Kinetic and Related Models
doi:10.3934/krm.2011.4.735

Volume **4**, Number **3**, September **2011**
pp. **735–766**

NON–LOCAL MACROSCOPIC MODELS BASED ON GAUSSIAN CLOSURES FOR THE SPIZER-HÄRM REGIME

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(Communicated by Ansgar Jüngel)

Abstract. The Spitzer-Härm regime arising in plasma physics leads asymptotically to a nonlinear diffusion equation for the electron temperature. In this work we propose a hierarchy of models intended to retain more features of the underlying modeling based on kinetic equations. These models are of non–local type. Nevertheless, owing to energy discretization they can lead to coupled systems of diffusion equations. We make the connection between the different models precise and bring out some mathematical properties of the models. A numerical scheme is designed for the approximate models, and simulations validate the proposed approach.

1. **Introduction.** The modeling of plasmas arising in Inertial Confinement Fusion leads to the following integro-differential equation

$$\partial_t F + v \cdot \nabla_x F + E \cdot \nabla_v F = C(F) \tag{1.1}$$

where the unknown $F(t, x, v)$ is interpreted as the number density of electrons in phase space. It depends on the time variable $t \geq 0$, the space variable $x \in \mathbb{R}^3$ and the velocity variable $v \in \mathbb{R}^3$. The problem is completed by an initial data

$$F_{|t=0} = F_{\mathrm{Init}}.$$

The right hand side in (1.1) describes interactions between particles. Both electrons/electrons and ions/elec-trons collisions are embodied into the operator C, which involves certain averages with respect to the variable v. We will detail below examples of operators C that arise in the physical context which motivates this work. For the time being, let us present the crucial requirements on which our analysis is based:

(H1). *Mass and energy conservation:*

$$\int \begin{pmatrix} 1 \\ v^2 \end{pmatrix} C(F)\,\mathrm{d}v = 0.$$

Note however that we do not require the momentum conservation: indeed, for the models we are interested in the charge current is not conserved by the collision processes. It arises when assuming that F is the distribution function of electrons subject to collisions with positive charges. We adopt the simplifying assumption

2000 *Mathematics Subject Classification.* 35Q99, 35B25, 82C70.
Key words and phrases. Plasma physics, hydrodynamic limits.

that the ion distribution is given. For such models mass and energy are conserved, but due to the electron/ion collision operator, the momentum is not conserved.

(H2). *Equilibrium:* $C(F) = 0$ *if and only if* F *is a centered Maxwellian* $M_{\rho,\theta}(v) = \frac{\rho}{(2\pi\theta)^{3/2}} e^{-|v|^2/(2\theta)}$, *with* $\rho \geq 0$, $\theta > 0$.

(H3). *Entropy dissipation:*

$$\int_{\mathbb{R}^3} C(F)\ln(F)\,\mathrm{d}v \leq 0,$$

and the entropy dissipation vanishes precisely when F *is a Maxwellian.*

To the particle distribution function $F(t,x,v)$ we associate the following macroscopic quantities

Density: $\rho(t,x) = \int_{\mathbb{R}^3} F(t,x,v)\,\mathrm{d}v,$

Current (J) and Bulk Velocity (u): $J(t,x) = \rho u(t,x) = \int_{\mathbb{R}^3} vF(t,x,v)\,\mathrm{d}v,$

Total Energy ($\mathcal{E}$) and Temperature (θ):

$$2\mathcal{E}(t,x) = \rho(t,x)|u(t,x)|^2 + 3\rho(t,x)\theta(t,x) = \int_{\mathbb{R}^3} v^2 F(t,x,v)\,\mathrm{d}v. \tag{1.2}$$

The definition of the field E in (1.1) that we consider is slightly unusual: E is defined as to maintain the quasi-neutrality, that is to preserve the constraint

$$\int_{\mathbb{R}^3} vF(t,x,v)\,\mathrm{d}v = 0. \tag{1.3}$$

To obtain the expression of E by means of the particles distribution F, we proceed as follows. Multiply (1.1) by v and integrate. Taking into account (1.3), we obtain

$$E(t,x) = \frac{1}{\rho}\left[\nabla_x \cdot \left(\int_{\mathbb{R}^3} v \otimes vF\,\mathrm{d}v\right) - \int_{\mathbb{R}^3} vC(F)\,\mathrm{d}v\right]. \tag{1.4}$$

The system (1.1), (1.4) is widely used for modeling and simulating Inertial Confinment Fusion phenomena, see e.g. [13]. We refer to [4] for a detailed derivation from the standard Vlasov-Maxwell-Fokker-Planck system in a regime of small Debye length. Note that, regardless specific difficulties related to the collision operator, the mathematical analysis of the non standard constraint (1.4) might lead to interesting questions. It involves the second order moment of the unknown, which is just controlled by the energy conservation, so that applications of average lemma techniques is not so clear, while by contrast to the Vlasov–Poisson coupling we cannot expect any regularization effect in the definition of the electric field. However existence–uniqueness issues are beyond the scope of the present work, which is rather concerned with asymptotic problems.

The simulation of Inertial Confinment Fusion devices is highly demanding in numerical resources, because the problem involves many different scales. The quasi-neutrality constraint and (1.4) constitute already a relevant simplification in comparison to the full system of electromagnetism, but solving the kinetic equation at the physical scales of interest is simply not affordable. For this reason, we are interested in designing reduced models intended to capture the main features of (1.1), (1.4) in certain asymptotic regimes. This is the objective of the present work where we are concerned with small mean free path regimes. More precisely, we are interested in the so-called Spitzer-Härm regime where the dynamics reduces to a non linear diffusion equation for the electron temperature, see [18]. However, the limiting equation, which is amenable to a quite simple numerical treatment, might fail in reproducing the behavior of the plasma with the necessary accuracy, as pointed out for instance in [2, 11, 16]. This has motivated the introduction of intermediate models. In particular, non local corrections to the Spitzer-Härm equation have been introduced in [12] with several extensions [10, 11, 2, 13, 1, 16]. In this work, we develop a different viewpoint to derive from the kinetic equation a hydrodynamic model. The system we wish to design will still depend on the scaling parameter but we expect that it can be solved with a reduced cost in comparison to kinetic simulations, with a strengthened accuracy in comparison to the asymptotic model. Our approach is inspired by the reasoning introduced by Levermore in [8] for gas dynamic equations.

This work is organized as follows. First in section 2, we briefly review the derivation of the Spitzer-Härm regime. Next in section 3, we discuss approximated models obtained by projection on the classical spherical harmonic basis. In particular we present variants to the classical P_N models, that can be referred to as the D_N models, [3, 8]. Such models, even if they remain of kinetic nature, already allow to substantially reduce the dimension of the variables. In Section 4 we combine this approach to a "micro–macro decomposition" where the solution F of (1.1) is split into an equilibrium state and a remainder. It yields a new hierarchy of equations describing the evolution of the electron temperature. We explain in Section 5 that these models can be seen as relevant generalizations of the Schurtz-Nicolaï system [16]. In Section 5, we also investigate some mathematical properties of the hydrodynamic system we obtain by Gaussian closure and energy discretization. While additional simplifying assumptions could be useful to make the arguments tractable, the results bring out remarkable properties of the model. Section 6 ends the paper with the presentation of numerical schemes for the approximate models and discussion of simulation results that validate the proposed equations.

2. **Spitzer-Härm regime.**

The physical regime of interest is embodied into a positive parameter ε and we are led to investigate the behavior as $\varepsilon \to 0$ of the solutions $(F_\varepsilon, E_\varepsilon)$ of

$$\partial_t F_\varepsilon + \frac{1}{\varepsilon}\left(v \cdot \nabla_x F_\varepsilon + E_\varepsilon \cdot \nabla_v F_\varepsilon\right) = \frac{1}{\varepsilon^2} C(F_\varepsilon), \tag{2.1}$$

coupled with

$$E_\varepsilon(t,x) = \frac{1}{\rho_\varepsilon}\left[\nabla_x \cdot \left(\int_{\mathbb{R}^3} v \otimes v F_\varepsilon \,\mathrm{d}v\right) - \frac{1}{\varepsilon}\int_{\mathbb{R}^3} v C(F_\varepsilon)\,\mathrm{d}v\right]. \tag{2.2}$$

We refer to [4] for a thorough discussion on the scaling issues: ε is defined as the ratio of the velocity unit defined by the observation time and length scales over the thermal velocity, while assuming that the mean free path is of order ε compared to the observation length scale. We start by writing the evolution equation for the macroscopic quantities ρ_ε, u_ε and θ_ε defined by (see (1.2))

$$\rho_\varepsilon = \int_{\mathbb{R}^3} F_\varepsilon \,\mathrm{d}v, \quad u_\varepsilon = \frac{1}{\rho_\varepsilon}\int_{\mathbb{R}^3} v F_\varepsilon \,\mathrm{d}v, \quad \theta_\varepsilon = \frac{1}{3\rho_\varepsilon}\int_{\mathbb{R}^3} |v - u_\varepsilon|^2 F_\varepsilon \,\mathrm{d}v.$$

Multiplying (2.1) by 1, v and $|v|^2$ respectively, and integrating yield

$$\begin{cases} \partial_t \rho_\varepsilon + \dfrac{1}{\varepsilon}\nabla_x \cdot (\rho_\varepsilon u_\varepsilon) = 0, \\ \rho_\varepsilon \left(\partial_t u_\varepsilon + \dfrac{1}{\varepsilon}(u_\varepsilon \cdot \nabla_x) u_\varepsilon \right) + \dfrac{1}{\varepsilon}\nabla_x \cdot \mathsf{P}_\varepsilon = \dfrac{1}{\varepsilon^2}\displaystyle\int_{\mathbb{R}^3} vC(F_\varepsilon)\,\mathrm{d}v + \dfrac{1}{\varepsilon}\rho_\varepsilon E_\varepsilon, \\ \partial_t \left(3\rho_\varepsilon\theta_\varepsilon + \rho_\varepsilon u_\varepsilon^2\right) + 2\nabla_x \cdot Q_\varepsilon + \dfrac{1}{\varepsilon}\nabla_x \cdot (3\rho_\varepsilon\theta_\varepsilon u_\varepsilon + 2\mathsf{P}_\varepsilon u_\varepsilon) \\ \qquad\qquad = -2\rho_\varepsilon E_\varepsilon \cdot u_\varepsilon - \nabla_x \cdot \left(\rho_\varepsilon |u_\varepsilon|^2 u_\varepsilon\right), \end{cases} \tag{2.3}$$

which is still coupled to (2.2). The system (2.3) is not closed since the pressure tensor P_ε and the heat flux Q_ε are defined by

$$\mathsf{P}_\varepsilon = \int_{\mathbb{R}^3} (v-u_\varepsilon)\otimes(v-u_\varepsilon)\,F_\varepsilon\,\mathrm{d}v, \qquad Q_\varepsilon = \frac{1}{2\varepsilon}\int_{\mathbb{R}^3} |v-u_\varepsilon|^2\,(v-u_\varepsilon)\,F_\varepsilon\,\mathrm{d}v$$

respectively. The system also involves the distribution function through the integral term $\int_{\mathbb{R}^3} vC(F_\varepsilon)\,\mathrm{d}v$. These quantities cannot be expressed in general by means of ρ_ε, u_ε and θ_ε. However, the constraint (1.3) imposes $u_\varepsilon = 0$ and, thus, by using the charge conservation (the first equation in (2.3)), $\rho_\varepsilon(t,x) = \rho_\star(x)$, a fixed density defined by the initial data. Therefore (2.3) reduces to

$$3\rho_\star \partial_t \theta_\varepsilon + 2\nabla_x \cdot Q_\varepsilon = 0. \tag{2.4}$$

Based on Hilbert expansions, it is shown in [4] that the asymptotic regime $\varepsilon \to 0$ leads to the following limiting equation:

$$\begin{aligned} &F_\varepsilon(t,x,v) \to \frac{\rho_\star(x)}{(2\pi\theta(t,x))^{3/2}}\; e^{-v^2/(2\theta(t,x))}, \\ &3\rho_\star\partial_t\theta + \nabla_x \cdot Q_{SH} = 0, \\ &Q_{SH} = -\kappa\theta^{5/2}\nabla_x\theta, \end{aligned} \tag{2.5}$$

where $\kappa > 0$ is a constant depending only on the details of the collision operator, [4, Eq. (2.15)] The system (2.5) will be referred to as the Spitzer-Härm system, see [18]. We are interested in the derivation of "intermediate" models, bearing in mind the following requirements:

- We seek systems "simpler" than the mesoscopic model (2.1), (2.2), having in mind numerical purposes. Based on the fact that the limiting behavior is described by macroscopic quantities depending only on time and space variables, we can expect to reduce the dimension of the relevant variables.
- Since the models we wish to design are intended to retain some features of the non-zero ε behavior, they will still depend on the scaling parameter. However, as ε goes to 0, they should be consistent with the Spitzer-Härm equation (2.5).

Hence our approach consists in defining the approximate temperature $\tilde{\theta}_\varepsilon(t,x)$ and heat flux $\tilde{Q}_\varepsilon(t,x)$ satisfying

$$3\rho_\star\partial_t\tilde{\theta}_\varepsilon + 2\nabla_x \cdot \tilde{Q}_\varepsilon(t,x) = 0,$$

where $\tilde{Q}_\varepsilon$ is related to the moment of a certain quantity of mesoscopic nature $\tilde{F}_\varepsilon(t,x,v)$,

$$\begin{aligned} \tilde{Q}_\varepsilon(t,x) &= \int_{\mathbb{R}^3} v\,\frac{|v|^2}{2}\,\tilde{F}_\varepsilon(t,x,v)\,\mathrm{d}v \\ &= \frac{8\pi\left(\tilde{\theta}_\varepsilon(t,x)\right)^3}{3}\int_0^\infty \left(\int_{\mathbb{S}^2} \Omega\;\tilde{F}_\varepsilon(t,x,\Omega\sqrt{2\tilde{\theta}_\varepsilon(t,x)\xi})\,\mathrm{d}\Omega\right)\xi^2\,\mathrm{d}\xi. \end{aligned}$$

We use the change of variable $v \mapsto \Omega\sqrt{2\theta_\varepsilon\xi}$, with $\Omega = v/|v| \in \mathbb{S}^2$ and $\theta_\varepsilon\xi = |v|^2/2$ interpreted as an energy variable. We shall detail in the following sections possible construction of the approximate flux.

3. **Spherical harmonics decomposition.** We start by introducing a few useful notations. For $A = (A_p)_{p\in\{1,2,3\}^P}$ a tensor of order P we define recursively $A^{\otimes 2} = A \otimes A$ with $\otimes$, the Kronecker product (that is $(A^{\otimes 2})_{p=(k,l)} = A_k A_l$ a tensor of order $2P$) and $A^{\otimes n} = A \otimes A^{\otimes(n-1)}$, a tensor of order nP. Given integers $i \le P$, it is convenient to write the set of indices of order P as a product: $\{1,2,3\}^P = \{1,2,3\}^i \times \{1,2,3\}^{P-i}$ so that we can write accordingly the components of a tensor A of order P as $A_p = A_{jl}$ with $j \in \{1,2,3\}^i$ and $l \in \{1,2,3\}^{P-i}$. Now, let A and B be tensors of order P and Q respectively. For $i \in \mathbb{N}$, such that $2i \le P+Q$, we define $A \overset{i}{\cdot} B$, the i-contracted product of A and B as to be the tensor of order $P+Q-2i$ defined by

$$\big(A \overset{i}{\cdot} B\big)_{jk} = \sum_{l\in\{1,2,3\}^i} A_{jl}B_{lk},$$

for $j \in \{1,2.3\}^{P-i}$ and $k \in \{1,2,3\}^{Q-i}$. For instance, when A and B are usual vectors ($P = Q = 1$), the ($i = 1$)-contracted product is nothing but the usual euclidean product; when A is a matrix and B a vector ($P = 2$, $Q = 1$), the ($i = 1$)-contracted product is nothing but the usual matrix–vector product; when A and B are matrices ($P = Q = 2$), the ($i = 1$)-contracted product is nothing but the usual matrix–matrix product while the ($i = 2$)-contracted product is what is commonly named the contracted product of A and B that is the sum of all the products of the components of A and B. By the way, in what follows when $i = 1$ we will simply note $A \cdot B$ instead of $A \overset{1}{\cdot} B$ and when $i = 2$ we will note $A : B$ instead of $A \overset{2}{\cdot} B$.

Next, we consider the spherical harmonics basis $\{Y_i,\ i \in \mathbb{N}\}$, defined by $Y_0(\Omega) = 1$ and

$$Y_i : \Omega \in \mathbb{S}^2 \longmapsto Y_i(\Omega) = \Omega^{\otimes i} - \sum_{j=0}^{i-1} P_j\left(\Omega^{\otimes i}\right) \overset{j}{\cdot} Y_j(\Omega),$$

with P_i defined by

$$P_i : f \in L^2(\mathbb{S}^2) \longmapsto P_i(f) = \left[\int_{\mathbb{S}^2} Y_i \otimes Y_i \,\mathrm{d}\Omega\right]^{-1} \overset{i}{\cdot} \int_{\mathbb{S}^2} f Y_i \,\mathrm{d}\Omega.$$

Note that Y_i is a tensor of order i and for a scalar function f, $P_i(f)$ is a tensor of order i too[1]. Given a tensor T of order j, we generalize the definition by setting $P_i(T)$ as to be the tensor of order $i+j$ defined by $(P_i(T))_k = P_i(T_k)$ for any $k \in \{1,2,3\}^j$. The first three spherical harmonic tensors are

$$Y_0(\Omega) = 1, \qquad Y_1(\Omega) = \Omega, \qquad Y_2(\Omega) = \Omega^{\otimes 2} - \frac{Id}{3}.$$

For functions defined on $\mathbb{R}^3$, we set similarly

$$P_i(f)(|v|) = \left[\int_{\mathbb{S}^2} Y_i \otimes Y_i \,\mathrm{d}\Omega\right]^{-1} \overset{i}{\cdot} \int_{\mathbb{S}^2} f(|v|\Omega)\ Y_i(\Omega)\,\mathrm{d}\Omega$$

[1]We warn the reader that the notation P_i has not to be confused with the ith Legendre polynomial.

and, given a subset $\mathbf{n} \subset \mathbb{N}$, we define

$$P_{\mathbf{n}}^{\perp}(f)(v) = f(v) - \sum_{i\in\mathbf{n}} P_i(f)(|v|) \overset{i}{\cdot} Y_i\Big(\frac{v}{|v|}\Big).$$

This operator satisfies the following orthogonality property

$$\text{For any } i \in \mathbf{n} \text{ and a.e. } r \geq 0, \int_{\mathbb{S}^2} P_{\mathbf{n}}^{\perp}(f)(r\Omega) Y_i(\Omega)\,\mathrm{d}\Omega = 0.$$

Therefore, it makes sense to split a function $F : \mathbb{R}^3 \to \mathbb{R}$ as follows

$$\begin{aligned}
&F(v) = \Pi_{\mathbf{n}}(v) + F_{\mathbf{n}}^{\perp}(v),\\
&\Pi_{\mathbf{n}}(v) = \textstyle\sum_{i\in\mathbf{n}} F_i(|v|) \overset{i}{\cdot} Y_i\Big(\dfrac{v}{|v|}\Big), \qquad F_i(|v|) = P_i(F)(|v|),\\
&F_{\mathbf{n}}^{\perp}(v) = P_{\mathbf{n}}^{\perp}(F)(v).
\end{aligned}$$

We apply this decomposition to the solution of (2.1) and we denote $F_\varepsilon(t,x,v) = (\Pi_{\mathbf{n}} + F_{\mathbf{n}}^{\perp})(t,x,v)$. Accordingly, let us apply the operators P_i and $P_{\mathbf{n}}^{\perp}$ to (2.1). We obtain

$$\begin{aligned}
&\partial_t F_i + \frac{1}{\varepsilon} P_i\left(\nabla_x \cdot (vF_\varepsilon) + E_\varepsilon \cdot \nabla_v F_\varepsilon\right) = \frac{1}{\varepsilon^2} P_i C\left(F_\varepsilon\right), \qquad \text{for } i \in \mathbf{n},\\
&\partial_t F_{\mathbf{n}}^{\perp} + \frac{1}{\varepsilon} P_{\mathbf{n}}^{\perp}\left(\nabla_x \cdot (vF_\varepsilon) + E_\varepsilon \cdot \nabla_v F_\varepsilon\right) = \frac{1}{\varepsilon^2} P_{\mathbf{n}}^{\perp} C\left(F_\varepsilon\right),
\end{aligned} \tag{3.1}$$

coupled with (2.2).

The first idea to design approximate models consists in simplifying the equation for the component orthogonal to $\{Y_i\}_{i\in\mathbf{n}}$, which is implicitly assumed to be small compared to the F_i's. A classical closure is obtained by truncation: the orthogonal part $F_{\mathbf{n}}^{\perp}$ is simply disregarded. It leads to the so–called $P_{\mathbf{n}}$ system where F_ε is approximated by $\Pi_{\mathbf{n}}$, the components of which are defined according to (**M**1). Remark that the unknown depend on 1 (time) + 3 (space) + 1 ($|v|$) variables.

$$\begin{aligned}
&\Pi_{\mathbf{n}}(v) = \sum_{i\in\mathbf{n}} F_i(|v|) \overset{i}{\cdot} Y_i\Big(\frac{v}{|v|}\Big),\\
&\partial_t F_i + \frac{P_i}{\varepsilon}\left[\nabla_x \cdot (v\Pi_{\mathbf{n}})\right] + \tilde{E}_\varepsilon \cdot \frac{P_i}{\varepsilon} \nabla_v\left(\Pi_{\mathbf{n}}\right) = \frac{1}{\varepsilon^2} P_i C\left(\Pi_{\mathbf{n}}\right),\\
&\tilde{E}_\varepsilon(t,x) = \frac{1}{\rho_\star}\left[\nabla_x \cdot \left(\int_{\mathbb{R}^3} v \otimes v \Pi_{\mathbf{n}}\,\mathrm{d}v\right) - \frac{1}{\varepsilon}\int_{\mathbb{R}^3} vC\left(\Pi_{\mathbf{n}}\right)\,\mathrm{d}v\right].
\end{aligned}$$

M 1. $P_{\mathbf{n}}$ model.

More involved closures can be proposed, following the reasoning introduced in [7] for gas dynamics. To this end, we need further assumptions on the collision operator and the set $\mathbf{n} \subset \mathbb{N}$.

(H4). *The collision operator is stable on the basis $(Y_i)_{i\in\mathbf{n}}$ in the sense that for any $(F_i)_{i\in\mathbf{n}} \in \prod_{i\in\mathbf{n}} \left(L^2\left(\mathbb{R}_+\right)\right)^{3^i}$, we have*

$$P_{\mathbf{n}}^{\perp} C\left(F_i \overset{i}{\cdot} Y_i\right) = 0.$$

(H5). *The range of the basis* $(Y_i)_{i\in\mathbf{n}}$ *by the collision operator is orthogonal to* Y_1*: for any* $(F_i)_{i\in\mathbf{n}} \in \prod_{i\in\mathbf{n}} \left(L^2(\mathbb{R}_+)\right)^{3^i}$ *we have*

$$P_1 C\left(F_i \overset{i}{\cdot} Y_i\right) = 0.$$

These two hypothesis hold assuming that the collision operator is isotropic, in the following sense [5].

(H6). *The collisional operator is isotropic, which means that for any tensor–valued function* T *defined on* $\mathbb{S}^2$ *and any* $f \in L^2(\mathbb{R}_+)$ *, there exists* $\lambda : [0,\infty) \to \mathbb{R}$ *such that*

$$C\left(f(|v|)\, T\left(\frac{v}{|v|}\right)\right) = \lambda(|v|)\, T\left(\frac{v}{|v|}\right).$$

The derivation assumes that the solution F_ε of (2.1) is mainly described by the projected part $\Pi_\mathbf{n}$ while the remainder $F_\mathbf{n}^\perp$ is supposed to be of order ε compared to the leading term. Let us introduce the linearized operator

$$L_f(g) = \frac{\mathrm{d}}{\mathrm{d}s} C(f+sg)\Big|_{s=0}.$$

Then we simplify (3.1) by getting rid of the low order terms and linearizing the collision term. We are thus led to the so–called $D_\mathbf{n}$ model where F_ε is approached by $\tilde{F}_\varepsilon = \Pi_\mathbf{n} + \varepsilon F_\mathbf{n}^\perp$, solutions of the system in (**M** 2).

$$\begin{aligned}
&\partial_t F_i + \frac{P_i}{\varepsilon}\left[\nabla_x \cdot \left(v\left(\Pi_\mathbf{n} + \varepsilon F_\mathbf{n}^\perp\right)\right) + \tilde{E}_\varepsilon \cdot \nabla_v \left(\Pi_\mathbf{n} + \varepsilon F_\mathbf{n}^\perp\right)\right] = \frac{P_i}{\varepsilon^2} C(\Pi_\mathbf{n} + \varepsilon F_\mathbf{n}^\perp),\\
&P_\mathbf{n}^\perp\left[\nabla_x \cdot (v\Pi_\mathbf{n}) + \tilde{E}_0 \cdot \nabla_v \Pi_\mathbf{n}\right] = P_\mathbf{n}^\perp L_{\Pi_\mathbf{n}} F_\mathbf{n}^\perp = P_\mathbf{n}^\perp L_{\Pi_\mathbf{n}} P_\mathbf{n}^\perp F_\mathbf{n}^\perp,\\
&\tilde{E}_\varepsilon(t,x) = \frac{1}{\rho_\star}\left[\nabla_x \cdot \left(\int_{\mathbb{R}^3} v\otimes v\left(\Pi_\mathbf{n} + \varepsilon F_\mathbf{n}^\perp\right)\,\mathrm{d}v\right) - \frac{1}{\varepsilon}\int_{\mathbb{R}^3} vC\left(\Pi_\mathbf{n} + \varepsilon F_\mathbf{n}^\perp\right)\,\mathrm{d}v\right],\\
&\tilde{E}_0(t,x) = \frac{1}{\rho_\star}\left[\nabla_x \cdot \left(\int_{\mathbb{R}^3} v\otimes v\Pi_\mathbf{n}\,\mathrm{d}v\right) - \int_{\mathbb{R}^3} vL_{\Pi_\mathbf{n}}\left(F_\mathbf{n}^\perp\right)\,\mathrm{d}v\right].
\end{aligned}$$

M 2. $D_\mathbf{n}$ model.

Note that we have used the linearized operator only when approximating the equation for $F_\mathbf{n}^\perp$ in (3.1). Hypothesis (H4) and (H5) is used to get rid of $\mathscr{O}(\varepsilon)$ terms in the definition of the electric field. Note also that the equation for $F_\mathbf{n}^\perp$ can be recast as

$$\begin{aligned}
P_\mathbf{n}^\perp &\left[L_{\Pi_\mathbf{n}} F_\mathbf{n}^\perp + \frac{\nabla_v \Pi_\mathbf{n}}{\rho_\star}\int_{\mathbb{R}^3} wL_{\Pi_\mathbf{n}} F_\mathbf{n}^\perp(w)\,\mathrm{d}w\right]\\
&= P_\mathbf{n}^\perp\left[\nabla_x \cdot (v\Pi_\mathbf{n}) - \frac{1}{\rho_\star}\nabla_x \cdot \left(\int_{\mathbb{R}^3} v\otimes v\,(\Pi_\mathbf{n})\,\mathrm{d}v\right)\right]
\end{aligned}$$

by taking into account the expression of $\tilde{E}_0$.

We can expect that such a model is simpler to solve numerically than the original kinetic equation. In particular, the leading unknowns F_i only depend on t, x and $|v|$, while the determination of the corrector $F_\mathbf{n}^\perp$ relies on the inversion of the linear operator $P_\mathbf{n}^\perp L_{\Pi_\mathbf{n}} P_\mathbf{n}^\perp$. This operator inherits the usual Fredholm properties satisfied by the linearized operator $L_{\Pi_\mathbf{n}}$, and in some circumstances it can be explicitely inverted. However, the number of unknowns grows exponentially with the number of elements in the set $\mathbf{n}$, which restricts the applicability of this approach to the few first elements. Furthermore, the cost of 3 dimensional simulations will remain quite high, and finally, it is not clear that the $P_{\mathbf{n}}$ and $D_{\mathbf{n}}$ closures preserve the non negativity of the corresponding approximate particles distribution function $\tilde{F}$. Nevertheless, we bear in mind the spherical harmonic decomposition to derive further approximate models.

4. **Derivation of hydrodynamic models based on micro-macro decomposition.** As said in the Introduction, we wish to derive hydrodynamic models, intended to describe intermediate regimes, for small but non zero ε. According to (2.5), the leading term is a Maxwellian, and we can start by expanding

$$F_\varepsilon(t,x,v) = M_{\rho_\star,\theta_\varepsilon(t,x)}(v) + \varepsilon f_\varepsilon(t,x,v), \qquad \int_{\mathbb{R}^3} (1, v, |v|^2) f_\varepsilon \,\mathrm{d}v = 0$$

in the spirit of the Chapman-Enskog procedure. Here and below we denote

$$M_{\rho,\theta}(v) = \frac{\rho}{\big(2\pi\theta\big)^{3/2}} \exp\Big(-\frac{|v|^2}{2\theta}\Big).$$

By writing a relevant closure equation for the fluctuation f_ε, we will obtain an evolution equation for the approximate temperature that we denote in the sequel $\tilde{\theta}$. Indeed, we rewrite (2.1) as follows

$$\partial_t f_\varepsilon + \frac{1}{\varepsilon} v\cdot\nabla_x f_\varepsilon + \frac{1}{\varepsilon} E_\varepsilon\cdot\nabla_v f_\varepsilon + \frac{1}{\varepsilon}\Big(\partial_t + \frac{1}{\varepsilon} v\cdot\nabla_x + \frac{1}{\varepsilon} E_\varepsilon\cdot\nabla_v\Big) M_{\rho_\star,\theta_\varepsilon} = \frac{1}{\varepsilon^3} C(M_{\rho_\star,\theta_\varepsilon} + \varepsilon f_\varepsilon).$$

We can expand the right hand side, taking into account $C(M_{\rho_\star,\theta_\varepsilon}) = 0$; the leading term is

$$\frac{1}{\varepsilon^2} L_{M_{\rho_\star,\theta_\varepsilon}} f_\varepsilon.$$

We remind that integration of the equation yields, see (2.4)

$$3\rho_\star \partial_t\theta_\varepsilon + 2\nabla_x\cdot \int_{\mathbb{R}^3} v\frac{|v|^2}{2} f_\varepsilon \,\mathrm{d}v = 3\rho_\star\partial_t\theta_\varepsilon + 2\nabla_x\cdot Q_\varepsilon = 0.$$

We can now reproduce the manipulations of the previous section to obtain approximate equations for the fluctuation f_ε. Following the reasoning of the previous section we set $f_\varepsilon(t,x,v) = (\pi_{\mathbf{n}} + \varepsilon f_{\mathbf{n}}^\perp)(t,x,v)$ with

$$\pi_{\mathbf{n}}(t,x,v) = \sum_{i\in\mathbf{n}} f_i(t,x,|v|) \overset{i}{\cdot} Y_i\Big(\frac{v}{|v|}\Big), \qquad f_i(t,x,|v|) = P_i\left(f_\varepsilon(t,x,\cdot)\right)(|v|),$$
$$f_{\mathbf{n}}^\perp(t,x,v) = P_{\mathbf{n}}^\perp\left(f_\varepsilon(t,x,\cdot)\right)(v).$$

We remind that the leading part of the particle distribution function is the Maxwellian $M_{\rho_\star,\theta_\varepsilon(t,x)}$. Therefore, reproducing the manipulations of the previous Section, we are led to

$$P_{\mathbf{n}}^\perp\big(\nabla_x(v(M_{\rho_\star,\theta_\varepsilon(t,x)} + \varepsilon\pi_{\mathbf{n}}) + \tilde{E}_0\cdot\nabla_v(M_{\rho_\star,\theta_\varepsilon(t,x)} + \varepsilon\pi_{\mathbf{n}})\big) = \varepsilon P_{\mathbf{n}}^\perp L_{M_{\rho_\star,\theta_\varepsilon}} P_{\mathbf{n}}^\perp f_{\mathbf{n}}^\perp$$

which is the analog of the second equation in (**M** 2), with $\Pi_{\mathbf{n}} = M_{\rho_\star,\theta_\varepsilon(t,x)} + \varepsilon\pi_{\mathbf{n}}$. To obtain a balanced expression, we get rid of the singular term; it imposes $P_{\mathbf{n}}^\perp(v\cdot\nabla_x + E\cdot\nabla_v)M_{\rho_\star,\theta} = 0$. Since $(v\cdot\nabla_x + E\cdot\nabla_v)M_{\rho_\star,\theta}$ is proportional to Y_1 it leads to suppose

$$1 \in \mathbf{n}. \tag{4.1}$$

It is consistent with the fact that one seeks an approximation of the heat flux $Q_\varepsilon = \int_{\mathbb{R}^3} v|v|^2 f_\varepsilon \,\mathrm{d}v$ where $v|v|^2$ is proportional to Y_1, hence the integral involves f_1.

In addition we shall use analog of hypothesis (H4) for the linearized operator. We suppose that

(H7). *The linearized operator is stable on the basis* $(Y_i)_{i\in\mathbf{n}}$*: for any* $(f_i)_{i\in\mathbf{n}} \in \prod_{i\in\mathbf{n}} \left(L^2(\mathbb{R}_+)\right)^{3^i}$ *we have*

$$P_{\mathbf{n}}^{\perp} L_{M_{\rho,\theta}} \left(f_i \overset{i}{:} Y_i\right) = 0.$$

As for (H4) and (H5), these hypothesis are fulfilled when assuming the following more classical property, see [5].

(H8). *The linearized operator is isotropic in the sense that for any tensor–valued function* T *defined on* $\mathbb{S}^2$ *and any* $f \in L^2(\mathbb{R}_+)$*, there exists* $\lambda : [0,\infty) \to \mathbb{R}$ *such that*

$$L_{M_{\rho,\theta}} \left(f(|v|)\, T\left(\frac{v}{|v|}\right)\right) = \lambda(|v|)\, T\left(\frac{v}{|v|}\right).$$

Remark 4.1. This property is fulfilled by many relevant operators:
- BGK operator,
- Fokker-Planck operator in the approximation where we neglect particles with small velocity, see e.g. [17],
- Boltzmann operator, see e. g. [5, Annexe I.9, pp.95–103].

We construct an approximation of F_ε having the form $\tilde{F}_\varepsilon = M_{\rho_\star,\tilde{\theta}} + \varepsilon\pi_{\mathbf{n}} + \varepsilon^2 f_{\mathbf{n}}^{\perp}$. Owing to (4.1) and (H7), we can define the $D_{\mathbf{n}}$ approximation for the fluctuation. We are thus led to the model summarized in (**M** 3) where the unknowns are $\tilde{\theta}(t,x)$, $\{f_i(t,x,r),\ i \in \mathbf{n}\}$, $f_{\mathbf{n}}^{\perp}(t,x,v)$, $\tilde{E}(t,x)$.

$$\begin{aligned}
&\tilde{F}_\varepsilon = M_{\rho_\star,\tilde{\theta}} + \varepsilon\pi_{\mathbf{n}} + \varepsilon^2 f_{\mathbf{n}}^{\perp},\\
&3\rho_\star \partial_t\tilde{\theta} + 2\nabla_x \cdot \int_{\mathbb{R}^3} v\frac{|v|^2}{2}(\pi_{\mathbf{n}} + \varepsilon f_{\mathbf{n}}^{\perp})\,\mathrm{d}v = 0,\\
&\partial_t f_i + \frac{P_i}{\varepsilon}(\partial_t M_{\rho_\star,\tilde{\theta}}) + \frac{P_i}{\varepsilon^2}\left(v\cdot\nabla_x\tilde{F}_\varepsilon + \tilde{E}_\varepsilon\cdot\nabla_v\tilde{F}_\varepsilon\right) = \frac{P_i}{\varepsilon^3}C\left(\tilde{F}_\varepsilon\right),\\
&P_{\mathbf{n}}^{\perp} L_{M_{\rho_\star,\tilde{\theta}}} P_{\mathbf{n}}^{\perp} f_{\mathbf{n}}^{\perp} = P_{\mathbf{n}}^{\perp}\left[\partial_t M_{\rho_\star,\tilde{\theta}} + v\cdot\nabla_x\pi_{\mathbf{n}} + \tilde{E}_0\cdot\nabla_v\pi_{\mathbf{n}} - D^2C(M_{\rho_\star,\tilde{\theta}})(\pi_{\mathbf{n}},\pi_{\mathbf{n}})\right],\\
&\tilde{E}_\varepsilon(t,x) = \frac{1}{\rho_\star}\left[\nabla_x\cdot\left(\int_{\mathbb{R}^3} v\otimes v\tilde{F}_\varepsilon\,\mathrm{d}v\right) - \frac{1}{\varepsilon}\int_{\mathbb{R}^3} vC(\tilde{F}_\varepsilon)\,\mathrm{d}v\right],\\
&\tilde{E}_0(t,x) = \frac{1}{\rho_\star}\left[\nabla_x\cdot\left(\int_{\mathbb{R}^3} v\otimes vM_{\rho_\star,\tilde{\theta}}\,\mathrm{d}v\right) - \int_{\mathbb{R}^3} vL_{M_{\rho_\star,\tilde{\theta}}}(\pi_{\mathbf{n}})\,\mathrm{d}v\right]
\end{aligned}$$

M 3. $D_{\mathbf{n}}$ model for fluctuations.

We point out that the heat flux depends only on f_1 (which is of vectorial nature). Again, $f_{\mathbf{n}}^{\perp}$ is simply determined by inverting the linear operator $P_{\mathbf{n}}^{\perp} L_{M_{\rho_\star,\tilde{\theta}}} P_{\mathbf{n}}^{\perp}$, but the right hand side of this equation involves a term quadratic with respect to $\pi_{\mathbf{n}}$ through the second derivative of the collision operator.

4.1. **Intermediate hydro-kinetic models.** In this Section we take advantage of the fact that the heat flux involves the component f_1 of the spherical harmonic expansion only. Therefore, we wish to derive a relevant approximation of f_1. From now on we restrict to the simple case where we choose $\mathbf{n} = \{1\}$. For the sake of concreteness, we replace the collision operator by a BGK–like approximation defined as follows: for any $F \in L^2\left(\mathbb{R}^3\right)$ expanded on the spherical harmonic basis $F = \sum_{i\in\mathbb{N}} F_i \overset{i}{\cdot} Y_i$, we set

$$C(F) = \frac{M_{\rho_\star,\tilde\theta}}{\tau_0} - \sum_{i\in\mathbb{N}} \frac{F_i \overset{i}{\cdot} Y_i}{\tau_i}, \tag{4.2}$$

with $\tau_i(\rho_\star, \tilde\theta, |v|) \in \mathbb{R}$, the relaxation time associated to Y_i. From now on we work with this operator, which can be seen as a reasonable approximation of more realistic collision operators. In particular, it satisfies (H8). In addition, the second derivative satisfies this property too, so its contribution in (**M** 3) vanishes. Let us remark that the following identities hold:

$$\begin{aligned}
&\nabla_x \cdot \Big(v\left(f_1(|v|)\cdot Y_1(v/|v|)\right)\Big)\\
=&\sum_{i,j} r\partial_{x_i}\left(f_1\right)_j(r)\Omega_i\Omega_j\\
=&\frac{r}{3}\nabla_x \cdot f_1(r)Y_0(\Omega) + r\nabla_x f_1(r) : Y_2(\Omega),\\
&\nabla_v \cdot \Big(\tilde E_\varepsilon f_1(|v|)\cdot Y_1(v/|v|)\Big)\\
=&\sum_{i,j}\Big(\Omega_i\Omega_j\partial_r(f_1)_j(r)\tilde E_i + [\delta_{ij} - \Omega_i\Omega_j]\Big(\frac{(f_1)_j(r)}{r}\tilde E_i\Big)\Big)\\
=&\Big(\frac{\partial_r f_1(r)\cdot\tilde E_\varepsilon}{3} + \frac{2f_1(r)\cdot\tilde E_\varepsilon}{3r}\Big)Y_0(\Omega)\\
&+\Big(\partial_r f_1(r)\otimes\tilde E_\varepsilon - \frac{f_1(r)\otimes\tilde E_\varepsilon}{r}\Big) : Y_2(\Omega),
\end{aligned}$$

with $v = r\Omega$, $r = |v|$ and $\Omega = v/|v|$. Therefore, the equation for f_1 casts as

$$\partial_t f_1 + P_1\left(\nabla_x\cdot\left(vf_{\{1\}}^\perp\right) + \tilde E_\varepsilon\cdot\nabla_v f_{\{1\}}^\perp\right) + \frac{r}{\varepsilon^2}\left(\nabla_x\tilde M - \frac{\tilde M}{\tilde\theta}\tilde E_\varepsilon\right) = -\frac{f_1}{\varepsilon^2\tau_1}. \tag{4.3}$$

On the same token, the equation for $f_{\{1\}}^\perp$ reads

$$\begin{aligned}
P_{\{1\}}^\perp L_{M_{\rho_\star,\tilde\theta}}\left(f_{\{1\}}^\perp\right) = &\left(\partial_t M_{\rho_\star,\tilde\theta} + \frac{\partial_r f_1\cdot\tilde E_0}{3} + \frac{2f_1\cdot\tilde E_0}{3r} + \frac{r}{3}\nabla_x\cdot f_1\right)Y_0\\
&+\left(\partial_r f_1\otimes\tilde E_0 - \frac{f_1\otimes\tilde E_0}{r} + r\nabla_x f_1\right) : Y_2
\end{aligned} \tag{4.4}$$

where the right hand side has only components on Y_0 and Y_2. Of course, the time derivative of $M_{\rho_\star,\tilde\theta}$ can be computed by using the equation for $\tilde\theta$. Using (4.2), we arrive at

$$\begin{aligned}
&f_{\{1\}}^\perp(t,x,v) = \widetilde{f}_0(t,x,r)Y_0(\Omega) + \widetilde{f}_2(t,x,r) : Y_2(\Omega),\\
&\widetilde{f}_0 = -\tau_0\left(\frac{r}{3}\nabla_x\cdot f_1 + \frac{\tilde E_0}{3}\cdot\left(\partial_r f_1 + \frac{2f_1}{r}\right)\right.\\
&\qquad\qquad\left. - \frac{4\pi}{9\rho_\star}\left(\frac{r^2}{2\tilde\theta} - \frac{3}{2}\right)\frac{M_{\rho_\star,\tilde\theta}}{\tilde\theta}\nabla_x\cdot\int_0^\infty r^5 f_1\,\mathrm{d}r\right)\\
&\widetilde{f}_2 = -\tau_2\left(r\nabla_x f_1 + \tilde E_0\otimes\left(\partial_r f_1 - \frac{f_1}{r}\right)\right)
\end{aligned} \tag{4.5}$$

We can summarize the model we have derived so far: the approximate particle distribution function reads $\tilde{F}_\varepsilon = M_{\rho_\star,\tilde\theta} + \varepsilon f_1 \cdot Y_1 + \varepsilon^2 \left(\tilde{f}_0 Y_0 + \tilde{f}_2 : Y_2\right)$ where the unknowns are governed by the equations in (**M** 4). (Here $\overset{3}{\cdot}$ is the 3−contracted product of a tensor a order 4 with a tensor of order 3.)

$$\begin{aligned}
3\rho_\star \partial_t \tilde\theta &= -\frac{4\pi}{3}\nabla_x \cdot \int_0^\infty r^5 f_1 \,\mathrm{d}r,\\
\partial_t f_1 + \frac{f_1}{\varepsilon^2 \tau_1} &= -r\nabla_x \widetilde{f}_0 - \partial_r \widetilde{f}_0 \tilde{E}_0 - 2\widetilde{f}_2 \cdot \tilde{E}_0 - \frac{r}{\varepsilon^2}\left(\nabla_x M_{\rho_\star,\tilde\theta} - \frac{M_{\rho_\star,\tilde\theta}}{\tilde\theta}\tilde{E}_\varepsilon\right)\\
&\quad - \frac{3}{4\pi}\int_{\mathbb{S}^2} Y_2 \otimes Y_2 \,\mathrm{d}\Omega \overset{3}{\cdot} \left(r\nabla_x \widetilde{f}_2 + \tilde{E}_\varepsilon \otimes \left(\partial_r \widetilde{f}_2 - \frac{2\widetilde{f}_2}{r}\right)\right),\\
\tilde{E}_\varepsilon(t,x) &= \frac{1}{\rho_\star}\left[\frac{4\pi}{3}\nabla_x \int_{\mathbb{R}} r^4 \left(M_{\rho_\star,\tilde\theta} + \varepsilon^2 \tilde{f}_0\right)\mathrm{d}r\right.\\
&\quad + \varepsilon^2 \int_{\mathbb{S}^2} Y_2 \otimes Y_2 \,\mathrm{d}\Omega \overset{3}{\cdot} \nabla_x \int_{\mathbb{R}} r^4 \tilde{f}_2 \,\mathrm{d}r\\
&\quad \left. + \frac{4\pi}{3}\int_{\mathbb{R}} \frac{r^3 f_1(t,x,r)}{\tau_1}\,\mathrm{d}r\right],\\
\tilde{E}_0(t,x) &= \frac{1}{\rho_\star}\left[\frac{4\pi}{3}\nabla_x \int_{\mathbb{R}} r^4 M_{\rho_\star,\tilde\theta}\,\mathrm{d}r + \frac{4\pi}{3}\int_{\mathbb{R}} \frac{r^3 f_1(t,x,r)}{\tau_1}\,\mathrm{d}r\right],\\
&\quad \widetilde{f}_0 \text{ and } \widetilde{f}_2 \text{ defined by (4.5).}
\end{aligned}$$

M 4. D_1 model for fluctuations.

We still wish to derive simpler models, fully of hydrodynamic nature. Proceeding with a naive Chapman-Enskog expansion might lead to ill-posed problems. Nevertheless, we shall introduce a relevant expansion of the solution, which will lead to interesting generalization of the non-local models proposed in the physics literature. For the next step, it is convenient to introduce the change of variable $r = \sqrt{2\xi\tilde\theta(t,x)}$, and to define $\frac{2\pi}{3} r^5 f_1(t,x,r)\,\mathrm{d}r = \mathcal{Q}(t,x,\xi)\,\mathrm{d}\xi$. In other words we set

$$\mathcal{Q}(t,x,\xi) = \frac{16\pi}{3}\tilde\theta^3(t,x)\xi^2\; f_1\Big(t,x,\sqrt{2\xi\tilde\theta(t,x)}\Big). \tag{4.6}$$

The new variable ξ can be interpreted as the ratio between the kinetic energy of the particle $\frac{1}{2}r^2$ and the thermal energy $\tilde\theta$. Consequently, for the heat flux we have

$$\tilde{Q}(t,x) = \frac{2\pi}{3}\int_0^\infty r^5 f_1(t,x,r)\,\mathrm{d}r = \int_0^\infty \mathcal{Q}(t,x,\xi)\,\mathrm{d}\xi$$

while the current constraint becomes

$$\begin{aligned}
0 &= \int_{\mathbb{R}^3} v f_1(t,x,|v|)\cdot Y_1(v/|v|)\,\mathrm{d}v = \frac{4\pi}{3}\int_0^\infty r^3 f_1(t,x,r)\,\mathrm{d}r\\
&= \frac{1}{\tilde\theta(t,x)}\int_0^\infty \frac{\mathcal{Q}(t,x,\xi)}{\xi}\,\mathrm{d}\xi.
\end{aligned}$$

Let us now check the consistency of the model (**M** 4). We expand the field $\tilde{E}_\varepsilon$ as follows:

$$\tilde{E}_\varepsilon = E_0 + \varepsilon E_1 + \varepsilon^2 \tilde{\mathscr{E}}.$$

Identifying the leading order terms in the equation for the deviation f_1 leads to

$$\begin{aligned} f_1 \simeq f_{1,0} &= -r\tau_1 \left(\nabla_x M_{\rho_\star,\tilde\theta} - \frac{M_{\rho_\star,\tilde\theta}}{\tilde\theta} E_0 \right) \\ &= -r\tau_1 M_{\rho_\star,\tilde\theta} \left(\frac{\nabla_x \rho_\star}{\rho_\star} + \left(\frac{v^2}{2\tilde\theta} - \frac{3}{2} \right) \frac{\nabla_x \tilde\theta}{\tilde\theta} - \frac{E_0}{\tilde\theta} \right). \end{aligned} \tag{4.7}$$

Hence the quasi-neutrality constraint

$$\int_0^\infty r^3 f_1^0(t,x,r)\,\mathrm{d}r = 0$$

yields

$$E_0 = \tilde\theta \, \frac{\displaystyle\int_0^\infty r^4 \tau_1 e^{-r^2/(2\tilde\theta)} \left(\nabla_x \ln \rho_\star + \left(\frac{r^2}{2\tilde\theta} - \frac{3}{2} \right) \nabla_x \ln \tilde\theta \right) \mathrm{d}r}{\displaystyle\int_0^\infty r^4 \tau_1 e^{-r^2/(2\tilde\theta)}\,\mathrm{d}r}, \tag{4.8}$$

by integration of (4.7). Continuing further the expansion and identifying terms arising with the same power with respect to ε, we get

$$E_1 = 0$$

and

$$\begin{aligned} \tilde{\mathscr{E}} =& \frac{1}{\varepsilon^2} \left(\frac{\nabla_x (\rho_\star \theta_\varepsilon)}{\rho_\star} - E_0 + \frac{4\pi}{3\rho_\star} \int_{\mathbb{R}_+} \frac{r^3}{\tau_1} f_1 \,\mathrm{d}r \right) \\ &+ \frac{1}{\rho_\star} \left(\frac{4\pi}{3} \int_{\mathbb{R}_+} r^4 \nabla_x \widetilde{f}_0 \,\mathrm{d}r + \int_{\mathbb{S}^2} Y_2 \otimes Y_2 \,\mathrm{d}\Omega \vdots \int_{\mathbb{R}_+} r^4 \nabla_x \widetilde{f}_2 \,\mathrm{d}r \right). \end{aligned} \tag{4.9}$$

Coming back to the heat flux, we obtain at leading order

$$\begin{aligned} \mathcal{Q}(t,x,\xi) \simeq \mathcal{Q}_0(t,x,\xi) &= \frac{16\pi}{3} \tilde\theta^3(t,x)\ \xi^2 f_{1,0}\big(t,x,\sqrt{2\xi\tilde\theta(t,x)}\big) \\ &= -\frac{4\sqrt{2}}{3\sqrt{\pi}} \rho_\star \tilde\theta(t,x)\ \tau_1\ e^{-\xi}\ \nabla_x \tilde\theta(t,x) \\ &\quad \times \xi^{5/2} \left(\xi - \tfrac{3}{2} - \frac{\displaystyle\int_0^\infty \zeta^{3/2} \left(\zeta - \frac{3}{2} \right) \tau_1 e^{-\zeta}\,\mathrm{d}\zeta}{\displaystyle\int_0^\infty \zeta^{3/2} \tau_1 e^{-\zeta}\,\mathrm{d}\zeta} \right). \end{aligned} \tag{4.10}$$

We keep in mind in all these expressions that τ_1 depends on $\big(\rho_\star(x), \tilde\theta(t,x), \sqrt{2\xi\tilde\theta(t,x)}\big)$. If we stop the expansion at this order, we recover the classical Spitzer-Härm heat flux after integration with respect to the variable ξ:

$$3\rho_\star \partial_t \theta = 2\nabla_x \cdot \int_0^\infty \mathcal{Q}_0(t,x,\xi)\,\mathrm{d}\xi = \nabla_x \cdot (\kappa \theta^{5/2} \nabla_x \theta) \tag{4.11}$$

with κ as in (2.5). The model we wish to design now can be seen as a correction of this equation, depending on ε.

$$3\rho_\star \partial_t \tilde\theta = 2\nabla_x \cdot \int_0^\infty (\mathcal{Q}_0 + \varepsilon \tilde{\mathcal{Q}})(t,x,\xi)\,\mathrm{d}\xi = \nabla_x \cdot (\kappa \tilde\theta^{5/2} \nabla_x \tilde\theta) + 2\varepsilon \nabla_x \cdot \int_0^\infty \tilde{\mathcal{Q}}(t,x,\xi)\,\mathrm{d}\xi$$

where the corrective term, which formally appears at order $\mathscr{O}(\varepsilon^2)$, is obtained by taking into account the perturbation on the equation (4.3).

Using the notation introduced above, we can go back to (4.3), which, by using (4.6), recasts as

$$\partial_t \mathcal{Q} + \frac{\mathcal{Q}}{\varepsilon^2 \tau_1} - \mathscr{D}_x(\mathcal{Q}) + \mathscr{T}(\mathcal{Q}) - \frac{4\rho_\star \tilde{\theta}}{3\sqrt{\pi}} \xi^{5/2} e^{-\xi} \tilde{\mathscr{E}} = \frac{\mathcal{Q}_0}{\varepsilon^2 \tau_1}, \tag{4.12}$$

with $\mathscr{D}_x$ the second order differential operator defined by

$$\begin{aligned} \mathscr{D}_x(\mathcal{Q}) =& \nabla_x \left[\frac{2\tilde{\theta}}{3} \tau_0 \xi \nabla_x \cdot \mathcal{Q} - \frac{8\tilde{\theta}}{9\sqrt{\pi}} \tau_0 \xi^{5/2} \left(\xi - \frac{3}{2} \right) e^{-\xi} \nabla_x \cdot \int_0^\infty \mathcal{Q}(t,x,\zeta)\,\mathrm{d}\zeta \right] \\ &+ \nabla_x \cdot \left[\frac{3\tilde{\theta}}{2\pi} \tau_2 \xi \int_{\mathbb{S}^2} Y_2 \otimes Y_2 \,\mathrm{d}\Omega : \nabla_x \mathcal{Q} \right], \end{aligned} \tag{4.13}$$

(with : the 2−contracted product $\overset{2}{\cdot}$ between a tensor of order 4 and a matrix, as defined in Section 3) and $\mathscr{T}$ is a pseudo-differential operator with respect to the variable ξ and a first order differential operator with respect to the space variable

$$\mathscr{T}(\mathcal{Q}) = \sum_{k=0}^{1} \sum_{j=1}^{3} \partial_{x_j}^k \int_0^\infty a_j^k(t,x,\xi,\zeta) \widehat{\mathcal{Q}}(t,x,\zeta)\; e^{i\xi\zeta}\,\mathrm{d}\zeta \tag{4.14}$$

where $\widehat{\mathcal{Q}}$ stands for the Fourier transform of $\mathcal{Q}$ and the kernels a_j^k depend on the field $\tilde{E}_\varepsilon$ (the derivation of their expression is tedious and the detailed formula is unexciting).

This term in (4.14) appears as a serious numerical bottleneck because its numerical approximation would require many discretization points of the energy variable, computations of integrals, and inversion of non–sparse matrices. Motivated by numerical purposes, the model we propose is now based on the following approximations:

- We replace $\mathscr{T}(\mathcal{Q}) = \mathscr{T}(\mathcal{Q}_0 + \varepsilon\tilde{\mathcal{Q}})$ by $\mathscr{T}(\mathcal{Q}_0)$. (Note that we have already neglected some terms of order $\mathscr{O}(\varepsilon)$ in equation (4.12).) The advantage is that the dependence of $\mathscr{T}(\mathcal{Q}_0)$ with respect to ξ is simple and explicit and $\mathscr{T}(\mathcal{Q}_0)$ is only a first order differential operator with respect to the space variable, the coefficients of which can be evaluated by computing a couple of simple integrals.
- We account for this modification in the definition of the field that comes from the current constraint, bearing in mind (4.8).

We are thus led to the set of equations collected in (**M** 5).

The system (**M** 5) is still not purely hydrodynamic because it involves the energy variable ξ. The final step relies on energy-discretization.

4.2. **Towards hydrodynamic models: Energy discretization.** Since the hydrodynamic equation (4.11) involves an integral over the energy variable, we are going to approximate the heat flux by using Gaussian quadrature. Proceeding that way, we shall obtain a system of coupled hydrodynamic equations. To this end we

$$
\begin{aligned}
&3\rho_\star\partial_t\tilde{\theta}+2\nabla_x\cdot\int_0^\infty \mathcal{Q}\,\mathrm{d}\xi=0,\\
&\partial_t\mathcal{Q}+\frac{\mathcal{Q}}{\varepsilon^2\tau_1}-\mathscr{D}_x(\mathcal{Q})-\frac{4\rho_\star\tilde{\theta}}{3\sqrt{\pi}}\xi^{5/2}e^{-\xi}\tilde{\mathscr{E}}=\frac{\mathcal{Q}_0}{\varepsilon^2\tau_1}-\mathscr{T}(\mathcal{Q}_0),\\
&\tilde{\mathscr{E}}=\frac{1}{\rho_\star\tilde{\theta}}\int_0^\infty\frac{1}{\xi}\left[\frac{\mathcal{Q}-\mathcal{Q}_0}{\varepsilon^2\tau_1}-\mathscr{D}_x(\mathcal{Q})+\mathscr{T}(\mathcal{Q}_0)\right]\mathrm{d}\xi,\\
&\text{with } \mathcal{Q}_0,\mathscr{D}_x,\mathscr{T} \text{ given respectively by (4.10), (4.13) and (4.14).}
\end{aligned}
$$

M 5. Approximate D_1 model for fluctuations: formulation with the generalized heat flux.

consider $g \in \mathbb{N}\setminus\{0\}$ and for any $i\in\{1,...,g\}$, $\alpha>-1$, we define $(\xi_1,...,\xi_g)$ as to be the zeroes of the gth Laguerre generalized polynomials

$$\mathscr{L}_g^{(\alpha)}(\xi)=\sum_{i=1}^{g}(-1)^i\binom{g+\alpha}{g-i}\frac{\xi^i}{i!}.$$

We shall see below how the parameter α can be chosen depending on the collision operator. Therefore, assuming enough regularity on the function $\xi\mapsto\mathcal{Q}(\xi)$ we have the following Gaussian quadrature formula

$$\int_0^\infty \mathcal{Q}(\xi)\,\mathrm{d}\xi=\sum_{i=1}^{g}\omega_i\mathcal{Q}(\xi_i)+\frac{g!\Gamma(g+\alpha+1)}{(2g)!}\partial_\xi^{2g}\left(\mathcal{Q}(\xi)\xi^{-\alpha}e^{\xi}\right)(c),$$

with $c\geq 0$ and weight ω_i associated to the point ξ_i defined by

$$\omega_i=\frac{\Gamma(g+\alpha)\xi_i}{g!(g+\alpha)\left(\mathscr{L}_{g-1}^{(\alpha)}(\xi_i)\right)^2}\xi_i^{-\alpha}e^{\xi_i}.$$

Based on this formula we are going to use a discrete model where only evaluations of $\mathcal{Q}$ at the quadrature points appear, replacing $\xi-$integrals by the corresponding weighted sums. Implicitly, using such a discretization presumes the regularity of the solution $\mathcal{Q}(t,x,\xi)$. Considering regimes close to the Spitzer-Härm regime, where the dynamics is determined by a (non–linear) diffusion equation, such a regularity assumption might look reasonable (we also refer to the regularity analysis of solutions close to equilibrium [6]). Furthermore, we are interested in a macroscopic description, which means that we accept to disregard the local details of the behavior of the solution with respect to the energy variable. Eventually, we write the hydrodynamic model by considering (**M** 5) at the quadrature points ξ_i, see (**M** 6) below.

Let us discuss the choice of the quadrature weight. To this end, let us assume that

(H9). *The coefficient τ_1 is given by*

$$\tau_1\left(\rho_\star,\tilde{\theta},r=\sqrt{2\xi\tilde{\theta}}\right)=\bar{\tau}_1\frac{r^3}{\rho_\star}=\bar{\tau}_1\frac{\left(2\tilde{\theta}\xi\right)^{3/2}}{\rho_\star},$$

with $\bar{\tau}_1>0$, a constant.

$$3\rho_\star\partial_t\tilde{\theta}+2\nabla_x\cdot\sum_{j=1}^{g}\omega_j\mathcal{Q}_j=0,$$

$$\tilde{\mathscr{E}}=\frac{1}{\rho_\star\tilde{\theta}}\sum_{j=1}^{g}\frac{\omega_j}{\xi_j}\left[\frac{\mathcal{Q}_j-\mathcal{Q}_0(\xi_j)}{\varepsilon^2\mu_j}-\mathscr{D}_j\left((\mathcal{Q}_k)_{k\in[\![1,g]\!]}\right)+\mathscr{T}_j\left(-\tilde{\kappa}\nabla_x\tilde{\theta}\right)\right],$$

$$\partial_t\mathcal{Q}_j+\frac{\mathcal{Q}_j}{\varepsilon^2\mu_j}-\mathscr{D}_j\left((\mathcal{Q}_k)_{k\in[\![1,g]\!]}\right)-\frac{4\rho_\star\tilde{\theta}}{3\sqrt{\pi}}\xi_j^{5/2}e^{-\xi_j}\tilde{\mathscr{E}}=-\frac{\kappa_j}{\varepsilon^2\mu_j}\nabla_x\tilde{\theta}-\mathscr{T}_j\left(-\tilde{\kappa}\nabla_x\tilde{\theta}\right),$$

$$\mathscr{D}_j\left((\mathcal{Q}_k)_{k\in[\![1,g]\!]}\right)=\nabla_x\left[\nu_j^1\nabla_x\cdot\mathcal{Q}_j-\nu_j^2\nabla_x\cdot\sum_{k=1}^{g}\omega_k\mathcal{Q}_k\right]+\nabla_x\cdot\left[\nu_j^3\int_{\mathbb{S}^2}Y_2\otimes Y_2\,\mathrm{d}\Omega:\nabla_x\mathcal{Q}_j\right],$$

$$\mathscr{T}_j(\mathcal{Q})=\sum_{k=0}^{1}\sum_{l=1}^{3}\partial_{x_l}^k\int_0^\infty a_l^k(t,x,\xi_j,\zeta)\mathcal{Q}(t,x,\zeta)\;e^{i\xi_j\zeta}\,\mathrm{d}\zeta,$$

with for any $j\in[\![1,g]\!]$, $\mathcal{Q}_j(t,x)=\mathcal{Q}(t,x,\xi_j)$,

$$\kappa_j(t,x)=\frac{4\sqrt{2}}{3\sqrt{\pi}}\rho_\star(x)\tilde{\theta}(t,x)\tau_1(t,x,\xi_j)\xi_j^{5/2}\times\left(\xi_j-\frac{3}{2}-\frac{\displaystyle\int_0^\infty\zeta^{3/2}\left(\zeta-\frac{3}{2}\right)\tau_1(t,x,\zeta)e^{-\zeta}\,\mathrm{d}\zeta}{\displaystyle\int_0^\infty\zeta^{3/2}\tau_1(t,x,\zeta)e^{-\zeta}\,\mathrm{d}\zeta}\right)e^{-\xi_j},$$

$$\mu_j(t,x)=\tau_1(t,x,\xi_j),$$

$$\nu_j^1(t,x)=\frac{2\tilde{\theta}(t,x)}{3}\xi_j\tau_0(t,x,\xi_j),$$

$$\nu_j^2(t,x)=\frac{8\tilde{\theta}(t,x)}{9\sqrt{\pi}}\xi_j^{5/2}\left(\xi_j-\frac{3}{2}\right)e^{-\xi_j}\tau_0(t,x,\xi_j),$$

$$\nu_j^3(t,x)=\frac{3\tilde{\theta}(t,x)}{2\pi}\xi_j\tau_2(t,x,\xi_j).$$

M 6. Hydrodynamic system based on energy discretization and quadrature approximation.

This assumption is physically relevant, in particular as far as we are concerned with the homogeneity with respect to the energy variable. Indeed, we remind that, coming back to physical quantities, the relaxation time for ion-electron collision has the form $\tau_{ei}=\frac{m^2}{4\pi c^4\ln\Lambda}\frac{r^3}{\rho_\star}$, with m, the electron mass, c, the electron charge and $\ln\Lambda$, the Coulomb logarithm. Within this framework (4.10) becomes

$$\mathcal{Q}_0=-\tilde{\kappa}(\xi)\nabla_x\tilde{\theta},\qquad \tilde{\kappa}(t,x,\xi)=\frac{16\bar{\tau}_1}{3\sqrt{\pi}}\xi^4\left(\xi-4\right)e^{-\xi}\tilde{\theta}^{5/2}(t,x).\tag{4.15}$$

Similarly, the electric field (4.8) reads

$$\tilde{E}_0 = \tilde{\theta}\left(\nabla_x \ln(\rho_\star) + \frac{5}{2}\nabla_x \ln(\tilde{\theta})\right).$$

With these formulae we can compute the Spitzer-Härm flux

$$Q_{SH} = \int_0^\infty \mathcal{Q}_0 \,\mathrm{d}\xi = -\kappa \nabla_x \tilde{\theta},$$

where

$$\kappa(t,x) = \int_0^\infty \tilde{\kappa}(t,x,\xi)\,\mathrm{d}\xi = \frac{128\bar{\tau}_1}{\sqrt{\pi}}\tilde{\theta}^{5/2}(t,x).$$

Then, we can distinguish two relevant choices for the parameter α :

- From a numerical point of view, we want the quadrature formula as exact as possible for a reduced number of discretization points. According to (4.15) and with $\alpha = 4$, we recover the Spitzer-Härm flux when $g \geq 1$.
- From a physical point of view, we would like to preserve the quasi-neutrality constraint (1.3). Bearing in mind the relation

  $$\rho_\star u = \frac{1}{\tilde{\theta}}\int_0^\infty \frac{\mathcal{Q}}{\xi}\,\mathrm{d}\xi,$$

  at the leading order we are led to

  $$0 = \int_0^\infty \frac{\mathcal{Q}_0}{\xi}\,\mathrm{d}\xi = \frac{16\bar{\tau}_1}{3\sqrt{\pi}}\int_0^\infty \xi^3\,(\xi - 4)\,e^{-\xi}\,\mathrm{d}\xi\ \tilde{\theta}^{5/2}\nabla_x\tilde{\theta}.$$

  With $\alpha = 3$, we recover the constraint when $g \geq 1$. However, since we also wish to recover the Spitzer-Härm heat flux, we will need $g \geq 2$.

5. **Nonlocal (Schurtz-Nicolaï) model revisited.** The purpose of this Section is two–fold. First, we shall see that (**M** 6) generalizes the non-local model introduced by Schurtz and Nicolaï [16]. Second, adopting further simplifications, we will discuss some mathematical properties of the model, that can be reasonably expected to extend to the complete system.

5.1. **The Schurtz-Nicolaï model.** The Schurtz-Nicolaï system is obtained from (**M** 6) with the following manipulations:

a) The field deviation $\tilde{\mathscr{E}}$ is neglected,
b) The operator $\mathscr{T}$ is disregarded,
c) The second order operator $\mathscr{D}_x$ is replaced by a mere Laplacian,
d) The time derivative is neglected in the equation for the generalized fluxes $\mathcal{Q}_j$.

Therefore, we are led to (**M** 7), with $\kappa_j(t,x) = \tilde{\kappa}(t,x,\xi_j)$ and ν_j a coefficient coming from the approximation of $\mathscr{D}_x$.

$$3\rho_\star\partial_t\tilde{\theta} + 2\nabla_x \cdot \sum_{j=1}^{g}\omega_j\mathcal{Q}_j = 0,$$
$$\mathcal{Q}_j - \varepsilon^2\tilde{\nu}_j\Delta_x\mathcal{Q}_j = -\kappa_j\nabla_x\tilde{\theta}.$$

M 7. Schurtz-Nicolaï model.

This model has been studied in the specific case $g = 1$ in [4] where well-posedness and maximum principle are established. A numerical scheme is also proposed which preserves these fundamental properties of the model. However, the model (**M** 7) suffers several drawbacks. Since the current is defined as $\sum_{i=1}^{g} \frac{\omega_i \mathcal{Q}_i}{\xi_i}$, the quasi-neutrality constraint cannot be satisfied with only one energy group ($g = 1$) in the case $\alpha = 4$, or the model lead to the trivial relation $\mathcal{Q}_1 = 0$ in the case $\alpha = 3$. However, dealing with $g > 1$ quadrature points might lead to ill-posed problems since $\widetilde{\kappa}(\xi)$ takes negative values as it has been argued in view of severe instabilities reported in numerical simulations, see [11], [16, Section III.C], [2, Section III].

For this reason, Schurtz and Nicolaï have suggested to replace $\widetilde{\kappa}$ by $\frac{16\bar{\tau}_1}{3\sqrt{\pi}}\xi^4 e^{-\xi}\tilde{\theta}^{5/2}$. This quantity is non negative and its integral over $(0, \infty)$ coincides with the integral of $\tilde{\kappa}$. This rough approximation looks highly questionable. In particular the quasineutrality constraint is not satisfied whatever the choice of the integration method and on a physical point of view, this approximation means that all particles are moving with the same orientation.

5.2. **Nonlocal models with flux defined by evolution equations.** The path we propose consists in going back to the system (**M** 6), where we use the simplifications a) and b), but we keep the time derivative in the flux equation. In order to match better with the derivation, we also slightly generalize the simplified second order operator which approaches $\mathscr{D}_x$ in c). It leads to (**M** 8), where we have set $\kappa_j(t,x) = \tilde{\kappa}(t,x,\xi_j)$. We shall see that this is crucial to restore the expected properties of the model.

$$
\begin{aligned}
3\rho_\star \partial_t \tilde{\theta} + 2\nabla_x \cdot \sum_{j=1}^{g} \omega_j \mathcal{Q}_j &= 0, \\
\partial_t \mathcal{Q}_j + \frac{\mathcal{Q}_j}{\varepsilon^2 \mu_j} - \nabla_x \left(\nu_j \nabla_x \cdot \mathcal{Q}_j \right) &= -\frac{\kappa_j}{\varepsilon^2 \mu_j} \nabla_x \tilde{\theta}.
\end{aligned}
$$

M 8. Hydrodynamic model with evolution equation on the heat flux.

To start with, we observe that the model has remarkable dissipation properties.

Proposition 5.1. *Assume that for any $j \in [\![1, g]\!]$, the coefficients μ_j, ν_j and κ_j are positive and constant. We introduce the following entropy functional $\mathcal{H} = \frac{1}{2}\left(\frac{3\rho_\star}{2}\tilde{\theta}^2 + \sum_{j=1}^{g} \frac{\omega_j \mu_j}{\kappa_j} |\mathcal{Q}_j|^2\right)$. This quantity is dissipated by the system (**M** 8)*

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \mathcal{H} \, \mathrm{d}x \leq 0
$$

Proof. We have

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \mathcal{H} \, \mathrm{d}x = \int_{\mathbb{R}^3} \left[\frac{3\rho_\star}{2} \tilde{\theta} \partial_t \tilde{\theta} + \sum_{j=1}^{g} \frac{\omega_j \mu_j}{\kappa_j} \mathcal{Q}_j \cdot \partial_t \mathcal{Q}_j \right] \mathrm{d}x.
$$

Using ($\mathbf{M}$ 8), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\mathcal{H}\,\mathrm{d}x &= -\sum_{j=1}^{g}\int_{\mathbb{R}^3}\Big[\theta\nabla_x\cdot(\omega_j\mathcal{Q}_j)+\nabla_x\theta\cdot(\omega_j\mathcal{Q}_j)\\ &\qquad +\frac{\omega_j}{\kappa_j}|\mathcal{Q}_j|^2-\frac{\omega_j\mu_j\nu_j}{\kappa_j}\mathcal{Q}_j\nabla_x(\nabla_x\cdot\mathcal{Q}_j)\Big]\,\mathrm{d}x.\end{aligned}$$

Integrating by part yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\mathcal{H}\,\mathrm{d}x = -\sum_{j=1}^{g}\omega_j\int_{\mathbb{R}^3}\left[\frac{1}{\kappa_j}|\mathcal{Q}_j|^2+\frac{\mu_j\nu_j}{\kappa_j}|\nabla_x\cdot\mathcal{Q}_j|^2\right]\mathrm{d}x\le 0.$$

□

However, the hypothesis of Proposition (5.1) are not satisfied with the coefficients $\kappa_j=\widetilde{\kappa}(t,x,\xi_j)$ obtained according to the derivation detailed above. Indeed for small ξ, the function $\xi\mapsto\widetilde{\kappa}(\xi)$ is non positive. In what follows we are interested in the well-posedness and stability issues for ($\mathbf{M}$ 8), in the one-dimension framework. For the sake of simplicity we suppose that the coefficients κ_j,ν_j,μ_j and $\rho_\star$ are constant. The coefficients κ_j are not supposed to be non negative, but we assume that they satisfy

$$\sum_{j=1}^{g}\omega_j\kappa_j>0. \tag{5.1}$$

This property is natural because it corresponds to the fact that the integral of the function $\xi\mapsto\tilde{\kappa}(\xi)$ is positive. It turns out that (5.1) is the crucial property guaranteeing the well–posedness of the Spitzer-Härm regime. We start by rewriting ($\mathbf{M}$ 8) as the following reaction, advection, diffusion system

$$\partial_t X+RX+A\partial_x X-D\partial_x^2 X=0, \tag{5.2}$$

where the unknown is the vector–valued function $X(t,x)=\big(\tilde{\theta}(t,x),\mathcal{Q}_1(t,x),...,\mathcal{Q}_g(t,x)\big)\in\mathbb{R}^{g+1}$ and the matrices R,A,D are defined by

$$R=\left(\begin{array}{c|cccc}0&0&\dots&0\\\hline \vdots&\ddots&0&0\\0&0&\dfrac{1}{\varepsilon^2\mu_j}&0\\\vdots&0&0&\ddots\end{array}\right),\qquad A=\left(\begin{array}{c|ccc}0&\dots&\dfrac{2\omega_j}{3\rho_\star}&\dots\\\hline \vdots&0&\dots&0\\\dfrac{\kappa_j}{\varepsilon^2\mu_j}&\vdots&\ddots&\vdots\\\vdots&0&\dots&0\end{array}\right),$$

and

$$D=\left(\begin{array}{c|ccc}0&\dots&0&\dots\\\hline \vdots&\ddots&0&0\\0&0&\nu_j&0\\\vdots&0&0&\ddots\end{array}\right).$$

Let us introduce a few notations which are needed to derive a well–posedness statement. We set $\alpha_0^0 = 1$, $\beta_0^0 = 0$,

$$\begin{aligned}
&\forall i \in [\![1, g]\!],\ \forall m \in [\![0, i]\!], && \alpha_i^m = \sum_{\substack{k_1,\dots,k_i \\ k_\alpha \neq k_\beta}}^{[\![1,g]\!]} \left[\prod_{l=1}^{m} \nu_{k_l}\right] \left[\prod_{l=m+1}^{i} \frac{1}{\varepsilon^2 \mu_{k_l}}\right], \\
&\forall m \in [\![0, g+1]\!], && \alpha_{g+1}^m = 0, \\
&\forall i \in [\![1, g+1]\!], && \beta_i^0 = 0, \qquad \beta_i^i = 0,
\end{aligned} \tag{5.3}$$

and

$$\forall i \in [\![1, g+1]\!],\ \forall m \in [\![1, i-1]\!],\ \beta_i^m = \sum_{\substack{k_1,\dots,k_{i-1} \\ k_\alpha \neq k_\beta}}^{[\![1,g]\!]} \frac{2\omega_{k_{i-1}} \kappa_{k_{i-1}}}{3\varepsilon^2 \rho_\star \mu_{k_{i-1}}} \left[\prod_{l=1}^{m-1} \nu_{k_l}\right] \left[\prod_{l=m}^{i-2} \frac{1}{\varepsilon^2 \mu_{k_l}}\right]. \tag{5.4}$$

We next define the sequence $\gamma_n^m(i)$ by

- for any $i \in [\![0, \lfloor \frac{g+1}{2} \rfloor]\!]$, $m \in [\![0, M_0(i) = 2i]\!]$, we set $\gamma_0^m(i) = \alpha_{2i}^m + \beta_{2i}^m$,
- for any $i \in [\![0, \lfloor \frac{g}{2} \rfloor]\!]$, $m \in [\![0, M_1(i) = 2i+1]\!]$, we set $\gamma_1^m(i) = \alpha_{2i+1}^m + \beta_{2i+1}^m$,
- and finally, for any $n \in [\![0, g-1]\!]$, $i \in [\![0, \lfloor \frac{g-1-n}{2} \rfloor]\!]$,
  $m \in [\![0, M_{n+2}(i) = \max(M_n(0) + M_{n+1}(i+1), M_n(i+1) + M_{n+1}(0))]\!]$, we have the iteration formula

$$\gamma_{n+2}^m(i) = \sum_{\substack{l_1=0 \\ l_2+l_1=m}}^{M_{n+1}} \sum_{l_2=0}^{M_n} \begin{vmatrix} \gamma_{n+1}^{l_1}(0) & \gamma_{n+1}^{l_1}(i+1) \\ \gamma_n^{l_2}(0) & \gamma_n^{l_2}(i+1) \end{vmatrix}. \tag{5.5}$$

Theorem 5.1. *We assume that*

(H10). *For any $n \in [\![0, g+1]\!]$, and $m \in [\![0, M_n(0)]\!]$, the coefficients $\gamma_n^m(0)$ are non negative and there exists an index $m \in [\![0, M_n(0)]\!]$ such that $\gamma_n^m(0) > 0$,*

(H11). *The coefficients μ_i, $i \in [\![1, g]\!]$, are positive, pairwise distinct, and there holds $\sum_{i=1}^{g} \omega_i \kappa_i > 0$,*

(H12). *The coefficients ν_i, $i \in [\![1, g]\!]$, are positive, pairwise distinct, and there holds $\sum_{i=1}^{g} \frac{\omega_i \kappa_i \nu_i}{\varepsilon^2 \mu_i} > 0$.*

Then, (5.2) is well-posed in $L_2(\mathbb{R})$: there exists a constant $C > 0$ such that, for any initial data $X_0 \in \big(L_2(\mathbb{R})\big)^{g+1}$, the problem (5.2) has a unique solution $X = \big(\tilde{\theta}, \mathcal{Q}_1, \dots, \mathcal{Q}_g\big) \in C^0\big([0,\infty), \big(L_2(\mathbb{R})\big)^{g+1}\big)$ which verifies the following stability estimate for any $t \geq 0$

$$\|X(t)\|_{L^2} \leq C \|X_0\|_{L^2}.$$

Furthermore, $\lim_{t\to\infty} \|X(t)\|_{L^2} = 0$.

The proof is inspired from arguments introduced by Liu and Zeng [9, Chapter 6], even though it is definitely not a direct application of Theorem 6.2 in [9]. Indeed, the system (5.2) does not admit a symmetric form: we cannot find a symmetric matrix S such that SA and SD are symmetric. By using the Fourier Transform, the solution of (5.2) can be cast as

$$X(t,x) = \mathscr{F}^{-1}\big(\Gamma(t,\xi) \mathscr{F} X(0)\big)$$

where

$$\Gamma(t,\xi) = \exp(-t(R + i\xi A + \xi^2 D)).$$

Therefore the stability estimate will appear as a consequence of the spectral properties of the matrix $\Lambda(\xi) = -R - i\xi A - \xi^2 D$. To this end, we remind the following classical statements.

Lemma 5.1 (Lyapounov stability estimate)**.** *Let $\mathcal{A}$ be a bounded subset of $\mathcal{M}_n(\mathbb{C})$. Pick $\delta > 0$. We suppose that for any $A \in \mathcal{A}$ and for any eigenvalue λ of A, we have $\mathrm{Re}(\lambda) \le -\delta$. Then, there exists $c > 0$ such that for any $t > 0$ and any $A \in \mathcal{A}$,*

$$\|e^{tA}\| \le \frac{1}{c} e^{-ct}.$$

Lemma 5.2 (Routh-Hurwitz stability criterion)**.** *Let P be a polynomial function with real coefficients. Let Q be a polynomial function the zeroes of which are the sum of two zeroes of P. The coefficients of P and Q are positive iff the real part of the zeroes of P is negative.*

The proof of this claim can be found in [15]. We shall use these basic statements to investigate the spectral properties of the system (5.2). We consider the characteristic polynomial of the matrix $\Lambda(\xi)$

$$\begin{aligned} P_g(\lambda,\xi) &= \det\left(\lambda\mathbb{I} + R + i\xi A + \xi^2 D\right) \\ &= \begin{vmatrix} \lambda & i\dfrac{2\omega_1}{3\rho_\star}\xi & \dots & \dots & i\dfrac{2\omega_g}{3\rho_\star}\xi \\ i\dfrac{\kappa_1}{\varepsilon^2\mu_1}\xi & \lambda + \dfrac{1}{\varepsilon^2\mu_1} + \nu_1\xi^2 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ i\dfrac{\kappa_g}{\varepsilon^2\mu_g}\xi & 0 & \dots & 0 & \lambda + \dfrac{1}{\varepsilon^2\mu_g} + \nu_g\xi^2 \end{vmatrix}. \end{aligned}$$

This determinant can be evaluated by expanding from the last line and the last column. We obtain the recursion formula

$$P_g(\lambda,\xi) = \left(\lambda + \frac{1}{\varepsilon^2\mu_g} + \nu_g\xi^2\right) P_{g-1}(\lambda,\xi) + \frac{2\omega_g\kappa_g}{3\varepsilon^2\rho_\star\mu_g}\xi^2 \prod_{k=1}^{g-1}\left[\lambda + \frac{1}{\varepsilon^2\mu_k} + \nu_k\xi^2\right].$$

Observe that $P_1(\lambda,\xi) = \lambda^2 + \left(\frac{1}{\varepsilon^2\mu_1} + \nu_1\xi^2\right)\lambda + \frac{2\omega_1\kappa_1}{3\varepsilon^2\rho_\star\mu_1}\xi^2$. Then, we are going to discuss the behavior of the eigenvalues depending whether $|\xi|$ is large or small while an estimate can be derived for the intermediate regimes.

Lemma 5.3. *Let assumption (H11) in Theorem 5.1 be fulfilled. Then, $\Lambda(0) = -R$ has $g+1$ distinct eigenvalues: 0 and g real and negative eigenvalues. When $\xi \ne 0$ lies in a sufficiently small neighborhood of 0, $\xi \mapsto \Lambda(\xi)$ admits $g+1$ distinct eigenvalues, which are holomorphic functions of ξ; all those eigenvalues have a negative real part and the eigenvalue which tends to 0 as ξ goes to 0 is asymptotically equivalent to $-\frac{2}{3\rho_\star}\sum_{j=1}^{g}\omega_j\kappa_j\xi^2$. More precisely, in this neighborhood of 0 we have*

$$\Gamma(t,\xi) = e^{-\frac{2}{3\rho_\star}\sum_{j=1}^{g}\omega_j\kappa_j\xi^2 t + \mathscr{O}(\xi^3 t)}\mathcal{P}_0(\xi) + \sum_{j=1}^{g} e^{-\frac{t}{\varepsilon^2\mu_j} + \mathscr{O}(\xi t)}\mathcal{P}_j(\xi),$$

where $\mathcal{P}_j(\xi)$ are the eigenprojectors of $\Lambda(\xi)$.

Proof. Let us set $P_g(\lambda,\xi) = \sum_{k=0}^{g+1} a_k^g(\xi)\lambda^{g+1-k}$. The induction formula for P_g yields an induction formula for the a_k^g's. More precisely we have

$$a_0^g = a_0^{g-1}$$

$$a_1^g = a_1^{g-1} + a_0^{g-1}\left(\frac{1}{\varepsilon^2\mu_g} + \nu_g\xi^2\right)$$

$$a_2^g = a_2^{g-1} + a_1^{g-1}\left(\frac{1}{\varepsilon^2\mu_g} + \nu_g\xi^2\right) + \frac{2\omega_g\kappa_g}{3\varepsilon^2\rho_\star\mu_g}\xi^2$$

$$a_i^g = a_i^{g-1} + a_{i-1}^{g-1}\left(\frac{1}{\varepsilon^2\mu_g} + \nu_g\xi^2\right) + \frac{2\omega_g\kappa_g}{3\varepsilon^2\rho_\star\mu_g}\xi^2 \sum_{\substack{k_1,\dots,k_{i-2}\\ k_\alpha\neq k_\beta}}^{[\![1,g-1]\!]} \prod_{l=1}^{i-2}\left[\frac{1}{\varepsilon^2\mu_{k_l}} + \nu_{k_l}\xi^2\right] \quad \forall i\in[\![3,g]\!],$$

$$a_{g+1}^g = a_g^{g-1}\left(\frac{1}{\varepsilon^2\mu_g} + \nu_g\xi^2\right) + \frac{2\omega_g\kappa_g}{3\varepsilon^2\rho_\star\mu_g}\xi^2\prod_{l=1}^{g-1}\left[\frac{1}{\varepsilon^2\mu_l} + \nu_l\xi^2\right],$$

with $a_0^1 = 1$, $a_1^1 = \frac{1}{\varepsilon^2\mu_1} + \nu_1\xi^2$ and $a_2^1 = \frac{2\omega_1\kappa_1}{3\varepsilon^2\rho_\star\mu_1}\xi^2$. We seek an expression of a_i^g as a polynomial with respect to the variable ξ. We can check that a_i^g is defined for any $i\in[\![1,g+1]\!]$ by $a_i^g = \sum_{m=0}^{i}(\alpha_i^m + \beta_i^m)\xi^{2m}$ and (5.3)– (5.4) satisfies the induction formula. Then, for $|\xi|\ll 1$, we seek an expansion of the zeroes $\lambda(\xi)$ of $\lambda\mapsto P_g(\lambda,\xi)$ as $\lambda(\xi) = \sum_{i=0}^{\infty}\lambda_i\xi^i$:

- Of course, we have

$$P_g(\lambda,0) = \sum_{i=0}^{g}\alpha_i^0\lambda^{g+1-i} = \lambda\sum_{i=1}^{g}\sum_{\substack{k_1,\dots,k_i\\ k_\alpha\neq k_\beta}}^{[\![1,g]\!]}\prod_{l=1}^{i}\frac{1}{\varepsilon^2\mu_{k_l}}\lambda^{g-i} = \lambda\prod_{i=1}^{g}\left(\lambda + \frac{1}{\varepsilon^2\mu_i}\right).$$

  Therefore, as $\xi\to 0$ it is clear that the eigenvalues of $\Lambda(\xi)$ tend to the eigenvalues of $-R$ which are $\{0, -\frac{1}{\varepsilon^2\mu_1},\dots,-\frac{1}{\varepsilon^2\mu_g}\}$. It defines the leading term λ_0. However, because 0 belongs to the spectrum of R, we need to describe more precisely the asymptotic behavior of the eigenvalues $\lambda(\xi)$ for small ξ's.
- At first order, we have

$$\lim_{\xi\to 0}\xi^{-1}P_g(\lambda(\xi),\xi) = \left(\alpha_g^0 + \beta_g^0\right)\lambda_1 = \prod_{i=1}^{g}\frac{1}{\varepsilon^2\mu_i}\lambda_1 = 0.$$

  Thus we get $\lambda_1 = 0$.
- At next order, we have

$$\begin{aligned}\lim_{\xi\to 0}\xi^{-2}P_g(\lambda(\xi),\xi) &= \left(\alpha_g^0 + \beta_g^0\right)\lambda_2 + \left(\alpha_{g+1}^1 + \beta_{g+1}^1\right)\\ &= \prod_{i=1}^{g}\frac{1}{\varepsilon^2\mu_i}\lambda_2 + \frac{2}{3\rho_\star}\sum_{j=1}^{g}\frac{\omega_j\kappa_j}{\varepsilon^2\mu_j}\prod_{\substack{i=1\\ i\neq j}}^{g}\frac{1}{\varepsilon^2\mu_i}.\end{aligned}$$

  Thus we get $\lambda_2 = -\frac{2}{3\rho_\star}\sum_{j=1}^{g}\omega_j\kappa_j$, which is negative by (5.1). □

Lemma 5.4. *Let assumption (H12) in Theorem 5.1 be fulfilled. There exists $K>0$ such that for any ξ verifying $|\xi|\geq K$, $\Lambda(\xi)$ admits $g+1$ distinct eigenvalues, which are holomorphic functions of ξ, all having a negative real part: the smallest* *eigenvalue is asymptotically equivalent to* $\frac{2}{3\rho_\star}\sum_{j=1}^g \frac{\omega_j\kappa_j\nu_j}{\varepsilon^2\mu_j}$ *while the others behave like* $-\nu_j\xi^2$*. More precisely, for* $|\xi|\geq K$ *we have*

$$\Gamma(t,\xi) = e^{-\frac{2}{3\rho_\star}\sum_{j=1}^g \frac{\omega_j\kappa_j\nu_j}{\varepsilon^2\mu_j}t+\mathscr{O}(t/\xi)}\mathcal{P}_0(\xi) + \sum_{j=1}^g e^{-\nu_j\xi^2 t+\mathscr{O}(\xi t)}\mathcal{P}_j(\xi),$$

where $\mathcal{P}_j(\xi)$ *are the eigenprojectors of* $\Lambda(\xi)$*.*

Proof. We use the notation introduced in the proof of Lemma 5.3. We are now interested in the asymptotic behavior for large ξ's. As $|\xi|$ goes to infinity we remark that $\Lambda(\xi)$ is equivalent to $-\xi^2 D$. Since 0 is a (simple) eigenvalue of D we need to investigate more precisely the asymptotic behavior of the spectrum of $\Lambda(\xi)$ in order to prevent the occurrence of an eigenvalue with positive real part. We remark that, as a function of ξ, $P_g(\lambda,\xi)$ involves only even powers of ξ, with highest degree $2g$. We have

$$\lim_{|\xi|\to\infty}\xi^{-2g}P_g(\lambda,\xi) = \left(\alpha_g^g+\beta_g^g\right)\lambda + \left(\alpha_{g+1}^g+\beta_{g+1}^g\right) = \prod_{i=1}^g \nu_i\lambda + \frac{2}{3\rho_\star}\sum_{j=1}^g \frac{\omega_j\kappa_j}{\varepsilon^2\mu_j}\prod_{\substack{i=1\\ i\neq j}}^g \nu_i.$$

It determines the behavior of the last eigenvalue of $\Lambda(\xi)$ as to given by $\lambda_\infty = -\frac{2}{3\rho_\star}\sum_{j=1}^g \frac{\omega_j\kappa_j\nu_j}{\varepsilon^2\mu_j} < 0$ when $|\xi|$ goes to ∞. □

It remains to study the behavior of the spectrum for intermediate $|\xi|$.

Lemma 5.5. *Let assumption (H10) in Theorem 5.1 be fulfilled. Then, for any* $\bar\xi>0$ *there exists* $\bar\lambda>0$ *such that for any* $\bar\xi\leq|\xi|\leq\frac{1}{\bar\xi}$*, the eigenvalues* $\lambda(\xi)$ *of* $\Lambda(\xi)$ *satisfy*

$$\mathrm{Re}\left(\lambda(\xi)\right)\leq-\bar\lambda$$

Proof. This is a direct application of Lemma 5.2. For any $\xi\neq0$, the coefficients of the polynomial function $\lambda\mapsto P_g(\lambda,\xi)$ are real positive. We are led to construct a polynomial Q the zeroes of which are the sum of two zeroes of $P_g(\cdot,\xi)$. Such a construction is well–established, and we shall use a practical criterion. Let us introduce for any $n\in[\![0,g+1]\!]$ and $i\in[\![0,\lfloor\frac{g+1-n}{2}\rfloor]\!]$

$$R_n(i)(\xi) = \frac{\displaystyle\sum_{m=0}^{M_n(i)}\gamma_n^m(i)\xi^{2m}}{\displaystyle\prod_{k=0}^{n-1}\left(R_k(0)(\xi)\right)^{F_{n-k}}},$$

where the $F(j)$'s are the Fibonacci numbers (defined by the recursion $F_0=0$, $F_1=1$ and $F_{n+2}=F_n+F_{n+1}$) and γ_n^m and M_n are given in (5.5). The coefficients R_n are the elements of the Routh array (see [15]). Then assumption (H10) implies that all $g+2$ elements $R_n(0)$, $n\in[\![0,g+1]\!]$, of the first column of the Routh array are positive. We deduce that for any $\xi\neq0$ fixed, the zeroes of $\lambda\mapsto P_g(\lambda,\xi)$ have negative real parts. We conclude by a continuity argument. □

Then we are ready to end the proof of Theorem 5.1.

Proof of Theorem 5.1. We start by using Lemma 5.3 and Lemma 5.4: we can find $0<\tilde\lambda_0\leq\frac{2}{3\rho_\star}\sum_{j=1}^g\omega_j\kappa_j$, $0<\tilde\lambda_\infty\leq\frac{2}{3\rho_\star}\sum_{j=1}^g\frac{\omega_j\kappa_j\nu_j}{\varepsilon^2\mu_j}$ and $\bar\xi>0$ such that for any $|\xi| \le \bar{\xi}$ (resp. $|\xi| \ge \frac{1}{\bar{\xi}}$) $\Lambda(\xi)$ is diagonalizable and the eigenvalues satisfy $\operatorname{Re}\lambda(\xi) \le -\tilde{\lambda}_0\xi^2$ (resp. $\operatorname{Re}\lambda(\xi) \le -\tilde{\lambda}_\infty$). In addition, by using Lemma 5.5 and the Lyapounov stability Lemma 5.1, there exists $\tilde{\lambda} > 0$ satisfying for any $\bar{\xi} < |\xi| < \frac{1}{\bar{\xi}}$, $|\Gamma(t,\xi)| \le \frac{1}{\tilde{\lambda}} e^{-\tilde{\lambda}t}$ (but for such ξ the matrix $\Lambda(\xi)$ might admit Jordan blocks). Then

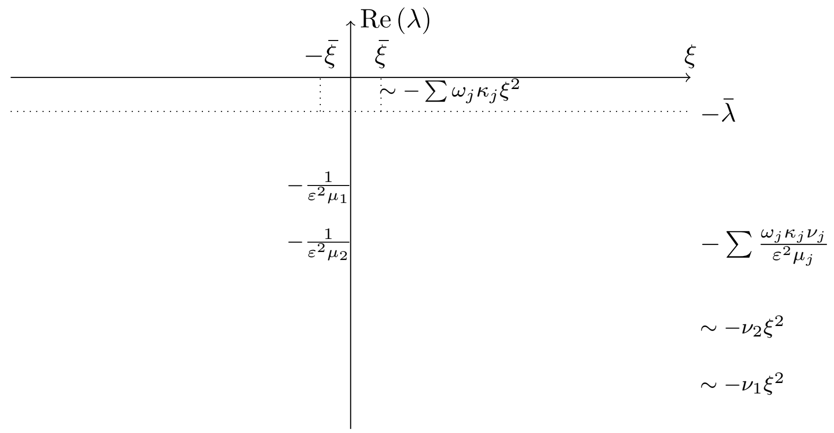

FIGURE 1. Example of real part of eigenvalues of Λ in function of ξ for two quadrature points.

we shall use these estimates to evaluate the Green function

$$\mathscr{G}(t,x) = \frac{1}{2\pi} \lim_{r\to\infty} \int_{-r}^{r} \Gamma(t,\xi) e^{i\xi x}\, \mathrm{d}\xi,$$

the limit being understood in the Schwartz class $\mathscr{S}'(\mathbb{R})$. We introduce the following approximate Green kernel

$$\mathscr{G}^\star(t,x) = \frac{1}{\sqrt{4\pi \sum_{j=1}^{g} \omega_j\kappa_j t}} e^{-\frac{x^2}{4\sum_{j=1}^{g}\omega_j\kappa_j t}} \mathcal{P}_0(0) + e^{-\frac{2}{3\rho_\star}\sum_{j=1}^{g}\frac{\omega_j\kappa_j\nu_j}{\varepsilon^2\mu_j}t} \delta_{x=0} \lim_{\zeta\to\infty} \mathcal{P}_0(\zeta).$$

It corresponds, by using the Inverse Fourier Transform, to the leading terms of Γ, associated to the eigenvalue 0. It remains to evaluate the remainder $\mathscr{R} = \mathscr{G} - \mathscr{G}^\star$. Following [9, Chapters 5– 6, sp. Lemma 5.5, 5.6 & 5.7], we show that the remainder is a function and there exists $C > 0$ such that for any $x \in \mathbb{R}$ and $t > 0$,

$$|\mathscr{R}(t,x)| \le C\,(t+1)^{-\frac{1}{2}}\, t^{-\frac{1}{2}} e^{-\frac{x^2}{Ct}}.$$

We conclude by writing the solution as a convolution product between the Green function and the initial condition. Qualitatively, as time becomes large, the solution mainly behaves like the solution of the heat equation with diffusion coefficient $\sum_{j=1}^{g} \omega_j\kappa_j$. □

Remark 5.1. The set of assumptions in Theorem 5.1 might look quite complicated. However, we can readily perform a computer-assisted check-up to determine whether or not the model is well-posed. For $g = 1$ we need to assume $(\kappa_1, \mu_1, \nu_1) \in \left(\mathbb{R}_+^*\right)^3$.

For $g = 2$ the Theorem applies when assuming $\mu_{1,2} \in \left(\mathbb{R}_+^*\right)^2$ with $\mu_1 \neq \mu_2$, $\nu_{1,2} \in \left(\mathbb{R}_+^*\right)^2$ with $\nu_1 \neq \nu_2$, and $\kappa_{1,2} \in \mathbb{R}^2$, with

$$\begin{cases} \omega_1\kappa_1 + \omega_2\kappa_2 > 0, \\ \dfrac{\omega_1\kappa_1\nu_1}{\mu_1} + \dfrac{\omega_2\kappa_2\nu_2}{\mu_2} > 0, \\ \dfrac{2}{3\rho_\star}\left(\dfrac{\omega_1\kappa_1}{\mu_1^2} + \dfrac{\omega_2\kappa_2}{\mu_2^2}\right) \geq -\dfrac{\mu_1\nu_1 + \mu_2\nu_2 + 2\varepsilon^2\mu_1\mu_2\left(\nu_1+\nu_2\right)}{\varepsilon^2\mu_1^2\mu_2^2}. \end{cases}$$

Anyway, assuming $\sum_{i=1}^{g} \omega_i\kappa_i > 0$ is very natural because it corresponds to the condition of well-posedness of the asymptotic model obtained by letting ε go to 0. It is also consistent with the definition of the classical diffusion coefficient in the Spitzer-Härm model. The coefficients $(\mu_i)_{i\in[\![1,g]\!]}$ are values of the isotropic relaxation time evaluated at the Gauss-Laguerre quadrature points. Thus they are naturally positive.

6. **Numerical schemes for the reduced models.** We propose a numerical scheme to solve simplified models as introduced in Section 5. The simulation will validate the interest of the new evolution models we have proposed in the previous Section. Dealing with the one dimension framework and with a single energy quadrature point, the model (**M** 7) has been numerically investigated in [4], where stability issues of the proposed scheme are discussed in details. Here we wish to study on numerical grounds the models (**M** 7) or (**M** 8) in the case where we consider many energy quadrature points. By contrast to the situation investigated in [4], it is likely there is no maximum principle for (**M** 8). Hence, we do not care of the L^∞ stability of the scheme. According to the modeling viewpoint, we are interested in solutions that verify the constraint (1.3). In discrete form, it casts as

$$\sum_{j=1}^{g} \frac{\omega_j \mathcal{Q}_j}{\xi_j} = 0, \tag{6.1}$$

and the scheme will be addressed to satisfy this constraint. Therefore, we need to reintroduce a field $\mathscr{E}$, and we are finally led to simulate the system

$$3\rho_\star\partial_t\theta + 2\sum_{j=1}^{g}\omega_j\nabla_x\cdot\mathcal{Q}_j = S(t,x), \tag{6.2}$$

$$\partial_t\mathcal{Q}_j + \frac{\mathcal{Q}_j}{\varepsilon^2\mu_j} - \nabla_x\left(\nu_j\nabla_x\cdot\mathcal{Q}_j\right) - \eta_j\mathscr{E} = -\frac{\kappa_j}{\varepsilon^2\mu_j}\nabla_x\theta, \tag{6.3}$$

with $\mathscr{E}$ associated to the constraint (6.1). Defining physically relevant initial data $(\theta^0, Q_1^0, ..., Q_g^0)$ for such a reduced model is far from clear, because there is no natural initialization for the generalized heat flux. In order to observe relevant evolution, we can start form a global equilibrium where θ^0 is constant and $\mathcal{Q}_j^0 = 0$, but the system is perturbed by the action of the source term $S(t,x) \neq 0$ in (6.2).

6.1. **Discretization.** Here we adopt the Finite Difference point of view. Hence we work on a cartesian homogeneous grid: we assume that the space variable is d-dimensional, we denote by $h_i > 0$ the space step in the i-th direction and we set $\mathbf{H} = \text{diag}(h_1, ..., h_d)$, $h_x = \max(h_1, ..., h_d)$. Let $h_t > 0$ be the time step. Given $\mathbf{k} = (k_1, ..., k_d) \in \mathbb{Z}^d$, the unknowns $\theta_{\mathbf{k}}^n$ are intended to be approximations of $\theta(nh_t, \mathbf{Hk})$. Since the density $\rho_\star$ is given, we denote $\rho_{\mathbf{k}} = \rho_\star(\mathbf{Hk})$. Following the ideas introduced in [4], the generalized heat fluxes are evaluated on a staggered grid, see Fig. 2: let $\mathbf{e}^i$ be the ith element of the canonic basis in $\mathbb{R}^d$, then for any $j \in \{1, ..., g\}$, $Q_{j,i}\big|^n_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}$ is intended to be an approximation of the ith component of the vector $Q_j \in \mathbf{R}^d$, evaluated at time nh_t and position $\mathbf{H}(\mathbf{k} + \frac{1}{2}\mathbf{e}^i)$. Dealing with such a regular grid in certainly not appropriate for physically realistic simulations where, due to the combination with mesh refinements strategies, unstructured meshes are used. We will go back to the design of adapted Finite Volume schemes for such kind of situations elsewhere [14].

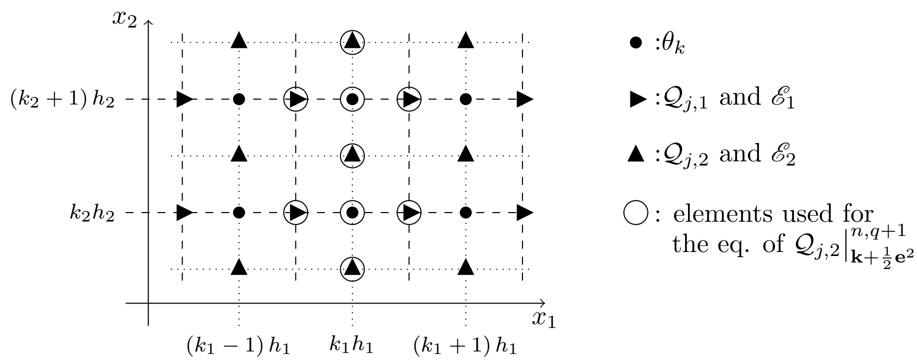

FIGURE 2. Staggered grids in dimension 2 ($\mathcal{Q}_j = (\mathcal{Q}_{j,1}, \mathcal{Q}_{j,2})$).

It is convenient to introduce the following operators:

- grad^D acts on scalar unknowns and returns a flux quantity: given $\big(\psi_{\mathbf{k}}\big)_{\mathbf{k}\in\mathbb{Z}^d}$, we define the vector with components

$$\text{grad}^D_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\,(\psi) = \frac{\psi_{\mathbf{k}+\mathbf{e}^i} - \psi_{\mathbf{k}}}{h_i},$$

  It is intended to approximate the gradient $\nabla_x \psi\big(\mathbf{H}(\mathbf{k} + \frac{1}{2}\mathbf{e}^i)\big)$.
- div^D acts of discrete fluxes and returns a scalar quantity:

$$\big(\phi_{1,\mathbf{k}+\frac{1}{2}\mathbf{e}^1}, ..., \phi_{d,\mathbf{k}+\frac{1}{2}\mathbf{e}^d}\big)_{\mathbf{k}\in\mathbb{Z}^d} \in \mathbf{R}^d \longmapsto \text{div}^D_{\mathbf{k}}\,(\phi) = \sum_{i=1}^{d} \frac{\phi_{i,\mathbf{k}+\frac{1}{2}\mathbf{e}^i} - \phi_{i,\mathbf{k}-\frac{1}{2}\mathbf{e}^i}}{h_i}.$$

  It is intended to approximate the divergence $\nabla_x \cdot \phi(\mathbf{H}\mathbf{k})$.

In this framework, given $0 \le \tau \le 1$, (6.2) is approximated by

$$\theta^{n+1}_{\mathbf{k}} = \theta^n_{\mathbf{k}} - \frac{2h_t}{3\rho_{\mathbf{k}}} \sum_{j=1}^{g} \omega_j \text{div}^D_{\mathbf{k}}\left(\tau \mathcal{Q}^n_j + (1-\tau)\mathcal{Q}^{n+1}_j\right) + h_t S^{n+1}_{\mathbf{k}}. \tag{6.4}$$

Similarly, the evolution of the generalized flux obeys, in discrete form,

$$\begin{aligned}&\frac{\mathcal{Q}_{j,i}\big|^{n+1}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}-\mathcal{Q}_{j,i}\big|^{n}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}}{h_t}+\frac{\tau\mathcal{Q}_{j,i}\big|^{n+1}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}+(1-\tau)\,\mathcal{Q}_{j,i}\big|^{n}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}}{\varepsilon^2\mu_j^{n+\tau}\big|_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}}\\&\quad-\mathrm{grad}^D_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\left(\nu_j^{n+\tau}\mathrm{div}^D\left(\tau\mathcal{Q}_j^{n+1}+(1-\tau)\,\mathcal{Q}_j^n\right)\right)-\eta_j\big|^{n+\tau}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\mathscr{E}_i\big|^{n+\tau}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\\&\quad=-\frac{\kappa_j\big|^{n+\tau}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}}{\mu_j\big|^{n+\tau}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}}\mathrm{grad}^D_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\left(\tau\theta^{n+1}+(1-\tau)\,\theta^n\right).\end{aligned}\tag{6.5}$$

Eventually the field $\mathscr{E}\big|^{n,n+1}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}$ has to be defined so that the following discrete form of the constraint (6.1):

$$\sum_{j=1}^{g}\frac{\omega_j\mathcal{Q}_{j,i}\big|^{n+1}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}}{\xi_j}=0\tag{6.6}$$

is fulfilled. With $\tau=0$ the scheme is explicit. But, according to the analysis in [4] we are led in this case to a stability constraint of parabolic type, $h_t=\mathscr{O}(h_x^2)$ which makes the computational cost prohibitive, especially for multi-dimensional simulations. Hence, using $0<\tau\leq 1$ makes the scheme implicit and improves the stability properties. The coefficients in (6.5) are thus defined as follows. First we define by interpolation the temperature on the staggered grid, by setting

$$\theta^n_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}=\frac{\theta^n_{\mathbf{k}}+\theta^n_{\mathbf{k}+\mathbf{e}^i}}{2}.$$

Then, we set

$$\begin{aligned}\mu_j\big|^{n+\tau}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}&=\mu_j\left(\tau\theta^{n+1}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}+(1-\tau)\,\theta^n_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\right),\\\kappa_j\big|^{n+\tau}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}&=\kappa_j\left(\tau\theta^{n+1}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}+(1-\tau)\,\theta^n_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\right),\\\eta_j\big|^{n+\tau}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}&=\eta_j\left(\tau\theta^{n+1}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}+(1-\tau)\,\theta^n_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\right),\\\nu_j\big|^{n+\tau}_{\mathbf{k}}&=\nu_j\left(\tau\theta^{n+1}_{\mathbf{k}}+(1-\tau)\,\theta^n_{\mathbf{k}}\right).\end{aligned}$$

In order to treat the non linear system (6.4)-(6.6), we proceed iteratively. Knowing the numerical unknowns at time nh_t, we set $\left(\theta^{n,0}_{\mathbf{k}},\mathcal{Q}_{j,i}\big|^{n,0}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\right)=\left(\theta^n_k,\mathcal{Q}_{j,i}\big|^{n}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\right)$.

Then, we define the following coefficients

$$
\begin{aligned}
&\theta\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i} = \frac{\theta^{n,q}_{\mathbf{k}} + \theta^{n,q}_{\mathbf{k}+\mathbf{e}^i}}{2},\\
&\mu_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i} = \mu_j\left(\tau\theta\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i} + (1-\tau)\,\theta^n_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\right),\\
&\kappa_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i} = \kappa_j\left(\tau\theta\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i} + (1-\tau)\,\theta^n_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\right),\\
&\eta_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i} = \eta_j\left(\tau\theta\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i} + (1-\tau)\,\theta^n_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\right),\\
&\nu_j\big|^{n,q}_{\mathbf{k}} = \nu_j\left(\tau\theta\big|^{n,q}_{\mathbf{k}} + (1-\tau)\,\theta^n_{\mathbf{k}}\right),\\
&\tilde{\mu}_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i} = \frac{\tau h_t \mu_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}}{\varepsilon^2\mu_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i} + \tau h_t},\\
&\tilde{\kappa}_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i} = \frac{\tau h_t \kappa_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}}{\varepsilon^2\mu_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i} + \tau h_t},\\
&\tilde{r}_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i} = \frac{\varepsilon^2\mu_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i} - (1-\tau)\,h_t}{\varepsilon^2\mu_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i} + \tau h_t}.
\end{aligned}
\tag{6.7}
$$

Now, we define $\left(\mathcal{Q}_{j,i}\big|^{n,q+1}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}, \mathscr{E}_i\big|^{n,q+1}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\right)$ as to be the solution of the *linear* system

$$
\begin{aligned}
&\mathcal{Q}_{j,i}\big|^{n,q+1}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i} - \varepsilon^2\tilde{\mu}_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\mathrm{grad}^D_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\left(\nu^{n,q}_j \mathrm{div}^D \mathcal{Q}_j\big|^{n,q+1}\right)\\
&\quad -\frac{2}{3}\tilde{\kappa}_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\tau h_t\ \mathrm{grad}^D_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\left(\frac{1}{\rho}\mathrm{div}^D\sum_{l=1}^{g}\omega_l\mathcal{Q}_l\big|^{n,q+1}\right)\\
&\quad -\varepsilon^2\tilde{\mu}_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\eta_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\mathscr{E}_i\big|^{n,q+1}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\\
&\qquad = -\frac{\tilde{\kappa}_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}}{\tau}\mathrm{grad}^D_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\left(\theta^n + \tau h_t S^{n+1}\right) + \tilde{r}_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\mathcal{Q}_{j,i}\big|^{n}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\\
&\qquad\quad +\varepsilon^2\tilde{\mu}_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\frac{1-\tau}{\tau}\mathrm{grad}^D_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\left(\nu^{n,q}_j\mathrm{div}^D\mathcal{Q}_j\big|^{n}\right)\\
&\qquad\quad +\frac{2}{3}\tilde{\kappa}_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}(1-\tau)\,h_t\ \mathrm{grad}^D_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\left(\frac{1}{\rho}\mathrm{div}^D\sum_{l=1}^{g}\omega_l\mathcal{Q}_l\big|^{n}\right),\\
&\sum_{j=1}^{g}\frac{\omega_j\mathcal{Q}_j\big|^{n,q+1}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}}{\xi_j} = 0.
\end{aligned}
\tag{6.8}
$$

Note that, unfortunately, this linear system is non-symmetric in general. Finally the temperature is obtained by solving

$$
\theta^{n,q+1}_{\mathbf{k}} = \theta^n_{\mathbf{k}} - \frac{2h_t}{3\rho_{\mathbf{k}}}\sum_{l=1}^{g}\omega_l\mathrm{div}^D_{\mathbf{k}}\left(\tau\mathcal{Q}_l\big|^{n,q+1} + (1-\tau)\,\mathcal{Q}_l\big|^{n}\right) + h_t S^{n+1}_{\mathbf{k}}. \tag{6.9}
$$

The updated quantities $\left(\theta^{n+1}_{\mathbf{k}}, \mathcal{Q}_j\big|^{n+1}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}, \mathscr{E}_i\big|^{n+1}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\right)$, solution of the non-linear system (6.4)–(6.6) are seen as the limit when $q\to\infty$ of $\left(\theta^{n,q}_{\mathbf{k}}, \mathcal{Q}_j\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}, \mathscr{E}_i\big|^{n,q}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}\right)$. In practice, we stop the iteration when the relative error between the two last iterates is below a certain threshold. Usually, a few iterations are enough to construct the fixed point. The performances of the scheme are discussed below through a series of numerical experiments in the one dimension framework.

Remark 6.1. The method might appear as too costly for the purposes of industrial simulations, because we have to solve a coupled system of $g+1$ vectorial equations (fluxes and field) at each position. A relevant simplification consists in uncoupling the definition of the heat fluxes. To this end we can make use of Jacobi methods where coupling terms are treated explicitly: only terms involving the current energy group are treated implicitly. It greatly reduces the complexity of the underlying linear system to be solved when updating the fluxes. In this approach, the constraint needs to be evaluated explicitly. Summing over the quadrature points in (6.8) after having multiplied it by $\frac{\omega_j}{\xi_j}$, yields an expression of $\mathscr{E}\big|^{n,q+1}_{\mathbf{k}+\frac{1}{2}\mathbf{e}^i}$. Then, by inserting it in the implicit equations of the generalized heat flux we write g linear systems of vectorial equations (fluxes) at each position. These linear systems are independent, so the resolution could easily be parallelized. Eventually, we estimate the temperature using (6.9). The mean field perturbation does not need to be estimated at each iteration. This adaptation has been introduced in the code `Aladin` developed by CEA/DAM: two-dimensional simulations and further comparisons with a kinetic P_1 code (see (**M** 1)) look promising but modeling efforts are still necessary to define correctly many physical parameters.

6.2. **Asymptotic consistency.** We start by investigating the behavior of the scheme (6.7)–(6.9) as the parameter ε vary. The test case we propose is quite close to the simulation performed in [4]. The problem is set on the slab $(0,1)$ endowed with periodic boundary conditions. As an initial condition we consider $\theta^0(x) = f(x-[x])$ with, for any $x \in (0,1)$

$$\begin{aligned} f(x) \quad = \quad & \frac{3}{2} + \frac{1}{2}\sin\left(\frac{2x}{2\delta}\frac{\pi}{2}\right)\mathbb{1}_{[0,\frac{1}{2}-\delta]} \\ & +\frac{1}{2}\sin\left(\frac{1-2x}{2\delta}\frac{\pi}{2}\right)\mathbb{1}_{]\frac{1}{2}-\delta,\frac{1}{2}+\delta[} + \frac{1}{2}\sin\left(\frac{2\,(x-1)}{2\delta}\frac{\pi}{2}\right)\mathbb{1}_{[\frac{1}{2}+\delta,1]} \end{aligned}$$

and $\delta = 0.01$. The generalized heat flux vanish initially $\mathcal{Q}_j^0(x) = 0$ (note that the initialisation of $\mathcal{Q}_j$ is physically not realistic) and we do not bring any energy to the system $S(t,x) = 0$. We assume an homogeneous density $\rho_\star = 1$. The number of energy groups is $g = 5$, and $\alpha = 3$. The relaxation times are defined as $\tau_0 = 5.10^2$ and τ_1 is given by (H9) with $\bar{\tau}_1 = 1$. We use the scheme (6.7)–(6.9) with $\tau = 1$ (fully implicit version), the space step $h_1 = 1/500$, and the time step $h_t = 10^{-6}$.

Formally, as ε goes to 0 the system (6.1)–(6.3) tends to the the Spitzer-Härm equation. As ε goes to infinity, the flux equation degenerates to

$$\partial_t \mathcal{Q}_j - \nabla_x\left(\nu_j \nabla_x \cdot \mathcal{Q}_j\right) - \eta_j \mathscr{E} = 0,$$

Initializing the generalized heat flux by $\mathcal{Q}_j^0(x) = 0$, we are led to the trivial solution is $\mathcal{Q}_j(t,x) = 0$.
Fig. 3 shows the solutions of (6.7)–(6.9), at a fixed time, for several values of ε. As ε decreases we observe that the discontinuity smears out, and the solution tends to the one given by the Spitzer-Härm model. This numerical experiment confirms that the model is consistent with the Spitzer-Härm regime as $\varepsilon \to 0$.

6.3. **Simulations of relaxation.** Here we wish to compare, for a fixed value of ε (here $\varepsilon = 10^{-4}$), the solutions provided by several models we have discussed so far. Precisely, we compare the solution of (6.1)-(6.3) to the solution of the model (**M** 7). To this end, we use the same scheme than for solving (6.7)–(6.9) but we drop the terms corresponding to the time derivative, as well as the terms associated to the constraint. Furthermore, since the problem is unstable (and likely ill-posed) when the coefficients $\tilde{\kappa}$ take negative values, we use the correction proposed in (5.1). Fig.

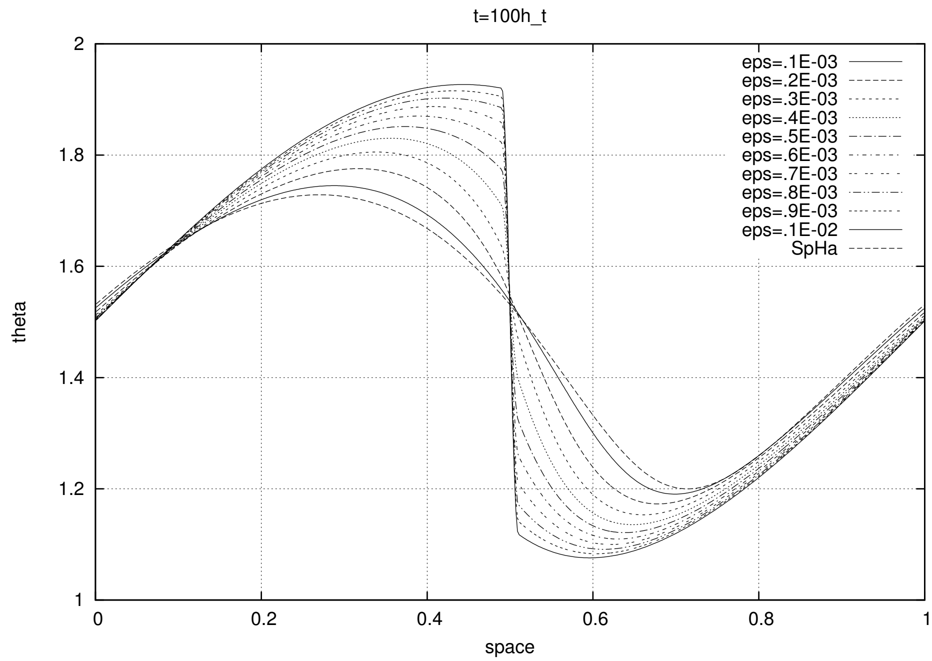

FIGURE 3. Asymptotic consistency: Temperature profile for different values of ε at time $t = 10^{-4}$.

6.3 displays the solutions, compared to the Spitzer-Härm model that corresponds to the asymptotic case $\varepsilon = 0$.

On this simulation, the temperature profiles are very similar. As said above, the initial condition is not physically realistic but its impact on the results becomes negligible after a few time steps. Indeed, the profiles of the heat flux become similar as time increases. Nevertheless the initial temperature profile is constructed so that the temperature gradient varies brutally, which brings out the delocalization effects: the non-local models capture the so–called "antidiffusive effect", as presented in [2]: for time $t = h_t$ and $t = 10h_t$, the direction of the heat flux is not necessarily dictated by the temperature gradient and $Q \cdot \nabla_x \theta$ might change sign. However, with the model (**M** 7) the current does not vanish: on the contrary, it is far from negligible in the area of delocalization where we observe discrepancies in the temperature profiles provided by the different models. By contrast (6.1)–(6.3) preserves a vanishing current due to the numerical constraint (6.6).

6.4. **Simulation of laser beam.** The next simulation is physically more relevant: we start from the global equilibrium, $\theta^0 = 1$ and $\mathcal{Q}_j^0 = 0$, but we bring locally some energy in the system through the source term $S \neq 0$. More precisely we set

$$S(t, x) = 10^5 \exp\left(-\frac{|x - 1/2|^2}{10^{-3}}\right) \mathbb{1}_{t \leq 2.10^{-6}}.$$

This source could be interpreted as a simple modeling of a laser beam in the plan orthogonal to the x direction. The other parameters are the same as in the previous simulations. As shown in Fig. 6.4, the temperature profiles are steeper with the non-local models, especially for short times. We also observe sensible differences

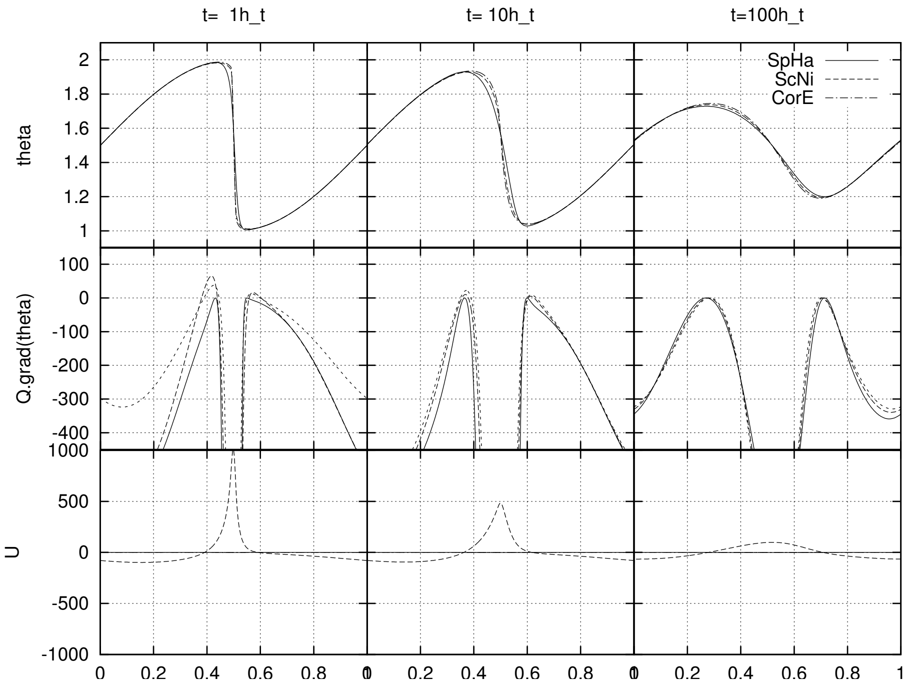

Figure 4. Relaxation test case. SpHa curves are performed with the Spitzer-Härm model, ScNi curves are performed with the Schurtz-Nicolaï model (**M** 7) and CorE curves are performed with the scheme (6.7)–(6.9): Temperature profiles (Up), scalar product between the heat flux and the temperature gradient (Middle), and current profiles (Bottom).

in the amplitude of the heat fluxes. Note also that the model (6.1)–(6.3) produces steeper profiles than (**M** 7). Like for the relaxation test case, the current does not vanish with (**M** 7) whereas it is exactly zero for (6.1)–(6.3).

Conclusion. We have revisited the derivation of the Spitzer-Härm regime in plasma physics and bring out a hierarchy of intermediate models. In particular, we have introduced a new class of non–local models which generalizes the Schurtz–Nicolaï models. By defining the generalized heat fluxes with evolution equations, we have shown that well–posedness and stability properties of the model are restored, without introducing spurious truncation of the coefficients. Furthermore, the physical constraint on the current can be readily incorporated in the system. A numerical scheme of Finite Difference type has been introduced to solve the non-local models, and we have checked on numerical grounds the consistency of the new model with the Spitzer-Härm regime and its ability in capturing anti–diffusive effects.

Acknowledgments. This work is supported by CEA/DAM. We thank J.-F. Clouet, D. Deck and J.-F. Coulombel for kind encouragements and fruitful advices.

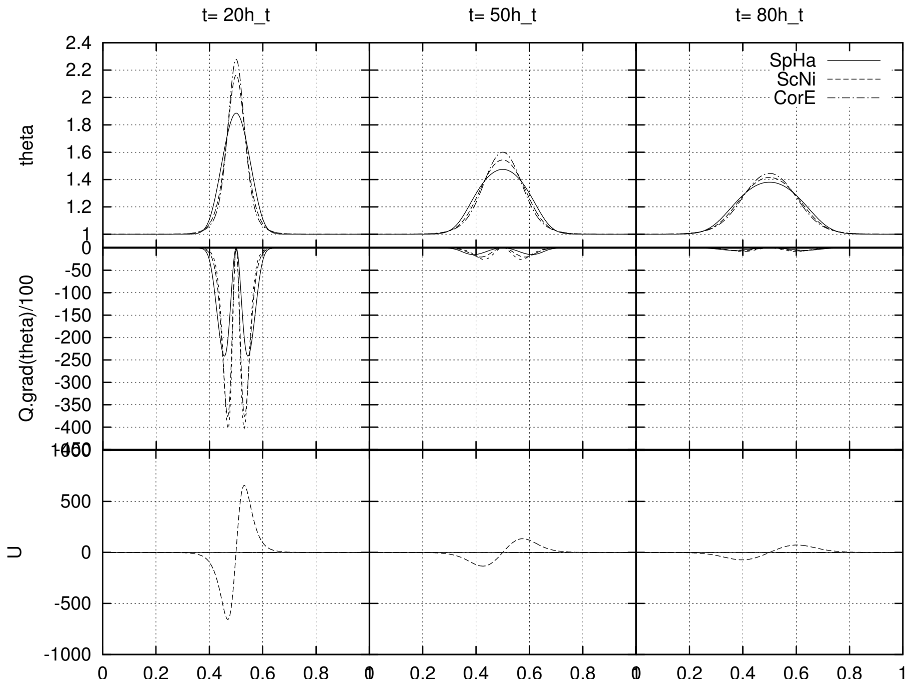

FIGURE 5. Simulation of a laser beam. `SpHa` curves are performed with the Spitzer-Härm model, `ScNi` curves are performed with the Schurtz-Nicolaï model (**M** 7) and `CorE` curves are performed with the scheme (6.7)–(6.9): Temperature profiles (Up), scalar product between the heat flux and the temperature gradient (Middle), and current profiles (Bottom).

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Received March 2011; revised June 2011.

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