STATISTICAL STABILITY FOR TRANSPORT IN RANDOM MEDIA *

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Abstract. We go back to the analysis of the transport of (classical or quantum) particles in time-dependent random media as proposed in [F. Poupaud, A. Vasseur, J. Math. Pures Appl., 82, 711–748, 2003], where the convergence of the expectation of solutions of the Liouville equation (resp. Wigner equation) to solution of the Fokker-Planck equation (resp. Boltzmann equation) is established. Here we show that the convergence holds in probability. This result relies on certain scale separation properties of the scaling, as indicated by several counter-examples.

Key words. Random homogenization. Particle dynamics. Quantum transport. Wigner transform. Fokker–Planck equation. Boltzmann equation

AMS subject classifications. 82C31, 82C70, 70F45, 74A25

1. Introduction.

The motivation and positioning of this work can be explained by going back to the Boltzmann-Grad limit for the Lorentz gas. The Lorentz gas is a simple dynamical system describing the motion of a single point particle, which flights freely through an array of scatterers; when the particle hits a scatterer it bounces back to the domain according to Descartes' reflection law. H. Lorentz suggested [26] that this system, in a certain regime that prescribes the size of the scatterers and the average distance between the scatterers (Boltzmann-Grad regime), can be suitably described by a linear Boltzmann equation, where the collision operator retains the information of the interaction process between the particle and the scatterer. The justification of this asymptotic regime relies on the homogenization topics. Considering a *random* distribution of scatterers, the analysis of this problem dates back to G. Galavotti [16] who proved the convergence of the expectation of the particle distribution function associated to an initially given probability density on the single-particle phase-space. The result has been strengthened in [34], and next in [5] where the almost sure convergence of the particle distribution function is established. The situation is completely different when dealing with a *periodic* distribution of obstacles; a very elegant comparison argument pointed out by F. Golse [17] shows that a linear Boltzmann equation cannot describe the asymptotic Lorentz gas, thus contradicting Lorentz' conjecture in this specific framework. In fact, the limiting behavior in this case can be described by means of a "collisional" equation in an extended phase space [7, 27], that involves an additional time variable.

A different version of the problem replaces the scatterers by a potential (which can be interpreted as a "soft obstacle", see for instance [9] in this direction). Again, the question can be seen through the viewpoint of dynamical systems, considering the trajectories

$$\frac{\mathrm{d}}{\mathrm{d}t}X = V, \qquad \frac{\mathrm{d}}{\mathrm{d}t}V = F_\epsilon(t, X, V),$$

where (X, V) is the position/velocity pair of the particle, subjected to the force field F_ϵ. The features of the variations of the applied force are embodied into the parameter $0 < \epsilon \ll 1$. This viewpoint is developed in the seminal paper [22], see also [12]. Since $f(t, x, v) = \delta(x = X(t)) \otimes \delta(v = V(t))$ satisfies the Liouville equation

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F_\epsilon f) = 0,$$

it is equally relevant to consider the asymptotic behavior as $\epsilon \to 0$ of the solutions of this PDE, associated to an initial data $f\big|_{t=0} = f_0$, which describes the distribution of the initial states. The behavior still depends on the nature of the oscillations of the force field. The periodic case leads to effective transport equations, see [15, 8]. In contrast, when the applied force is a random variable, the asymptotic regime is driven by a Boltzmann or a Fokker-Planck equation, with coefficients characterized by the correlations of the randomly oscillating field, in the spirit of the pioneering computations of Taylor for the transport of passive particles

*Submitted to the editors DATE.

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[35], see also [21]. It is worth pointing out that the scaling issues are critical in this analysis: we refer the reader for instance to [11, 32, 1] for various aspects of these issues and their connection to the so-called anomalous diffusion regimes.

Similar questions can be addressed in the framework of quantum particles. Then, the Liouville equation is replaced by a system of Schrödinger equations, subjected to the random oscillating potential. H. Spohn derived this way a radiative transport equation, which thus describes the transport of electrons over a time-independent random array of impurities [33]. The result has been extended on arbitrarily large time interval by L. Erdös and H. T. Yau [13], while G. Bal, G. Papanicolaou and L. Ryzhik [3] considered the case of time-dependent random potentials satisfying the Markov property. We refer the reader to the review [2] for a thorough presentation of the physical motivations of this problem, and of the mathematical methods designed to handle the asymptotic analysis.

Most of these results are obtained by using a representation of the solution of the PDE by means of a series, iterating the Duhamel formula ([3] being an exception, with an approach based on the construction of approximate martingales and compactness arguments). Passing to the limit in this formula gives a quantity which turns out to define a solution of a Boltzmann (or Fokker-Planck) equation. This question has been revisited in [30] with a different viewpoint, exploiting a suitable time-decorrelation property of the oscillating potential. The approach, which works directly on the PDE and does not use any representation formula, is quite systematic and it has been applied to various contexts: plasma physics [25], semi-classical regimes and hyperbolic systems [6], quantum particles subjected to white noise [4], modeling of laden-flows [19], turbulent transport [20], etc. In these papers, the convergence (in a suitable weak sense) of the expectation of the particle distribution is established. One may wonder whether or not the convergence holds pointwise with respect to the random variable (or at least in probability).

This is the question addressed in this paper, which is organized as follows. In Section 2 we remind the basis of Poupaud-Vasseur's approach [30]. The simple example presented in Section 2 also shows that, in general, the almost sure convergence does not hold, and the convergence in probability neither. In Section 3, we go back to the Liouville equation as dealt with in [30]. We show that, for this problem, a *scale separation* property allows us to establish the convergence in probability, which strengthens the original result, a property which is referred to as the *statistical stability* of the limit particle distribution. The idea of the proof consists in considering non linear quantities, constructed by doubling the phase space variable. Identifying the limiting equation shows that non linear quantities pass to the limit, thus proving the statistical stability (see [2] for a similar approach). We describe further examples in Section 4; in particular we establish the strengthened convergence for the case of quantum transport.

2. Overview of the Poupaud-Vasseur (PV) approach, and a counter-example to the almost sure convergence. We shall deal with random fields. Namely, given a probability space $(\Omega, \mathscr{T}, \mu)$, with μ a σ-finite probability measure, we are going to consider random variables $X : \omega \to \mathbb{R}$. The expectation of such a random variable is nothing but

$$\mathbb{E}(X) = \int_\Omega X(\omega)\,\mathrm{d}\mu(\omega).$$

We will consider evolution equations where the coefficients are such random variables, considered as given. Accordingly, the solutions of the evolution equations are random variables too. As it is usual we will omit to denote the dependence with respect to the randomness variable $\omega \in \Omega$.

Roughly speaking, the problem we are interested in has the form of an evolution equation, whose unknown is u_ε, and which involves an oscillating quantity a_ε. The parameter $0 < \varepsilon \ll 1$ characterizes the scales of variation of the field. The equation is linear, but the analysis of the regime $\varepsilon \to 0$, leads to consider the limit of the product $a_\varepsilon u_\varepsilon$. We set up the basis of the PV approach [30] which relies on

- the Duhamel formula in order to analyze the behavior of $\mathbb{E}[a_\varepsilon u_\varepsilon]$; it leads to identify a leading term which is quadratic with respect to a_ε;
- an assumption on the time decorrelation of a_ε.

We start with a toy model, inspired from F. Poupaud's lecture notes [29]. Consider the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} u_\varepsilon(t) = \frac{\mathrm{i}}{\varepsilon} a(t/\varepsilon^2) u_\varepsilon(t) \tag{2.1}$$

where $a(t):\Omega\to\mathbb{R}$ is a random variable with zero mean

$$\mathbb{E}a(t)=0 \quad \text{for any } t\geq 0. \tag{2.2}$$

We suppose it has finite variance, and fulfils the following stationarity property

$$\mathbb{E}\big[a(t)a(s)\big]=R(s-t) \quad \text{for any } t,s\geq 0. \tag{2.3}$$

Obviously, the equation preserves the modulus

$$|u_\varepsilon(t)|=\Big|\exp\Big(\mathrm{i}\int_0^t a(s/\varepsilon^2)\,\mathrm{d}s\Big)\Big|\ |u(0)|=|u(0)|. \tag{2.4}$$

We assume that the initial data $u(0)$ is deterministic (it does not depend on the "hidden" variable $\omega\in\Omega$) and we wish to determine the asymptotic behavior of the expection value $\mathbb{E}u_\varepsilon(t)$ as ε goes to 0. The crucial assumption consists in the following finite time decorrelation hypothesis:

$$\begin{cases} a(t) \text{ and } a(s) \text{ decorrelate when } |t-s|\geq 1: \\ \mathbb{E}\big[a(t)\ a(s)\big]=R(s-t)=0 \quad \text{for } |t-s|\geq 1. \end{cases} \tag{2.5}$$

The analysis is based on the Duhamel formula:

$$u_\varepsilon(t)=u_\varepsilon(s)+\frac{\mathrm{i}}{\varepsilon}\int_s^t a(\sigma/\varepsilon^2)u_\varepsilon(\sigma)\,\mathrm{d}\sigma. \tag{2.6}$$

It has two immediate consequences:

a) first of all, with $s=0$ and bearing in mind that $u_\varepsilon(0)$ is deterministic, we realize that $u_\varepsilon(t)$ depends only on realization of $a(s/\varepsilon^2)$ for $0\leq s\leq t$; hence due to (2.5), $u_\varepsilon(t)$ and $a(t'/\varepsilon^2)$ are independent when $t'-t\geq\varepsilon^2$;

b) second of all, it already provides the continuity estimate

$$|u_\varepsilon(t)-u_\varepsilon(s)|\leq\|a\|_{L^\infty}|u(0)|\ \frac{|t-s|}{\varepsilon}, \tag{2.7}$$

where we have used (2.4).

Then, let us specify (2.6) to $s=t-\varepsilon^2$, which involves the decorrelation time scale:

$$u_\varepsilon(t)=u_\varepsilon(t-\varepsilon^2)+\frac{1}{\varepsilon}\int_{t-\varepsilon^2}^t \mathrm{i}a(\sigma/\varepsilon^2)u_\varepsilon(\sigma)\,\mathrm{d}\sigma. \tag{2.8}$$

It allows us to rewrite

$$\frac{a(t/\varepsilon^2)}{\varepsilon}u_\varepsilon(t)=\frac{a(t/\varepsilon^2)}{\varepsilon}u_\varepsilon(t-\varepsilon^2)+\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^t \mathrm{i}a(t/\varepsilon^2)a(\sigma/\varepsilon^2)u_\varepsilon(\sigma)\,\mathrm{d}\sigma.$$

In the right hand side, due to (2.2) and property a) stated above, the expectation of the first term vanishes

$$\begin{aligned}\mathbb{E}\Big(\frac{a(t/\varepsilon^2)}{\varepsilon}u_\varepsilon(t-\varepsilon^2)\Big) &= \mathbb{E}\frac{a(t/\varepsilon^2)}{\varepsilon}\ \mathbb{E}u_\varepsilon(t-\varepsilon^2) \quad \text{by decorrelation}\\ &= 0 \qquad\qquad\qquad\qquad\qquad \text{by (2.2)},\end{aligned}$$

while the second term is of order

$$\Big(\frac{1}{\varepsilon^2}\times\text{length of the integration interval}\Big)=\Big(\frac{1}{\varepsilon^2}\times\varepsilon^2\Big)=\mathcal{O}(1).$$

In particular, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}u_\varepsilon(t)=\mathrm{i}\mathbb{E}\frac{a(t/\varepsilon^2)}{\varepsilon}u_\varepsilon(t)=-\mathbb{E}\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^t a(t/\varepsilon^2)a(\sigma/\varepsilon^2)u_\varepsilon(\sigma)\,\mathrm{d}\sigma=\mathcal{O}(1).$$

To be more specific, by (2.4), we have

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}u_\varepsilon(t)\Big| \le \|a\|_{L^\infty}^2 |u(0)|.$$

Therefore, the family $\{t \mapsto \mathbb{E}u_\varepsilon(t),\ \varepsilon > 0\}$ is equibounded and equicontinuous and, by applying the Arzela-Ascoli theorem, we deduce that it belongs to a compact set of $C^0([0,T])$ for any $0 < T < \infty$. The next step consists in replacing in the last integral $u_\varepsilon(\sigma)$, with $t-\varepsilon^2 \le \sigma \le t$, by $\mathbb{E}u_\varepsilon(t)$. The error can indeed be controlled and vanishes as ε goes to 0, as a consequence of (2.5) and (2.7). To this end, we write

$$\begin{aligned}
&\mathbb{E}\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^t a(t/\varepsilon^2)a(\sigma/\varepsilon^2)u_\varepsilon(\sigma)\,\mathrm{d}\sigma\\
&\quad= \mathbb{E}\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^t a(t/\varepsilon^2)a(\sigma/\varepsilon^2)\,\mathrm{d}\sigma\ \mathbb{E}u_\varepsilon(t)\\
&\qquad+\mathbb{E}\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^t a(t/\varepsilon^2)a(\sigma/\varepsilon^2)\big(u_\varepsilon(\sigma)-u_\varepsilon(t-2\varepsilon^2)\big)\,\mathrm{d}\sigma\\
&\qquad+\mathbb{E}\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^t a(t/\varepsilon^2)a(\sigma/\varepsilon^2)\big(u_\varepsilon(t-2\varepsilon^2)-\mathbb{E}u_\varepsilon(t-2\varepsilon^2)\big)\,\mathrm{d}\sigma\\
&\qquad+\mathbb{E}\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^t a(t/\varepsilon^2)a(\sigma/\varepsilon^2)\big(\mathbb{E}u_\varepsilon(t-2\varepsilon^2)-\mathbb{E}u_\varepsilon(t)\big)\,\mathrm{d}\sigma.
\end{aligned}\tag{2.9}$$

Since, by (2.5), $u_\varepsilon(t-2\varepsilon^2)$ is independent of $\{a(\sigma/\varepsilon^2),\ t-\varepsilon^2\le\sigma\le t\}$, the third term can be recast as

$$\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^t \mathbb{E}\Big(a(t/\varepsilon^2)a(\sigma/\varepsilon^2)\Big)\mathbb{E}\big(u_\varepsilon(t-2\varepsilon^2)-\mathbb{E}u_\varepsilon(t-2\varepsilon^2)\big)\,\mathrm{d}\sigma = 0$$

and it vanishes. The second term is estimated by using (2.7): for any $t-\varepsilon^2\le\sigma\le t$, we have

$$|u_\varepsilon(\sigma)-u_\varepsilon(t-2\varepsilon^2)| \le \|a\|_{L^\infty}|u(0)|\frac{|\sigma-(t-2\varepsilon^2)|}{\varepsilon} \le 2\|a\|_{L^\infty}|u(0)|\varepsilon.$$

Hence the second term in (2.9) is dominated by

$$2\|a\|_{L^\infty}^3|u(0)| \times \frac{1}{\varepsilon^2}\times\varepsilon\times\varepsilon^2 = 2\|a\|_{L^\infty}^3|u(0)|\varepsilon.$$

A similar estimate holds for the last term in (2.9). We thus have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}u_\varepsilon(t) = \mathrm{i}\mathbb{E}\Big(\frac{a(t/\varepsilon^2)}{\varepsilon}u_\varepsilon(t)\Big) = -\mathbb{E}\left(\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^t a(t/\varepsilon^2)a(\sigma/\varepsilon^2)\,\mathrm{d}\sigma\right)\ \mathbb{E}u_\varepsilon(t) + r_\varepsilon,\tag{2.10}$$

with $r_\varepsilon \xrightarrow[\varepsilon\to 0]{} 0$.

We end up with the following statement.

Theorem 2.1. *The expectation $\mathbb{E}u_\varepsilon$ converges uniformly on $[0,T]$ to u, solution of the ODE*

$$\frac{\mathrm{d}}{\mathrm{d}t}u = -\lambda u$$

where the effective coefficient is

$$\lambda = \frac{1}{2}\int_{-\infty}^{+\infty} R(\tau)\,\mathrm{d}\tau \ge 0.$$

Proof. It only remains to identify the coefficient λ. First, let us check the positivity of λ which is not completely direct. The proof relies on the following observation: for any $F \in L^1(\mathbb{R})$, we have

$$\int_{\mathbb{R}} F(\tau)\,\mathrm{d}\tau = \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{+L}\int_{-L}^{+L} F(\sigma-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau.\tag{2.11}$$

Therefore, λ becomes

$$\begin{aligned}\lambda &= \lim_{L\to\infty}\frac{1}{4L}\int_{-L}^{+L}\int_{-L}^{+L} R(\sigma-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau = \lim_{L\to\infty}\frac{1}{4L}\int_{-L}^{+L}\int_{-L}^{+L}\mathbb{E}\big[a(\tau)a(\sigma)\big]\,\mathrm{d}\sigma\,\mathrm{d}\tau\\ &= \lim_{L\to\infty}\frac{1}{4L}\mathbb{E}\left(\int_{-L}^{+L} a(\sigma)\,\mathrm{d}\sigma\right)^2 \geq 0.\end{aligned}$$

Consequently, the modulus of the limit u is not conserved anymore, but it decays as time grows. It indicates that the passage to the limit and the stochasticity effects have induced a loss of irreversibility. We prove (2.11) by writing

$$\int_{\mathbb{R}} F(s)\,\mathrm{d}s = \frac{1}{2L}\int_{-L}^{+L}\int_{\mathbb{R}} F(s)\,\mathrm{d}s\,\mathrm{d}t = \frac{1}{2L}\int_{-L}^{+L}\left(\int_{\mathbb{R}} F(\sigma-t)\,\mathrm{d}\sigma\right)\,\mathrm{d}t.$$

Therefore, it suffices to show

$$\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{+L}\left(\int_{|\sigma|\geq L}|F(\sigma-t)|\,\mathrm{d}\sigma\right)\,\mathrm{d}t = 0.$$

Changing variables again, we reduce the problem to investigating the behavior of

$$\frac{1}{2L}\int_{-L}^{+L}\left(\int_{s+t\geq L}|F(s)|\,\mathrm{d}s\right)\,\mathrm{d}t,$$

for large L's, and similarly for the quantity obtained by replacing $s+t\geq L$ by $s+t\leq -L$. The Fubini theorem yields

$$\begin{aligned}&\frac{1}{2L}\int_0^{\infty}|F(s)|\left(\int_{\mathbb{R}}\mathbf{1}_{-L\leq t\leq L}\mathbf{1}_{L-s\leq t}\,\mathrm{d}t\right)\,\mathrm{d}s\\ &\quad= \frac{1}{2L}\int_0^{2L}|F(s)|\left(\int_{L-s}^{L}\mathrm{d}t\right)\,\mathrm{d}s + \frac{1}{2L}\int_0^{2L}|F(s)|\left(\int_{-L}^{L}\mathrm{d}t\right)\,\mathrm{d}s\\ &\quad= \int_0^{2L}|F(s)|\frac{s}{2L}\,\mathrm{d}s + \int_{2L}^{\infty}|F(s)|\,\mathrm{d}s,\end{aligned}$$

and we conclude by applying the Lebesgue theorem.

We go back to the definition of the effective coefficient. In (2.10), we make the following quantity appear

$$\frac{1}{\varepsilon^2}\mathbb{E}\int_{t-\varepsilon^2}^{t} a(t/\varepsilon^2)a(s/\varepsilon^2)\,\mathrm{d}s = \int_{t-\varepsilon^2}^{t} R\Big(\frac{s-t}{\varepsilon^2}\Big)\frac{\mathrm{d}s}{\varepsilon^2} = \int_{-1}^{0} R(\tau)\,\mathrm{d}\tau.$$

as a consequence of (2.3). Hence this quantity does not depend on ε anymore and it can be rewritten as

$$\begin{aligned}\int_0^1 R(-\tau)\,\mathrm{d}\tau &= \mathbb{E}\int_0^1 a(\tau)a(0)\,\mathrm{d}\tau = \mathbb{E}\int_0^1 a(0)a(\tau)\,\mathrm{d}\tau = \mathbb{E}\int_0^1 R(\tau)\,\mathrm{d}\tau\\ &= \frac{1}{2}\int_{-1}^{+1} R(\tau)\,\mathrm{d}\tau = \frac{1}{2}\int_{-\infty}^{+\infty} R(\tau)\,\mathrm{d}\tau\end{aligned}$$

due to the support property of the function R in (2.5). □

However with this simple example, we immediately see an obstruction to the almost sure convergence: we have

$$\mathbb{E}|u_\varepsilon(t)|^2 = 1$$

whose limit as $\varepsilon\to 0$ does not coincide with

$$|u(t)|^2 = \lim_{\varepsilon\to 0}|\mathbb{E}u_\varepsilon(t)|^2 = e^{-2\lambda t}|u(0)|^2.$$

For the same reason, the convergence does not even hold in probability (since $|u_\varepsilon(t) - u(t)| \geq |u_0|(1 - e^{-\lambda t})$, for any given $t > 0$, the set $\{\omega \in \Omega,\ |u_\varepsilon(t) - u(t)| \geq \delta\}$ has full measure provided δ is large enough).

The general strategy described on this basic example — use of time-compactness, the Duhamel formula and the decorrelation property of the oscillating field — applies to further, more intricate, contexts. In particular, we will be interested in partial differential equations describing the transport of classical or quantum particles. We shall see that the convergence in probability can be obtained, owing to certain scale separation induced by the scaling.

3. The Liouville equation with a randomly oscillating potential. We are interested in the Liouville equation

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \mathscr{E}_\varepsilon \cdot \nabla_v f_\varepsilon = 0, \tag{3.1}$$

endowed with the initial data

$$f_\varepsilon\big|_{t=0} = f_0.$$

The initial data f_0 is deterministic, while the force field $(t, x) \mapsto \mathscr{E}_\varepsilon(t, x)$ is a random variable described through three scales:

- η which measures the strength of the force field,
- τ which is the scale of the time variation of the force field,
- λ which is the scale of the space variation of the force field.

At the end of the day, certain relations will be imposed between these parameters, so that we can assume they are all functions of $\varepsilon > 0$, which is intended to become small. We thus denote

$$\mathscr{E}_\varepsilon(t, x) = \eta E(t/\tau, x/\lambda).$$

It is interesting to keep for a while all the parameters, but we shall consider the situation where

$$\eta \to +\infty, \qquad \tau \to 0, \qquad \lambda \to 0 \tag{3.2}$$

assuming that

$$\tau\eta^2 = 1, \qquad \tau = \lambda. \tag{3.3}$$

In other words, we can set

$$\eta = \frac{1}{\varepsilon}, \quad \tau = \varepsilon^2, \quad \lambda = \varepsilon^2, \quad 0 < \varepsilon \ll 1.$$

We recover the scaling dealt with in [30]: as $\varepsilon \to 0$, the sequence f_ε converges to the solution of a linear Boltzmann equation, where the collision kernel depends on R.

We shall assume throughout the discussion

(H1) $E \in W^{1,\infty}(\mathbb{R} \times \mathbb{R}^N)$,

(H2) $\mathbb{E}E(t, x) = 0$ for any $(t, x) \in \mathbb{R} \times \mathbb{R}^N$,

(H3) $\mathbb{E}\big[E(t, x) \otimes E(s, y)\big] = 0$ when $|t - s| \geq 1$, for any $x, y \in \mathbb{R}^N$; more precisely we have

$$\mathbb{E}\big[E(s, y) \otimes E(t, x)\big] = R(t - s, x - y)$$

with

$$\operatorname{supp}(R) \subset [-1, +1] \times \mathbb{R}^N, \qquad \lim_{|z|\to\infty} R(s, z) = 0, \qquad \partial_x^\alpha R \in C^0(\mathbb{R}; L^1 \cap L^\infty(\mathbb{R}^N)) \text{ for } |\alpha| \leq 3.$$

(H4) f_0 is deterministic and it lies in $L^1 \cap L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$.

Note that **(H3)** is two fold. On the one hand, it contains the decorrelation property, saying that $E(t, x)$ and $E(s, y)$ become independent when $|t - s|$ is large enough; this is crucial in the approach of [30]. As time evolves, a given configuration of the force field can be visited only once; hence the decorrelation property satisfied by E induces a decorrelation property on the solution. On the other hand, **(H3)** also contains a stationarity assumption, saying that the correlations between $E(t,x)$ and $E(s,y)$ depends on the differences $t-s$ and $x-y$; we only need a decay assumption of the correlations with respect to the space variable, which are not necessarily compactly supported in space. This decay assumption becomes important to investigate the statistical stability of the limit.

Remark 3.1. We point out that the Poupaud-Vasseur approach has been recently revisited to consider time-independent stochastic acceleration [37].

We start by writing the Duhamel formula

$$
\begin{aligned}
f_\varepsilon(t,x,v) &= f_\varepsilon(t-s,x-sv,v) + \int_{t-s}^{t} \mathscr{E}_\varepsilon(\sigma, x-(t-\sigma)v)\cdot\nabla_v f_\varepsilon(\sigma,x-(t-\sigma)v,v)\,\mathrm{d}\sigma \\
&= S_s f_\varepsilon(t-s,x,v) + \int_{t-s}^{t} S_{t-\sigma}\big[\mathscr{E}_\varepsilon\cdot\nabla_v f_\varepsilon\big](\sigma,x,v)\,\mathrm{d}\sigma,
\end{aligned}
\tag{3.4}
$$

where S_t stands for the semi-group associated to the transport operator

$$S_t g(x,v) = g(x-tv,v).$$

For further purposes, note that its adjoint is defined by

$$S_t^\star \phi(x,v) = \phi(x+tv,v),$$

and we have

$$\nabla_v\big[S_t^\star\phi(x,v)\big] = (\nabla_v\phi)(x+tv,v) + t(\nabla_x\varphi)(x+tv,v) = S_t^\star(\nabla_v + t\nabla_x)\varphi(x,v).$$

By using (3.4) with $s=t$, we see that $f_\varepsilon(t)$ only depends on the realizations of $\mathscr{E}_\varepsilon(\sigma)$ for $0\le\sigma\le t$. Since the initial data is deterministic, we deduce the following independence property (which does not use (3.2)–(3.3)).

LEMMA 3.2. *Assume* ***(H3)****. For any $h\ge\tau$, $f_\varepsilon(t)$ and $\mathscr{E}_\varepsilon(t+h)$ are independent.*

Owing to (3.4), we can compute the force term in the Liouville equation with the formula

$$\mathscr{E}_\varepsilon(t,x)f_\varepsilon(t,x,v) = \mathscr{E}_\varepsilon(t,x)S_s f_\varepsilon(t-s,x,v) + \int_{t-s}^{t}\mathscr{E}_\varepsilon(t,x)S_{t-\sigma}\big[\mathscr{E}\cdot\nabla_v f_\varepsilon\big](\sigma,x,v)\,\mathrm{d}\sigma.$$

In fact, we shall work on a weak form of this relation. Let $\varphi\in C_c^\infty(\mathbb{R}^N\times\mathbb{R}^N)$. We have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\iint f_\varepsilon(t,x,v)\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x \\
&\quad= \iint f_\varepsilon(t,x,v)v\cdot\nabla_x\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x + \iint f_\varepsilon(t,x,v)\mathscr{E}_\varepsilon(t,x)\cdot\nabla_v\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x,
\end{aligned}
$$

where only the last term can present difficulties for the asymptotic issues $\varepsilon\to 0$. It can be cast as follows

$$
\begin{aligned}
&\iint \mathscr{E}_\varepsilon(t,x)f_\varepsilon(t,x,v)\cdot\nabla_v\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x \\
&= \iint f_\varepsilon(t-s,x,v)S_s^\star\big[\mathscr{E}_\varepsilon(t)\cdot\nabla_v\varphi\big](x,v)\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad + \int_{t-s}^{t}\iint \nabla_v\cdot\big(\mathscr{E}_\varepsilon(\sigma)S_{t-\sigma}^\star\big[\mathscr{E}_\varepsilon(t)\cdot\nabla_v\varphi\big]\big)(x,v)\ f_\varepsilon(\sigma,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\sigma \\
&= \iint f_\varepsilon(t-s,x,v)S_s^\star\big[\mathscr{E}_\varepsilon(t)\cdot\nabla_v\varphi\big](x,v)\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad + \int_{t-s}^{t}\iint \mathscr{E}_\varepsilon(\sigma,x)\cdot S_{t-\sigma}^\star\big[(\nabla_v+(t-\sigma)\nabla_x)(\mathscr{E}_\varepsilon(t)\cdot\nabla_v\varphi)\big](x,v)\ f_\varepsilon(\sigma,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\sigma.
\end{aligned}
\tag{3.5}
$$

We observe that

$$
\begin{aligned}
&(\nabla_v+(t-\sigma)\nabla_x)\big(\mathscr{E}_\varepsilon(t)\cdot\nabla_v\varphi\big)(x,v) \\
&\quad= \sum_{j=1}^{N}(\mathscr{E}_\varepsilon)_j(t,x)(\nabla_v+(t-\sigma)\nabla_x)\partial_{v_j}\varphi(x,v) + (t-\sigma)\sum_{j=1}^{N}\partial_{v_j}\varphi\nabla_x(\mathscr{E}_\varepsilon)_j(t,x).
\end{aligned}
\tag{3.6}
$$

Let us discuss the order of the different terms with respect to the scaling parameters, and the role of (3.2)–(3.3). We shall use (3.5) with $s = \tau$ and take the expectation. By virtue of Lemma 3.2 and **(H2)**, the expectation of $\iint f_\varepsilon(t-\tau,x,v) S^\star_\tau\big[\mathscr{E}_\varepsilon(t)\varphi\big](x,v)\,\mathrm{d}v\,\mathrm{d}x$ vanishes. The second term in the right hand side of (3.5) is quadratic with respect to the force field and integrated over a time interval with amplitude τ, which yields a factor $\tau\eta^2$. This is the order of the term with a velocity derivative coming from (3.6). Concerning the term involving $\nabla_x\varphi$, it is multiplied by $t-\sigma$, which, again, is of order τ; it yields a contribution of order $\tau^2\eta^2$. Finally, we have

$$(t-\sigma)\nabla_x\mathscr{E}_\varepsilon(t,x) = \eta\frac{t-\sigma}{\lambda}\ (\nabla_x E)\Big(\frac{t}{\tau},\frac{x}{\lambda}\Big).$$

It yields a contribution of order $\frac{\tau^2\eta^2}{\lambda}$.

From now on, we assume the scaling (3.2)–(3.3). With these relations, the leading term when taking the expectation in (3.5) thus reads

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathbb{E}\int_{t-\varepsilon^2}^t\iint E\Big(\frac{\sigma}{\varepsilon^2},\frac{x}{\varepsilon^2}\Big)\otimes E\Big(\frac{t}{\varepsilon^2},\frac{x+(t-\sigma)v}{\varepsilon^2}\Big)\\
&\qquad\qquad : (D_v^2\varphi)(x+(t-\sigma)v,v) f_\varepsilon(\sigma,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\sigma\\
&+\frac{1}{\varepsilon^2}\mathbb{E}\int_{t-\varepsilon^2}^t\iint\frac{t-\sigma}{\varepsilon^2}(\nabla_x E)\Big(\frac{t}{\varepsilon^2},\frac{x+(t-\sigma)v}{\varepsilon^2}\Big)\ E\Big(\frac{\sigma}{\varepsilon^2},\frac{x}{\varepsilon^2}\Big)\\
&\qquad\qquad \cdot(\nabla_v\varphi)(x+(t-\sigma)v,v) f_\varepsilon(\sigma,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\sigma.
\end{aligned}\tag{3.7}$$

In order to identify the limit equation, we combine two ingredients. First of all, we replace $f_\varepsilon(\sigma,x,v)$ in (3.7), where $t-\tau\leq\sigma\leq t$, by $\mathbb{E}f_\varepsilon(t,x,v)$; this is possible at the price of error terms that tend to 0 as $\tau\to 0$, in the same spirit as in (2.9) and (2.10). Second of all, we use the information properties assumed on the statistics of the force field, see **(H3)**. We have already used the fact that the correlation function R is supported in $[-1,+1]\times\mathbb{R}^N$; here we additionally make use of the stationarity assumption with respect to the space variable, with a correlation function that decays at infinity. We refer the reader to [30, Example 2] for the construction of potentials that satisfy such an assumption (for instance with R compactly supported in both time and space variables).

We are thus led to compute the coefficients

$$\mathrm{A}_\varepsilon[\varphi](x,v) = \frac{1}{\varepsilon^2}\mathbb{E}\int_{t-\varepsilon^2}^t E\Big(\frac{\sigma}{\varepsilon^2},\frac{x}{\varepsilon^2}\Big)\otimes E\Big(\frac{t}{\varepsilon^2},\frac{x+(t-\sigma)v}{\varepsilon^2}\Big) : (D_v^2\varphi)(x+(t-\sigma)v,v)\,\mathrm{d}\sigma,$$

and

$$\mathrm{B}_\varepsilon[\varphi](x.v) = \frac{1}{\varepsilon^2}\mathbb{E}\int_{t-\varepsilon^2}^t\frac{t-\sigma}{\varepsilon}(\nabla_x E)\Big(\frac{t}{\varepsilon^2},\frac{x+(t-\sigma)v}{\varepsilon^2}\Big)\ E\Big(\frac{\sigma}{\varepsilon^2},\frac{x}{\varepsilon^2}\Big)\cdot(\nabla_v\varphi)(x+(t-\sigma)v,v)\,\mathrm{d}\sigma.$$

With **(H3)**, we obtain

$$\begin{aligned}
\mathrm{A}_\varepsilon[\varphi](x,v) &= \frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^t R\Big(\frac{t-\sigma}{\varepsilon^2},\frac{(t-\sigma)v}{\varepsilon^2}\Big) : (D_v^2\varphi)(x+(t-\sigma)v,v)\,\mathrm{d}\sigma\\
&= \int_0^1 R(\theta,\theta v) : (D_v^2\varphi)(x+\varepsilon^2\theta v,v)\,\mathrm{d}\theta.
\end{aligned}$$

We conclude that

$$\mathrm{A}_\varepsilon[\varphi](x,v)\xrightarrow[\varepsilon\to 0]{}\mathrm{A}[\varphi](x,v) = \int_0^1 R(\theta,\theta v)\,\mathrm{d}\theta : D_v^2\varphi(x,v).$$

Next, we remark that

$$\begin{aligned}
\partial_{v_i}\Big[R\Big(\frac{t-\sigma}{\varepsilon^2},\frac{(t-\sigma)v}{\varepsilon^2}\Big)\Big] &= \frac{(t-\sigma)}{\varepsilon}(\partial_{x_i}R)\Big(\frac{t-\sigma}{\varepsilon^2},\frac{(t-\sigma)v}{\varepsilon^2}\Big)\\
&= \partial_{v_i}\mathbb{E}\Big[E\Big(\frac{\sigma}{\varepsilon^2},\frac{x}{\varepsilon^2}\Big)\otimes E\Big(\frac{t}{\varepsilon^2},\frac{x+(t-\sigma)v}{\varepsilon^2}\Big)\Big]\\
&= \mathbb{E}\Big[E\Big(\frac{\sigma}{\varepsilon^2},\frac{x}{\varepsilon^2}\Big)\otimes \partial_{v_i}\Big(E\Big(\frac{t}{\varepsilon^2},\frac{x+(t-\sigma)v}{\varepsilon^2}\Big)\Big)\Big]\\
&= \mathbb{E}E\Big(\frac{\sigma}{\varepsilon^2},\frac{x}{\varepsilon^2}\Big)\otimes\frac{(t-\sigma)}{\lambda}(\partial_{x_i}E)\Big(\frac{t}{\varepsilon^2},\frac{x+(t-\sigma)v}{\varepsilon^2}\Big).
\end{aligned}$$

We deduce that

$$\begin{aligned}
\mathrm{B}_\varepsilon[\varphi](x,v) &= \sum_{i,j=1}^N \int_0^1 \partial_{v_i}\Big[R_{ij}\Big(\theta,\frac{\tau\theta v}{\lambda}\Big)\Big](\partial_{v_j}\varphi)(x+\varepsilon^2\theta v,v)\,\mathrm{d}\theta\\
&\xrightarrow[\tau\to 0]{} \mathrm{B}[\varphi](x,v)=\sum_{i,j=1}^N \partial_{v_i}\Big[\int_0^1 R_{ij}(\theta,\theta v)\,\mathrm{d}\theta\Big]\partial_{v_j}\varphi(x,v).
\end{aligned}$$

It only remains to explain how the compactness of $\mathbb{E}f_\varepsilon$ can be obtained. We now make use of **(H4)**. Since the field $(t,x,v)\mapsto(v,\mathscr{E}_\varepsilon(t,x))$ is smooth, we can define the associated characteristics $(t,s,x,v)\mapsto(\xi(t,s,x,v),\Theta(t,s,x,v))$, solution of the ODE system

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\xi(t)=\Theta(t), \quad &&\frac{\mathrm{d}}{\mathrm{d}t}\Theta(t)=\mathscr{E}_\varepsilon(t,\xi(t))\\
&\xi(s)=x, &&\Theta(s)=v.
\end{aligned}$$

Then, the solution of (3.1) is simply given by

$$f_\varepsilon(t,x,v)=f_0\big(\xi(0,t,x,v),\Theta(0,t,x,v)\big),$$

which clearly implies

$$\|f_\varepsilon(t,\cdot)\|_{L^p(\mathbb{R}^N\times\mathbb{R}^N)}\le\|f_0\|_{L^p(\mathbb{R}^N\times\mathbb{R}^N)}$$

(using the fact that $\mathrm{div}_{x,v}(v,\mathscr{E}_\varepsilon(t,x))=0$). Next, let $\varphi\in C_c^\infty(\mathbb{R}^N\times\mathbb{R}^N)$, with $\mathrm{supp}(\varphi)\subset B(0,M)\times B(0,M)$. We have

$$\Big|\iint f_\varepsilon(t,x,v)\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x\Big|\le\|f_0\|_{L^p}\|\varphi\|_{L^{p'}} \tag{3.8}$$

for any $1\le p\le\infty$, $1/p+1/p'=1$, and, by using (3.5) and Lemma 3.2,

$$\begin{aligned}
&\Big|\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\iint f_\varepsilon(t,x,v)\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x\Big|\\
&\quad\le\Big|\mathbb{E}\iint f_\varepsilon(t,x,v)v\cdot\nabla_x\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x\Big|+\Big|\mathbb{E}\iint f_\varepsilon(t,x,v)\mathscr{E}_\varepsilon(t,x)\cdot\nabla_v\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x\Big|\\
&\quad\le\|f_\varepsilon(t,\cdot)\|_{L^1}M\|\nabla_x\varphi\|_{L^\infty}\\
&\qquad+\Big|\mathbb{E}\int_{t-\varepsilon^2}^t\iint f_\varepsilon(\sigma,x,v)\mathscr{E}_\varepsilon(\sigma,x)\cdot S^\star_{t-\sigma}\big[(\nabla_v+(t-\sigma)\nabla_x)\mathscr{E}_\varepsilon(t)\cdot\nabla_v\varphi\big](x,v)\,\mathrm{d}v\,\mathrm{d}x\Big|\\
&\quad\le\|f_\varepsilon(t,\cdot)\|_{L^1}M\|\nabla_x\varphi\|_{L^\infty}+3\varepsilon^2\|\varphi\|_{W^{2,\infty}}\frac{\|E\|^2_{W^{1,\infty}}}{\varepsilon^2}\|f_\varepsilon(t,\cdot)\|_{L^1}\\
&\quad\le\|f_0\|_{L^1}\|\varphi\|_{W^{2,\infty}}\|E\|^2_{W^{1,\infty}}(3+M).
\end{aligned}$$

By virtue of the Arzela-Ascoli theorem, the family $\big\{t\mapsto\mathbb{E}\iint f_\varepsilon(t,x,v)\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x,\varepsilon>0\big\}$ lies in a compact set of $C^0([0,T])$ for any $0<T<\infty$. Coming back to (3.8), this conclusion applies for a trial function $\varphi\in L^{p'}(\mathbb{R}^N\times\mathbb{R}^N)$. We finally appeal to the diagonal argument and we conclude that we can find $f\in$

$L^\infty(0,T;L^p(\mathbb{R}^N\times\mathbb{R}^N))$ and we can extract a subsequence, still labelled by ε, such that for any $\varphi\in L^{p'}(\mathbb{R}^N\times\mathbb{R}^N)$, we have

$$\lim_{\varepsilon\to 0}\mathbb{E}\iint f_\varepsilon(t,x,v)\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x=\iint f(t,x,v)\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x\quad\text{uniformly on }[0,T].$$

We conclude that the limit of the leading term (3.7) reads

$$\iint\big(A[\varphi]+B[\varphi]\big)(x,v)f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x=\iint\nabla_v\cdot\Big(\int_0^1 R(\theta,\theta v)\,\mathrm{d}\theta\nabla_v\varphi(x,v)\Big)f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x.$$

We recap the analysis in the following statement [30, Theorem 3.1 and Proposition 3.1 for the properties of the diffusion matrix].

Theorem 3.3. *We suppose* ***(H1)–(H4)***. *Then, up to a subsequence,* $\mathbb{E}f_\varepsilon$ *converges in* $C^0([0,T];L^p(\mathbb{R}^N\times\mathbb{R}^N)-\text{weak})$ *to* $f\in L^\infty(0,T;L^p(\mathbb{R}^N\times\mathbb{R}^N))$ *solution of*

$$\partial_t f+v\cdot\nabla_x f-\nabla_v\cdot(D\nabla_v f)=0,\qquad f\big|_{t=0}=f_0,$$

where

$$D(v)=\int_0^1 R(\theta,\theta v)\,\mathrm{d}\theta.$$

The matrix D is symmetric, and non negative; its coefficients are even and belong to $L^\infty(\mathbb{R}^N)$. Furthermore, $\partial_v^\alpha D_{ij}\in L^\infty(\mathbb{R}^N)$ for $|\alpha|\le 3$ and $\lim_{|v|\to\infty}D(v)=0$.

The regularity of the diffusion matrix and its behavior for large velocities is a direct consequence of **(H3)**. We turn to discuss the statistical stability: the main result of the paper states as follows.

Theorem 3.4. *In Theorem 3.3,* f_ε^ω *converges in probability in* $C^0([0,T];L^p(\mathbb{R}^N\times\mathbb{R}^N)-\text{weak})$ *to* f.

Let us explain the meaning of the convergence we are using in this statement. Pick $\varphi\in L^{p'}(\mathbb{R}^N\times\mathbb{R}^N)$ and set

$$\mathscr{X}_\varepsilon(t)=\iint f_\varepsilon(t,x,v)\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x,\tag{3.9}$$

which is a random variable. We shall establish that $\mathscr{X}_\varepsilon(t)$ converges in probability to $\mathscr{X}(t)$ as $\varepsilon\to 0$ which means that

$$\sup_{0\le t\le T}\mathbb{P}(|\mathscr{X}_\varepsilon(t)-\mathscr{X}(t)|\ge\delta)\xrightarrow[\varepsilon\to 0]{}0$$

holds for any $\delta>0$. The key of the proof consists in studying the correlations between $f_\varepsilon(t,x,v)$ and $f_\varepsilon(t,y,w)$, remarking that the product of these quantities satisfies a Liouville equation in the phase space obtained by doubling the variables (see also [2] for a similar approach). We can thus apply the previous analysis to this extended equation, which will allow us to justify that, at the limit, "the product of the expectations coincides with the expectation of the product".

Proof. Let us introduce the extended phase-space variables

$$X=(x,y)\in\mathbb{R}^N\times\mathbb{R}^N,\qquad V=(v,w)\in\mathbb{R}^N\times\mathbb{R}^N$$

and set

$$F_\varepsilon(t,X,V)=f_\varepsilon(t,x,v)f_\varepsilon(t,y,w).$$

It satisfies

$$\partial_t F_\varepsilon+V\cdot\nabla_x F_\varepsilon-\mathfrak{E}_\varepsilon\cdot\nabla_v F=0,$$

where

$$\mathfrak{E}_\varepsilon(t,X)=\begin{pmatrix}\mathscr{E}_\varepsilon(t,x)\\ \mathscr{E}_\varepsilon(t,y)\end{pmatrix}\in\mathbb{R}^{2N}.$$

The $(2N \times 2N)$ correlation matrix is thus defined blockwise by

$$\mathbb{E}\big[\mathfrak{E}_\varepsilon(t,X)\otimes\mathfrak{E}_\varepsilon(s,X')\big] = \eta^2 \begin{pmatrix} R\big(\frac{t-s}{\varepsilon^2},\frac{x-x'}{\varepsilon^2}\big) & R\big(\frac{t-s}{\varepsilon^2},\frac{x-y'}{\varepsilon^2}\big) \\ R\big(\frac{t-s}{\varepsilon^2},\frac{y-x'}{\varepsilon^2}\big) & R\big(\frac{t-s}{\varepsilon^2},\frac{y-y'}{\varepsilon^2}\big) \end{pmatrix} = \mathfrak{R}\Big(\frac{t-s}{\varepsilon^2},\frac{X}{\varepsilon^2},\frac{X'}{\varepsilon^2}\Big).$$

In particular, the decorrelation property in **(H3)** still holds for the field $\mathfrak{E}_\varepsilon$. However, the stationarity property in space in **(H3)** is not satisfied by the extended matrix $\mathfrak{R}$: the extra-diagonal terms do not depend on the difference $(X-X') = (x-x', y-y')$.

Nevertheless, we can repeat the analysis made above, that includes the identification of the leading term, with the same form as (3.7). We can still replace $F_\varepsilon(\sigma,X,V)$ by $\mathbb{E}F_\varepsilon(t,X,V)$, up to small error terms and we are thus led to study

$$\begin{aligned}
&\mathfrak{A}_\varepsilon[\varphi](X,V) = \\
&\quad \frac{1}{\varepsilon^2}\mathbb{E}\int_{t-\varepsilon^2}^t \mathfrak{E}\Big(\frac{\sigma}{\varepsilon^2},\frac{X}{\varepsilon^2}\Big)\otimes\mathfrak{E}\Big(\frac{t}{\varepsilon^2},\frac{X+(t-\sigma)V}{\varepsilon^2}\Big) : (D_V^2\varphi)(X+(t-\sigma)V,V)\,\mathrm{d}\sigma \\
&\quad = \frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^t \mathfrak{R}\Big(\frac{t-\sigma}{\varepsilon^2},\frac{X}{\varepsilon^2},\frac{X+(t-\sigma)V}{\varepsilon^2}\Big) : (D_V^2\varphi)(X+(t-\sigma)V,V)\,\mathrm{d}\sigma \\
&\quad = \int_0^1 \mathfrak{R}\Big(\theta,\frac{X}{\varepsilon^2},\frac{X+\varepsilon^2\theta V}{\varepsilon^2}\Big) : (D_V^2\varphi)(X+\varepsilon^2\theta V,V)\,\mathrm{d}\theta.
\end{aligned}$$

The matrix reads

$$\mathfrak{R}\Big(\theta,\frac{X}{\varepsilon^2},\frac{X+\varepsilon^2\theta V}{\varepsilon^2}\Big) = \begin{pmatrix} R(\theta,\theta v) & R\big(\theta,\frac{x-y}{\varepsilon^2}-\theta w\big) \\ R\big(\theta,\frac{y-x}{\varepsilon^2}-\theta v\big) & R(\theta,\theta w)\end{pmatrix}.$$

For almost every $(X,V) = (x,y,v,w)\in\mathbb{R}^{2N}\times\mathbb{R}^{2N}$, it tends to the diagonal matrix

$$\begin{pmatrix} R(\theta,\theta v) & 0 \\ 0 & R(\theta,\theta w)\end{pmatrix}$$

as $\varepsilon\to 0$, owing to the fact that $\lim_{|z|\to\infty} R(t,z) = 0$ from **(H3)**. (Note also that the stationarity property with respect to the space variable plays a role to obtain the expression of the diagonal terms.) Thus, we obtain

$$\lim_{\varepsilon\to 0}\mathfrak{A}_\varepsilon[\varphi](X,V) = \mathfrak{D}(V) : D_V^2\varphi(X,V),$$

with

$$\mathfrak{D}(V) = \int_0^1 \begin{pmatrix} R(\theta,\theta v) & 0 \\ 0 & R(\theta,\theta w)\end{pmatrix}\mathrm{d}\theta.$$

Similarly, we check that

$$\begin{aligned}
&\mathfrak{B}_\varepsilon[\varphi](X,V) \\
&\quad = \frac{1}{\varepsilon^2}\mathbb{E}\int_{t-\varepsilon^2}^t \frac{t-\sigma}{\varepsilon^2}(\nabla_X\mathfrak{E})\Big(\frac{t}{\varepsilon^2},\frac{X+(t-\sigma)V}{\varepsilon^2}\Big)\ \mathfrak{E}\Big(\frac{\sigma}{\varepsilon^2},\frac{x}{\varepsilon^2}\Big)\cdot(\nabla_V\varphi)(X+(t-\sigma)v,V)\,\mathrm{d}\sigma,
\end{aligned}$$

tends to

$$\sum_{i,j=1}^N \partial_{V_i}\big[\mathfrak{D}_{ij}(V)\big]\partial_{V_j}\varphi(X,V).$$

We conclude that, possibly at the price of extracting a suitable subsequence, $\mathbb{E}F_\varepsilon(t,X,V)$ converges to $F(t,X,V)$ which satisfies

$$\partial_t F + V\cdot\nabla_X F - \nabla_V\cdot(\mathfrak{D}\nabla_V F) = 0, \quad F(0,X,V) = f_0(x,v)f_0(y,w).$$

Then we observe that $f(t,x,v)f(t,y,w)$ satisfies the same equation, with the same initial data. By uniqueness (see the discussion below), we thus get $F(t,X,V) = f(t,x,v)f(t,y,w)$.

We explain how this result implies the convergence in probability. Pick $\varphi \in L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$ and consider the random variable defined by (3.9). We have shown that, on the one hand,

$$\mathbb{E}\mathscr{X}_\varepsilon(t) \xrightarrow[\varepsilon\to 0]{} \mathscr{X}(t) = \iint f(t,x,v)\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x \quad \text{uniformly on } [0,T],$$

and, on the other hand

$$\mathbb{E}\big[\mathscr{X}_\varepsilon(t)^2\big] = \mathbb{E}\Big[\iiiint F_\varepsilon(t,X,V)\varphi(x,v)\varphi(y,w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}w\Big] \xrightarrow[\varepsilon\to 0]{} \mathscr{X}(t)^2 = \Big(\iint f(t,x,v)\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x\Big)^2.$$

It follows that

$$\mathbb{E}\big[|\mathscr{X}_\varepsilon(t) - \mathbb{E}\mathscr{X}_\varepsilon(t)|^2\big] = \mathbb{E}\big[\mathscr{X}_\varepsilon(t)^2 - \big(\mathbb{E}\mathscr{X}_\varepsilon(t)\big)^2\big] \xrightarrow[\varepsilon\to 0]{} 0.$$

We infer that

$$\mathbb{E}\big[|\mathscr{X}_\varepsilon(t) - \mathscr{X}(t)|^2\big] \leq 2\Big(\mathbb{E}\big[|\mathscr{X}_\varepsilon(t) - \mathbb{E}\mathscr{X}_\varepsilon(t)|^2\big] + \big[|\mathbb{E}\mathscr{X}_\varepsilon(t) - \mathscr{X}(t)|^2\big]\Big) \xrightarrow[\varepsilon\to 0]{} 0.$$

In other words $\mathscr{X}_\varepsilon(t)$ converges to $\mathscr{X}(t)$ as $\varepsilon \to 0$ in $L^2(\Omega, \mathrm{d}\mu(\omega))$. By using the Bienaymé-Tchebyschev inequality

$$\mathbb{P}(|\mathscr{X}_\varepsilon(t) - \mathscr{X}(t)| \geq \delta) \leq \frac{\mathbb{E}[|\mathscr{X}_\varepsilon(t) - \mathscr{X}(t)|^2]}{\delta^2},$$

we deduce that $\mathscr{X}_\varepsilon(t)$ converges to $\mathscr{X}(t)$ in probability.

The validity of the analysis thus relies on a uniqueness statement for the limiting equation, which can be cast as a Fokker-Planck equation on $(0,\infty) \times \mathbb{R}^{2d}$

$$\partial_t f + \nabla_y \cdot (\mathcal{C} f) - \nabla_y \cdot (\mathcal{A} \nabla_y f) = 0$$

where $y \in \mathbb{R}^{2d}$ stands for the extended variable (x,v), and

$$\mathcal{C}(y) = \begin{pmatrix} v \\ 0 \end{pmatrix}, \qquad \mathcal{A}(y) = \left(\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & D(v) \end{array}\right).$$

Note that $\nabla_y \cdot \mathcal{C} = 0$. The difficulty is two-fold; it relies

- on the degeneracy of the diffusion coefficient $\mathcal{A}$: it is a non-negative matrix, but it does not fulfil an ellipticity criterion, even locally (the best that can be expected is to find, for any $0 < M < \infty$, a constant α_M such that $D(v)\xi \cdot \xi \geq \alpha_M |\xi|^2$ holds for any $|v| \leq M$ and $\xi \in \mathbb{R}^N$);
- on the fact that the limiting process provides a solution in, say, $L^\infty((0,\infty); L^1 \cap L^\infty(\mathbb{R}^{2d}))$, satisfying some weak continuity with respect to time, but it tells nothing about $\mathcal{A}^{1/2}\nabla_y f$, a quantity which would naturally appear in energy estimates for the Fokker-Planck equation.

The issue of the uniqueness in such a framework has been addressed quite recently: we refer the reader for instance to [23] where conditions on $\mathcal{A}^{1/2}\nabla_y f$ appeared, and to [14] where a connection with SDEs is established, with statements that do not require the a priori estimate on $\nabla_y f$ (an issue for uniqueness even when ellipticity is enforced). For our purposes, we directly use the uniqueness results in [36, Theorem 3.1 & Theorem A.7], see also [31, Theorem 1.1]. The analysis relies on renormalization techniques, in the spirit of the pioneering work [10] on transport equations. (Here, the coefficients of $\mathcal{C}$ are unbounded, but keep a linear growth, while first, second and third order derivatives of D are bounded, which permits to adapt the arguments of these references.) ∎

Remark 3.5. We warn the reader that the obtained convergence is not enough to establish the almost sure convergence, even for a subsequence. Indeed, it would be tempting from the knowledge of $\mathbb{E}[|\mathscr{X}_\varepsilon(t) - \mathscr{X}(t)|^2) \to 0$, to extract a subsequence which converges almost surely, by virtue of the partial reciprocal to the Lebesgue theorem, see e. g. [18, Lemma 3.31]. However, the drawback of the reasoning is that the extracted subsequence would depend on the time variable, and it is not clear that we can find an extraction that works for all $t \in [0,T]$. We can only say that $\int_0^T \mathbb{E}[|\mathscr{X}_\varepsilon(t) - \mathscr{X}(t)|^2]\,\mathrm{d}t = \mathbb{E}\big[\int_0^T |\mathscr{X}_\varepsilon(t) - \mathscr{X}(t)|^2\,\mathrm{d}t\big] \to 0$ as $\varepsilon \to 0$. This allows us to extract a subsequence such that $\int_0^T |\mathscr{X}_\varepsilon(t) - \mathscr{X}(t)|^2\,\mathrm{d}t \to 0$ almost surely. The convergence can be slightly improved by adding further assumptions of the force field in order to get some uniform continuity. Namely, let us assume that $E \in L^2(\mathbb{R}; L^\infty(\mathbb{R}^N))$. Then, we get, for instance,

$$\begin{aligned}
|\mathscr{X}_\varepsilon(t+h) - \mathscr{X}_\varepsilon(t)| &\leq \|f_\varepsilon\|_{L^1}(\|v\cdot\nabla_x\phi\|_{L^\infty} + \|\nabla_v\phi\|_{L^\infty})\left(h + \int_t^{t+h}\Big\|E\Big(\frac{\sigma}{\varepsilon^2},\cdot\Big)\Big\|_{L^\infty}\frac{\mathrm{d}\sigma}{\varepsilon}\right)\\
&\leq \|f_\varepsilon\|_{L^1}(\|v\cdot\nabla_x\phi\|_{L^\infty} + \|\nabla_v\phi\|_{L^\infty})\left(h + \sqrt{h}\left(\int_{\mathbb{R}}\Big\|E(s,\cdot)\Big\|^2_{L^\infty}\,\mathrm{d}s\right)^{1/2}\right)\\
&\leq C(h+\sqrt{h}).
\end{aligned}$$

From this, we can use a diagonal argument, and find a subsequence such that $\mathscr{X}_\varepsilon(t)$ converges to $\mathscr{X}(t)$ almost surely, for any $t \in [0,T]$.

The analysis performed on (3.1) can be reproduced when we do not assume that the length scale vanishes; namely, we replace (3.2)–(3.3) by

$$\tau\eta^2 = 1, \qquad \tau = \varepsilon^2 \to 0, \qquad \lambda > 0 \text{ fixed}.$$

In this regime the term involving

$$\begin{aligned}
\frac{1}{\varepsilon^2}\mathbb{E}\int_{t-\varepsilon^2}^{t}\iint \frac{t-\sigma}{\lambda}(\nabla_x E)\Big(\frac{t}{\varepsilon^2}, \frac{x+(t-\sigma)v}{\lambda}\Big)\, E\Big(\frac{\sigma}{\varepsilon^2}, \frac{x}{\lambda}\Big)&\\
\cdot(\nabla_v\varphi)(x+(t-\sigma)v, v) f_\varepsilon(\sigma,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\sigma.&
\end{aligned}$$

is of order ε^2 and it vanishes when $\varepsilon \to 0$. The asymptotic regime is still described by a Fokker-Planck, like in Theorem 3.3, but now the diffusion coefficient does not depend on the velocity variable: it becomes

$$D = \int_0^1 R(\theta, 0)\,\mathrm{d}\theta.$$

We refer the reader to [19, Theorem 2.8] for the analysis of a very similar situation. However, with this scaling we do not get the statistical stability. Indeed, the correlation matrix becomes

$$\mathfrak{R}\Big(\theta, \frac{X}{\lambda}, \frac{X+\varepsilon^2\theta V}{\lambda}\Big) = \begin{pmatrix} R(\theta, \frac{\varepsilon^2}{\lambda}\theta v) & R\big(\theta, \frac{x-y}{\lambda} - \frac{\varepsilon^2}{\lambda}\theta w\big) \\ R\big(\theta, \frac{y-x}{\lambda} - \frac{\varepsilon^2}{\lambda}\theta v\big) & R(\theta, \frac{\varepsilon^2}{\lambda}\theta w) \end{pmatrix},$$

and the off-diagonal terms do not disappear when $\varepsilon \to 0$. The limit equation for $F = \lim_{\varepsilon\to 0}\mathbb{E}F_\varepsilon$ is a Fokker-Planck equation with the (space-dependent) diffusion coefficient

$$\mathfrak{D}(X) = \int_0^1 \begin{pmatrix} R(\theta,0) & R\big(\theta, \frac{x-y}{\lambda}\big) \\ R\big(\theta, \frac{y-x}{\lambda}\big) & R(\theta,0) \end{pmatrix}\mathrm{d}\theta.$$

In particular $F(t,X,V) \neq f(t,x,v)f(t,y,w)$, and we cannot prove this way the statistical stability.

4. Further examples.

4.1. Very thin spray model.

The discussion adapts readily to the following model for very thin sprays

$$\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \nabla_v\cdot\big((\eta u(t/\varepsilon^2, x/\lambda) - v) f_\varepsilon\big) = 0.$$

Here f_ε is the distribution function of particles immersed in a "turbulent" fluid: the function $(t,x) \mapsto \eta u(t/\tau, x/\lambda) \in \mathbb{R}^N$ represents the local velocity of the carrier fluid and the particles' dynamics is governed by the drag force exerted by the fluid on the particles. With

$$\eta = \frac{1}{\varepsilon} \quad \text{and} \quad \frac{\varepsilon^2}{\lambda} \to \kappa \in [0,\infty)$$

the limiting equation has the form of a Fokker-Planck equation with friction

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot \big(vf + D(v)\nabla_v f\big) = 0,$$

see [19, Theorem 2.8]. When $\kappa > 0$ the convergence holds in probability. The proof is exactly the same as above.

4.2. Quantum transport. The starting point of the problem we address is the following set of Schrödinger equations: for $n \in \mathbb{N}$,

$$i\varepsilon\partial_t\psi_{n,\varepsilon} + \frac{\varepsilon^2}{2}\Delta_x\psi_{n,\varepsilon} = \sqrt{\varepsilon}\underbrace{V(t/\varepsilon, x/\varepsilon)}_{V_\varepsilon(t,x)}\psi_{n,\varepsilon}$$

that describe the dynamics of quantum particles. The equations are completed by the initial data

$$\psi_{n,\varepsilon}\big|_{t=0} = \psi^{\text{init}}_{n,\varepsilon}.$$

The particles are subjected to a fluctuating environment described through the potential V. Here we adopt the same scaling as in [30] where the short time and length scales of variation of the potential have the same order of magnitude than the Planck constant: $0 < \varepsilon \ll 1$ (compared to the scales of observation). We assume that $\big\{\psi^{\text{init}}_{n,\varepsilon},\ n \in \mathbb{N}\big\}$ is an orthonormal system in $L^2(\mathbb{R}^N)$. It follows that, for any $t \geq 0$, $\big\{\psi_{n,\varepsilon}(t,\cdot),\ n \in \mathbb{N}\big\}$ is an orthonormal system in $L^2(\mathbb{R}^N)$, too. To each index n, we associate an occupation probability $\lambda_{n,\varepsilon} \geq 0$ such that

$$\sum_{n\in\mathbb{N}} \lambda_{n,\varepsilon} = 1, \qquad \sum_{n\in\mathbb{N}} |\lambda_{n,\varepsilon}|^2 \leq C_0\varepsilon^N.$$

In order to investigate the semi-classical regime $\varepsilon \to 0$, we introduce the Wigner transform associated to the solutions of the Schrödinger equations, namely

$$f_\varepsilon(t,x,v) = \sum_{n\in\mathbb{N}} \lambda_{n,\varepsilon} \int e^{-iy\cdot v}\psi_{n,\varepsilon}(t, x+\varepsilon y/2)\overline{\psi_{n,\varepsilon}(t, x-\varepsilon y/2)}\,\mathrm{d}y.$$

It satisfies the following Wigner equation

$$(\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon)(t,x,v) = L_\varepsilon(t)[f_\varepsilon(t,\cdot)](x,v) \tag{4.1}$$

where, for a function f that depends on x, v, we denote

$$L_\varepsilon(t)[f](x,v) = \frac{i}{\sqrt{\varepsilon}} \int e^{i\xi\cdot v}\Big(V_\varepsilon(t, x+\varepsilon\xi/2) - V_\varepsilon(t, x-\varepsilon\xi/2)\Big)\widehat{f}(x,\xi)\,\text{đ}\xi.$$

In this formula, $\widehat{\cdot}$ stands for the Fourier transform $v \to \xi$, that is

$$\widehat{f}(t,x,\xi) = \int e^{-i\xi\cdot v} f(t,x,v)\,\mathrm{d}v,$$

and $\text{đ}\xi$ stands for the Lebesgue measure with the normalization arising in the inverse Fourier transform $\text{đ}\xi = \frac{\mathrm{d}\xi}{(2\pi)^N}$. Note that, as usual in semi-classical analysis, we use the abuse of notation where the integral symbol has actually a distributional meaning. We refer the reader to [24] for the analysis of the semi-classical limit which leads from the Wigner equation to the Liouville equation, with a force field given by the gradient of the potential. For our purposes, we bear in mind the following properties.

LEMMA 4.1. *For any $\varepsilon > 0$, the operator L_ε is a bounded operator on $L^2(\mathbb{R}^N \times \mathbb{R}^N)$, with*

$$|\!|\!|L_\varepsilon|\!|\!| \le \frac{2}{\sqrt{\varepsilon}} \|V\|_{L^\infty(\mathbb{R}\times\mathbb{R}^N)}.$$

LEMMA 4.2. *The functions f_ε are real-valued and the sequence $\big(f_\varepsilon\big)_{\varepsilon>0}$ is bounded in $L^\infty((0,\infty); L^2(\mathbb{R}^N \times \mathbb{R}^N))$. Moreover, if $f_\varepsilon \rightharpoonup f$ weakly$-\star$ in $L^\infty([0,\infty); L^2(\mathbb{R}^N \times \mathbb{R}^N))$, then $f \ge 0$.*

We keep the same assumptions as in [30] on the random potential.

(h1) $V \in L^\infty(\mathbb{R} \times \mathbb{R}^N)$,

(h2) $\mathbb{E}V = 0$,

(h3) $\mathbb{E}\big(V(t,x)V(s,y)\big) = R(t-s, x-y)$, where $R \in L^\infty(\mathbb{R}; L^1(\mathbb{R}^N))$ is such that $\mathrm{supp}(R) \subset [-r,r] \times \mathbb{R}^N$.

(h4) The Fourier transform

$$Q(\tau, v) = \int e^{-ix\cdot v} R(\tau, x)\,\mathrm{d}x$$

lies in $L^\infty(\mathbb{R}; L^1(\mathbb{R}^N, (1+|v|)\,\mathrm{d}v))$.

Let us collect a few formulae and results form [30] that will be useful for our analysis. It is convenient to observe that

$$\begin{aligned}
& L_\varepsilon(t)[f](x,v) \\
&= \frac{i}{\sqrt{\varepsilon}} \int f(x,w) \left(\int e^{i\xi\cdot(v-w)} \Big(V_\varepsilon(t, x+\varepsilon\xi/2) - V_\varepsilon(t, x-\varepsilon\xi/2) \Big) \text{đ}\xi \right) \mathrm{d}w \\
&= \frac{i2^N}{\sqrt{\varepsilon}} \int f(x,w)\ \big(e^{2i(z-x/\varepsilon)\cdot(v-w)} - e^{-2i(z-x/\varepsilon)\cdot(v-w)} \big)\ V(t/\varepsilon, z) \text{đ}z\,\mathrm{d}w, \\
&\qquad \text{by using the change of variables } \tfrac{1}{\varepsilon}(x \pm \varepsilon\xi/2) \to z \\
&= \frac{i2^N}{\sqrt{\varepsilon}} \int f(x,w) \left\{ e^{-2ix\cdot(v-w)/\varepsilon} \widehat{V}(t/\varepsilon, 2(w-v)) - e^{2ix\cdot(v-w)/\varepsilon} \widehat{V}(t/\varepsilon, 2(v-w)) \right\} \text{đ}w \\
&= \frac{i}{\sqrt{\varepsilon}} \int \big(f(x, v+w'/2) - \int f(x, v-w'/2) \big)\ e^{iw'\cdot x/\varepsilon}\ \widehat{V}(t/\varepsilon, w') \text{đ}w', \\
&\qquad \text{by using the change of variables } \pm 2(v-w) \to w'.
\end{aligned} \tag{4.2}$$

The analysis still relies on an intensive use of the Duhamel formula, based on the free-transport semi-group

$$f_\varepsilon(t,x,v) = f_\varepsilon(t-s, x-sv, v) + \int_{t-s}^t L_\varepsilon(\sigma)[f(\sigma,\cdot)](x-(t-\sigma)v, v)\,\mathrm{d}\sigma. \tag{4.3}$$

As a matter of fact, (4.3) with $s = t$ already shows that $f_\varepsilon(t,\cdot)$ only depends on the realizations of $V_\varepsilon(\sigma,\cdot)$ for $0 \le \sigma \le t$, Owing to the decorrelation assumption **(h3)**, and the initial data being deterministic, we deduce that $f_\varepsilon(t-s,\cdot)$ and $V_\varepsilon(t,\cdot)$ are independent random variables when $s \ge \varepsilon$, see [30, Lemma 4.6 & 4,7].

Therefore, the strategy consists in applying the Duhamel formula (4.3) with $s = \varepsilon$ in order to compute $\mathbb{E}\big(L_\varepsilon(t)[f_\varepsilon(t,\cdot)](x,v)\big)$. It turns out that the singular term $\mathbb{E}\big(L_\varepsilon(t)[f_\varepsilon(t-\varepsilon,\cdot)](x,v)\big)$ vanishes, owing to the decorrelation property and the fact that V_ε is centered, see **(h2)**. Therefore, we are led to compute

$$\mathbb{E}\int_{t-\varepsilon}^t L_\varepsilon(t) S_{t-\sigma} L_\varepsilon(\sigma)[f(\sigma,\cdot)](x,v)\,\mathrm{d}\sigma$$

where S_t denotes the semi-group of the transport equation

$$S_t[f](x,v) = f(x-tv, v).$$

Since in the integration domain $0 \le t-\sigma \le \varepsilon$, we can replace, up to small error terms $f_\varepsilon(\sigma,\cdot)$ by $f_\varepsilon(t,\cdot)$ and even by $\mathbb{E}f_\varepsilon(t,\cdot)$, see [30, Lemma 4.8]. We are left with the task of finding the limit as $\varepsilon \to 0$ of (see [30, formula (58)])

$$
\begin{aligned}
&\mathbb{E}L_\varepsilon(t)S_{t-\sigma}L_\varepsilon(\sigma)[f](x,v)\\
&= -\frac{1}{\varepsilon}\mathbb{E}\int_{t-\varepsilon}^{t}\iint e^{iw\cdot x/\varepsilon}\ \widehat{V}(t/\varepsilon,w)\ \widehat{V}(\sigma/\varepsilon,w')\\
&\qquad\times\Big(e^{iw'\cdot(x-(t-\sigma)(v+w/2))/\varepsilon}\Big[f(x-(t-\sigma)(v+w/2),v+w/2+w'/2)\\
&\qquad\qquad -f(x-(t-\sigma)(v+w/2),v+w/2-w'/2)\Big]\\
&\qquad -e^{iw'\cdot(x-(t-\sigma)(v-w/2))/\varepsilon}\Big[f(x-(t-\sigma)(v-w/2),v-w/2+w'/2)\\
&\qquad\qquad -f(x-(t-\sigma)(v-w/2),v-w/2-w'/2)\Big]\Big)\,đw'đw\,\mathrm{d}\sigma\\
&= -\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\iint \mathbb{E}\big[\widehat{V}(t/\varepsilon,w)\ \widehat{V}(\sigma/\varepsilon,w')\big]e^{i(w+w')\cdot x/\varepsilon}\\
&\qquad\times\Big(e^{-i(t-\sigma)w'\cdot(v+w/2)/\varepsilon}\Big[f(x-(t-\sigma)(v+w/2),v+w/2+w'/2)\\
&\qquad\qquad -f(x-(t-\sigma)(v+w/2),v+w/2-w'/2)\Big]\\
&\qquad -e^{-i(t-\sigma)w'\cdot(v-w/2)/\varepsilon}\Big[f(x-(t-\sigma)(v-w/2),v-w/2+w'/2)\\
&\qquad\qquad -f(x-(t-\sigma)(v-w/2),v-w/2-w'/2)\Big]\Big)\,đw'đw\,\mathrm{d}\sigma.
\end{aligned}
\tag{4.4}
$$

Note that the factor $1/\varepsilon$ is compensated by the fact that the time integral is taken over an interval of size ε, so that $\mathbb{E}L_\varepsilon(t)S_{t-\sigma}L_\varepsilon(\sigma)[f](x,v)$ remains of order $\mathscr{O}(1)$. By virtue of the decorrelation assumption **(h3)**, we obtain (see [30, formula (61)]

$$
\begin{aligned}
\mathbb{E}\big(\widehat{V}(t,w)\widehat{V}(s,w')\big) &= \mathbb{E}\iint e^{-ix\cdot w}e^{-iy\cdot w'}V(t,x)V(s,y)\,\mathrm{d}y\,\mathrm{d}x\\
&= \iint e^{-ix\cdot w}e^{-iy\cdot w'}R(t-s,x-y)\,\mathrm{d}y\,\mathrm{d}x\\
&= \int e^{-ix\cdot(w+w')}\underbrace{\left(\int e^{iz\cdot w'}R(t-s,z)\,\mathrm{d}z\right)}_{Q(t-s,w')}\mathrm{d}x\\
&= Q(t-s,w')\ \delta_0(w+w').
\end{aligned}
\tag{4.5}
$$

Note that

$$R(-\tau,-z)=R(\tau,z)$$

implies

$$Q(-\tau,-w)=Q(\tau,w).$$

In order to continue the computation, we go back to (4.4) with this information. As a matter of fact, we observe that the factor $e^{i(w+w')\cdot x/\varepsilon}$ in the integrand of (4.4) is always equal to $e^0=1$. We now make gain and loss terms appear (see [30, Lemma 4.9]). We are led to

$$\mathbb{E}L_\varepsilon(t)S_{t-\sigma}L_\varepsilon(\sigma)[f](x,v)=\mathrm{q}_\varepsilon^+-\mathrm{q}_\varepsilon^-$$

with

$$
\begin{aligned}
\mathrm{q}_\varepsilon^+(t,x,v) &= \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\int Q\Big(\frac{t-\sigma}{\varepsilon},w\Big)\Big[f(x-(t-\sigma)(v-w/2),v-w)e^{i(t-\sigma)(v-w/2)\cdot w/\varepsilon}\\
&\qquad +f(x-(t-\sigma)(v+w/2),v+w)e^{i(t-\sigma)(v+w/2)\cdot w/\varepsilon}\Big]đw'\frac{\mathrm{d}\sigma}{(2\pi)^N},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{q}_\varepsilon^-(t,x,v) &= \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\int Q\Big(\frac{t-\sigma}{\varepsilon},w\Big)\Big[f(x-(t-\sigma)(v+w/2),v)e^{i(t-\sigma)(v+w/2)\cdot w/\varepsilon}\\
&\qquad +f(x-(t-\sigma)(v-w/2),v)e^{i(t-\sigma)(v-w/2)\cdot w/\varepsilon}\Big]đw\frac{\mathrm{d}\sigma}{(2\pi)^N}.
\end{aligned}
$$

With the change of variables

$$\frac{t-\sigma}{\varepsilon} = \tau, \qquad w = \pm(v - v'),$$

and letting ε go to 0, we are led to

$$\mathrm{q}^+[f](x,v) = \int f(x,v')q(v,v')\mathrm{đ}v'$$

with

$$\begin{aligned} q(v,v') &= \frac{1}{(2\pi)^N}\int_0^1 Q(\tau, v-v')e^{i\tau(|v|^2-|v'|^2)/2}\,\mathrm{d}\tau \\ &\quad + \frac{1}{(2\pi)^N}\int_0^1 Q(\tau, v'-v)e^{i\tau(|v'|^2-|v|^2)/2}\,\mathrm{d}\tau \\ &= \frac{1}{(2\pi)^N}\int_{-1}^1 Q(\tau, v-v')e^{i\tau(|v|^2-|v'|^2)/2}\,\mathrm{d}\tau \end{aligned}$$

since Q is an even function of its arguments. Similarly, we obtain

$$\mathrm{q}^-[f](x,v) = f(x,v)\int q(v',v)\mathrm{đ}v.$$

We refer the reader to [30, Proposition 4.5] for more precise estimates. In particular, we bear in mind the following property which makes the definition of $\mathrm{q}^\pm$ consistent with the usual form of the gain and loss terms of the linear Boltzmann operator.

Lemma 4.3. *[30, Lemma 4.5] The cross section*

$$q(v,v') = \frac{1}{(2\pi)^N}\int_{-1}^1 Q(\tau, v-v')e^{i\tau(|v|^2-|v'|^2)/2}\,\mathrm{d}\tau$$

is real-valued and in fact non-negative; it is also invariant by changing the role of v and v'.

We conclude this presentation with the following statement.

Theorem 4.4. *[30, Theorem 4.2] The expectation $\mathbb{E}f_\varepsilon$ converges to f in $C^0([0,T];L^2(\mathbb{R}^N\times\mathbb{R}^N)-\text{weak})$; the limit f is non negative, it lies in $L^\infty(0,T;L^2(\mathbb{R}^N\times\mathbb{R}^N))$ and it satisfies*

$$(\partial_t + v\cdot\nabla_x)f = C[f] \tag{4.6}$$

where C is the operator

$$C[f] = \mathrm{q}^+[f] - \mathrm{q}^-[f].$$

Due to the time-decorrelation of the potential and the scaling of its strength, the process is very close to a Markovian dynamics with instantaneous generator, which here corresponds to a jump process in velocity. We refer the reader to [28] and to the review [2] for a thorough discussion of scaling issues and relevant asymptotic regimes, motivated by the modeling of wave propagation in random media; by the way, [2, 28] already brought out that decorrelations might occur when considering quantities evaluated at different location of the phase space. It is also worth mentioning [4] where the analysis is performed with a scaling that keeps the semi-classical parameter fixed: the asymptotic regime thus incorporates a jump process in the Wigner equation. We turn to prove that, with the same assumptions as in [30], the convergence holds in probability.

Theorem 4.5. *In Theorem 4.4, f_ε^ω converges in probability in $C^0([0,T];L^2(\mathbb{R}^N\times\mathbb{R}^N)-\text{weak})$ to f .*

Proof. We are going to apply the same analysis to the density obtained by doubling the variables. Namely, denoting

$$X = (x_1,x_2)\in\mathbb{R}^N\times\mathbb{R}^N, \quad V = (v_1,v_2)\in\mathbb{R}^N\times\mathbb{R}^N$$

we set

$$F_\varepsilon(t,X,V) = f_\varepsilon(t,x_1,v_1)f_\varepsilon(t,x_2,v_2).$$

It satisfies

$$(\partial_t + V\cdot\nabla_X)F_\varepsilon(t,X,V) = \mathfrak{L}_\varepsilon(t)[F_\varepsilon(t,\cdot)](X,V) \tag{4.7}$$

where the extended Wigner operator is defined with the same formulae as above, in the extended variables and with the potential

$$\mathfrak{V}_\varepsilon(t,X) = V_\varepsilon(t,x_1) + V_\varepsilon(t,x_2) = \mathfrak{V}(t/\varepsilon, X/\varepsilon).$$

Namely, we have, see (4.2)

$$\begin{aligned}&\mathfrak{L}_\varepsilon(t)[F](X,V)\\ &= \frac{i}{\sqrt{\varepsilon}}\int e^{iW\cdot X/\varepsilon}\big(F(X,V+W/2)-F(X,V-W/2)\big)\underbrace{\widehat{\mathfrak{V}}(t/\varepsilon,W)}_{\delta(w_1)\widehat{V}(t/\varepsilon,w_2)+\delta(w_2)\widehat{V}(t/\varepsilon,w_1)}\,\text{đ}W.\end{aligned}$$

Indeed, we start by observing

$$\mathfrak{L}_\varepsilon(t)[F](X,V) = f(x_1,v_1)L_\varepsilon(t)[f](x_2,v_2) + f(x_2,v_2)L_\varepsilon(t)[f](x_1,v_1).$$

It yields the asserted expression by writing $f(x,v)$ as $\mathscr{F}^{-1}_{\xi\to v}\mathscr{F}_{y\to\xi}f(x,v)$. In order to identify the analog of (4.4), by using (4.5), we compute

$$\begin{aligned}&\mathbb{E}\big[\widehat{\mathfrak{V}}(t,W)\widehat{\mathfrak{V}}(s,W')\big]\\ &\quad= \underbrace{\delta(w_1)\delta(w_1')\delta(w_2+w_2')Q(t-s,w_2)+\delta(w_2)\delta(w_2')\delta(w_1+w_1')Q(t-s,w_1)}_{\mathrm{G}(t-s,W,W')}\\ &\qquad+\underbrace{\delta(w_1)\delta(w_2')\delta(w_1'+w_2)Q(t-s,w_2)+\delta(w_2)\delta(w_1')\delta(w_1+w_2')Q(t-s,w_1)}_{\mathrm{B}(t-s,W,W')}\end{aligned}$$

The first line, denoted $\mathrm{G}(t-s,W,W')$, corresponds to the good terms that leads to the expected cross section. It can be cast as

$$\mathrm{G}(t-s,W,W') = \mathfrak{Q}(t-s,W)\delta(W+W')$$

with

$$\mathfrak{Q}(\tau,W) = \delta(w_1)Q(\tau,w_2) + \delta(w_2)Q(\tau,w_1). \tag{4.8}$$

The second line, denoted $\mathrm{B}(t-s,W,W')$, contains the correlations that are expected to go away in the limit $\varepsilon\to 0$.

Indeed, if $f(t,x,v)$ denotes the limit of $\mathbb{E}f_\varepsilon(t,x,v)$, which thus satisfies the kinetic equation (4.6), then let us set $\widetilde{F}(t,X,V) = f(t,x_1,v_1)f(t,x_2,v_2)$. It satisfies

$$(\partial_t + V\cdot\nabla_X)\widetilde{F}(t,X,V) = \underbrace{f(t,x_2,v_2)C[f](t,x_1,v_1)+f(t,x_1,v_1)C[f](t,x_2,v_2)}_{\mathfrak{C}[\widetilde{F}](t,X,V)}$$

where the extended collision operator $\mathfrak{C}$ is associated to the cross section

$$\begin{aligned}\mathfrak{q}(V,V') &= \delta(v_1-v_1')q(v_2,v_2')+\delta(v_2-v_2')q(v_1,v_1')\\ &= \delta(v_1-v_1')\int_{-1}^{1}Q(\tau,v_2-v_2')e^{i\tau(|v_2|^2-|v_2'|^2)/2}\frac{\mathrm{d}\tau}{(2\pi)^N}\\ &\quad+\delta(v_2-v_2')\int_{-1}^{1}Q(\tau,v_1-v_1')e^{i\tau(|v_1|^2-|v_1'|^2)/2}\frac{\mathrm{d}\tau}{(2\pi)^N}\\ &= \int_{-1}^{1}\mathfrak{Q}(\tau,V-V')e^{i\tau(|V|^2-|V'|^2)/2}\frac{\mathrm{d}\tau}{(2\pi)^N}\end{aligned}$$

with $\mathfrak{Q}(\tau, V)$ given by (4.8). Going back to (4.4) the kernel $\mathrm{G}(\tau, W, W')$ is multiplied by

$$e^{iX\cdot(W+W')/\varepsilon}e^{-i\tau W'\cdot(V\pm W/2)}.$$

In particular in this product the fast oscillation term $e^{iX\cdot(W+W')/\varepsilon}$ becomes $e^0 = 1$. Then, we can reproduce the same calculations as above, and this part leads to the collision operator $\mathfrak{C}$.

We are left with the task of investigating the contribution of $\mathrm{B}(\tau, W, W')$, which thus contains the correlations. In (4.4) it is multiplied by

$$e^{2ix_1\cdot(w_1+w_1')/\varepsilon}e^{2ix_2\cdot(w_2+w_2')/\varepsilon} = e^{2iX\cdot(W+W')/\varepsilon} \tag{4.9}$$

and

$$\begin{aligned}
&F(X-(t-\sigma)(V+W/2), V+W/2+W'/2)e^{-i(t-\sigma)W'\cdot(V+W)/\varepsilon}\\
&\quad -F(X-(t-\sigma)(V-W/2), V-W/2+W'/2)e^{-i(t-\sigma)W'\cdot(V-W)/\varepsilon}\\
&\quad +F(X-(t-\sigma)(V+W/2), V+W/2-W'/2)e^{-i(t-\sigma)W'\cdot(V+W)/\varepsilon}\\
&\quad -F(X-(t-\sigma)(V-W/2), V-W/2-W'/2)e^{-i(t-\sigma)W'\cdot(V-W)/\varepsilon}.
\end{aligned}$$

Therefore, due to the form of the kernel B, in the corresponding expression, one always have $W\cdot W' = 0$ since either

$$\text{(a)}\qquad W' = (0, w_2'),\quad W = (w_1, 0) = (-w_2', 0)$$

or

$$\text{(b)}\qquad W' = (w_1', 0),\quad W = (0, w_2) = (0, -w_1').$$

Accordingly, the factor in (4.9) becomes

$$\text{either (a) } e^{2iw_1\cdot(x_1-x_2)/\varepsilon} \text{ or (b) } e^{2iw_2\cdot(x_2-x_1)/\varepsilon},$$

while

$$W'\cdot(V\pm W) = W'\cdot V = \begin{cases} w_2'\cdot v_2 = -w_1\cdot v_2 & \text{case (a)},\\ w_1'\cdot v_1 = -w_2\cdot v_1 & \text{case (b)}.\end{cases}$$

Therefore, we have to deal with terms which have the following form

$$\int_0^1\int Q(\tau, w_1)e^{iw_1(v_2+(x_2-x-1))/\varepsilon}F\left(X-\varepsilon\tau\left(V+\frac{1}{2}\begin{pmatrix}w_1\\-w_1\end{pmatrix}\right), V\pm\frac{1}{2}\begin{pmatrix}w_1\\-w_1\end{pmatrix}\right)\text{đ}w_1\,\mathrm{d}\tau. \tag{4.10}$$

In the approach of [30], the asymptotic regime is obtained by multiplying (4.1) by a trial function $\varphi\in C_c^\infty(\mathbb{R}^N\times\mathbb{R}^N)$. The linear part with the convection term clearly passes to the limit, the difficulty comes only from the Wigner term. The latter is treated by using the Duhamel formula, as explained above, and thus we only have to discuss the fact that

$$\mathbb{E}\left[\frac{1}{\varepsilon}\int_{t-\varepsilon}^t\left[L_\varepsilon(t)S_{t-\sigma}L_\varepsilon(\sigma)\right]^\star[\varphi](x,v)\,\mathrm{d}\sigma\right]$$

tends to $C^\star[\varphi](x,v)$ in $L^2((\mathbb{R}^N\times\mathbb{R}^N))$ as $\varepsilon\to 0$, see [30, Proposition 2.5]. We wish to apply the same strategy for (4.7). Since the structure of the operator $\left[\mathfrak{L}_\varepsilon(t)S_{t-\sigma}\mathfrak{L}_\varepsilon(\sigma)\right]$ and its adjoint $\left[\mathfrak{L}_\varepsilon(t)S_{t-\sigma}\mathfrak{L}_\varepsilon(\sigma)\right]^\star$ are the same, we are going to justify that

$$\mathbb{E}\left[\frac{1}{\varepsilon}\int_{t-\varepsilon}^t\mathfrak{L}_\varepsilon(t)S_{t-\sigma}\mathfrak{L}_\varepsilon(\sigma)[\Phi](X,V)\,\mathrm{d}\sigma\right]$$

tends to $\mathfrak{C}[\Phi]$ as $\varepsilon\to 0$, for a given $\Phi\in C_c^\infty(\mathbb{R}^{2N}\times\mathbb{R}^{2N})$. With the analog of (4.4), the analysis of [30] applies directly to prove that the contribution with $\mathrm{G}(\tau, W, W')$ leads to $\mathfrak{C}[\Phi]$, as sketched above. We are just left with the task of proving that the contribution with $\mathrm{B}(\tau, W, W')$ tends to 0, which reduces to show that

$$\int_0^1\int Q(\tau, w_1)e^{iw_1\cdot(v_2+(x_2-x_1))/\varepsilon}\Phi\left(X-\varepsilon\tau\left(V+\frac{1}{2}\begin{pmatrix}w_1\\-w_1\end{pmatrix}\right), V\pm\frac{1}{2}\begin{pmatrix}w_1\\-w_1\end{pmatrix}\right)\text{đ}w_1\,\mathrm{d}\tau$$

tends to 0 in $L^2(\mathbb{R}_X^{2N} \times \mathbb{R}_V^{2N})$ as $\varepsilon \to 0$. It is clear that we can safely remove the shift with respect to the space variable (at the price of an error of order $\mathscr{O}(\varepsilon)$), and we only address the question for

$$\mathrm{I}_\varepsilon(X,V) = \int_0^1 \int Q(\tau, w_1) e^{iw_1\cdot(v_2+(x_2-x_1))/\varepsilon} \Phi\left(X, V \pm \frac{1}{2}\begin{pmatrix} w_1 \\ -w_1\end{pmatrix}\right) \mathrm{đ}w_1 \,\mathrm{d}\tau.$$

Let us introduce the following function, seen as a function of the variable $w_1 \in \mathbb{R}^N$, parametrized by $(X,V) \in \mathbb{R}^{2N} \times \mathbb{R}^{2N}$:

$$\Psi_{X,V}(w_1) = \int_0^1 Q(\tau, w_1)\,\mathrm{d}\tau \times \Phi\left(X, V \pm \frac{1}{2}\begin{pmatrix} w_1 \\ -w_1\end{pmatrix}\right).$$

Then, we have

$$\mathrm{I}_\varepsilon(X,V) = \widehat{\Psi}_{X,V}\Big(v_2 + \frac{x_2 - x_1}{\varepsilon}\Big).$$

where $\widehat{\cdot}$ denotes here the Fourier transform with respect to the variable w_1. By virtue of **(h4)**, for any $(X,V) \in \mathbb{R}^{2N} \times \mathbb{R}^{2N}$, the function $w_1 \mapsto \Psi_{X,V}(w_1)$ is integrable. Hence the Riemann-Lebesgue implies that $\lim_{\varepsilon\to 0} \mathrm{I}_\varepsilon(X,V)$ for a. e. $(X,V) \in \mathbb{R}^{2N} \times \mathbb{R}^{2N}$ (precisely as far as (x_1,x_2) does not lie on the diagonal of $\mathbb{R}^{2N}$, which is a negligible set). Moreover for any $n \in \mathbb{N}$, there exists C_n such that

$$(1+|X|^2)^n(1+|V|^2)^n \ \left|\Phi\left(X, V \pm \frac{1}{2}\begin{pmatrix} w_1 \\ -w_1\end{pmatrix}\right)\right| \leq C_n$$

holds for every $(X,V) \in \mathbb{R}^{2N} \times \mathbb{R}^{2N}$ and $w_1 \in \mathbb{R}^N$. Thus, using **(h4)** again, we can dominate

$$|\mathrm{I}_\varepsilon(X,V)| \leq \|Q\|_{L^\infty(\mathbb{R};L^1(\mathbb{R}^N))} \frac{C_n}{(1+|X|^2)^n(1+|V|^2)^n}$$

and we select n large enough so that the function on the right hand side lies in $L^2(\mathbb{R}^{2N}\times\mathbb{R}^{2N})$. Finally, we can conclude by applying the Lebesgue theorem. Note that for the white noise limit considered in [4], the scaling is such that there is no fast oscillating phase in the expression of B, and thus it is likely that the present approach, which strongly relies on the scale interactions, does not help in improving the convergence statement.

We deduce that the limit $F(t,X,V)$ of $\mathbb{E}F_\varepsilon(t,X,v) = \mathbb{E}[f_\varepsilon(t,x_1,v_1)f_\varepsilon(t,x_2,v_2)]$ satisfies

$$(\partial_t + V\cdot\nabla_X)F = \mathfrak{C}(F),$$

with initial data $F(0,X,V) = f(0,x_1,v_1)f(0,x_2,v_2)$. By uniqueness of the solution of the Cauchy problem, see [30, Lemma 4.11], it follows that

$$F(t,X,V) = f(t,x_1,v_1)f(t,x_2,v_2),$$

(denoted $\widetilde{F}(t,X,V)$ above) with f the limit of $\mathbb{E}f_\varepsilon$. Applying the same reasoning as in the previous section, it allows us to establish the asserted statistical stability. ∎

Acknowledgements. We thank Dario Trevisan for helpful comments about the uniqueness theory for Fokker-Planck equations. We are also gratefully indebted to Leo Vivion who helped us in clarifying the preliminary versions of this article.

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