^N$, and $(\partial_t)^r$ stands for r^{th} derivative with respect to $t \ge 0$,
- $(\partial_\tau)^r(\partial_y)^s u$ exist and are continuous for $2r + s \le q$ where $(\partial_y)^s$ denotes any partial derivative of order s with respect to $y \in \mathbb{R}^N$, and $(\partial_\tau)^r$ stands for r^{th} derivative with respect to $\tau \in \mathbb{R}$,
- $(\partial_\tau)^r(\partial_y)^s u$ for $2r + s = q$ are Hölder continuous with exponent α with respect to y and with exponent $\alpha/2$ with respect to τ.

Finally, we also need $C^{[p;q,\alpha]}_{c,\#}([0,\infty) \times \Omega \times \mathbb{R}^N)$, the subspaces of functions with compact support with respect to the first two variables.

The formal limit obtained in the previous Section can be rigorously justified by using the double scale techniques as developed by N'GUETSENG [30] and ALLAIRE [1]. This technique is quite equivalent to the method of oscillating test functions of TARTAR [11, 31] and EVANS [14, 15] and can be seen as a systematic way (but restricted to periodic situations) to choose the "good" test functions. In the present problem there is no a priori L^2-estimates, except if the u^i's are divergence free. Then, the natural framework is a L^1-setting. We will adapt the results of [1, 30] to this setting. Then, let us introduce some definitions and basic results about family of parametrized measures.

DEFINITION 4.1. *Let I be an interval of $\mathbb{R}$. A family $\{\mu(t);\ t \in I\}$ of Radon measures on $\mathbb{R}^N$ is said to be vaguely continuous if and only if*

$$\forall \varphi \in C^0_c(\mathbb{R}^N), \qquad t \longmapsto \int_{\mathbb{R}^N} \varphi(x)\ \mu(t,x)\,\mathrm{d}x \ \ \textit{is a continuous function on } I.$$

DEFINITION 4.2. *A sequence $\{\mu_n(t);\ t \in I,\ n \in \mathbb{N}\}$ is said to be equibounded and vaguely equicontinuous on I if and only if*

i) There exists $M > 0$ such that

$$\sup_{t\in I,\, n\in\mathbb{N}} |\mu_n|(t,\mathbb{R}^N) \le M,$$

ii) For any $\varphi \in C^0_c(\mathbb{R}^N)$, the sequence of functions $\left(t \longmapsto \int_{\mathbb{R}^N} \varphi(x)\ \mu_n(t,x)\,\mathrm{d}x\right)_{n\in\mathbb{N}}$ is equicontinuous on I.

We recall the following classical compactness result.

PROPOSITION 4.3. *Let I be an interval of $\mathbb{R}$. Let $\big(\mu_n(t)\big)_{n\in\mathbb{N}}$ be a sequence of Radon measures on $\mathbb{R}^N$, equibounded and vaguely equicontinuous on I. Then, there exists a measure $\mu(t)$ vaguely continuous on I and a subsequence $\big(\mu_{n_k}(t)\big)_{k\in\mathbb{N}}$ such that*

$$\forall\varphi\in C_c^0(I\times\mathbb{R}^N),\qquad \int_{\mathbb{R}^N}\varphi(t,x)\ \mu_{n_k}(t,x)\,\mathrm{d}x \xrightarrow[k\to\infty]{} \int_{\mathbb{R}^N}\varphi(t,x)\ \mu(t,x)\,\mathrm{d}x,$$

uniformly with respect to $t\in I$. We say that sequence $\big(\mu_{n_k}(t)\big)_{k\in\mathbb{N}}$ converges vaguely to $\mu(t)$ locally, uniformly on I.

Then, we also have the existence of a double scale limit, in the spirit of ALLAIRE [1], N'GUETSENG [30].

PROPOSITION 4.4. *Let $\big(\epsilon_n\big)_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to 0. Let $\big(\mu_n(t)\big)_{n\in\mathbb{N}}$ be a sequence of measures on $\mathbb{R}^N$, equibounded on an interval $I\subset\mathbb{R}$. Then, there exist a subsequence $\big(\mu_{n_k}(t)\big)_{k\in\mathbb{N}}$ and a measure M on $I\times\mathbb{R}\times\mathbb{R}^N\times Y$ such that for any $\varphi\in C^0_{c,\#}(I\times\mathbb{R}^N\times\mathbb{R}^{N+1})$, we have*

$$\begin{aligned}\int_I\int_{\mathbb{R}^N}&\varphi(t,x;t/\epsilon_{n_k}^2,x/\epsilon_{n_k})\ \mu_{n_k}(t,x)\,\mathrm{d}x\,\mathrm{d}t\\ &\xrightarrow[k\to\infty]{}\int_I\int_{\mathbb{R}^N}\int_Y\varphi(t,x;\tau,y)\ M(t,x;\tau,y)\,\mathrm{d}(\tau,y)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We say that the measure M is the double scale limit of the sequence $\big(\mu_{n_k}(t)\big)_{k\in\mathbb{N}}$.

This proposition is nothing but a consequence of the Banach-Alaoglu theorem applied to the sequence of measures M_n defined by

$$\int_I\int_{\mathbb{R}^N}\int_Y\varphi(t,x;\tau,y)\ M_n(t,x;\tau,y)\,\mathrm{d}(\tau,y)\,\mathrm{d}x\,\mathrm{d}t := \int_I\int_{\mathbb{R}^N}\varphi(t,x;t/\epsilon_n^2,x/\epsilon_n)\ \mu_n(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

The double scale limit captures the periodic oscillations of μ_n with frequency $1/\epsilon_n$ in x and $1/\epsilon_n^2$ in t. By using test functions of the form $\varphi(t,x;\tau,y)=\psi(t,x)$, we also obtain immediately that the vague limit $\mu(t)$ of $\mu_{n_k}(t)\,\mathrm{d}t$ is given by the marginal

$$\mu(t,x)=\int_Y M(t,x;\tau,y)\,\mathrm{d}(\tau,y)\in\mathcal{M}^1(I\times\mathbb{R}^N).$$

4.2. Cell problems. Next, we are concerned with the cell problems. In particular we aim at discussing properties (A) and (B) stated in the previous section.

PROPOSITION 4.5. *Let $u^0\in C_\#^{[2;1,\alpha]}([0,\infty)\times\mathbb{R}^N\times\mathbb{R}^{N+1})$. Then, the following assertions hold:*

i) There exists a unique function $\Theta\in C_\#^{[2;2,\alpha]}([0,\infty)\times\mathbb{R}^N\times\mathbb{R}^{N+1})$ such that

$$\left\{\begin{array}{l}\mathcal{T}_0(\Theta)=\Big(\partial_\tau\Theta+\mathrm{div}_y(u^0\ \Theta)-\eta\ \Delta_y\Theta\Big)(t,x;\tau,y)=0,\\ \qquad\forall(t,x;\tau,y)\in[0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N,\\ \displaystyle\int_Y\Theta(t,x;\tau,y)\,\mathrm{d}(\tau,y)=1,\qquad\forall(t,x)\in[0,\infty)\times\mathbb{R}^N.\end{array}\right.$$

Furthermore, we have $\Theta(t,x;\tau,y)>0$ for all $(t,x;\tau,y)$.

ii) For any $S\in C_\#^{[p;0,\alpha]}([0,\infty)\times\mathbb{R}^N\times\mathbb{R}^{N+1})$ with $p=0,1,2$ the cell problem

$$\mathcal{T}_0(R)=\partial_\tau R+\mathrm{div}_y(u^0R)-\eta\Delta_yR=S,\qquad\int_Y R\,\mathrm{d}(\tau,y)=0,$$

has a unique solution $R \in C^{[p;2,\alpha]}_{\#}([0,\infty) \times \mathbb{R}^N \times \mathbb{R}^{N+1})$ *under the necessary and sufficient orthogonality condition*

$$\int_Y S(t,x;\tau,y)\,\mathrm{d}(\tau,y) = 0, \quad \forall (t,x) \in [0,\infty) \times \mathbb{R}^N. \tag{4.1}$$

iii) For any $H \in C^{[p;0,\alpha]}_{\#}([0,\infty) \times \mathbb{R}^N \times \mathbb{R}^{N+1})$ *with* $p = 0, 1, 2$ *the cell problem*

$$-\partial_\tau \Phi - u^0 \cdot \nabla_y \Phi - \eta \Delta_y \Phi = H, \quad \int_Y \Phi\,\mathrm{d}(\tau,y) = 0,$$

has a unique solution $\Phi \in C^{[p;2,\alpha]}_{\#}([0,\infty) \times \mathbb{R}^N \times \mathbb{R}^{N+1})$ *under the necessary and sufficient orthogonality condition*

$$\int_Y H(t,x;\tau,y)\ \Theta(t,x;\tau,y)\,\mathrm{d}(\tau,y) = 0, \quad \forall (t,x) \in [0,\infty) \times \mathbb{R}^N. \tag{4.2}$$

Proof. Let $(t,x) \in [0,\infty) \times \mathbb{R}^N$ be fixed. We denote by

$$\mathcal{T}_0^\star(\Phi) = -\partial_\tau \Phi - u^0 \cdot \nabla_y \Phi - \eta \Delta_y \Phi$$

the adjoint operator of $\mathcal{T}_0$. Obviously, constants belong to the kernel of $\mathcal{T}_0^\star$. For $\lambda > 0$ large enough we shall see that the resolvant $R_\lambda = (\lambda + \mathcal{T}_0^\star)^{-1}$ (resp. $S_\lambda = (\lambda + \mathcal{T}_0)^{-1}$) is well defined and we rewrite the problems $\mathcal{T}_0^\star(\Phi) = H$ (resp. $\mathcal{T}_0(R) = S$) as

$$(I - \lambda R_\lambda)\Phi = R_\lambda H \tag{4.3}$$

(resp. $(I - \lambda S_\lambda)\Phi = S_\lambda H$). We wish to conclude by applying the Fredholm alternative. We are thus led to investigate compactness and spectral properties of the resolvant operator R_λ (resp. S_λ).

Hence, we are interested in the problem

$$(\mathcal{T}_0^\star + \lambda)\Phi = H \in L^2_{\#}(Y) \tag{4.4}$$

for some $\lambda > 0$ and domain $D(\mathcal{T}_0^\star)$ which will be made precise later on. Assuming existence in $H^1_{\#}(Y)$, uniqueness of the solution follows from the positivity of the energy functional

$$\begin{aligned} a_0(\Phi,\Phi) &= \int_Y (\mathcal{T}_0^\star + \lambda)\Phi\ \Phi\,\mathrm{d}(\tau,y) \\ &= \int_Y (\lambda + \frac{1}{2}\mathrm{div}_y u^0)\Phi^2\,\mathrm{d}(\tau,y) + \eta \int_Y |\nabla_y \Phi|^2\,\mathrm{d}(\tau,y), \end{aligned}$$

provided λ satisfies $\lambda > \max_{(\tau,y)\in Y} \max(0, -\frac{1}{2}\mathrm{div}_y u^0)$. From now on, we fix λ which fulfills this relation. It remains to justify the existence of a solution of (4.4).

To this end, we introduce an elliptic regularization of $\mathcal{T}_0^\star$. Let ${\mathcal{T}_0^\star}_\mu$, $\mu > 0$, be the operator

$${\mathcal{T}_0^\star}_\mu(\Phi) = -\mu \partial^2_{\tau,\tau}\Phi - \partial_\tau \Phi - u^0 \cdot \nabla_y \Phi - \eta \Delta \Phi$$

with domain $D({\mathcal{T}_0^\star}_\mu) = H^2_{\#}(Y)$, the space of functions Y periodic, belonging to $H^2_{\mathrm{loc}}(\mathbb{R}^{N+1})$. For $\lambda > \frac{1}{2}\max_{\tau,y}\{0, -\mathrm{div}_y u_0\}$, the associated energy functional

$$\begin{aligned} a_\mu(\Phi,\Phi) &= \int_Y \Phi({\mathcal{T}_0^\star}_\mu(\Phi) + \lambda\Phi)\,\mathrm{d}(\tau,y) \\ &= \mu \int_Y |\partial_\tau \Phi|^2\,\mathrm{d}(\tau,y) + \int_Y (\lambda + \frac{1}{2}\mathrm{div}_y(u^0))\ \Phi^2\,\mathrm{d}(\tau,y) + \eta \int_Y |\nabla_y \Phi|^2\,\mathrm{d}(\tau,y) \end{aligned}$$

is coercive on the space $H^1_\#(Y)$. A direct application of the Lax-Milgram theorem guarantees, for any data $H \in L^2_\#(Y)$, the existence-uniqueness of a solution $\Phi \in H^1_\#(Y)$ of the cell problem ${\mathcal{T}_0^\star}_\mu(\Phi_\mu) + \lambda\Phi_\mu = H$ in $H^1_\#(Y)$. Then $\mu\partial^2_{\tau,\tau}\Phi + \eta\Delta\Phi \in L^2_\#(Y)$ which implies that Φ_μ belongs to $H^2_\#(Y) = D({\mathcal{T}_0^\star}_\mu)$. We also have $\Phi_\mu \geq 0$ when $H \geq 0$ by the maximum principle. Then, (4.4) will be solved by passing to the limit $\mu \to 0$.

By using the relation $a_\mu(\Phi_\mu, \Phi_\mu) = \int_Y H\ \Phi_\mu \,\mathrm{d}(\tau, y)$, we obtain that Φ_μ and $\nabla_y\Phi_\mu$ are uniformly bounded in $L^2_\#(Y)$. Multiplying the equation ${\mathcal{T}_0^\star}_\mu(\Phi_\mu) + \lambda\Phi_\mu = H$ by $\partial_\tau\Phi_\mu$ and integrating by parts give

$$\int_Y |\partial_\tau\Phi_\mu|^2 \,\mathrm{d}(\tau, y) = -\int_Y u_0 \cdot \nabla_y\Phi_\mu\partial_\tau\Phi_\mu \,\mathrm{d}(\tau, y) - \int_Y H\ \partial_\tau\Phi_\mu \,\mathrm{d}(\tau, y).$$

We deduce that $\partial_\tau\Phi_\mu$ is also uniformly bounded in $L^2_\#(Y)$. Hence, Φ_μ is bounded in $H^1_\#(Y)$. The cluster points of the sequence Φ_μ as $\mu \to 0$ are solutions in $H^1_\#(Y)$ of (4.4). Such a solution satisfies $\Delta_y\Phi \in L^2_\#(Y)$. Defining $D(\mathcal{T}_0^\star) = \{\Phi \in H^1_\#(Y),\, \Delta_y\Phi \in L^2_\#(Y)\}$, we have obtained the existence of a compact resolvant $R_\lambda = (\mathcal{T}_0^\star + \lambda)^{-1}$.

Then, in order to apply the Fredholm alternative for (4.3), it remains to determine the compatibility relation, and in particular the dimension of the eigenspace $\mathrm{Ker}(\mathcal{T}_0^\star) = \mathrm{Ker}(R_\lambda - 1/\lambda))$ (resp. $\mathrm{Ker}(\mathcal{T}_0) = \mathrm{Ker}(S_\lambda - 1/\lambda)))$. We use the fact that R_λ preserves non negativity: for $H \geq 0$, $\Phi = R_\lambda(H) \geq 0$. The Krein-Rutman theorem (see [25]) applies and the spectral radius ρ of R_λ is an eigenvalue, associated to a non negative eigenfunction. Reasoning similarly for the adjoint operator, we obtain the existence of a function $\Theta \in H^1_\#(Y)$, $\Theta \geq 0$, verifying $S_\lambda\Theta = \rho\Theta$. However, we have remarked that $(1/\lambda, \mathbb{1})$ is an eigenpair for R_λ. Hence, we get

$$\rho\int_Y \Theta \,\mathrm{d}(\tau, y) = \int_Y \mathbb{1}\ S_\lambda(\Theta) \,\mathrm{d}(\tau, y) = \int_Y R_\lambda(\mathbb{1})\ \Theta \,\mathrm{d}(\tau, y) = \frac{1}{\lambda}\int_Y \Theta \,\mathrm{d}(\tau, y),$$

i.e. $\rho = 1/\lambda$ is the principal eigenvalue of R_λ, S_λ and Θ satisfies $\mathcal{T}_0(\Theta) = 0$.

Then, we make use of regularity results and maximum principle for parabolic equations. We first remark that the functions of $D(\mathcal{T}_0) = D(\mathcal{T}_0^\star)$ are continuous with respect to time τ with value in $L^2_\#([0,1]^N)$ and their first derivatives with respect to y belong to $L^2_\#(Y)$. With the notation of [27], such functions belong to $V_2^{1,0}$. Therefore, since the coefficients are smooth we can apply Theorem 12.1 p. 223 of [27]. We deduce that $R_\lambda(H) = \Phi \in C^{[2,\alpha]}_\#(Y)$ ($H^{2+\alpha,1+\alpha/2}(Y)$ in the terminology of [27]) for $H \in C^{[0,\alpha]}_\#(Y)$. Similarly, we have $\Theta \in C^{[2,\alpha]}_\#(Y)$. Thus, the maximum principle applies (see [16], Th. 5, p. 39) and we obtain $\Theta > 0$, and $R_\lambda(H) = \Phi > 0$ for $H \geq 0$. Suppose now there exists a $\Phi \in \mathrm{Ker}(\mathcal{T}_0^\star)$ with $\int_Y \Phi \,\mathrm{d}(\tau, y) = 0$. Then, Φ is also in $C^{[2,\alpha]}_\#(Y)$ and the positive and negative parts $\Phi_\pm$ are continuous functions non identically zero. By the maximum principle $R_\lambda\Phi_\pm > 0$. We deduce that $|\Phi| < \lambda R_\lambda(|\Phi|)$. It would imply $\int_Y \Theta|\Phi| \,\mathrm{d}(\tau, y) = \lambda\int_Y \Theta R_\lambda(|\Phi|) \,\mathrm{d}(\tau, y) > \int_Y \Theta|\Phi| \,\mathrm{d}(\tau, y)$, a contradiction. Thus, the kernel of $\mathcal{T}_0^\star$ (resp. $\mathcal{T}_0$) is mono-dimensional and spanned by $\mathbb{1}$ (resp. $\Theta > 0$).

We conclude by applying the Fredholm alternative. The solution of $\mathcal{T}_0(R) = S$ exists if and only if S is orthogonal to the kernel of the adjoint that is $\int_Y S \,\mathrm{d}(\tau, y) = 0$. The solution is unique if we impose $\int_Y R\ \mathrm{d}(\tau, y) = 0$. The same is true for $\mathcal{T}_0^\star$: the equation $\mathcal{T}_0^\star(\Phi) = S$ has a solution if and only if $\int_Y S\ \Theta\ \mathrm{d}(\tau, y) = 0$ and the uniqueness is assured by the condition $\int_Y R\ \mathrm{d}(\tau, y) = 0$. By applying Theorem 12.1 p. 223 of [27], we obtain that Θ, R and Φ belong to $C^{[2,\alpha]}_\#(Y)$, if S, $H \in C^{[0,\alpha]}_\#(Y)$ for fixed (t, x).

We study now the dependency with respect to these parameters (t, x). Let δ_h be a finite difference operator in the variables (t, x) we have

$$\mathcal{T}_0^\star(\delta_h(\Phi)) = -\delta_h(u^0) \cdot \nabla_y(\Phi) + \delta_h(H).$$

The right hand side is bounded in $C_{\#}^{[0,\alpha]}(Y)$ uniformly with respect to h. Then the sequence $\big(\delta_h(\Phi)\big)_{h>0}$ is bounded in $C_{\#}^{2,\alpha}(\mathbb{R}^N)$ uniformly with respect h. It implies that $(t,x)\longmapsto \Phi(t,x,.)$ is differentiable with respect to (t,x) with values in $C_{\#}^{[2,\alpha]}(Y)$. In particular $\Phi\in C^0\Big([0,\infty)\times\mathbb{R}^N; C_{\#}^{[2,\alpha]}(Y)\Big)$. If $\partial\Phi$ denotes a derivative with respect to t or x, we have, letting $h\to 0$,

$$\mathcal{T}_0^\star(\partial\Phi) = -\partial u^0\cdot\nabla_y(\Phi)+\partial H.$$

Since, the right hand side is continuous with respect to the parameter (t,x), it yields the continuity of $\partial\Phi$. We obtain the same result for the derivatives up to order 2 by repeating the argument once. The regularity of the solution R and of the eigenfunction Θ are obtained in the same way. □

REMARK 5. *Of course, when the velocity field u^0 does not oscillate with respect to time (i.e. $u^{0,\epsilon} = u^0(t,x;x/\epsilon)$), the corresponding solution of the cell problem Θ does not depend on the fast time variable τ.*

4.3. Effective coefficients. Our proof of the homogenization result is based on the expansion of the dual equation. Therefore, we obtain the diffusion matrix and the drift velocity, given by (3.3) and (3.4) respectively, by means of dual formulae. Let us introduce $\chi^\star$, solution of the dual problem of (3.1), namely,

$$\mathcal{T}_0^\star(\chi^\star) = -\partial_\tau\chi^\star - u^0\cdot\nabla_y\chi^\star - \eta\Delta_y\chi^\star = u^0, \qquad \int_Y \chi^\star\,\mathrm{d}(\tau,y)=0. \tag{4.5}$$

We can express D and v with $\chi^\star$ as follows. On the one hand, we have

$$\begin{aligned} D &= \int_Y u^0\otimes\chi\,\mathrm{d}(\tau,y) = \int_Y \mathcal{T}_0^\star(\chi^\star)\otimes\chi\,\mathrm{d}(\tau,y)\\ &= \int_Y \chi^\star\otimes\mathcal{T}_0(\chi)\,\mathrm{d}(\tau,y) = \int_Y \chi^\star\otimes(\Theta u^0 - 2\eta\nabla_y\Theta)\,\mathrm{d}(\tau,y)\end{aligned} \tag{4.6}$$

and on the other hand

$$\begin{aligned} v-\int_Y u^1\Theta\,\mathrm{d}(\tau,y) &= \int_Y u^0\ \kappa\,\mathrm{d}(\tau,y) = \int_Y \mathcal{T}_0^\star(\chi^\star)\ \kappa\,\mathrm{d}(\tau,y) = \int_Y \chi^\star\mathcal{T}_0(\kappa)\,\mathrm{d}(\tau,y)\\ &= \int_Y \chi^\star\Big(-\operatorname{div}_x(\Theta u^0) - \operatorname{div}_y(\Theta u_1) + 2\eta\nabla_x\cdot\nabla_y\Theta\Big)\,\mathrm{d}(\tau,y).\end{aligned} \tag{4.7}$$

Of course, the crucial question relies on the positiveness of the diffusion coefficient. Actually, we shall see that (the symmetric part of) D can be non positive, while the sum $\eta + D$ remains non negative. To this end, it is particularly illuminating to consider the following fundamental examples.

The divergence free case.
Suppose that $\operatorname{div}_y(u^0)=0$.
In this case, $\Theta=1$ and the vanishing ballistic velocity condition is nothing but

$$\int_Y u^0\,\mathrm{d}(\tau,y)=0.$$

The function χ satisfies

$$\mathcal{T}_0(\chi) = u^0\Theta = u^0.$$

Then, for $\xi\in\mathbb{R}^N\backslash\{0\}$, we have

$$\begin{aligned} D\xi\cdot\xi &= \int_Y u^0\cdot\xi\ \chi\cdot\xi\,\mathrm{d}(\tau,y) = \int_Y \mathcal{T}_0(\chi\cdot\xi)\ \chi\cdot\xi\ \ \mathrm{d}(\tau,y)\\ &= \eta\int_Y |\nabla_y(\chi\cdot\xi)|^2\ \ \mathrm{d}(\tau,y)\geq 0.\end{aligned}$$

The potential case.
Suppose that u_0 does not depend on the variable τ and is of the form $u^0(t,x;y) = \nabla_y V(t,x;y)$, for some potential function $V : \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $[0,1]^N-$periodic with respect to the last variable. Then, the operator $\mathcal{T}_0$ recasts in the following symmetric form

$$\mathcal{T}_0(R) = \partial_\tau R - \eta \ \mathrm{div}_y\Big(e^{V/\eta}\ \nabla_y(e^{-V/\eta}\ R)\Big)$$

so that

$$\int_Y \mathcal{T}_0(R)R\ e^{-V/\eta}\,\mathrm{d}(\tau,y) = \eta \int_Y e^{V/\eta}\ |\nabla_y(Re^{-V/\eta})|^2\,\mathrm{d}(\tau,y).$$

We deduce that

$$\mathrm{Ker}(\mathcal{T}_0) = \mathrm{Span}\{e^{V/\eta}\}, \qquad \Theta = Ze^{V/\eta},$$

$Z = (\int_Y e^{V/\eta}\,\mathrm{d}(\tau,y))^{-1}$ being a normalization constant. Notice that the vanishing ballistic velocity condition is fulfilled since

$$\int_Y u^0\ \Theta\ \ \mathrm{d}(\tau,y) = \eta Z \int_Y \nabla_y(e^{V/\eta})\ \ \mathrm{d}(\tau,y) = 0.$$

Finally, the equation for χ reads

$$\mathcal{T}_0(\chi) = u^0\Theta - 2\eta\nabla_y\Theta = -Z\nabla_y V\ e^{V/\eta} = -u^0\Theta.$$

(Notice the change of sign in the right hand side in comparison to the divergence free case.) Then, for $\xi \in \mathbb{R}^N\backslash\{0\}$, we have

$$\begin{aligned} D\xi\cdot\xi &= \int_Y u^0\cdot\xi\ \chi\cdot\xi\ \ \mathrm{d}(\tau,y) = -\int_Y (-u^0\Theta\cdot\xi)\ \chi\cdot\xi\ \Theta^{-1}\ \ \mathrm{d}(\tau,y) \\ &= -\int_Y \mathcal{T}_0(\chi\cdot\xi)\ \chi\cdot\xi\ \Theta^{-1}\,\mathrm{d}(\tau,y) = -\eta\int_Y \Theta\ |\nabla_y(\chi\cdot\xi\Theta^{-1})|^2\,\mathrm{d}(\tau,y) \le 0, \end{aligned}$$

which reveals an anti-diffusive effect. However, Avellaneda-Vergassola [5] pointed out that, in this potential case, the total diffusivity remains non negative $(\eta + D)\xi\cdot\xi \ge 0$. Their argument can be generalized, as shown by the following statement.

Proposition 4.6. *Let D the matrix defined in (3.3). Then, for any $\xi \in \mathbb{R}^N\backslash\{0\}$, one has*

$$(\eta + D)\xi\cdot\xi > 0.$$

Proof. The proof starts with the a dissipativity property of $\mathcal{T}_0^\star$, using a inner product with weight Θ. It was pointed out to us by J.F. Collet [10] that this property is general and applies for any elliptic operator of order two and more generally for any operator coming from Markovian processes (see Kubo

[26]). We have

$$\begin{aligned}\int_Y \mathcal{T}_0^\star(\Phi)\ \Phi\ \Theta\,\mathrm{d}(\tau,y) &= \int_Y (-\partial_\tau\Phi - u^0\cdot\nabla_y\Phi - \eta\Delta_y\Phi)\Phi\ \Theta\,\mathrm{d}(\tau,y)\\ &= -\int_Y \Big(\partial_\tau\Big(\frac{\Phi^2}{2}\Big) + u^0\cdot\nabla_y\Big(\frac{\Phi^2}{2}\Big)\Big)\ \Theta\ \ \mathrm{d}(\tau,y)\\ &\quad +\eta\int_Y \nabla_y\Big(\frac{\Phi^2}{2}\Big)\cdot\nabla_y\Theta\ \ \mathrm{d}(\tau,y) + \eta\int_Y |\nabla_y\Phi|^2\ \Theta\ \ \mathrm{d}(\tau,y)\\ &= \int_Y \frac{\Phi^2}{2}\ \mathcal{T}_0\Theta\,\mathrm{d}(\tau,y) + \eta\int_Y |\nabla_y\Phi|^2\ \Theta\,\mathrm{d}(\tau,y)\\ &= \eta\int_Y |\nabla_y\Phi|^2\ \Theta\,\mathrm{d}(\tau,y) \geq 0.\end{aligned}$$

Then, let us use the expression of D obtained in (4.6). We get

$$\begin{aligned}D\xi\cdot\xi &= \int_Y \chi^\star\cdot\xi\ u^0\cdot\xi\ \Theta\ \ \mathrm{d}(\tau,y) - 2\eta\int_Y \chi^\star\cdot\xi\ \nabla_y\Theta\cdot\xi\ \ \mathrm{d}(\tau,y)\\ &= \int_Y \chi^\star\cdot\xi\ \mathcal{T}_0^\star(\chi^\star\cdot\xi)\ \Theta\ \ \mathrm{d}(\tau,y) + 2\eta\int_Y \Theta\ \xi\cdot\nabla_y\big(\chi^\star\cdot\xi\big)\,\mathrm{d}(\tau,y)\\ &= \eta\int_Y |\nabla_y(\chi^\star\cdot\xi)|^2\ \Theta\,\mathrm{d}(\tau,y) + 2\eta\int_Y \Theta\ \xi\cdot\nabla_y\big(\chi^\star\cdot\xi\big)\,\mathrm{d}(\tau,y).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}(\eta + D)\xi\cdot\xi &= \eta\int_Y |\xi + \nabla_y(\chi^\star\cdot\xi)|^2\Theta\,\mathrm{d}(\tau,y) \geq 0\\ &= \eta\Big(\int_Y (I + \nabla_y\chi^\star)\ (I + \nabla_y\chi^\star)^T\ \Theta\,\mathrm{d}(\tau,y)\Big)\xi\cdot\xi.\end{aligned}$$

since $\int_Y \Theta\,\mathrm{d}(\tau,y) = 1$. The above integral vanishes when for a. e. $(\tau,y)\in Y$, $\nabla_y(\chi^\star\cdot\xi) = -\xi$ does not depend on τ, y. By using the periodicity, it follows that $0 = \int_Y \nabla_y(\chi^\star\cdot\xi)\,\mathrm{d}(\tau,y) = -\xi|Y|$; thus $\xi = 0$. We conclude that the symmetric part of D is definite positive. ☐

REMARK 6. *It should be pointed out that it is crucial to take into account a positive diffusivity coefficient $\eta > 0$, even for the simple divergence free case. In this case, we have $\Theta = 1$. Suppose there exists a vector valued function χ solution of $\partial_\tau\chi + u^0\cdot\nabla_y\chi = u^0$. Then, if $\eta = 0$ it is readily checked that $D\xi\cdot\xi = 0$, i.e. the "effective diffusion matrix" vanishes! Turbulent diffusivity strongly depends on the molecular diffusivity.*

We have two kinds of results depending on the fact that u^0 satisfies or not the orthogonality relation $\int_Y u^0\ \Theta\,\mathrm{d}(\tau,y) = 0$. We have seen that if u^0 is divergence free this condition is nothing but the fact that u^0 has a zero mean value and that the condition is always satisfied in the potential case. When the orthogonality relation is satisfied the result is complete. When it is not, the result is not complete in the sense that we do not know if the effective equations we obtain give rise to a well posed problem.

4.4. Vanishing ballistic velocity. Let us first investigate the case of a vanishing ballistic velocity.

THEOREM 4.7. *Let $\eta > 0$. Let $u^0,\ u^1 \in C_\#^{[2;1,\alpha]}([0,\infty)\times\mathbb{R}^N\times\mathbb{R}^{N+1})$. Let Θ be defined as in Proposition 4.5 and suppose that*

$$\int_Y u^0(t,x;\tau,y)\ \Theta(t,x;\tau,y)\ \ \mathrm{d}(\tau,y) = 0, \qquad \forall (t,x)\in[0,\infty)\times\mathbb{R}^N \tag{4.8}$$

holds. Let $\big(\rho_I^\epsilon\big)_{\epsilon>0}$ be a bounded sequence of non negative measures on $\mathbb{R}^N$. We suppose that $\big(\rho_I^\epsilon\big)_{\epsilon>0}$ converges vaguely to ρ_I. Let $\rho^\epsilon(t)$ be the solutions of the advection diffusion problem (2.1), (2.2) with

$$u^\epsilon(t,x) = \frac{1}{\epsilon}\big(u^0(t,x,\frac{t}{\epsilon^2},\frac{x}{\epsilon}) + \epsilon u^1(t,x,\frac{t}{\epsilon^2},\frac{x}{\epsilon})\big).$$

Then, up to the extraction of a subsequence, $\rho^\epsilon(t) \geq 0$ converges vaguely to $\rho(t) \geq 0$ locally, uniformly and the limit $\rho(t)$ is a solution of

$$\partial_t \rho(t,x) + \mathrm{div}_x\big(\rho\ v\big)(t,x) - \mathrm{div}_x\left((D + \eta I_N)\cdot \nabla_x \rho\right)(t,x) = 0, \tag{4.9}$$

in the sense of distributions with the Cauchy data $\rho(0) = \rho_I$. The coefficients are given by

$$\begin{aligned} v(t,x) &= \int_Y u^1\ \Theta(t,x;\tau,y)\ \mathrm{d}(\tau,y) \\ &\quad - \int_Y \chi^\star \Big(\mathrm{div}_x(\Theta\ u^0) + \mathrm{div}_y(\Theta\ u^1) - 2\eta\nabla_x\cdot\nabla_y\Theta\Big)(t,x;\tau,y)\ \mathrm{d}(\tau,y), \\ D(t,x) &= \int_Y \chi^\star \otimes (\Theta\ u^0 - 2\eta\nabla_y\Theta)(t,x;\tau,y)\ \mathrm{d}(\tau,y). \end{aligned}$$

and $\chi^\star$ is the solution of

$$-\partial_\tau \chi^\star - u^0\cdot\nabla_y\chi^\star - \eta\Delta_y\chi^\star = u^0, \quad \int_Y \chi^\star\,\mathrm{d}(\tau,y) = 0.$$

Remark 7. *The statement can be strengthened when the velocity fields satisfy the divergence free condition $\mathrm{div}_{x,y}u^{0,1} = 0$. In this case, recall that $\Theta = 1$. Assume the initial condition is converging in $L^2(\mathbb{R}^N)$: $\rho_I^\epsilon \to \rho_I$. Then, one has L^2 estimates on both ρ^ϵ and $\nabla_x\rho^\epsilon$. Consequently, one has convergence of the solutions ρ^ϵ to ρ in $L^2(0,T;H^1(\mathbb{R}^N))$ weakly, and in $C^0([0,T];L^2(\mathbb{R}^N)-weak)$, and for a. e. $t\in[0,T]$, $\rho^\epsilon(t)\to\rho(t)$ strongly in $L^2(K)$ for any compact set $K\subset\mathbb{R}^N$.*

Proof. By using standard results for parabolic equations, the solutions ρ^ϵ satisfy

$$\left\{\begin{array}{l} \forall (t,x)\in[0,\infty)\times\mathbb{R}^N,\ \rho^\epsilon(t,x)\geq 0, \\ \forall t\in[0,\infty),\ \displaystyle\int_{\mathbb{R}^N}\rho^\epsilon(t,x)\,\mathrm{d}x \leq \int_{\mathbb{R}^N}\rho_I^\epsilon(x)\,\mathrm{d}x \leq M \end{array}\right. \tag{4.10}$$

where $M = \sup_{\epsilon>0}\ \rho_I^\epsilon(\mathbb{R}^N)$ is finite by assumption. Remark that the conservation of mass $\int_{\mathbb{R}^N}\rho^\epsilon(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^N}\rho_I^\epsilon(x)\,\mathrm{d}x$, that can be formally expected from the equation, is not obvious at all. To obtain this conservation law an additional bound on $|u^\epsilon|$ is necessary. For instance we have to assume $|u^0(t,x;\tau,y)| + |u^1(t,x;\tau,y)| \leq C|x|$ for some constant $C>0$. We will not address this problem in this work.

We cannot immediately use Proposition 4.3 for $\rho^\epsilon(t)$ since the equicontinuity with respect to time is far from clear: there is no obvious bound for $\partial_t\rho^\epsilon$ because of the singular term $\frac{1}{\epsilon}\mathrm{div}_x(u^0\rho^\epsilon)$. Actually we will use crucially the condition (4.8) to prove that the sequence $\rho^\epsilon(t)$ is equicontinuous. The idea is to recover the formal asymptotics of the previous subsection but working on the test functions for the adjoint equation. Hence, let us define for any $\varphi\in C^2_\#(\mathbb{R}^N\times\mathbb{R}^{N+1})$ the following operators:

$$\begin{aligned} \mathcal{T}_0^\star(\varphi)(t,x;\tau,y) &= -\partial_\tau\varphi(x;\tau,y) - u^0(t,x;\tau,y)\cdot\nabla_y\varphi(x;\tau,y) - \eta\Delta_y\varphi(x;\tau,y), \\ \mathcal{T}_1^\star(\varphi)(t,x;\tau,y) &= -u^1(t,x;\tau,y)\cdot\nabla_y\varphi(x;\tau,y) - u^0(t,x;\tau,y)\cdot\nabla_x\varphi(x;\tau,y) \\ &\quad -2\eta\nabla_x\cdot\nabla_y\varphi(x;\tau,y), \\ \mathcal{T}_2^\star(\varphi)(t,x;\tau,y) &= -u^1(t,x;\tau,y)\cdot\nabla_x\varphi(x;\tau,y) - \eta\Delta_x\varphi(x;\tau,y). \end{aligned}$$

Step 1
Let $\varphi\in C^2_{c,\#}([0,\infty)\times\mathbb{R}^N\times\mathbb{R}^{N+1})$. By multiplying the equation by $\epsilon^2\,\varphi(t,x;t/\epsilon^2,x/\epsilon)$, we get

$$\begin{aligned} &\int_{\mathbb{R}^N}\mathcal{T}_0^\star(\varphi)(t,x;t/\epsilon^2,x/\epsilon)\ \rho^\epsilon(t,x)\,\mathrm{d}x = -\,\epsilon\int_{\mathbb{R}^N}(\mathcal{T}_1^\star+\epsilon\mathcal{T}_2^\star)(\varphi)(t,x;t/\epsilon^2,x/\epsilon)\ \rho^\epsilon(t,x)\,\mathrm{d}x \\ &-\epsilon^2\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\varphi(t,x;t/\epsilon^2,x/\epsilon)\ \rho^\epsilon(t,x)\,\mathrm{d}x + \epsilon^2\int_{\mathbb{R}^N}\partial_t\varphi(t,x;t/\epsilon^2,x/\epsilon)\ \rho^\epsilon(t,x)\,\mathrm{d}x. \end{aligned} \tag{4.11}$$

Then, we obtain

$$\lim_{\epsilon\to 0}\left(\int_{\mathbb{R}^N}\mathcal{T}_0^\star(\varphi)(t,x;t/\epsilon^2,x/\epsilon)\ \rho^\epsilon(t,x)\,\mathrm{d}x\right)=0,\qquad \text{in } \mathcal{D}'(0,\infty) \tag{4.12}$$

for any $\varphi\in C^2_{c,\#}([0,\infty)\times\mathbb{R}^N\times\mathbb{R}^{N+1})$.

Step 2
In order to get rid of the leading term in the dual equation (4.11), we choose the test function of the form $\varphi(t,x;t/\epsilon^2,x/\epsilon)=\psi(x)+\epsilon\ \phi(t,x;t/\epsilon^2,x/\epsilon)$. We obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}&(\psi+\epsilon\ \phi)\,(t,x;t/\epsilon^2,x/\epsilon)\rho^\epsilon(t,x)\,\mathrm{d}x\\ &=-\frac{1}{\epsilon}\int_{\mathbb{R}^N}\left(\mathcal{T}_1^\star(\psi)+\mathcal{T}_0^\star(\phi)\right)(t,x;t/\epsilon^2,x/\epsilon)\ \rho^\epsilon(t,x)\,\mathrm{d}x\\ &\quad-\int_{\mathbb{R}^N}\left(\mathcal{T}_2^\star(\psi)+\mathcal{T}_1^\star(\phi)\right)(t,x;t/\epsilon^2,x/\epsilon)\ \rho^\epsilon(t,x)\,\mathrm{d}x\\ &\quad+\epsilon\int_{\mathbb{R}^N}\left(\partial_t\phi-\mathcal{T}_2^\star(\phi)\right)(t,x;t/\epsilon^2,x/\epsilon)\ \rho^\epsilon(t,x)\,\mathrm{d}x.\end{aligned} \tag{4.13}$$

Since ψ does not depend on the fast variables (τ,y), we remark that $\mathcal{T}_1^\star(\psi)=-u^0\cdot\nabla_x\psi$. Consequently, multiplying by ϵ, passing to the limit and using (4.12) yield

$$\begin{aligned}\lim_{\epsilon\to 0}&\left(\int_{\mathbb{R}^N}\left(\mathcal{T}_1^\star(\psi)+\mathcal{T}_0^\star(\phi)(t,x;t/\epsilon^2,x/\epsilon)\right)\ \rho^\epsilon(t,x)\,\mathrm{d}x\right)=0\\ &=\lim_{\epsilon\to 0}\left(\int_{\mathbb{R}^N}u^0(t,x;t/\epsilon^2,x/\epsilon)\cdot\nabla_x\psi(x)\ \rho^\epsilon(t,x)\,\mathrm{d}x\right).\end{aligned} \tag{4.14}$$

Up to now, condition (4.8) has not been used. We shall use the condition to write $u^0\cdot\nabla_x\psi$ as $\mathcal{T}_0^\star(\phi)$ for a convenient choice of the function ϕ. Therefore, we realize that (4.14) is already contained in (4.12).

In order to go further we have to get rid of the term of order $1/\epsilon$ in (4.13). Let us introduce the vector-valued function $\chi^\star(t,x;\tau,y)\in\left(C^{[2;2,\alpha]}_{\#}([0,\infty)\times\mathbb{R}^N\times\mathbb{R}^{N+1})\right)^N$ solution of the adjoint cell problem

$$\mathcal{T}_0^\star(\chi^\star)=-\partial_\tau-u^0\cdot\nabla_y\chi^\star-\eta\Delta_y\chi^\star=u^0,\qquad \int_Y\chi^\star\,\mathrm{d}(\tau,y)=0.$$

The function $\chi^\star$ is well defined thanks to Proposition 4.5 and to the condition (4.8). We choose the test function ϕ depending on ψ as follows

$$\phi(t,x;\tau,y)=\chi^\star(t,x;\tau,y)\cdot\nabla_x\psi(x).$$

Assuming $\psi\in C^3_c(\mathbb{R}^N)$, we have $\phi\in C^{[2;2,\alpha]}_{c,\#}([0,T]\times\mathbb{R}^N\times\mathbb{R}^{N+1})$, for any $T>0$. The crucial fact is that $\mathcal{T}_0^\star(\phi)=\mathcal{T}_0^\star(\chi^\star)\cdot\nabla_x\psi=u^0\cdot\nabla_x\psi=-\mathcal{T}_1^\star(\psi)$. Consequently, we recover (4.12) from (4.14). Furthermore, the $\frac{1}{\epsilon}$ term in (4.13) vanishes. We deduce the estimate

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}(\psi+\epsilon\ \phi)\,(t,x;t/\epsilon^2,x/\epsilon)\ \rho^\epsilon(t,x)\,\mathrm{d}x\right|\le C(\psi). \tag{4.15}$$

Moreover, passing to the limit $\epsilon\to 0$, we are led to

$$\lim_{\epsilon\to 0}\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\psi(x)\ \rho^\epsilon(t,x)\,\mathrm{d}x+\int_{\mathbb{R}^N}\left(\mathcal{T}_2^\star(\psi)+\mathcal{T}_1^\star(\phi)\right)(t,x;t/\epsilon^2,x/\epsilon)\ \rho^\epsilon(t,x)\,\mathrm{d}x\right)=0 \tag{4.16}$$

for any $\psi \in C_c^3(\mathbb{R}^N)$ with $\phi(t,x;\tau,y) = \chi^\star(t,x;\tau,y)\cdot\nabla_x\psi(x)$.

Step 3
We are now ready to obtain the equicontinuity of $\rho^\epsilon(t)$. Indeed, the bound (4.15) shows that for any $\psi \in C_c^3(\mathbb{R}^N)$ the sequence of functions

$$t \longmapsto \int_{\mathbb{R}^N} (\psi + \epsilon\ \phi)\,(t,x;t/\epsilon^2, x/\epsilon)\ \rho^\epsilon(t,x)\,\mathrm{d}x$$

is equicontinuous on $[0,\infty)$. Since the family of functions

$$t \longmapsto \int_{\mathbb{R}^N} \psi(x)\ \rho^\epsilon(t,x)\,\mathrm{d}x$$

is close, up to ϵ, to the previous sequence, it is also equicontinuous. The density of $C_c^3(\mathbb{R}^N)$ in $C_c^0(\mathbb{R}^N)$ allows to conclude that $\rho^\epsilon(t)$ is vaguely equicontinuous on $[0,\infty)$. Thanks to (4.10) we can apply Propositions 4.3 and 4.4 to the sequence $\rho^\epsilon(t)$.

Step 4
Up to a subsequence there is a double scale limit R of $\rho^\epsilon(t)$ and the vague limit of $\rho^\epsilon(t)$ is given by the marginal $\rho(t,x) = \int_Y R(t,x;\tau,y)\,\mathrm{d}(\tau,y)$. The limits (4.12) and (4.16) now become

$$\int_0^\infty \int_{\mathbb{R}^N}\int_Y \mathcal{T}_0^\star(\varphi)(t,x;\tau,y)\ R(t,x;\tau,y)\,\mathrm{d}(\tau,y)\,\mathrm{d}x\,\mathrm{d}t = 0, \qquad \forall\varphi \in C^2_{c,\#}([0,\infty)\times\mathbb{R}^N\times\mathbb{R}^{N+1}), \tag{4.17}$$

and, in the distribution sense on $[0,\infty)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\psi(x)\ \rho(t,x)\,\mathrm{d}x + \int_{\mathbb{R}^N}\int_Y \left(\mathcal{T}_2^\star(\psi) + \mathcal{T}_1^\star(\phi)\right)\ R\ (t,x;\tau,y)\,\mathrm{d}(\tau,y)\,\mathrm{d}x = 0 \qquad \forall\psi \in C_c^3(\mathbb{R}^N) \text{ with } \phi = \chi^\star(t,x;\tau,y)\cdot\nabla_x\psi(x), \tag{4.18}$$

respectively. By using formally Proposition 4.5, (4.17) means that the double scale limit R lies in the orthogonal set of $\mathrm{Ran}(\mathcal{T}_0^\star) = (\mathrm{Ker}(\mathcal{T}_0))^\perp = (\mathrm{Span}\{\Theta\})^\perp$; thus $R \in \mathrm{Span}\{\Theta\}$ which corresponds to the first step of the formal asymptotics. Let us make this argument rigorous (it does not work due to the lack of regularity of the limit R).

Thanks to Proposition 4.5, we obtain from (4.17) that, for any $H \in C_\#^{[0;0,\alpha]}([0,\infty)\times\mathbb{R}^N\times\mathbb{R}^{N+1})$ verifying $\int_Y H\ \Theta\,\mathrm{d}(\tau,y) = 0$, we have

$$\int_0^\infty\int_{\mathbb{R}^N}\int_Y H(t,x;\tau,y)\ R(t,x;\tau,y)\,\mathrm{d}(\tau,y)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Let $\varphi \in C_\#^{[0;0,\alpha]}([0,\infty)\times\mathbb{R}^N\times\mathbb{R}^{N+1})$. We write

$$\varphi(t,x;\tau,y) = c_\varphi(t,x)\Theta(t,x;\tau,y) + (\varphi(t,x;\tau,y) - c_\varphi(t,x)\Theta(t,x;\tau,y))$$

where

$$c_\varphi(t,x) = \Big(\int_Y \varphi(t,x;z)\ \Theta(t,x;z)\,\mathrm{d}z\Big)\ /\ \Big(\int_Y\ \Theta(t,x;\tau,y)^2\,\mathrm{d}(\tau,y)\Big).$$

Since $\int_Y (\varphi - c_\varphi \Theta)\ R\ \mathrm{d}(\tau,y) = 0$, we obtain

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^N} \int_Y \varphi(t,x;\tau,y)\ R(t,x;\tau,y)\,\mathrm{d}(\tau,y)\,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^\infty \int_{\mathbb{R}^N} c_\varphi(t,x)\Big(\int_Y \Theta(t,x;\tau,y)R(t,x;\tau,y)\,\mathrm{d}(\tau,y)\Big)\,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^\infty \int_{\mathbb{R}^N} \int_Y \varphi(t,x;\tau,y)\ \widetilde{\rho}(t,x)\Theta(t,x;\tau,y)\,\mathrm{d}(\tau,y)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where $\widetilde{\rho}(t,x) = \int_Y \Theta(t,x;\tau,y)R(t,x;\tau,y)\,\mathrm{d}(\tau,y) \times \big(\int_Y \Theta(t,x;\tau,y)^2\,\mathrm{d}(\tau,y)\big)^{-1}$. Hence, it shows that $R(t,x;\tau,y) = \widetilde{\rho}(t,x)\Theta(t,x;\tau,y)$. Furthermore, using $\int_Y \Theta\,\mathrm{d}(\tau,y) = 1$, we have $\widetilde{\rho} = \rho$, the weak limit of ρ^ϵ. Plugging this result in (4.18) gives the desired equation for $\rho(t,x)$ and ends the proof of Theorem 4.7. Note that we obtain in this way the diffusion matrix and the drift velocity with the dual formulae (4.6) and (4.7) respectively. ∎

4.5. Non vanishing ballistic velocity. We can follow step by step the same strategy without assuming the condition (4.8). Of course, we cannot define in this situation $\chi^\star$ by (4.5) since the right hand side does not fulfill the solvability condition. We have to take into account the ballistic velocity

$$c(t,x) = \int_Y u^0\ \Theta(t,x;\tau,y)\,\mathrm{d}(\tau,y) \neq 0.$$

Then, we are led to the following statement.

THEOREM 4.8. *Let the assumptions of Theorem 4.7 be fulfilled, except (4.8). Then, up to a subsequence, $\rho^\epsilon \geq 0$ converges to ρ in the vague sense for measures on $[0,\infty) \times \mathbb{R}^N$. The limit ρ satisfies*

$$\mathrm{div}_x(c\ \rho) = 0,$$

in the sense of distributions on $[0,\infty) \times \mathbb{R}^N$. Moreover, for any $\psi \in C^2_c([0,\infty) \times \mathbb{R}^N)$ which satisfies $c(t,x) \cdot \nabla_x \psi(t,x) = 0$ for all $(t,x) \in [0,\infty) \times \mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N} \psi(t,x)\ \rho^\epsilon(t,x)\,\mathrm{d}x \xrightarrow[\epsilon\to 0]{} \int_{\mathbb{R}^N} \psi(t,x)\ \rho(t,x)\,\mathrm{d}x,$$

uniformly on any interval $[0,T]$, $0 < T < \infty$. Furthermore, for any such test function ψ, the limit satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^N} \psi\rho(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^N} \big(\partial_t \psi\ + v \cdot \nabla_x \psi + \mathrm{div}_x\big(D^T \cdot \nabla_x \psi\big)\ + \eta \Delta_x \psi\big)\ \rho\,\mathrm{d}x, \\ \int_{\mathbb{R}^N} \psi\rho(0,x)\,\mathrm{d}x = \int_{\mathbb{R}^N} \psi\rho_I(x)\,\mathrm{d}x\end{aligned}$$

where D^T is the transpose of the matrix D. Here D and v are defined as in Theorem 4.7 but with $\chi^\star$ solution of

$$\mathcal{T}_0^\star(\chi^\star) = u^0 - c, \qquad \int_Y \chi^\star\,\mathrm{d}(\tau,y) = 0. \tag{4.19}$$

Proof. The $L^\infty(0,T;L^1(\mathbb{R}^N)) \subset L^1((0,T)\times\mathbb{R}^N)$ estimate on ρ^ϵ allows us to assume that, for a subsequence, $\rho^\epsilon \rightharpoonup \rho$ vaguely in $\mathcal{M}^1((0,T)\times\mathbb{R}^N)$. In Step 2, we rewrite (4.14) as

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^N} \Big((c-u^0)\cdot\nabla_x\psi + \mathcal{T}_0^\star(\phi)\Big)(t,x;t/\epsilon^2,x/\epsilon)\ \rho^\epsilon\,\mathrm{d}x\,\mathrm{d}t \\ - \int_0^T \int_{\mathbb{R}^N} c\cdot\nabla_x\psi\ \rho^\epsilon\,\mathrm{d}x\,\mathrm{d}t \to 0,\ \text{as } \epsilon \to 0.\end{aligned}$$

For a given function ψ, we choose ϕ such that the first integral in this expression vanishes. Namely, we set $\phi(t,x;\tau,y) = \chi^\star(t,x;\tau,y)\cdot\nabla_x\psi(t,x)$, $\chi^\star$ being defined by (4.19). We deduce that

$$\int_0^T \int_{\mathbb{R}^N} c\cdot\nabla_x\psi\ \rho\,\mathrm{d}x\,\mathrm{d}t = 0$$

for any test function ψ. Then, considering now a test function verifying $c\cdot\nabla_x\psi = 0$ we can reproduce the arguments of the proof of Theorem 4.7. □

Let q be a (scalar) distribution on $(0,T)\times\mathbb{R}^N$. Clearly, $T = \mathrm{div}_x(c\ q)$ belongs to the orthogonal of $E = \{\psi : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$, such that $c\cdot\nabla_x\psi = 0\}$. At least formally, the elements T of $E^\perp$ have always this form. Indeed, let us introduce the symmetric degenerate elliptic operator $\mathcal{A}(\psi) = -\mathrm{div}_x(c\otimes c\nabla_x\psi)$. Remarking that $\int_{\mathbb{R}^N}\mathcal{A}(\psi)\ \psi\,\mathrm{d}x = \int_{\mathbb{R}^N}|c\cdot\nabla_x\psi|^2\,\mathrm{d}x$, we have $E = \mathrm{Ker}(\mathcal{A})$. Thus, formally, a distribution $T\in E^\perp$ lies in the range of $\mathcal{A}$. It means that $T = \mathcal{A}(p) = \mathrm{div}_x(cq)$, with $q = -c\nabla_x p$. In these conditions the equations for ρ read

$$\begin{cases} \mathrm{div}_x(c\ \rho) = 0, \\ \partial_t\rho + \mathrm{div}_x(v\ \rho) - \mathrm{div}_x\Big((D+\eta I_N)\nabla_x\rho\Big) = \mathrm{div}_x(c\ q) \end{cases}$$

and q appears as a Lagrange multiplier associated to the constraint $\mathrm{div}_x(c\ \rho) = 0$.

Actually it is possible that the second part of the statement is meaningless. For instance, consider the simple case $c = (1,0,...)\in\mathbb{R}^N$. Thus, $\mathrm{div}_x(c\rho) = 0 = \partial_{x_1}\rho = 0$ means that ρ does not depend on the first variable. Since ρ is a bounded measure on $\mathbb{R}^N$, this implies that $\rho = 0$! According to this example, the set of compactly supported test functions verifying $c\cdot\nabla_x\psi = 0$ can reduce to $\{0\}$ (this is the case in the autonomous case with characteristics curves verifying $|X(s,x)|\to\infty$ as $s\to\infty$...).

The simplest way to treat the difficulty consists of changing the time scale. Let us define $s = t/\epsilon$. Then, we study

$$\frac{1}{\epsilon}\partial_s\rho^\epsilon(s,x) + \frac{1}{\epsilon}\mathrm{div}_x\big((u^0+\epsilon u^1)(\epsilon s,x;s/\epsilon,x/\epsilon)\ \rho^\epsilon(s,x)\big) = \eta\ \Delta_x\rho^\epsilon(s,x),$$

instead of (2.1), (2.3). At least formally we can replace $u^i(\epsilon s)$ by $u^i(0)$. In this case we have only to consider velocities fields independent on the time scale s. For the sake of generality we consider the problem

$$\frac{1}{\epsilon}\partial_s\rho^\epsilon(s,x) + \frac{1}{\epsilon}\mathrm{div}_x\big((u^0+\epsilon u^1)(s,x;s/\epsilon,x/\epsilon)\ \rho^\epsilon(s,x)\big) = \eta\ \Delta_x\rho^\epsilon(s,x), \tag{4.20}$$
$$\forall (s,x)\in(0,\infty)\times\mathbb{R}^N,$$
$$\rho^\epsilon(0,x) = \rho_I^\epsilon(x), \qquad \forall x\in\mathbb{R}^N. \tag{4.21}$$

The initial data ρ_I^ϵ is still assumed to be a bounded sequence of non negative measures which converges vaguely to ρ_I. Therefore, the sequence of solution satisfies the uniform estimate (4.10). The behavior as ϵ goes to 0 is then simply described by the transport equation with velocity $c(s,x) = \int_Y u^0\ \Theta\,\mathrm{d}(\tau,y)$.

THEOREM 4.9. *Let ρ^ϵ be the solution of (4.20), (4.21). We assume that ρ_I^ϵ is a bounded sequence of non negative measures which converges vaguely to ρ_I. Then, ρ^ϵ converges vaguely locally, uniformly on $\mathbb{R}^+$ to ρ, solution of the transport equation*

$$\partial_s\rho + \mathrm{div}_x(c\rho) = 0$$

with initial data ρ_I.

Proof. We still follow the strategy of proof of Theorem 4.7.
Multiplying (4.20) by $\varphi(s,x;s/\epsilon,x/\epsilon)$ yields

$$\begin{aligned}&\int_{\mathbb{R}^N} \mathcal{T}_0^\star(\varphi)(s,x;s/\epsilon,x/\epsilon)\ \rho^\epsilon(s,x)\,\mathrm{d}x = -\epsilon\int_{\mathbb{R}^N}(\mathcal{T}_1^\star+\epsilon\mathcal{T}_2^\star)(\varphi)(s,x;s/\epsilon,x/\epsilon)\ \rho^\epsilon(s,x)\,\mathrm{d}x\\ &-\epsilon\,\frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}^N}\varphi(s,x;s/\epsilon,x/\epsilon)\ \rho^\epsilon(s,x)\,\mathrm{d}x+\epsilon\int_{\mathbb{R}^N}\partial_s\varphi(s,x;s/\epsilon,x/\epsilon)\ \rho^\epsilon(s,x)\,\mathrm{d}x.\end{aligned}\tag{4.22}$$

Hence, we recover (4.12). Then, we use $\varphi(s,x;\tau,y)=\psi(x)+\epsilon\phi(s,x;\tau,y)$ as test function so that the ϵ^0 term in (4.22) disappears. We get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}^N}(\psi+\epsilon\phi)(s,x;s/\epsilon,x/\epsilon)\ \rho^\epsilon(s,x)\,\mathrm{d}x-\epsilon\int_{\mathbb{R}^N}\partial_s\phi(s,x;s/\epsilon,x/\epsilon)\ \rho^\epsilon(s,x)\,\mathrm{d}x\\ &=\int_{\mathbb{R}^N}(u^0\cdot\nabla_x\psi-\mathcal{T}_0^\star(\phi))(s,x;s/\epsilon,x/\epsilon)\ \rho^\epsilon(s,x)\,\mathrm{d}x\\ &+\epsilon\int_{\mathbb{R}^N}\Big((u^1\cdot\nabla_x\psi+\eta\Delta_x\psi)+(\mathcal{T}_1^\star+\epsilon\mathcal{T}_2^\star)(\phi)\Big)(s,x;s/\epsilon,x/\epsilon)\ \rho^\epsilon(s,x)\,\mathrm{d}x.\end{aligned}\tag{4.23}$$

We deduce that $\Big(s\mapsto\int_{\mathbb{R}^N}\psi\rho^\epsilon\,\mathrm{d}x\Big)_{\epsilon>0}$ is equicontinuous on $\mathbb{R}^+$, and thus the compactness of ρ^ϵ. Next, the last term in (4.23) goes to 0 as $\epsilon\to 0$. The first term in the right hand side recasts as

$$\int_{\mathbb{R}^N}\Big((u^0-c)\cdot\nabla_x\psi-\mathcal{T}_0^\star(\phi)\Big)(s,x;s/\epsilon,x/\epsilon)\ \rho^\epsilon(s,x)\,\mathrm{d}x+\int_{\mathbb{R}^N}c(s,x)\cdot\nabla_x\psi\ \rho^\epsilon(s,x)\,\mathrm{d}x.$$

Hence, we choose ϕ depending on ψ so that the first integral vanishes. Namely, we set $\phi=\chi^\star\cdot\nabla_x\psi$, with $\mathcal{T}_0^\star(\chi^\star)=(u^0-c)$. Then, letting $\epsilon\to 0$, we obtain the relation

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}^N}\psi\ \rho(s,x)\,\mathrm{d}x-\int_{\mathbb{R}^N}c(s,x)\cdot\nabla_x\psi\ \rho(s,x)\,\mathrm{d}x=0,$$

which ends the proof. □

REMARK 8. *The situation we observe is reminiscent to the hydrodynamic limits in kinetic theory. There, the small parameter is related to the Knudsen number, i.e. the ratio of the mean free path over some characteristic length. When the flux associated to the equilibrium states of the equation vanishes, a parabolic scaling has to be considered. It corresponds to the vanishing ballistic velocity in this work. On the converse when the flux does not vanish, (non vanishing ballistic velocity) an hyperbolic scaling has to be used which correspond to a faster time scale. We refer to the lecture notes of* GOLSE *[17] for a presentation of these questions.*

More precise information can be given when we restrict to purely periodic velocity fields. Namely, we assume that $u^{0,1}$ do not depend on t,x but only on the fast variables $(\tau,y)\in Y$. Consequently, the ballistic velocity c is constant. In such a case, there is no mixing of the time scales and we can give a result incorporating both the transport through the ballistic velocity and the diffusion at the parabolic time scale. A similar approach has been introduced by MELLET in kinetic theory, see [18]. We also mention the interesting attempts due to CAPDEBOSCQ [8, 9].

THEOREM 4.10. *Let $\eta>0$. Let $u^0,\ u^1\in C_\#^{[1,\alpha]}(\mathbb{R}^{N+1})$. Let $\big(\rho_I^\epsilon\big)_{\epsilon>0}$ be a bounded sequence of non negative measures on $\mathbb{R}^N$. We suppose that $\big(\rho_I^\epsilon\big)_{\epsilon>0}$ converges vaguely to ρ_I. Let $\rho^\epsilon(t)$ be the solutions of the advection diffusion problem (2.1), (2.2) with*

$$u^\epsilon(t,x)=\frac{1}{\epsilon}\big(u^0(\frac{t}{\epsilon^2},\frac{x}{\epsilon})+\epsilon u^1(\frac{t}{\epsilon^2},\frac{x}{\epsilon})\big).$$

Then, up to a subsequence, $\widetilde{\rho}^\epsilon(t,x)=\rho^\epsilon(t,x+ct/\epsilon)$ converges vaguely locally, uniformly on $\mathbb{R}^+$ to $\widetilde{\rho}$, solution of the drift-diffusion equation

$$\partial_t\widetilde{\rho}+\mathrm{div}_x(v\widetilde{\rho}-D\nabla_x\widetilde{\rho})=0$$

with initial data ρ_I. The velocity v is defined as in Theorem 4.7, while

$$D = \int_Y \chi^\star \otimes \big(\Theta(u^0 - c) - 2\eta \nabla_y \Theta\big)\,\mathrm{d}(\tau, y)$$

with $\chi^\star$ solution of $\mathcal{T}_0^\star(\chi^\star) = u^0 - c$.

Proof. We remark that $\widetilde{\rho}^\epsilon$ is the solution of the advection diffusion problem (2.1), (2.2) with u^ϵ replaced by

$$\widetilde{u}^\epsilon(t,x) = \frac{1}{\epsilon}\big(\widetilde{u}^0(\frac{t}{\epsilon^2}, \frac{x}{\epsilon}) + \epsilon \widetilde{u}^1(\frac{t}{\epsilon^2}, \frac{x}{\epsilon})\big)$$
$$\widetilde{u}^0(\tau, y) = u^0(\tau, y + c\tau) - c, \quad \widetilde{u}^1(\tau, y) = u^1(\tau, y + c\tau).$$

The functions $\widetilde{u}^i$, $i = 1, 2$ are not Y periodic. But if $(e_0, e_1, ..., e_N)$ is the canonical basis of $\mathbb{R}^{N+1}$ the new cell basis is $(e_0 - c, e_1, ..., e_N)$ corresponding to the cell

$$\widetilde{Y} = \{(\tau, y)/\tau \in (0,1),\ y \in -c\ \tau + (0,1)^N\}.$$

The functions $\widetilde{u}^i$, $i = 1, 2$ are $\widetilde{Y}$ periodic. It is easily checked that the null function of the new cell problem is then $\widetilde{\Theta}(\tau, y) = \Theta(\tau, y + c\tau)$. As a consequence we have

$$\int_{\widetilde{Y}} \widetilde{u}^0\ \widetilde{\Theta}(\tau, y)\ \mathrm{d}(\tau, y) = \int_Y (u^0 - c)\Theta(\tau, y)\ \mathrm{d}(\tau, y) = 0.$$

Then $\widetilde{u}^0$ satisfies the vanishing ballistic velocity condition and we can apply Theorem 4.7. It leads to the result. Remark that it is crucial that $u^{0,1}$ depends only on (τ, y) and c is constant. □

5. Dissipation properties. The only immediate estimate on $\rho^\epsilon(t)$ is in the space L^1, by using the conservative form of the equation. The asymptotic behavior stated in Theorem 4.7, is obtained for a limit $\rho(t)$ which is *a priori* only a measure. In this functional framework, there is no uniqueness of the solution of the limit effective advection diffusion problem. In this Subsection we want to recover uniqueness (and regularity) of the limit. Assuming more bounds on the initial data, it is possible to obtain more involved dissipation properties of the equation. Actually, quantities like

$$\int H(\rho^\epsilon/\Psi^\epsilon)\ \Psi^\epsilon\,\mathrm{d}x$$

can be uniformly bounded for convex functions H, and a suitable choice of $\Psi^\epsilon > 0$.

As a preliminary, we establish a general dissipation property which is a consequence of dissipativity of Markovian processes (see [26]) as it is explained by J.F. Collet in [10].

Proposition 5.1. *Let*

$$\mathcal{T} = \partial_t \cdot + \sum_{i=1}^N \partial_{x_i}(u_i(t,x)\cdot) - \sum_{i,j=1}^N \partial_{x_i}(a_{ij}(t,x)\partial_{x_j}\cdot)$$

with bounded coefficients u_i, a_{ij} verifying $\sum_{ij} a_{ij}\xi_i\xi_j \geq 0$ for any $\xi \in \mathbb{R}^N$. Let $H : \mathbb{R} \to \mathbb{R}$ be a C^2 convex function. Let ρ and Ψ in $H^1_{\mathrm{loc}}(\mathbb{R}^{N+1})$, with $\Psi > 0$. Then, we have

$$\mathcal{T}\Big(H\big(\frac{\rho}{\Psi}\big)\ \Psi\Big) - H'\big(\frac{\rho}{\Psi}\big)\mathcal{T}(\rho) - G\big(\frac{\rho}{\Psi}\big)\mathcal{T}(\Psi) = -\sum_{i,j=1}^N H''\big(\frac{\rho}{\Psi}\big)\ \Psi\ a_{ij}\partial_i\big(\frac{\rho}{\Psi}\big)\ \partial_j\big(\frac{\rho}{\Psi}\big) \leq 0,$$

with $G(s) = H(s) - sH'(s)$.

The assumptions on ρ and Ψ allow to use the chain rule for the functions ρ, Ψ and $\frac{\rho}{\Psi}$. Then the above Proposition is a consequence of two computations whose results are given in the following Lemma.

Lemma 5.2. *Let $\rho, \Psi : \mathbb{R}^{N+1} \to \mathbb{R}$ as in Proposition 5.1. Let $a : \mathbb{R}^{N+1} \to \mathbb{R}$ be a bounded coefficient. Let $H : \mathbb{R} \to \mathbb{R}$ be a C^2 function. Denote by ∂ any derivative in $\mathbb{R}^{N+1}$. We have*

$$\partial\Big(H\big(\frac{\rho}{\Psi}\big)\ a\ \Psi\Big) = G\big(\frac{\rho}{\Psi}\big)\ \partial(a\Psi) + H'\big(\frac{\rho}{\Psi}\big)\ \partial(a\rho)$$

with $G(s) = H(s) - sH'(s)$.

Proof. We compute

$$\begin{aligned}
\partial\Big(H\big(\tfrac{\rho}{\Psi}\big)\ a\ \Psi\Big) &= H'\big(\tfrac{\rho}{\Psi}\big)\ \partial\big(\tfrac{\rho}{\Psi}\big)\ a\ \Psi + H\big(\tfrac{\rho}{\Psi}\big)\ \partial(a\Psi) \\
&= H'\big(\tfrac{\rho}{\Psi}\big)\ \Big(\partial\big(\tfrac{\rho}{\Psi}\ a\ \Psi\big) - \tfrac{\rho}{\Psi}\partial(a\Psi)\Big) + H\big(\tfrac{\rho}{\Psi}\big)\ \partial(a\Psi) \\
&= \Big(H\big(\tfrac{\rho}{\Psi}\big) - \tfrac{\rho}{\Psi}H'\big(\tfrac{\rho}{\Psi}\big)\Big)\partial(a\Psi) + H'\big(\tfrac{\rho}{\Psi}\big)\ \partial(a\rho).
\end{aligned}$$

□

Lemma 5.3. *We keep the notation of Lemma 5.2. Denote by ∂_i, ∂_j any derivatives in $\mathbb{R}^{N+1}$. We have*

$$\partial_i\Big(a\partial_j\big[H\big(\frac{\rho}{\Psi}\big)\ \Psi\big]\Big) = G\big(\frac{\rho}{\Psi}\big)\ \partial_i(a\partial_j\Psi) + H'\big(\frac{\rho}{\Psi}\big)\ \partial_i(a\partial_j\rho) + H''\big(\frac{\rho}{\Psi}\big)\ a\Psi\ \partial_i\big(\frac{\rho}{\Psi}\big)\ \partial_j\big(\frac{\rho}{\Psi}\big).$$

Proof. We compute

$$\begin{aligned}
&\partial_i\Big(a\partial_j\big[H\big(\tfrac{\rho}{\Psi}\big)\ \Psi\big]\Big) = \partial_i\Big(aH'\big(\tfrac{\rho}{\Psi}\big)\ \partial_j\big(\tfrac{\rho}{\Psi}\big)\ \Psi + aH\big(\tfrac{\rho}{\Psi}\big)\partial_j\Psi\Big) \\
&= a\Psi\ H''\big(\tfrac{\rho}{\Psi}\big)\partial_i\big(\tfrac{\rho}{\Psi}\big)\ \partial_j\big(\tfrac{\rho}{\Psi}\big) + H'\big(\tfrac{\rho}{\Psi}\big)\ \Big[\partial_i\Big(a\partial_j\big(\tfrac{\rho}{\Psi}\big)\ \Psi\Big) + \partial_i\big(\tfrac{\rho}{\Psi}\big)\ a\partial_j\Psi\Big] + H\big(\tfrac{\rho}{\Psi}\big)\ \partial_i\big(a\partial_j\Psi\big).
\end{aligned}$$

Then, we observe that

$$\partial_i\Big(a\partial_j\big(\frac{\rho}{\Psi}\big)\ \Psi\Big) = \partial_i\Big(a\partial_j\big(\frac{\rho}{\Psi}\ \Psi\big)\Big) - a\frac{\rho}{\Psi}\partial_j\Psi\Big) = \partial_i(a\partial_j\rho) - \frac{\rho}{\Psi}\partial_i(a\partial_j\Psi) - \partial_i\big(\frac{\rho}{\Psi}\big)\ a\partial_j\Psi.$$

Hence, we get

$$\begin{aligned}
\partial_i\Big(a\partial_j\big[H\big(\tfrac{\rho}{\Psi}\big)\ \Psi\big]\Big) &= a\Psi\ H''\big(\tfrac{\rho}{\Psi}\big)\partial_i\big(\tfrac{\rho}{\Psi}\big)\ \partial_j\big(\tfrac{\rho}{\Psi}\big) \\
&\qquad + H'\big(\tfrac{\rho}{\Psi}\big)\ \Big[\partial_i(a\partial_j\rho) - \tfrac{\rho}{\Psi}\partial_i(a\partial_j\Psi)\Big] + H\big(\tfrac{\rho}{\Psi}\big)\ \partial_i\big(a\partial_j\Psi\big) \\
&= a\Psi\ H''\big(\tfrac{\rho}{\Psi}\big)\partial_i\big(\tfrac{\rho}{\Psi}\big)\ \partial_j\big(\tfrac{\rho}{\Psi}\big) + H'\big(\tfrac{\rho}{\Psi}\big)\ \partial_i(a\partial_j\rho) + \Big(H\big(\tfrac{\rho}{\Psi}\big) - \tfrac{\rho}{\Psi}H'\big(\tfrac{\rho}{\Psi}\big)\Big)\ \partial_i\big(a\partial_j\Psi\big).
\end{aligned}$$

□

Then, we apply Proposition 5.1 with the operator $\mathcal{T}_\epsilon$, ρ^ϵ, and $\Psi^\epsilon = (\Theta + \epsilon\kappa)(t, x; t/\epsilon^2, x/\epsilon)$, where we recall that κ is the solution of the cell problem $\mathcal{T}_0(\kappa) = -\mathcal{T}_1(\Theta)$. We know (Proposition 4.5) that Θ and κ are continuous function and that Θ is positive. Therefore on every compact K of $\mathbb{R}^{n+1}$ they are bounded and Θ is bounded from below. It guarantees that on K, for ϵ small enough, Ψ^ϵ is bounded from above and below. Of course, positivity of Ψ^ϵ is not guaranteed in the whole space. For that we have to assume a uniform behavior of $u^{0,1}$ at infinity. This problem disappears if the problem (2.1), (2.2) is posed on a bounded domain with periodic or Dirichlet conditions. We deduce the following corollary.

Corollary 5.4. *With the same assumption as in Theorem 4.7, let ρ^ϵ be the solution of the problem (2.1), (2.2) on $[0, \infty) \times \Omega$. Let us assume that :*

- *$\Omega = \mathbb{R}^N$ and the functions Θ, κ and their derivatives are bounded and Θ is bounded from below, or*

- *Ω is a cell or a smooth bounded domain and the problem is completed by periodic or Dirichlet conditions.*

Set $\Psi^\epsilon(t,x) = (\Theta+\epsilon\kappa)(t,x;t/\epsilon^2,x/\epsilon) > 0$, which is bounded from above and below, for $t \in (0,T)$, $x \in \mathbb{R}^N$, $\epsilon \in (0,\epsilon_0)$. Let H be a C^2 convex function, satisfying $\forall s \geq 0,\ |sH'(s)| \leq CH(s)$ for some constant $C > 0$. Suppose that initially

$$\sup_{\epsilon>0} \int H\big(\frac{\rho_I^\epsilon}{\Psi^\epsilon}\big)\ \Psi^\epsilon \,\mathrm{d}x \leq C < \infty.$$

Then, the quantities

$$\begin{cases} \displaystyle\int H\big(\frac{\rho^\epsilon}{\Psi^\epsilon}\big)\ \Psi^\epsilon \,\mathrm{d}x, \\ \displaystyle\int_0^t \int H''\big(\frac{\rho^\epsilon}{\Psi^\epsilon}\big)\ \big|\nabla_x\big(\frac{\rho^\epsilon}{\Psi^\epsilon}\big)\big|^2 \,\mathrm{d}x\,\mathrm{d}s \end{cases}$$

are bounded on $(0,T)$, uniformly with respect to $0 < \epsilon < \epsilon_0$.

Proof. We integrate with respect to x the relation given by Proposition 5.1. We get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int H\big(\frac{\rho^\epsilon}{\Psi^\epsilon}\big)\ \Psi^\epsilon \,\mathrm{d}x + \int H''\big(\frac{\rho^\epsilon}{\Psi^\epsilon}\big)\ \big|\nabla_x\big(\frac{\rho^\epsilon}{\Psi^\epsilon}\big)\big|^2 \,\mathrm{d}x &= \int G\big(\frac{\rho^\epsilon}{\Psi^\epsilon}\big)\ \mathcal{T}_\epsilon(\Psi^\epsilon)\,\mathrm{d}x \\ &\leq (1+C)\int H\big(\frac{\rho^\epsilon}{\Psi^\epsilon}\big)\ \Psi^\epsilon\ \Big|\frac{\mathcal{T}_\epsilon(\Psi^\epsilon)}{\Psi^\epsilon}\Big|\,\mathrm{d}x. \end{aligned}$$

We have $\mathcal{T}_\epsilon(\Psi^\epsilon) = \Big(\mathcal{T}_2(\Theta+\epsilon\kappa) + \mathcal{T}_1(\kappa)\Big)(t,x;\frac{t}{\epsilon^2},\frac{x}{\epsilon})$ which is a bounded function. Then the Gronwall lemma concludes the proof. ☐

This result allows to improve the regularity of the limit function. Indeed, with $H(s) = s^2/2$, we get that $\mu^\epsilon(t,x) = \rho^\epsilon/\Psi^\epsilon$ is bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ (actually, we can obtain bound in any L^p). The double scale limit of μ^ϵ is

$$R(t,x;\tau,y)/\Theta(t,x;\tau,y) = \rho(t,x).$$

Hence μ^ϵ converges to $\rho(t,x)$, which thus belongs to $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$ (or $L^2(0,T;H_0^1(\Omega))$). In this class, there is uniqueness of the solution of the effective advection diffusion problem. As a consequence, we deduce that the whole sequence converges to a unique cluster point. Let us also remark that $\rho^\epsilon = \mu^\epsilon\ \Psi^\epsilon$ is bounded in $L^\infty(0,T;L^2(\Omega))$ which implies as in the proof of Theorem 4.7 the convergence of ρ^ϵ in $C^0([0,T];L^2(\Omega) - weak)$. We point out that we cannot have a convergence in L^2-strong due to the oscillating factor Ψ^ϵ.

Corollary 5.5. *With the same assumption as in Corollary 5.4, suppose that the initial data is bounded in $L^2(\Omega)$. Then, ρ^ϵ converges to ρ in $C^0([0,T];L^2(\Omega)-weak)$, unique solution in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$ of (4.9) with the corresponding boundary conditions and the initial condition ρ_I.*

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REFERENCES

[1] Allaire G., *Homogenization and two-scale convergence*, SIAM J. Math. Anal., 23 (1992) 1482–1518.
[2] Amirat Y., Hamdache K., Ziani A., *Homogénéisation d'équations hyperboliques du premier ordre et application aux écoulements miscibles en milieux poreux*, Ann. IHP, An. Non linéaire, 6 (1989) 397–417.
[3] Amirat Y., Hamdache K., Ziani A., *Some results on homogenization of convection-diffusion equations*, Arch. Rat. Mech. Anal., 114 (1989) 155–178.

[4] Avellaneda M., Majda A., *An integral representation and bounds on the effective diffusivity in passive advection by laminar and turbulent flows*, Comm. Math. Phys., 138 (1991) 339–391.

[5] Avellaneda M, Vergassola M., *Scalar transport in compressible flow*, Phys. D, 106 (1997) 148–166.

[6] Bensoussan A., Lions J.-L., Papanicolaou G., *Asymptotic analysis for periodic structures*, Studies in Math. and its Applications, vol. 5, North-Holland, 1978.

[7] Brenier Y., *Remarks on some linear hyperbolic equations with oscillatory coefficients*, in Third International Conference on Hyperbolic Problems: Theory, numerical methods and applications, Proceedings of the conference dedicated to Professor Heinz-Otto Kreiss on his 60th birthday held in Uppsala, June 11–15, 1990, Engquist B. and Gustafsson B. Eds., Vol. I, II, (Studentlitteratur, 1991) pp. 119–130.

[8] Capdeboscq Y., *Homogenization of a spectral problem with drift*, Proc. Roy. Soc. Edinburgh Sect. A, 132 (2002) 567–594; Announced in *Homogenization of a diffusion equation with drift*, C. R. Acad. Sci., 327 (1998) 807–812 .

[9] Capdeboscq Y., *Homogénéisation des modèles de diffusion en neutronique*, Thèse Université Paris 6, Juin 1999.

[10] Collet J.F., *Work in preparation.*

[11] Cioranescu D., Donato P., *An introduction to homogenization*, Oxford Lecture Series in Math. and Appl., 17, Oxford Univ. Press, 1999.

[12] Di Perna R., Majda A., *Oscillations and concentrations in weak solutions of the incompressible fluid equations*, Comm. Math. Phys., 108 (1987) 667–689.

[13] E W., *Homogenization of linear and nonlinear transport equations*, Comm. Pure Appl. Math., 65 (1990) 301–326.

[14] Evans L. C., *The perturbed test function method for viscosity solutions of nonlinear PDE*, Proc. Roy. Soc. Edinburgh Sect. A, 111 (1989) 359–375.

[15] Evans L. C., *Periodic homogenization of certain fully nonlinear partial differential equations*, Proc. Roy. Soc. Edinburgh Sect. A, 120 (1992) 245–265.

[16] Friedman A., *Partial differential equations of parabolic type*, Prentice-Hall, 1964.

[17] Golse F. *From kinetic to macroscopic models* in *Kinetic equations and asymptotic theory*, B. Perthame, L. Desvillettes Eds., Series in Appl. Math., 4, Gauthier-Villars, 2000, pp. 41-126.

[18] Goudon T., Mellet A., *Homogenization and diffusion asymptotics of the linear Boltzmann equation*, ESAIM-COCV, 9 (2003) 371–398.

[19] Hamdache K., *Homogénéisation non locale d'équations hyperboliques*, in *Non linear pde's and their applications, Collège de France seminar*, vol. XII., Pitman Res. Notes in Math., 302 (Longman Sci. Tech., Harlow, 1994) pp. 97–112.

[20] Hamdache K., *Equations de transport, homogénéisation*, Notes de cours, Université Bordeaux 1, 1994.

[21] Hou T., Xin X., *Homogenization of linear transport equations with oscillatory vector fields*, SIAM J. Appl. Math., 52 (1992) 34–45.

[22] Jikov V. V., Kozlov S. M., Oleinik O. A., *Homogenization of differential operators and integral functionals*, Springer-Verlag, 1994.

[23] Kesten H., Papanicolaou G., *A limit theorem for turbulent diffusion*, Comm. Math. Phys., 65 (1979) 97–128.

[24] Kramer P., Majda A., *Simplified models for turbulent diffusion: Theory, numerical modeling, and physical phenomena*, Physics Reports, 314 (1999) 237–574.

[25] Krein M., Rutman M., *Linear operator leaving invariant a cone in a Banach space*, AMS Transl., 10 (1962) 199–325.

[26] Kubo R., *H-Theorems for Markoffian Processes* in *Perspectives in Statistical Physics*, H.J. Raveché Ed., North Holland, 1981.

[27] Ladyzhenskaya O. A., Solonnikov V.A., Uralt'tseva N.N., *Linear and Quasilinear Equations of Parabolic Type*, Translation of mathematical monograph, 23, A.M.S., Providence, Rhode Island, 1968.

[28] Mascarenhas L., *A linear homogenization problem with time dependent coefficient*, Trans. AMS, 281 (1984) 179–195.

[29] Mc Laughlin D., Papanicolaou G., Pironneau O., *Convection of microstructure and related problems*, SIAM J. Appl. Math., 45 (1985) 780–797.

[30] N'Guetseng G., *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal., 20 (1989) 608–623.

[31] Tartar L., *Remarks on homogenization* in *Homogenization and effective moduli of material and media*, IMA Vol. in Math. and Appl., Springer, 1986, pp. 228–246.

[32] Tartar L., *Nonlocal effects induced by homogenization*, in *Partial differential equations and the calculus of variations, Essays of Mathematical analysis in honor of E. De Giorgi*, Vol. II, Progr. Nonlinear Differential Equations Appl., 2 , Birkhauser, 1989, pp. 925–938.